TWO FUNCTIONS IN NUMBER THEORY AND SOME UPPER BOUNDS FOR THE SMARANDACHE'S FUNCTION

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The aim of this article is to introduce two functions and to give some simple properties for one of them. The function's properties are studied in connection with the prime numbers. Finally, these functions are applied to obtain some inequalities concerning the Smarandache's function.

1. Introduction

In this section, the main results concerning the Smarandache and Euler's functions are review. Smarandache proposed [1980] a function $S : N^* \rightarrow N$ defined by $S(n) = \min\{k \mid k! \vdots n\}$. This function satisfies the following main equations:

1. $(n,m) = 1 \Rightarrow S(n \cdot m) = \min\{S(n), S(m)\}$

   (1)

2. $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_s^{k_s} \Rightarrow S(n) = \min\{S(p_1^{k_1}), S(p_2^{k_2}), \ldots, S(p_s^{k_s})\}$

   (2)

3. $(\forall\, n > 1)\ S(n) \leq n$

   (3)

and the equality in the inequality (3) is obtained if and only if n is a prime number. The research on the Smarandache's function has been carried out in several directions. One of these direction studies the average function $\overline{S} : N^* \rightarrow N$ defined by $\overline{S}(n) = \frac{\sum_{i=1}^{n} S(i)}{n}$. Tabirca [1997] gave the following two upper bounds for this function $(\forall\, n > 5)\ \overline{S}(n) \leq \frac{3}{8} \cdot n + \frac{1}{4} - \frac{2}{n}$ and $(\forall\, n > 23)\ \overline{S}(n) \leq \frac{21}{72} \cdot n + \frac{1}{12} - \frac{2}{n}$ and conjectured that $(\forall\, n > 1)\ \overline{S}(n) \leq \frac{2 \cdot n}{\ln}$.

Let $\varphi : N^* \rightarrow N$ be the Euler function defined by $\varphi(n) = card\{k = \overline{1,n} \mid (k,n) = 1\}$. The main properties of this function are review below:

1. $(n,m)=1 \Rightarrow \varphi(n\cdot m)=\varphi(n)\cdot\varphi(m)$

(4)

2. $n = p_1^{k_1}\cdot p_2^{k_2}\cdots p_s^{k_s}\cdot \Rightarrow \varphi(n) = n\cdot\prod_{i=1}^{s}(1-\frac{1}{p_i})$

(5)

3. $\varphi(\frac{n}{m}) = card\{k = \overline{1,n} | (k,n) = m\}$.

(6)

It is known that if $f : N^* \to N$ is a multiplicative function then the function $g : N^* \to N$ defined by $g(n) = \sum_{d|n} f(d)$ is multiplicative as well.

2. The functions ψ_1, ψ_2

In this section two functions are introduced and some properties concerning them are presented.

Definition 1.

Let ψ_1, ψ_2 be the functions defined by the formulas

1. $\psi_1 : N^* \to N$, $\psi_1(n) = \sum_{i=1}^{n}\frac{n}{(i,n)}$

(7)

2. $\psi_2 : N^* \to N$, $\psi_2(n) = \sum_{i=1}^{n}\frac{i}{(i,n)}$.

(8)

i	$\psi_1(i)$	$\psi_2(i)$	I	$\psi_1(i)$	$\psi_2(i)$	i	$\psi_1(i)$	$\psi_2(i)$
1	1	1	11	111	56	21	301	151
2	3	2	12	77	39	22	333	167
3	7	4	13	157	79	23	507	254
4	11	6	14	129	65	24	301	151
5	21	11	15	147	74	25	521	261
6	21	11	16	171	86	26	471	236

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7	43	22	17	273	137	27	547	274
8	43	22	18	183	92	28	473	237
9	61	31	19	343	172	29	813	407
10	63	32	20	231	116	30	441	221

Table 1. Table of the functions ψ_1, ψ_2

Remarks 1.

1. These function are correctly defined based on the implication $\frac{n}{(i,n)}, \frac{i}{(i,n)} \in N \Rightarrow \sum_{i=1}^{n} \frac{n}{(i,n)}, \sum_{i=1}^{n} \frac{i}{(i,n)} \in N$.
2. If p is prime number, then the equations $\psi_1(p) = p^2 - p + 1$ and $\psi_2(p) = \frac{p(p-1)}{2} + 1$ can be easy verified.
3. The values of these functions for the first 30 natural numbers are shown in Table 1. From this table, it is observed that the values of ψ_1 are always odd and moreover the equation $\psi_2(n) = \left\lceil \frac{\psi_1(n)}{2} \right\rceil$ seems to be true.

Proposition 1 establishes a connection between ψ_1 and φ.

Proposition 1

If $n>0$ is an integer number, then the equation

$$\psi_1(n) = \sum_{d|n} d \cdot \varphi(d) \tag{9}$$

holds.

Proof

Let $A_d = \{i = \overline{1,n} | (i,n) = d\}$ be the set of the elements which satisfy $(i,n)=d$.

The following transformations of the function ψ_1 holds.

$$\psi_1(n)=\sum_{i=1}^{n}\frac{n}{(i,n)}=\sum_{d|n}\sum_{i\in A_d}\frac{n}{(i,n)}=\sum_{d|n}\sum_{i\in A_d}\frac{n}{d}=\sum_{d|n}\frac{n}{d}\sum_{i\in A_d}1=\sum_{d|n}\frac{n}{d}\cdot|A_d|. \qquad (10)$$

Using (6) the equation (10) gives $\psi_1(n)=\sum_{d|n}\frac{n}{d}\cdot\varphi\left(\frac{n}{d}\right)$.

Changing the index of the last sum, the equation (9) is found true.♣

The function $g(n)=n\varphi(n)$ is multiplicative resulting in that the function $\psi_1(n)=\sum_{d|n}d\cdot\varphi(d)$ is multiplicative. Therefore, it is sufficiently to find a formula for $\psi_1(p^k)$, where p is a prime number.

Proposition 2.

If p is a prime number and $k\geq 1$ then the equation

$$\psi_1(p^k)=\frac{p^{2k+1}+1}{p+1} \qquad (11)$$

holds.

Proof

The equation (11) is proved based on a direct computation, which is described below.

$$\psi_1(p^k)=\sum_{d|p^k}d\cdot\varphi(d)=1+\sum_{i=1}^{k}p^i\cdot\varphi(p^i)=1+\left(1-\frac{1}{p}\right)\cdot\sum_{i=1}^{k}p^{2i}=$$

$$=1+\left(1-\frac{1}{p}\right)\cdot p^2\cdot\frac{p^{2k}-1}{p^2-1}=1+p\cdot\frac{p^{2k}-1}{p-1}=\frac{p^{2k+1}+1}{p-1}$$

Therefore, the equation (11) is true.♣

Theorem 1.

If $n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots p_s^{k_s}$ is the prime numbers decomposition of n, then the formula

$$\psi_1\left(\prod_{i=1}^{s}p_i^{k_i}\right)=\prod_{i=1}^{s}\frac{p_i^{2k_i+1}-1}{p_i+1} \qquad (12)$$

holds.

Proof

The proof is directly found based on Proposition1 1 and on the multiplicative property of ψ_1.

Obviously, if p is a prim number then $\psi_1(p) = \frac{p^3+1}{p+1} = p^2 - p + 1$ holds finding again the equation from Remark 1.2. If $n = p_1 \cdot p_2 \cdot \ldots \cdot p_s$ is a product of prime numbers then the following equation is true.

$$\psi_1(n) = \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_s) = \prod_{i=1}^{s}\left(p_i^2 - p_i + 1\right) \tag{13}$$

Proposition3.

$$(\forall\, n > 1) \quad \sum_{i=1,\,(i,n)=1}^{n} i = \frac{n \cdot \varphi(n)}{2} \tag{14}$$

Proof

This proof is made based on the *Inclusion & Exclusion* principle.

Let $D_p = \{i = 1,2,\ldots,n \mid p|n\}$ be the set which contains the multiples of p.

This set satisfies

$$D_p = p \cdot \left\{1,2,\ldots,\frac{n}{p}\right\} \text{ and } \sum_{i \in D_p} i = p \cdot \sum_{i=1}^{\frac{n}{p}} i = p \cdot \frac{\frac{n}{p} \cdot \left(\frac{n}{p}+1\right)}{2} = \frac{n}{2} \cdot \left(\frac{n}{p}+1\right).$$

Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \ldots p_s^{k_s}$ be the prime number decomposition of n.

The following intersection of sets

$$D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}} = \{i = 1, 2, \ldots, n \mid p_{j_1} | n \wedge p_{j_2} | n \wedge \ldots \wedge p_{j_m} | n\}$$

is evaluated as follows

$$D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}} = \{i = 1, 2, \ldots, n \mid p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m} | n\} = D_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}}$$

Therefore, the equation

$$\sum_{i \in D_{p_{j_1}} \cap D_{p_{j_2}} \cap \ldots \cap D_{p_{j_m}}} i = \sum_{i \in D_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}}} i = \frac{n}{2} \cdot \left(\frac{n}{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_m}} + 1\right) \tag{14}$$

holds.

The *Inclusion & Exclusion* principle is applied based on

$$D = \{i = 1, 2, \ldots, n \mid (i,n) = 1\} = \{1, 2, \ldots, n\} - \bigcup_{j=1}^{s} D_{p_k}$$

and it gives

$$\sum_{i<n,\,(i,n)=1} i = \sum_{i=1}^{n} i - \sum_{m=1}^{n}(-1)^{m+1} \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \quad \sum_{i\in D_{p_{j_1}}\cap D_{p_{j_2}}\cap\ldots\cap D_{p_{j_m}}} i \tag{15}$$

Applying (14), the equation (15) becomes

$$\sum_{i<n,\,(i,n)=1} i = \sum_{i=1}^{n} i + \sum_{m=1}^{n}(-1)^{m} \cdot \sum_{1\le j_1<j_2<\ldots<j_m\le n} \frac{n}{2}\cdot\left(\frac{n}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}+1\right) \tag{16}$$

The right side of the equation (16) is simplified by reordering the terms as follows

$$\sum_{i<n,\,(i,n)=1} i = \frac{n^2}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^{m}\cdot\sum_{1\le j_1<j_2<\ldots<j_m\le n}\frac{1}{p_{j_1}\cdot p_{j_2}\cdot\ldots\cdot p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^{m}\cdot\sum_{1\le j_1<j_2<\ldots<j_m\le n} 1\right)$$

$$\sum_{i<n,\,(i,n)=1} i = \frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)+\frac{n}{2}\cdot\left(1+\sum_{m=1}^{n}(-1)^{m}\cdot\binom{n}{m}\right)=\frac{n^2}{2}\cdot\prod_{m=1}^{s}\left(1-\frac{1}{p_{j_m}}\right)=\frac{n}{2}\cdot\varphi(n)\cdot$$

Therefore, the equation (14) holds.♣

Obviously, the equation (14) does not hold for $n=1$ because $\sum_{i=1,\,(i,1)=1}^{1} i = 1$ and $\frac{n\cdot\varphi(n)}{2}=\frac{1}{2}$.

Based on Proposition 3, the formula of the second function is found.

Proposition 4.

The following equation

$$(\forall\, n>1)\ \psi_2(n)=\frac{\psi_1(n)+1}{2} \tag{17}$$

holds.

Proof

Let $I_{n,d}=\{i=1,2,\ldots,n|\,(i,n)=d\}$ be the set of indices which satisfy $(i,n)=d$. Obviously, the following equation

$$(\forall\, d|n)\ I_{n,d}=d\cdot I_{\frac{n}{d},1} \tag{18}$$

holds. Based on (18), the sum $\sum_{i=1}^{n}\frac{i}{(i,n)}$ is transformed as follows

$$\psi_2(n)=\sum_{i=1}^{n} i\cdot(i,n)^{-1}=\sum_{d|n} d^{-1}\cdot\sum_{i\in I_{n,d}} i=\sum_{d|n} d^{-1}\cdot d\cdot\sum_{i_1\in I_{\frac{n}{d},1}} i_1=\sum_{d|n}\ \sum_{i_1\in I_{\frac{n}{d},1}} i_1\,. \tag{19}$$

Proposition 3 is applied for any divisor $d\neq n$ and the equation (19) becomes

$$\psi_2(n)=\sum_{d/n}\sum_{i_1\in I_{\frac{n}{d},1}} i_1 = 1+\sum_{n\neq d/n}\frac{\frac{n}{d}\varphi\left(\frac{n}{d}\right)}{2} = 1+\frac{1}{2}\sum_{n\neq d/n}\left(\frac{n}{d}\right)\cdot\varphi\left(\frac{n}{d}\right). \qquad (20)$$

Completing the last sum and changing the index, the equation (20) is transformed as follows

$$\psi_2(n)=1+\frac{1}{2}\sum_{1\neq d/n} d\cdot\varphi(d) = 1-\frac{1}{2}+\frac{1}{2}\sum_{d/n} d\cdot\varphi(d)=\frac{1}{2}+\frac{1}{2}\cdot\psi_1(n)$$

resulting in that (17) is true. ♣

Remarks 2

1. Based on the equation $\psi_2(n)=\frac{\psi_1(n)+1}{2}$, it is found that $\psi_1(n)=2\cdot\psi_2(n)-1$ is always an odd number and that the equation $\psi_2(n)=\left\lceil\frac{\psi_1(n)}{2}\right\rceil$ holds.

2. If $n=p_1^{k_1}\cdot p_2^{k_2}\cdot\ldots p_s^{k_s}>1$ is the prime numbers decomposition of n, then the formula $\psi_2\left(\prod_{i=1}^{s} p_i^{k_i}\right)=\frac{1}{2}+\frac{1}{2}\cdot\prod_{i=1}^{s}\frac{p_i^{2\cdot k_i+1}+1}{p_i+1}$ holds.

3. Upper bounds for the Smarandache's function

In this section, an application of the functions ψ_1,ψ_2 is presented. Based on these function an inequality concerning the Smarandache's function is proposed and some upper bounds for $\bar{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)$ are deduced.

Let $p_1=2$, $p_2=3$, ..., p_m, ... be the set of the prime numbers.

Proposition 5.

$$\left(\forall\, i\ge p_m\right)\left(\forall\, j=\overline{1,p_1\cdot p_2\cdot\ldots\cdot p_m}\right)\ S(p_1\cdot p_2\cdot\ldots\cdot p_m\cdot i+j)\le\frac{p_1\cdot p_2\cdot\ldots\cdot p_m\cdot i+j}{(p_1\cdot p_2\cdot\ldots\cdot p_m,j)} \qquad (21)$$

Proof

Let i, j be two numbers such that $i\ge p_m$ and $j=\overline{1,p_1\cdot p_2\cdot\ldots\cdot p_m}$.

Let us suppose that $(p_1\cdot p_2\cdot\ldots\cdot p_m,j)=p_{i_1}\cdot p_{i_2}\cdot\ldots\cdot p_{i_s}$ and $j=p_{i_1}\cdot p_{i_2}\cdot\ldots\cdot p_{i_s}\cdot j_1$.

Based on the inequality $\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1 \geq i + 1 \geq p_m + 1$, we find that the product $\left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right)! = 1 \cdot 2 \cdot \ldots \cdot \left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right)$ contains the factors $p_{i_1}, p_{i_2}, \ldots, p_{i_s}$ and $\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1$.

The following divisibility holds

$$\left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right)! \,\vdots\, p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s} \cdot \left(\frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_s}} i + j_1\right) = \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)}$$

therefore, the inequality (21) is found true.♣

Proposition 6.

$$(\forall\, i \geq p_m) \quad \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} S(p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j) \leq i \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \tag{22}$$

Proof

The equation (21) is applied for this proof as follows:

$$\sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} S(p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j) \leq \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)} =$$

$$= i \cdot \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)} + \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \frac{j}{(p_1 \cdot p_2 \cdot \ldots \cdot p_m, j)}$$

Applying the definitions of the functions ψ_1, ψ_2, the inequality (22) is found true.♣

Theorem 2.

The following inequality

$$\bar{S}(n) = \frac{1}{n} \cdot \sum_{j=1}^{n} S(i) \leq \frac{\psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m)}{2 \cdot (p_1 \cdot p_2 \cdot \ldots \cdot p_m)^2} \cdot n + \frac{2 \cdot \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) - \frac{1}{2}}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} + \frac{1}{n} \cdot C_m \tag{23}$$

is true for all $n > p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2$, where

$$C_m = \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) - \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot \frac{(p_m - 1) \cdot p_m}{2} - \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot (p_m - 1) \tag{24}$$

is a constant which does not depend on n.

Proof

Proposition 6 is used for this proof.

Let n be a number such that $n > p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2$. The sum $\sum_{i=1}^{n} S(i)$ is split into two sums as follows

$$\sum_{i=1}^{n} S(i) = \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2+1}^{n} S(i) \le \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2+1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m \left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil} S(i) =$$

$$= \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_m}^{\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil -1} \sum_{j=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_m} S(p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot i + j).$$

For the second sum the inequality (22) is applies resulting in the following inequality

$$\sum_{i=1}^{n} S(i) \le \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \sum_{i=p_m}^{\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil -1} \left[i \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \right]. \qquad (25)$$

Calculating the last sum, the inequality (25) becomes

$$\sum_{i=1}^{n} S(i) \le \sum_{i=1}^{p_1 \cdot p_2 \cdot \ldots \cdot p_{m-1} \cdot p_m^2} S(i) + \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot \frac{\left(\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1 \right) \cdot \left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil}{2} +$$

$$+ \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \cdot \left(\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1 \right) - \sum_{i=1}^{p_m - 1} \left[i \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m) \right].$$

Based on the double inequality $\left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil - 1 < \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \le \left\lceil \frac{n}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \right\rceil$, we find

$$\sum_{i=1}^{n} S(i) \le \frac{\psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m)}{2 \cdot (p_1 \cdot p_2 \cdot \ldots \cdot p_m)^2} \cdot n^2 + \frac{\frac{1}{2} \cdot \psi_1(p_1 \cdot p_2 \cdot \ldots \cdot p_m) + \psi_2(p_1 \cdot p_2 \cdot \ldots \cdot p_m)}{p_1 \cdot p_2 \cdot \ldots \cdot p_m} \cdot n + C_m.$$

Dividing by n and using Proposition 4, the equation (22) is found true.♣

4. Conclusions

The inequality (22) extends the results presented by Tabirca [1997] and generates several inequalities concerning the function S, which are presented in the following:

- m=1 $\Rightarrow (n>4)\ \overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.375\cdot n+0.75+\frac{5}{n}$

- m=2 $\Rightarrow (n>18)\ \overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.29167\cdot n+1.76+\frac{24}{n}$

- m=3 $\Rightarrow (n>150)\ \overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.245\cdot n+7.35-\frac{1052}{n}$

- m=4 $\Rightarrow (n>1470)\ \overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le 0.215\cdot n+45.15-\frac{176859}{n}$

The coefficients of n from the above inequalities are decreasing and the inequalities are stronger and stronger. Therefore, it is natural to investigate other upper bounds such us the bound proposed by Tabirca [1997] $\overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le\frac{2\cdot n}{\ln n}$. Ibstedt based on an UBASIC program [Ibstedt 1997] proved that the inequality $\overline{S}(n)=\frac{1}{n}\cdot\sum_{i=1}^{n}S(i)\le\frac{n}{\ln n}$ holds for natural numbers less than 5000000. A proof for this results has not been found yet.

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