Sankhyā B : The Indian Journal of Statistics
https://doi.org/10.1007/s13571-025-00369-9
© 2025, Indian Statistical Institute

A Family of Neutrosophic Estimators for Estimating Mean: An Application to Real Data

Bavita Singh
Amity University, Noida, India
Abhishek Singh
Dr. Vishwanath Karad MIT World Peace University, Pune, India
Florentin Smarandache
University of New Mexico, Gallup, USA

Abstract

In the realm of sample survey research, the classical statistics approach primarily deals with precise and definitive types of data to estimate population parameters when additional information is available. However, this approach fails when faced with data indeterminacies. To address such ambiguities, neutrosophic statistics emerges as an extension of both fuzzy and classical statistics. Thus, in light of the challenges posed by indeterminacy in sampling, we have introduced a proficient neutrosophic class of estimators, with and without Searls technique (optimization tool) for estimating the mean utilizing additional (ancillary) information under neutrosophic simple random sampling (NeSRS). The expression for the Bias and mean square error (MSE) of propounded estimators are derived up to the first order of approximation. The primary objective of this manuscript is to attain optimal estimates using our proposed neutrosophic estimators for unknown population mean values, while minimizing MSE and maximizing relative efficiency (RE). We accomplish this through an extensive study utilizing neutrosophic real data and simulations.

AMS (2000) subject classification. 62D05; 62A86.
Keywords and phrases. Neutrosophic statistics, Ancillary variable, Class of estimators, Neutrosophic simple random sampling, Mean square error.

1 Introduction

Classical statistics primarily deals with precise and well-defined numerical values for estimation. Numerous researchers have harnessed the principles of classical statistics within the realm of sampling theory, particularly for esti-

Abhishek Singh and Bavita Singh have contributed equally to this work.

Published online: 18 June 2025

mating population parameters while incorporating ancillary information. As a result, a substantial body of literature has emerged, encompassing diverse advancements in the field of estimators. Cochran (1940) delved into the intricacies of ratio estimation, while Murthy (1964) explored product estimation, both accompanied by illustrative examples. Building upon these foundations, Sisodia and Dwivedi (1981) devised a transformed ratio estimator, leveraging the coefficient of variation of the ancillary variable. In a similar vein, Upadhyaya and Singh (1999) introduced a methodology involving transformed ancillary information to estimate the population mean. Introducing enhanced techniques, Bahl and Tuteja (1991) advocated the employment of efficient ratio and product exponential methods for estimation purposes. Expanding upon these ideas, Upadhyaya et al. (2011) and Khan et al. (2014) contributed to the field by presenting improved estimators for mean estimation, capitalizing on ancillary information. In conclusion, classical statistics serves as the foundation for precise estimation using definite numerical values. Its integration with sampling theory and ancillary information has paved the way for a wealth of innovative estimator development, as evidenced by the contributions of various researchers over time. The estimation under classical statistics is free from uncertainty in the measurements of the observations but where there is uncertainty in data, we deal with these problems with fuzzy and neutrosophic statistics among these two methods neutrosophic statistics is better as it measures the indeterminacy in data too, see Aslam (2019). So, to deal with the data having some kind of uncertainty or indeterminacy, neutrosophic statistics is the most preferred method to analyze these kinds of data sets. Nowadays, neutrosophic statistics is very popular in statistical theory, and only because of its popularity, many forms or modified forms of estimators have also been proposed by several authors. An alternative definition characterizes it as an extension that amalgamates concepts from both fuzzy and classical statistics, designed to effectively manage the inherent data uncertainty. A lot of work has been carried out by many researchers based on neutrosophic statistics. Smarandache (2014) proposed neutrosophic statistics. In the previous few years, the development of new estimators or modification of estimators are much in trend and many authors have propounded various extended forms of classical as well as neutrosophic estimators with their applications in many applied fields. For a more detailed study including the applications of classical as well as neutrosophic statistics, one may refer to Chakraborty et al. (2021), andVishwakarma and Singh (2022a, b). The paper is presented as follows: In Section 1, we present an introduction followed by the subsequent sections. In Section 2, we have discussed neutrosophic data, the research gap, and the motivation of the study, in Subsection 2.1, we have defined the terminologies. In Section 3, we have shown some existing neutrosophic estimators by several authors. In Section 4, we have proposed the neutrosophic estimators motivated by several authors with and without the Searls technique. In Sections/Subsections 5, 5.1, and 5.2, we have shown efficiency comparisons theoretically as well as numerically. The results and discussion are shown in Section 6, and in Section 7, the conclusion of this manuscript is shown.

2 Neutrosophic Data, Research Gap, and Motivation of the Study

Zadeh (1996, 1965) introduced the foundational concept of fuzzy sets, marking a significant milestone in the field. To tackle uncertainty within data, neutrosophic logic emerged as an extension of fuzzy logic. Consequently, neutrosophic statistics, rooted in neutrosophic logic and sets, emerged as a more sophisticated iteration of fuzzy statistics designed to address uncertainty. The development of neutrosophic statistics has been enriched by extensive research endeavors. The inception of neutrosophic theory is attributed to Florentin Smarandache, with comprehensive literature available in Florentin Smarandache works (1999, 2001, 2013, 2014, 1998, 2005, 2019), offering an introduction to neutrosophic statistics as an extension of fuzzy sets. Smarandache (2010) emphasized the superiority of neutrosophic logic over fuzzy logic. Neutrosophic statistics not only extends fuzzy and classical statistics but also encompasses intuitionistic statistics, as explored by Atanassov (1986, 1999). Furthermore, Chen et al. (2017a, b) elucidated the application of neutrosophic numbers in rock engineering contexts. Aslam (2019) pioneered the novel neutrosophic analysis of variance technique for neutrosophic data. Alhabib et al. (2018) embarked on groundbreaking research in neutrosophic probability statistics. Building upon these foundations, a multitude of researchers, including Aslam (2019), Aslam et al. (2020), Aslam and Algarni (2020), and Rao and Aslam (2021), have harnessed the concept of neutrosophic statistics across diverse statistical procedures and fields. Recent contributions have extended to sampling theory, with Tahir et al. (2021) introducing neutrosophic ratio-type estimators for mean estimation and Vishwakarma and Singh (2022a, b) presenting computations for neutrosophic generalized estimators. AlAita and Aslam (2023) have given the analysis of covariance under neutrosophic statistics. Further, Singh et al. (2023a, b), Yadav and Smarandache (2023), Singh et al.

(2023a, b), Alomair and Shahzad (2023), Tahir et al. (2023), Raghav (2023), Singh et al. (2024a, b), Kumari et al. (2024), Abbasi et al. (2024), Khanna (2024), Singh et al. (2024a, b), Singh and Kumari (2024), Shahzad et al. (2025), Singh and Gupta (2025), and Singh et al. (2025a, b) have given neutrosophic estimation procedures for estimation population mean. Neutrosophic statistics offers a solution to address the inherent uncertainty within data. A comprehensive resource for exploring this concept can be found in the book "Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics" authored by Florentine Smarandache and Muhammed Aslam. Further in-depth study of neutrosophic statistics can also be undertaken by visiting the website (http://fs.unm.edu/NS/NeutrosophicStatistics.htm). In the last two decades or so, several studies have been carried out in survey sampling for certain, determined, and clear data. Nevertheless, in numerous scenarios, data exhibit a neutrosophic nature, characterized by uncertainty, vague interval values, and ambiguous observations, under some circumstances, at this instant, neutrosophic statistics is applied. In real life, the scenario is quite different, here data that is indeterminant are more available than determinant data therefore neutrosophic statistical techniques are much needed. This manuscript is one step in this area. Statistics is the compilation of the data collection, analysis, and interpretation. To amass data, it becomes imperative to employ statistical techniques capable of accommodating a spectrum of uncertain observations, encompassing the likelihood of incorporating actual measurements. Here, classical statistics failed to investigate the data in the indeterminant form and hence neutrosophic statistics replace classical statistics. Taking the motivation from the lots of work, Tahir et al. (2021), and Vishwakarma and Singh (2022a, b) have initiated estimation procedure in sampling theory, and also, taking motivation from them, this article aims to propose some class ofratio and product type estimators under the neutrosophic scheme.

2.1 Notations and Terminologies The neutrosophic number's potential range could extend over an unfamiliar interval [a, b], yet there exist various methods to express neutrosophic observations, each offering distinct representations of the interval values.Here, we are presenting neutrosophic values as $Z_N = Z_L + Z_U I_N$, where $I_N \epsilon [I_L, I_U]$, Z_L and Z_U are lower and upper values of neutrosophic observations, and N is here for neutrosophic number. Thus,the neutrosophic values are in the interval form $Z_N \epsilon [a, b]$, where `$a'$ and `b' are the lower and upper values of the Z_N. Taking motivation from Tahir et al. (2021), and Vishwakarma and Singh (2022a, b), we propose some classes of estimators with Searls and without Searls technique using ancillary variable. The technique proposed by Searls (1964) aims to minimize the mean square error or reduce the bias of an estimator by incorporating known values of the coefficient of variation (C.V.) on prior information related to the study variable. The prior knowledge is often accessible or available in advance to researchers in many research fields, like biology, agriculture, risk and finance analysis, quality control and reliability engineering, and medical and pharmaceutical research for conducting clinical trials, if not available in advance, then can be attained from the past data. This information plays a crucial role in designing experiments and determining optimal sample sizes, ultimately leading to the development of more precise estimators. With the Searls (1964) technique, the family of neutrosophic estimators appears to be more efficient than their member estimators. The whole procedure of estimation can be seen in a flowchart diagram placed in Appendix B.

Consider a neutrosophic random sample of size n_N drawn from a finite population comprising N_N units. Assume y_{iN} as the *i-th* sample observation of our neutrosophic data, which is of the form y_{iN} and similarly the auxiliary variable as x_{iN}. Here, $y_{iN}$is our neutrosophic variable of interest, and similarly is our auxiliary neutrosophic variable which is correlated to our study variable y_{iN}. In addition, $\bar{y}_N$ and $\bar{x}_N$ are sample means, $\bar{Y}_N$ and $\bar{X}_N$ are the overall averages, C_{yN} and C_{xN} are neutrosophic coefficient of variation for Y_N and X_N, respectively, ρ_{xyN} is the neutrosophic correlation between Y_N, and X_N, (neutrosophic variables), $\beta_{2(x)N}$ is the neutrosophic coefficient of kurtosis for auxiliary variable X_N, $\bar{e}_{yN}$, and $\bar{e}_{xN}$ are neutrosophic mean errors study and auxiliary variable respectively. In a similar vein, the calculation of bias and mean squared error (MSE) is extended to neutrosophic sets for analytical purposes. To formulate the expressions for Bias and MSE of the estimators, relevant error terms that are up to the first-order approximation are taken into account, contributing to the comprehensive analysis.

$\bar{e}_{yN} = (\bar{y}_N - \bar{Y}_N)$; $\bar{e}_{xN} = (\bar{x}_N - \bar{X}_N)$ and expected values are as: $E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0$;

$E(\bar{e}^2_{yN}) = \theta_N \bar{Y}^2_N C^2_{yN}$; $E(\bar{e}^2_{xN}) = \theta_N \bar{Y}^2_N C^2_{xN}$; $E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{Y}_N C_{yN} \bar{X}_N$; $C^2_{xN} = \frac{\sigma^2_{xN}}{\bar{X}^2_N}$; $C^2_{yN} = \frac{\sigma^2_{yN}}{\bar{y}^2_N}$; $\rho_{xyN} = \frac{\sigma_{xyN}}{\sigma_{xN}\sigma_{yN}}$; $\theta_N = \frac{1-f_N}{n_N}$; $f_N = \frac{n_N}{N_N}$.

3 Existing Estimators within the Framework of Neutrosophic Simple Random Sampling (NeSRS)

Here, we have transformed many existing estimators into neutrosophic estimators to defeat the complications of uncertainty in data.

The mean and variance under the neutrosophic estimator of the unbiased estimator are given below:

$$\bar{y}_N = \frac{1}{n}\sum_{i=1}^{n_N} y_{iN} \tag{1}$$

$$MSE(\bar{y}_N) = Var(\bar{y}_N) = \theta_N \bar{Y}_N^2 C_{yN}^2 \tag{2}$$

The neutrosophic ratio estimator for estimating the mean using ancillary variables is given below:

$$\bar{y}_{rN} = \frac{\bar{y}_N}{\bar{x}_N}\bar{X}_N \tag{3}$$

$$\bar{y}_{rN} = (\bar{Y}_N + \bar{e}_{yN})\left(1 + \frac{\bar{e}_{xN}}{\bar{X}_N}\right)^{-1} \tag{4}$$

Expanding Eq. 4 in neutrosophic error terms. We have expressions for the Bias and MSE of the estimator in Eq. 4 up to 1st order of approximation are given as

$$Bias(\bar{y}_{rN}) = \theta_N \bar{Y}_N \left[C_{xN}^2 - C_{xN}C_{yN}\rho_{xyN}\right] \tag{5}$$

$$MSE(\bar{y}_{rN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + C_{xN}^2 - 2C_{xN}C_{yN}\rho_{xyN}\right] \tag{6}$$

The neutrosophic product estimator, in the case of negative correlation, is given by:

$$\bar{y}_{rN} = \bar{y}_N\left(\frac{\bar{x}_N}{\bar{X}_N}\right) \tag{7}$$

$$\bar{y}_{rN} = (\bar{Y}_N + \bar{e}_{yN})\left(1 + \frac{\bar{e}_{xN}}{\bar{X}_N}\right) \tag{8}$$

Simplify Eq. 8 and then by taking expectation, the Bias of the estimator in Eq. 7, as given below:

$$Bias(\bar{y}_{rN}) = \theta_N C_{xN} \bar{Y}_N C_{yN}\rho_{xyN} \tag{9}$$

Squaring and taking the expectations of the simplified form of Eq. 8, the MSE of neutrosophic product estimator up to the 1st order approximation is given by:

$$MSE(\bar{y}_{rN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + C_{xN}^2 + 2\rho_{xyN} C_{xN} C_{yN}\right] \tag{10}$$

Motivated by Bahl and Tuteja (1991), we propose the neutrosophic exponential type ratio estimator by doing some transformation in the existing estimator, given as follows:

$$\bar{y}_{BTrN} = \bar{y}_N exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) \tag{11}$$

Using the expression of neutrosophic mean error terms, we have:

$$= (\bar{e}_{yN} + \bar{Y}_N) exp\left(\frac{-\bar{e}_{xN}}{2\bar{X}_N}\left(1 + \frac{\bar{e}_{xN}}{2\bar{X}_N}\right)^{-1}\right) \tag{12}$$

Expanding the above series in Eq. 12, we have:

$$\bar{y}_{BTrN} - \bar{Y}_N = \bar{e}_{yN} - \frac{\bar{e}_{xN}\bar{e}_{yN}}{2\bar{X}_N} - \frac{\bar{Y}_N \bar{e}_{xN}}{2\bar{X}_N} + \frac{3\bar{Y}_N \bar{e}_{xN}^2}{8\bar{X}_N^2} \tag{13}$$

To get the expression of Bias, we take the expectation of Eq. 13 on both sides:

$$Bias(\bar{y}_{BTrN}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{xN} C_{yN} \rho_{xyN}\right] \tag{14}$$

Taking the square on both sides in Eq. 13 and after applying expectation, we obtain the MSE as given below:

$$MSE(\bar{y}_{BTrN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{xN}^2 - \rho_{xyN} C_{xN} C_{yN}\right] \tag{15}$$

Motivated by Bahl and Tuteja (1991), we propose the neutrosophic exponential type product estimator, after doing some transformation in the existing estimator which is given as:

$$\bar{y}_{BTrN} = \bar{y}_N exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right) \tag{16}$$

Following the Case of the estimator in Eq. 11, the expressions for Bias and MSE for the neutrosophic exponential estimator in Eq. 16 are as:

$$Bias(\bar{y}_{BTpN}) = \theta_N \bar{Y}_N \left(\frac{-1}{8} C_{xN}^2 + \frac{\rho_{xyN} C_{xN} C_{yN}}{2} \right) \tag{17}$$

$$MSE(\bar{y}_{BTpN}) = \theta_N \bar{Y}_N^2 \left(C_{yN}^2 + \frac{1}{4} C_{xN}^2 + \rho_{xyN} C_{xN} C_{yN} \right) \tag{18}$$

The neutrosophic values can be represented in various formats, with the neutrosophic numbers potentially encompassing an unknown interval [a, b]. Here, we are illustrating neutrosophic values as $Z_N = Z_L + Z_U I_N$ with $I_N \epsilon [I_L, I_U]$, the symbol 'N' is used to represent a neutrosophic number. Consequently, our neutrosophic observations will fall within an interval $Z_N \in [a, b]$, where $`a'$ and $`b'$ denote the lower and upper values of the neutrosophic data.

4 Proposed Neutrosophic Class of Estimators with and without Searls Technique

Motivated by Singh and Espejo (2003), we have developed the following neutrosophic estimator by combining the ratio and product estimator, given below:

$$\bar{y}_{SErpN} = \bar{y}_N \left[\alpha_1 \left(\frac{\bar{X}_N}{\bar{x}_N} \right) + (1 - \alpha_1) \left(\frac{\bar{x}_N}{\bar{X}_N} \right) \right] \tag{19}$$

Here α_1 is a constant which is supposed to optimize the estimator $\bar{y}_{SErpN}$ for minimum MSE. Now putting the values of $\bar{y}_N$ and $\bar{x}_N$ in terms of the neutrosophic mean error in Eq. 19, we have the result below:

$$\bar{y}_{SErpN} = (\bar{e}_{yN} + \bar{Y}_N) \left[\alpha_1 \left(1 + \frac{\bar{e}_{xN}}{\bar{X}_N} \right)^{-1} (1 + \alpha_1) \left(1 + \frac{\bar{e}_{xN}}{\bar{X}_N} \right) \right] \tag{20}$$

The approximated Bias and MSE of $\bar{y}_{SErpN}$, up to 1st order are given by

$$Bias(\bar{y}_{SErpN}) = \theta_N \bar{Y}_N \big(\alpha_1 C_{xN}^2 + (1 - 2\alpha_1) \rho_{xyN} C_{xN} C_{yN} \big) \tag{21}$$

$$\begin{aligned} MSE(\bar{y}_{SErpN}) = \theta_N \bar{Y}_N^2 \big(& C_{yN}^2 + C_{xN}^2 + 4\alpha_1^2 C_{xN}^2 - 4\alpha_1 C_{xN}^2 \\ & + (2 - 4\alpha_1) \rho_{xyN} C_{xN} C_{yN} \big) \end{aligned} \tag{22}$$

The optimum value of the constant $\alpha_1$is given by

$$\alpha_1 = \frac{1}{2}\left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) \tag{23}$$

$$Bias(\bar{y}_{SErpN}) = \theta_N \bar{y}_N \left(\frac{1}{2}\left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) C_{xN}^2 - \rho_{xyN}^2 C_{yN}^2\right) \tag{24}$$

We have the optimum (minimum) MSE of $(\bar{y}_{SErpN})$ as follows:

$$MSE(\bar{y}_{SErpN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 \left(1 - \rho_{xyN}^2\right) \tag{25}$$

Inspired by Singh et al. (2008), we have propounded a neutrosophic ratio cum product exponential estimator, as given below:

$$\bar{y}_{SrpeN} = \bar{y}_N \left[\alpha_4 exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) + (1 - \alpha_4) exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right)\right] \tag{26}$$

Now emplacing the values of $\bar{y}_N$ and $\bar{x}_N$ in terms of the neutrosophic mean error in Eq. 26), the simplified expression is:

$$\bar{y}_{SrpeN} = (\bar{Y}_N + \bar{e}_{yN})\left[\alpha_4 exp\left(\frac{\bar{e}_{xN}}{2\bar{X}_N}\left(1 + \frac{\bar{e}_{xN}}{\bar{X}_N}\right)^{-1}\right)\right.$$

$$\left. + (1 - \alpha_4) exp\left(\frac{\bar{e}_{xN}}{2\bar{X}_N}\left(1 + \frac{\bar{e}_{xN}}{\bar{X}_N}\right)^{-1}\right)\right] \tag{27}$$

$$\bar{y}_{SrpeN} - \bar{Y}_N = \alpha_4 \left[\frac{-\bar{e}_{yN}\bar{e}_{xN}}{\bar{X}_N} - \frac{\bar{e}_{xN}\bar{Y}_N}{\bar{X}_N} + \frac{\bar{e}_{xN}^2 \bar{Y}_N}{\bar{X}_N}\right] + \bar{e}_{yN}$$

$$+ \frac{\bar{e}_{yN}\bar{e}_{xN}}{2\bar{X}_N} + \frac{\bar{e}_{yN}\bar{e}_{yN}}{2\bar{X}_N} - \frac{\bar{e}_{xN}^2 \bar{Y}_N}{8\bar{X}_N^2} \tag{28}$$

The approximated Bias and MSE of the estimator $\bar{y}_{SrpeN}$ up to 1storder are given by as

$$Bias(\bar{y}_{SrpeN}) = \theta_N \bar{Y}_N \left[\left(-\alpha_4 + \frac{1}{2}\right)\rho_{xyN} C_{yN} C_{xN}\right.$$

$$\left. + \left(\frac{\alpha_4}{2} - \frac{1}{8}\right) C_{xN}^2\right] \tag{29}$$

$$MSE(\bar{y}_{SrpeN}) = \theta_N \bar{Y}_N^2 \Big[C_{yN}^2 + \Big(-\alpha_4 + \frac{1}{2} \Big)^2 C_{xN}^2 + 2\Big(-\alpha_4 + \frac{1}{2} \Big) \rho_{xyN} C_{yN} C_{xN} \Big] \tag{30}$$

and α_4 is constant whose optimum value is obtained by minimizing (equating the derivative of $MSE(\bar{y}_{SrpeN})$ with respect to α_4,to zero), given by:

$$\hat{\alpha_4} = \frac{1}{2} + \rho_{xyN} \frac{C_{yN}}{C_{xN}} \tag{31}$$

The expressions of $Bias(\bar{y}_{SrpeN})$ and minimum MSE of $\bar{y}_{SrpeN}$ as follows:

$$Bias(\bar{y}_{SrpeN}) = \frac{-\theta_N \bar{Y}_N}{8} \big(-C_{xN}^2 + 8\rho_{xyN}^2 C_{yN}^2 - 4\rho_{xyN} C_{yN} C_{xN} \big) \tag{32}$$

$$MSE(\bar{y}_{SrpeN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{xyN}^2). \tag{33}$$

Influenced by Searls (1964), Rehman and Asif (2020), we propose the class of neutrosophic estimator
Case I:

$$\bar{y}_{R_1 rN} = k_1 \bar{y}_N \Big[\alpha_5 \Big(\frac{\bar{X}_N}{\bar{x}_N} \Big) + (1 - \alpha_5) \Big(\frac{\bar{x}_N}{\bar{X}_N} \Big) \Big] \tag{34}$$

where k_1 and α_5 are constants.

Substituting the values of $\bar{y}_N$ and $\bar{x}_N$ in terms of the neutrosophic mean error in Eq. 34, now the resulting expression is given here:

$$\bar{y}_{R_1 rN} = k_1 (\bar{Y}_N + \bar{e}_{yN}) \Big[\alpha_5 \Big(1 + \frac{\bar{e}_{xN}}{\bar{X}_N} \Big)^{-1} + (1 - \alpha_5) \Big(1 + \frac{\bar{e}_{xN}}{\bar{X}_N} \Big) \Big] \tag{35}$$

Now, we expand the neutrosophic mean error given in Eq. 35, up to first-order approximation,the simplified expression is given below:

$$\bar{y}_{R_1 rN} - \bar{Y}_N = (k_1 - 1)\bar{Y}_N + k_1 \Big[\bar{e}_{yN} + (1 - 2\alpha_5) \Big(\frac{\bar{e}_{xN} \bar{Y}_N}{\bar{X}_N} + \frac{\bar{e}_{xN} \bar{e}_{yN}}{\bar{X}_N} \Big) + \alpha_5 \bar{Y}_N \Big(\frac{\bar{e}_{xN}}{\bar{X}_N} \Big)^2 \Big] \tag{36}$$

Approximated Bias and MSE of $\bar{y}_{R_1rN}$, up to 1storder are given by:

$$Bias(\bar{y}_{R_1rN}) = (k_1-1)\bar{Y}_N + \theta_N k_1 \Big[(1-2\alpha_5)\bar{Y}_N \rho_{xyN} C_{yN} C_{xN} \quad (37)$$
$$+\bar{Y}_N \alpha_5 \theta_N C_{xN}^2\Big] \quad (38)$$

$$MSE(\bar{y}_{R_1rN}) = (k_1-1)^2\bar{Y}_N^2 + \theta_N k_1^2 \bar{Y}_N^2\Big[C_{yN}^2 + (1-2\alpha_5)^2 C_{xN}^2 + 2(1-2\alpha_5)\times \rho_{xyN} C_{yN} C_{xN}\Big]$$
$$+\theta_N k_1(k_1-1)\bar{Y}_N^2\Big[\alpha_5 C_{xN}^2 + (1-2\alpha_5) C_{xN} C_{yN} \rho_{xyN}\Big] \quad (39)$$

Now minimizing $MSE(\bar{y}_{R_1rN})$ (equating the derivatives of $MSE(\bar{y}_{R_1rN})$ w.r.to α_5 and k_1 to zero), we get the following two solution sets:
The solution set I:

$$k_1 = 0, \quad \alpha_5 = \frac{\theta_N \rho_{xyN} C_{yN} C_{xN} + 1}{\theta_N C_x (2\rho_{xyN} C_{yN} - C_{xN})}$$

The solution set II:

$$k_1 = \frac{8\theta_N \rho_{xyN}^2 C_{yN}^2 - 6\theta_N \rho_{xyN} C_{yN} C_{xN} - \theta_N C_{xN}^2 - 4}{16\theta_N \rho_{xyN}^2 C_{yN}^2 - 8\theta_N \rho_{xyN} C_{yN} C_{xN} - 3\theta_N C_{xN}^2 - 4\theta_N C_{yN}^2 - 4}$$

.

$$\alpha_5 = \frac{4\theta_N \rho_{xyN}^2 C_{xN} C_{yN}^2 - 3\theta_N \rho_{xyN} C_{yN} C_{xN}^2 + 2\theta_N \rho_{xyN} C_{yN}^3 - \theta_N C_{xN}^3 - \theta_N C_{xN} C_{yN}^2 - 2\rho_{xyN} C_{yN} - 2C_{xN}}{C_{xN}(8\theta_N \rho_{xyN}^2 C_{yN}^2 - 6\theta_N \rho_{xyN} C_{yN} C_{xN} - \theta_N C_{xN}^2 - 4)}$$

On putting the values from solution set IIin Eqs. 36 and 37, the Bias and minimum MSE of $\bar{y}_{R_1rN}$ as follows:

$$Bias(\bar{y}_{R_1rN}) = \frac{\theta_N \bar{Y}_N (4\theta_N \rho_{xyN}^2 C_{xN}^2 C_{yN}^2 - 4\theta_N \rho_{xyN}^2 C_{yN}^4 - 2\theta_N \rho_{xyN} C_{yN}^3 C_{xN} - \theta_N C_{xN}^4}{4 + 4\theta_N C_{yN}^2 + 3\theta_N C_{xN}^2 + 8\theta_N \rho_{xyN} C_{xN} C_{yN} - 16\theta_N \rho_{xyN}^2 C_{yN}^2} \quad (40)$$

$$MSE(\bar{y}_{R_1rN}) = \left[\frac{\theta_N \bar{Y}_N^2 (4\theta_N \rho_{xyN}^2 C_{xN}^3 C_{yN}^2 - 4\theta_N \rho_{xyN}^2 C_{yN}^4 - 2\theta_N \rho_{xyN} C_{yN} C_{xN}^3}{4 + 4\theta_N C_{yN}^2 + 3\theta_N C_{xN}^2 + 8\theta_N \rho_{xyN} C_{xN} C_{yN} C_{xN} - 16\theta_N \rho_{xyN}^2 C_{yN}^2}\right]$$
$$+\left[\frac{4\theta_N \rho_{xyN} C_{xN} C_{yN}^3 - \theta_N C_{xN}^4 - \theta_N C_{xN}^2 C_{yN}^2 - 4\rho_{xyN}^2 C_{yN}^2 + 4C_{yN}^2}{4 + 4\theta_N C_{yN}^2 + 3\theta_N C_{xN}^2 + 8\theta_N \rho_{xyN} C_{xN} C_{yN} C_{xN} - 16\theta_N \rho_{xyN}^2 C_{yN}^2}\right]. \quad (41)$$

Case II:

$$\bar{y}_{R_2rN} = k_2\bar{y}_N\left[\alpha_6\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N - \bar{x}_N}\right)+(1-\alpha_6)\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{X}_N + \bar{x}_N}\right)\right] \tag{42}$$

where k_2 and α_6 are constants.

Now, placing the expressions of neutrosophic mean error terms in Eq. 40 results in the expression below:

$$\bar{y}_{R_2rN} = k_2(\bar{Y}_N+\bar{e}_{yN})\left[\alpha_6\exp\left(\frac{-\bar{e}_{xN}}{2\bar{X}_N}\left(1+\frac{\bar{e}_{xN}}{2\bar{X}_N}\right)^{-1}\right)+(1-\alpha_6)\exp\left(\frac{\bar{e}_{xN}}{2\bar{X}_N}\left(1+\frac{\bar{e}_{xN}}{2\bar{X}_N}\right)^{-1}\right)\right] \tag{43}$$

Now, we expand the neutrosophic mean error given in Eq. 41, up to first-order approximation, the resulting expression given here:

$$\bar{y}_{R_2rN} - \bar{Y}_N = \left[\bar{Y}_N(k_2-1)+k_2\left(\bar{e}_{yN}+\left(\frac{1}{2}-\alpha_6\right)\left(\frac{\bar{e}_{yN}\bar{e}_{xN}}{\bar{X}_N}+\frac{\bar{e}_{xN}\bar{Y}_N}{\bar{X}_N}\right)+\left(\frac{\alpha_6}{2}-\frac{1}{8}\right)\frac{\bar{e}^2_{xN}\bar{Y}_N}{\bar{X}^2_N}\right)\right] \tag{44}$$

Approximated Bias and MSE of $\bar{y}_{R_2rN}$, up to 1st order are expressed by:

$$Bias(\bar{y}_{R_2rN}) = \bar{Y}_N\left[(k_2-1)+k_2\theta_N\left(\left(\frac{1}{2}-\alpha_6\right)\rho_{xyN}C_{yN}C_{xN}+\left(\frac{\alpha_6}{2}-\frac{1}{8}\right)C^2_{xN}\right)\right] \tag{45}$$

$$\begin{aligned} MSE(\bar{y}_{R_2rN}) = {} & (k_2-1)^2\bar{Y}^2_N+\theta_N k_2^2\bar{Y}^2_N\left[C^2_{yN}+\left(\frac{1}{2}-\alpha_6\right)^2C^2_{xN}+2\left(\frac{1}{2}-\alpha_6\right)\rho_{xyN}C_{yN}C_{xN}\right] \\ & +2\theta_N k_2(k_2-1)\bar{Y}^2_N\left[\left(\frac{\alpha_6}{2}-\frac{1}{8}\right)C^2_{xN}+\left(\frac{1}{2}-\alpha_6\right)C_{yN}C_{xN}\rho_{xyN}\right] \end{aligned} \tag{46}$$

Following the lines of case I, we minimize $MSE(\bar{y}_{R_2rN})$, and get the following two possible solution sets:
The solution set I:

$$k_2 = 0,\ \ \alpha_6 = \frac{8\theta_N\rho_{xyN}C_{yN}C_{xN} - \theta_N C^2_{xN} + 8}{4\theta_N C_x(2\rho_{xyN}C_{yN} - C_{xN})}$$

The solution set II:

$$k_2 = \frac{16\theta_N\rho^2_{xyN}C^2_{yN} - 12\theta_N\rho_{xyN}C_{yN}C_{xN} + \theta_N C^2_{xN} - 8}{8(4\theta_N\rho^2_{xyN}C^2_{yN} - 2\theta_N\rho_{xyN}C_{yN}C_{xN} - \theta_N C^2_{yN} - 1)}$$

and

$$\alpha_6 = \frac{2(4\theta_N\rho^2_{xyN}C_{xN}C^2_{yN} - 3\theta_N\rho_{xyN}C_{yN}C^2_{xN} + 4\theta_N\rho_{xyN}C^3_{yN} - 2\theta_N C_{xN}C^2_{yN} - 4\rho_{xyN}C_{yN} - 2C_{xN})}{C_{xN}(16\theta_N\rho^2_{xyN}C^2_{yN} - 12\theta_N\rho_{xyN}C_{yN}C_{xN} - \theta_N C^2_{xN} - 8)}$$

By putting values from the solution set II in Eq. 44, we have the following Bias and minimum MSE as

$$Bias(\bar{y}_{R_2rN}) = \frac{\theta_N\bar{Y}_N(16\theta_N\rho^2_{xyN}C^2_{xN}C^2_{yN} + 64\theta_N\rho^2_{xyN}C^4_{yN} - 16\theta_N\rho_{xyN}C_{yN}C^3_{xN} - \theta_N C^4_{xN})}{64(4\theta_N\rho^2_{xyN}C^2_{yN} - 2\theta_N\rho_{xyN}C_{yN}C_{xN} - \theta_N C^2_{yN} - 1)}$$

$$\frac{-16\theta_N\bar{Y}_N C^2_{xN}C^2_{yN} - 192\rho^2_{xyN}C^2_{yN} + 64C^2_{yN}}{64(4\theta_N\rho^2_{xyN}C^2_{yN} - 2\theta_N\rho_{xyN}C_{yN}C_{xN} - \theta_N C^2_{yN} - 1)} \tag{47}$$

$$MSE(\bar{y}_{R_2rN}) = \frac{\theta_N\bar{Y}^2_N(16\theta_N\rho^2_{xyN}C^2_{xN}C^2_{yN} - 64\theta_N\rho^2_{xyN}C^4_{yN} - 8\theta_N\rho_{xyN}C_{yN}C^3_{xN} + 64\theta_N\rho_{xyN}C_{xN}C^3_{yN})}{64(1 - 4\theta_N\rho^2_{xyN}C^2_{yN} + 2\theta_N\rho_{xyN}C_{yN}C_{xN} + \theta_N C^2_{yN})}$$

$$\frac{-\theta_N C^4_{xN} - 16\theta_N C^2_{yN}C^2_{xN} - 64\rho^2_{xyN}C^2_{yN} + 64C^2_{yN}}{64(1 - 4\theta_N\rho^2_{xyN}C^2_{yN} + 2\theta_N\rho_{xyN}C_{yN}C_{xN} + \theta_N C^2_{yN})} \tag{48}$$

For the above all expressions in this section, the neutrosophic observations can be represented in various formats, with the neutrosophic numbers potentially encompassing an unknown interval [a, b]. Here, we are illustrating neutrosophic values as $Z_N = Z_L + Z_U I_N$ with $I_N \epsilon [I_L, I_U]$, the symbol 'N' is used to represent a neutrosophic number. Consequently, our neutrosophic observations will fall within an interval $Z\epsilon[a, b]$, where 'a' and 'b' denote the lower and upper values of the neutrosophic data.

5 Efficiency Comparison

In this section, to show the properties of the proposed estimators theoretically, we compare the optimum MSE of proposed estimators with the other existing estimators, we have used the MSE and relative efficiency (RE) criteria to show the supremacy of our proposed estimators over other existing estimators Θ and proposed estimators will be better than to other existing estimators iff

(MSE(Φ)-MSE(Θ))$<$ 0 $\Rightarrow$ MSE(Φ)$<$ MSE(Θ) $\Rightarrow \frac{MSE(\Theta)}{MSE(\Phi)} > 1$, and we have a relative efficiency formula:

RE(Φ, Θ)=$\frac{MSE(\Theta)}{MSE(\Phi)}$ i.e., RE must be greater than 1.

Where, MSE(Φ) is the MSE of the proposed estimators $\Phi = \bar{y}_{R1rN}, \bar{y}_{R2rN}$ and MSE(Θ) is the MSE of the existing estimators $\Theta = \bar{y}_{nN}, \bar{y}_{rN}, \bar{y}_{pN}, \bar{y}_{BTrN}$, $\bar{y}_{BTpN}, \bar{y}_{SErpN}, \bar{y}_{SrpeN}$. The comparisons of proposed estimators over existing estimators are as follows:

(i) The estimator $\bar{y}_{R_2rN}$ from Eq. 46 will be more efficient than the estimator $\bar{y}_{nN}$ from Eq. 2 iff
(MSE($\bar{y}_{R_2rN}$)-MSE($\bar{y}_{nN}$))< 0 or RE($\bar{y}_{R_2rN}, \bar{y}_{nN}$)$= \frac{MSE(\bar{y}_{nN})}{MSE(\bar{y}_{R_2rN})} > 1$, i.e.,

$$\Bigg[\frac{\theta_N \bar{Y}_N^2(16\theta_N \rho_{xyN}^2 C_{xN}^2 C_{yN}^2 - 64\theta_N \rho_{xyN}^2 C_{yN}^4 - 8\theta_N \rho_{xyN} C_{yN} C_{xN}^3 + 64\theta_N \rho_{xyN}}{64(1 - 4\theta_N \rho_{xyN}^2 C_{yN}^2 + 2\theta_N \rho_{xyN} C_{yN} C_{xN} + \theta_N C_{yN}^2)}$$

$$\frac{\times C_{xN} C_{yN}^3) - \theta_N C_{xN}^4 - 16\theta_N C_{yN}^2 C_{xN}^2 - 64\rho_{xyN}^2 C_{yN}^2 + 64C_{yN}^2)}{64(1 - 4\theta_N \rho_{xyN}^2 C_{yN}^2 + 2\theta_N \rho_{xyN} C_{yN} C_{xN} + \theta_N C_{yN}^2)} - \theta_N \bar{Y}_N^2 C_{yN}^2\Bigg] < 0$$

$$\frac{8\theta_N \rho_{xyN} C_{yN} C_{xN}^3 + 64\theta_N C_{yN}^3 \rho_{xyN} C_{xN} + \theta_N C_{xN}^4 + 16\theta_N C_{yN}^2 C_{xN}^2 + 64\rho_{xyN}^2 C_{yN}^2 + 64\theta_N C_{yN}^4}{16\theta_N \rho_{xyN}^2 C_{xN}^2 C_{yN}^2 + 192\theta_N \rho_{xyN}^2 C_{yN}^4} > 1$$

(ii) The estimator $\bar{y}_{R_2rN}$ from Eq. 46 will be efficient than the estimator $\bar{y}_{rN}$ from Eq. 6 iff
(MSE($\bar{y}_{R_2rN}$)-MSE($\bar{y}_{rN}$))< 0 or RE($\bar{y}_{R_2rN}, \bar{y}_{rN}$)$= \frac{MSE(\bar{y}_{rN})}{MSE(\bar{y}_{R_2rN})} > 1$, i.e.,

$$\Bigg[\frac{\theta_N \bar{Y}_N^2(16\theta_N \rho_{xyN}^2 C_{xN}^2 C_{yN}^2 - 64\theta_N \rho_{xyN}^2 C_{yN}^4 - 8\theta_N \rho_{xyN} C_{yN} C_{xN}^3 + 64\theta_N \rho_{xyN} C_{xN} C_{yN}^3)}{64(1 - 4\theta_N \rho_{xyN}^2 C_{yN}^2 + 2\theta_N \rho_{xyN} C_{yN} C_{xN} + \theta_N C_{yN}^2)}$$

$$\frac{-\theta_N C_{xN}^4 - 16\theta_N C_{yN}^2 C_{xN}^2 - 64\rho_{xyN}^2 C_{yN}^2 + 64C_{yN}^2)}{64(1 - 4\theta_N \rho_{xyN}^2 C_{yN}^2 + 2\theta_N \rho_{xyN} C_{yN} C_{xN} + \theta_N C_{yN}^2)}$$
$$-\theta_N \bar{Y}_N^2 (C_{yN}^2 + C_{xN}^2 - 2C_{xN} C_{yN} \rho_{xyN})\Bigg] < 0, i.e.,$$

$$\Bigg[\frac{\theta_N C_{xN}^4 + 80\theta_N C_{yN}^2 C_{xN}^2 + 136\theta_N \rho_{xyN} C_{yN} C_{xN}^3 + 64\rho_{xyN}^2 C_{yN}^2}{528\theta_N \rho_{xyN}^2 C_{xN}^2 C_{yN}^2 + 192\theta_N \rho_{xyN}^2 C_{yN}^4 + 64\theta_N \rho_{xyN} C_{yN}^3 C_{xN} + 128\rho_{xyN} C_{yN} C_{xN}}$$

$$\frac{64\theta_N C_{yN}^4 + 64 C_{xN}^2 + 512\theta_N \rho_{xyN}^3 C_{yN}^3 C_{xN}}{528\theta_N \rho_{xyN}^2 C_{xN}^2 C_{yN}^2 + 192\theta_N \rho_{xyN}^2 C_{yN}^4 + 64\theta_N \rho_{xyN} C_{yN}^3 C_{xN} + 128\rho_{xyN} C_{yN} C_{xN}}\Bigg] > 1$$

(iii) The estimator $\bar{y}_{R_2rN}$ from Eq. 46 will be more efficient than the estimator $\bar{y}_{BTrN}$ from Eq. 15 iff
MSE($\bar{y}_{R_2rN}$)-MSE($\bar{y}_{BTrN}$)<0 or RE($\bar{y}_{R_2rN}$,$\bar{y}_{BTrN}$)$=\frac{MSE(\bar{y}_{BTrN})}{MSE(\bar{y}_{R_2rN})}>1$, i.e.,

$$\Bigg[\frac{\theta_N\bar{Y}_N^2(16\theta_N\rho_{xyN}^2C_{xN}^2C_{yN}^2-64\theta_N\rho_{xyN}^2C_{yN}^4-8\theta_N\rho_{xyN}C_{yN}C_{xN}^3+64\theta_N\rho_{xyN}C_{xN}C_{yN}^3)}{64(1-4\theta_N\rho_{xyN}^2C_{yN}^2+2\theta_N\rho_{xyN}C_{yN}C_{xN}+\theta_NC_{yN}^2)}$$

$$\frac{-\theta_NC_{xN}^4-16\theta_NC_{yN}^2C_{xN}^2-64\rho_{xyN}^2C_{yN}^2+64C_{yN}^2)}{64(1-4\theta_N\rho_{xyN}^2C_{yN}^2+2\theta_N\rho_{xyN}C_{yN}C_{xN}+\theta_NC_{yN}^2)}-\theta_N\bar{Y}_N^2(C_{yN}^2+\frac{C_{xN}^2}{4}-C_{xN}C_{yN}\rho_{xyN})\Bigg]<0$$

$$\Bigg[\frac{256\theta_N\rho_{xyN}^3C_{xN}C_{yN}^2+40\theta_N\rho_{xyN}C_{yN}C_{xN}^3+\theta_NC_{xN}^4}{208\theta_N\rho_{xyN}^2C_{xN}^2C_{yN}^2+192\theta_N\rho_{xyN}^2C_{yN}^4+64\theta_N\rho_{xyN}C_{yN}C_{xN}}$$

$$\frac{+32\theta_NC_{xN}^2C_{yN}^2-64C_{yN}^4+64\rho_{xyN}^2C_{xN}^2+16C_{xN}^2}{208\theta_N\rho_{xyN}^2C_{xN}^2C_{yN}^2+192\theta_N\rho_{xyN}^2C_{yN}^4+64\theta_N\rho_{xyN}C_{yN}C_{xN}}\Bigg]>1$$

Similarly, further efficiency comparison is done like Shukla et al. (2023).

(iv) $\bar{y}_{R_2rN}$ is more efficient compared to $\bar{y}_{BTpN}$ iff RE($\bar{y}_{R_2rN}$,$\bar{y}_{BTpN}$)$=\frac{MSE(\bar{y}_{BTpN})}{MSE(\bar{y}_{R_2rN})}>1$

(v) $\bar{y}_{R_2rN}$ is more efficient compared to $\bar{y}_{SErpN}$ iff RE($\bar{y}_{R_2rN}$,$\bar{y}_{SErpN}$)$=\frac{MSE(\bar{y}_{SErpN})}{MSE(\bar{y}_{R_2rN})}>1$

(vi) $\bar{y}_{R_2rN}$ is more efficient compared to $\bar{y}_{SrpeN}$ iff RE($\bar{y}_{R_2rN}$,$\bar{y}_{SrpeN}$)$=\frac{MSE(\bar{y}_{SErpN})}{MSE(\bar{y}_{R_2rN})}>1$

(vii) $\bar{y}_{R_2rN}$ is more efficient compared to $\bar{y}_{pN}$ iff RE($\bar{y}_{R_2rN}$,$\bar{y}_{pN}$)$=\frac{MSE(\bar{y}_{pN})}{MSE(\bar{y}_{R_2rN})}>1$

(viii) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{nN}$ iff RE($\bar{y}_{R_1rN}$,$\bar{y}_{nN}$)$=\frac{MSE(\bar{y}_{nN})}{MSE(\bar{y}_{R_1rN})}>1$

(ix) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{rN}$ iff RE($\bar{y}_{R_1rN}$,$\bar{y}_{rN}$)$=\frac{MSE(\bar{y}_{rN})}{MSE(\bar{y}_{R_1rN})}>1$

(ix) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{BTrN}$ iff RE($\bar{y}_{R_1rN}$,$\bar{y}_{BTrN}$)$=\frac{MSE(\bar{y}_{BTrN})}{MSE(\bar{y}_{R_1rN})}>1$

(x) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{BTpN}$ iff RE($\bar{y}_{R_1rN}$,$\bar{y}_{BTpN}$)= $\frac{MSE(\bar{y}_{BTpN})}{MSE(\bar{y}_{R_1rN})} > 1$

(xi) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{SErpN}$ iff RE($\bar{y}_{R_1rN}$,$\bar{y}_{SErpN}$)= $\frac{MSE(\bar{y}_{SErpN})}{MSE(\bar{y}_{R_1rN})} > 1$

(xii) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{SrpeN}$ iff RE($\bar{y}_{R_2rN}$,$\bar{y}_{SrpeN}$)= $\frac{MSE(\bar{y}_{SrpeN})}{MSE(\bar{y}_{R_1rN})} > 1$

(xiii) $\bar{y}_{R_1rN}$ is more efficient compared to $\bar{y}_{pN}$ iff RE($\bar{y}_{R_1rN}$,$\bar{y}_{pN}$)=$\frac{MSE(\bar{y}_{pN})}{MSE(\bar{y}_{R_1rN})}$ > 1

5.1 Empirical Study This section consists of applying real-life data to examine the performance of the proposed estimators under NeSRS. The data is taken from chapter 16 of the book titled "Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics", and is aboutnetworking and sales of medical representatives. We have considered two neutrosophic variables Percentage of networking before the pandemic and Percentage of sales after the pandemic. Here, Percentage of networking before the pandemic is the neutrosophic subsidiary variable, $X_N \in [X_L, X_U]$ and `, Percentage of sales after the pandemic$'$ is a neutrosophic study character $Y_N \in [Y_L, Y_U]$. The parameters for the given data are listed in Table 1.

Further, by the method of NeSRS, we have taken total $n_N = 10, 15$ samples from the population, $N_N = 30$ and from Table 1, we have REs values of the propounded neutrosophic estimators in Table 5.

5.2 Monte-Carlo Simulation This section consists of applying a Monte-Carlosimulationfor neutrosophic values given in Vishwakarma and Singh (2022a, b). The neutrosophic random variables (NRV) follow a neutro-

Table 1: Neutrosophic parameters for the data

Percentage of networking before the pandemic vs Percentage of sales after pandemic			
Parameters	Neutrosophic values	Parameters	Neutrosophic values
N_N	$[30, 30]$	S_{yN}	$[12.32, 12.32]$
n_N	$[10, 10], [15, 15]$	C_{xN}	$[0.441, 0.438]$
$\bar{X}_N$	$[34.16, 34.45]$	C_{yN}	$[0.354, 0.351]$
$\bar{Y}_N$	$[34.80, 35.04]$	$\beta_{2(x)N}$	$[1.793, 1.793]$
S_{xN}	$[15.08, 15.09]$	ρ_{yxN}	$[0.861, 0.862]$

sophic normal distribution (NND) i.e., $(X_N, Y_N) \sim NN[(\mu_{xN}, \sigma^2_{xN}), (\mu_{yN}, \sigma^2_{yN})]$. The neutrosophic data are sourced from a 4 variable multivariate normal distribution characterized by means $(\mu_{xL}, \mu_{yL}, \mu_{xU}, \mu_{yU})$ and a covariance matrix

$$\begin{bmatrix} \sigma^2_{xL} & \rho_{yxL}\sigma_{xL}\sigma_{yN} & 0 & 0 \\ \rho_{yxL}\sigma_{xL}\sigma_{yN} & \sigma^2_{yL} & 0 & 0 \\ 0 & 0 & \sigma^2_{uU} & \rho_{vuU}\sigma_{uU}\sigma_{vU} \\ 0 & 0 & \rho_{vuU}\sigma_{uU}\sigma_{vU} & \sigma^2_{vN} \end{bmatrix}$$

The neutrosophic parameters for the simulated uncertain data are in Tables 2 and 3, and for classical data corresponding in Table 2 is given in Table 4. In Table 2, the neutrosophic data is generated with a correlation coefficient [0.70, 0.70], and in Table 3, the data is generated with a correlation coefficient[0.90, 0.90], and in Table 4, classical data is generated with correlation coefficient 0.70 along with means and variances are taken as averages of lower and upper values of corresponding neutrosophic means and variances in Table 2 for classical evaluation along with neutrosophic evaluation.

Further, by the method of NeSRS and SRS for both positive and negative correlation coefficients, we have taken total $n = 42, 72$ samples from the population of size 150 along with the parameters stated in Tables 2, 3,

Table 2: Neutrosophic parameters for the estimation under NeSRS

For +ve correlation coefficient			
Parameters	Neutrosophic values	Parameters	Neutrosophic values
N_N	$[150, 150]$	S_{yN}	$[10.54, 12.56]$
n_N	$[42, 42], [72, 72]$	C_{xN}	$[0.250, 0.207]$
$\bar{X}_N$	$[45.42, 65.69]$	C_{yN}	$[0.233, 0.194]$
$\bar{Y}_N$	$[45.29, 64.86]$	$\beta_{2(x)N}$	$[2.702, 3.114]$
S_{xN}	$[11.33, 13.58]$	ρ_{yxN}	$[0.700, 0.700]$
For -ve correlation coefficient			
Parameters	Neutrosophic values	Parameters	Neutrosophic values
N_N	$[150, 150]$	S_{yN}	$[11.56, 13.75]$
n_N	$[42, 42], [72, 72]$	C_{xN}	$[0.263, 0.215]$
$\bar{X}_N$	$[44.4, 65.38]$	C_{yN}	$[0.259, 0.211]$
$\bar{Y}_N$	$[44.71, 65.14]$	$\beta_{2(x)N}$	$[2.977, 2.946]$
S_{xN}	$[11.68, 14.05]$	ρ_{yxN}	$[-0.70, -0.70]$

Table 3: Neutrosophic parameters for the estimation under NeSRS

For +ve correlation coefficient			
Parameters	Neutrosophic values	Parameters	Neutrosophic values
N_N	$[150, 150]$	S_{yN}	$[11.48, 13.59]$
n_N	$[42, 42], [72, 72]$	C_{xN}	$[0.243, 0.176]$
$\bar{X}_N$	$[46.08, 65.3]$	C_{yN}	$[0.296, 0.210]$
$\bar{Y}_N$	$[46.29, 64.65]$	$\beta_{2(x)N}$	$[3.268, 3.426]$
S_{xN}	$[11.22, 13.59]$	ρ_{yxN}	$[0.900, 0.900]$
For -ve correlation coefficient			
Parameters	Neutrosophic values	Parameters	Neutrosophic values
N_N	$[150, 150]$	S_{yN}	$[10.30, 14.43]$
n_N	$[42, 42], [72, 72]$	C_{xN}	$[0.236, 0.204]$
$\bar{X}_N$	$[44.41, 63.48]$	C_{yN}	$[0.224, 0.217]$
$\bar{Y}_N$	$[46.02, 66.38]$	$\beta_{2(x)N}$	$[3.970, 3.010]$
S_{xN}	$[10.46, 12.97]$	ρ_{yxN}	$[-0.90, -0.90]$

and 4, we have the REs values of the proposed neutrosophic estimators and some other existing estimators under both NeSRSand classical simple random sampling (SRS). The whole computation of the attaining REs values

Table 4: Neutrosophic parameters for the estimation under NeSRS

For +ve correlation coefficient			
Parameters	Classical values	Parameters	Classical values
N	150	S_y	13.02
n	$42, 72$	C_x	0.228
$\bar{X}$	56.19	C_{yN}	0.233
$\bar{Y}$	55.85	$\beta_{2(x)}$	2.892
S_x	12.81	ρ_{yx}	0.700
For -ve correlation coefficient			
Parameters	Classical values	Parameters	Classical values
N	150	S_y	13.02
n	$42, 72$	C_x	0.228
$\bar{X}$	54.99	C_{yN}	0.221
$\bar{Y}$	54.8	$\beta_{2(x)}$	2.571
S_x	13.36	ρ_{yx}	-0.70

of the estimators under both NeSRS and SRS is repeated 6000 times and the results are shown in Tables 6, 7, and 8

6 Results and Discussion

We have given the mathematical expressions for the proposed estimators approximated up to the 1st order under NeSRS. Also, we have shown theoretical efficiency comparisons and numerically, we have carried out real data application and a simulation study on generated data. The calculated REs are shown in Tables 5, 6, 7, and 8.

In Table 5, the REs of the propounded neutrosophic estimators and some other existing estimators under NeSRS are given for sample sizes 10 and 15 through neutrosophic real data. The highest REs of the estimators $\bar{y}_{SErpN}$, $\bar{y}_{SrpeN}$, $\bar{y}_{R_1rN}$, $\bar{y}_{R_2rN}$ are in bold font and we can see our proposed Searls estimators $\bar{y}_{R_1rN}$, and $\bar{y}_{R_2rN}$ are superior to other estimators.

In Table 6, the REs of the propounded estimators and some existing estimators are given through the Monte-Carlo simulation method for ρ_{yxN}=[0.70, 0.70] and ρ_{yxN}=[-0.70, -0.70]. In Table 6 like Table 5, the highest REs of the estimators $\bar{y}_{SErpN}$, $\bar{y}_{SrpeN}$, $\bar{y}_{R_1rN}$, $\bar{y}_{R_2rN}$ are in bold font. Theproposed neutrosophic Searls estimators $\bar{y}_{R_1rN}$, and $\bar{y}_{R_2rN}$ are superior to proposed andother existing estimators under NeSRS. We also see the REs of the propounded estimators and other existing estimators under NeSRS are increasing with the increasing value of sample size.

Table 5: REs of the proposed estimators under NeSRS over $\bar{y}_{nN}$

Percentage of networking before the pandemic vs Percentage of sales after pandemic			
Estimators	REs $n=10$	Estimators	REs $n=15$
$\bar{y}_{nN}$	$[1.000, 1.000]$	$\bar{y}_{nN}$	$[1.000, 1.000]$
$\bar{y}_{rN}$	$[2.454, 2.468]$	$\bar{y}_{rN}$	$[2.454, 2.468]$
$\bar{y}_{pN}$	$[0.213, 0.213]$	$\bar{y}_{pN}$	$[0.213, 0.213]$
$\bar{y}_{BTrN}$	$[3.173, 3.183]$	$\bar{y}_{BTrN}$	$[3.173, 3.183]$
$\bar{y}_{BTpN}$	$[0.406, 0.406]$	$\bar{y}_{BTpN}$	$[0.406, 0.406]$
$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	**[3.865, 3.887]**	$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	**[3.865, 3.887]**
$\bar{y}_{R_1rN}$	**[4.032, 4.054]**	$\bar{y}_{R_1rN}$	**[3.947, 3.969]**
$\bar{y}_{R_2rN}$	**[3.891, 3.913]**	$\bar{y}_{R_2rN}$	**[3.878, 3.900]**

The bold highlighted fonts are the estimators which performed better over other estimators

Table 6: REs of the proposed estimators under NeSRS over $\bar{y}_{nN}$

$\rho_{xyN} = [0.70, 0.70]$			
Estimators	REs	Estimators	REs
	$n = 42$		$n = 72$
$\bar{y}_{nN}$	$[1.000, 1.000]$	$\bar{y}_{nN}$	$[1.000, 1.000]$
$\bar{y}_{rN}$	$[1.459, 1.615]$	$\bar{y}_{rN}$	$[1.466, 1.940]$
$\bar{y}_{pN}$	$[0.288, 0.265]$	$\bar{y}_{pN}$	$[0.277, 0.266]$
$\bar{y}_{BTrN}$	$[1.744, 1.957]$	$\bar{y}_{BTrN}$	$[1.805, 2.115]$
$\bar{y}_{BTpN}$	$[0.509, 0.479]$	$\bar{y}_{BTpN}$	$[0.495, 0.477]$
$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**1.816,2.081**]	$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**1.880, 2.374**]
$\bar{y}_{R_1rN}$	[**1.818, 2.082**]	$\bar{y}_{R_1rN}$	[**1.882, 2.375**]
$\bar{y}_{R_2rN}$	[**1.817, 2.081**]	$\bar{y}_{R_2rN}$	[**1.881, 2.374**]
$\rho_{xyN} = [-0.70, -0.70]$			
Estimators	REs	Estimators	REs
	$n = 42$		$n = 72$
$\bar{y}_{nN}$	$[1.000, 1.000]$	$\bar{y}_{nN}$	$[1.000, 1.000]$
$\bar{y}_{rN}$	$[0.324, 0.305]$	$\bar{y}_{rN}$	$[0.285, 0.283]$
$\bar{y}_{pN}$	$[1.326, 1.826]$	$\bar{y}_{pN}$	$[1.779, 1.863]$
$\bar{y}_{BTrN}$	$[0.552, 0.523]$	$\bar{y}_{BTrN}$	$[0.501, 0.498]$
$\bar{y}_{BTpN}$	$[1.545, 1.835]$	$\bar{y}_{BTpN}$	$[1.915, 1.964]$
$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**1.585, 2.044**]	$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**2.101, 2.185**]
$\bar{y}_{R_1rN}$	[**1.585, 2.045**]	$\bar{y}_{R_1rN}$	[**2.099, 2.185**]
$\bar{y}_{R_2rN}$	[**1.585, 2.045**]	$\bar{y}_{R_2rN}$	[**2.102, 2.187**]

The bold highlighted fonts are the estimators which performed better over other estimators

Similarly, in Table 7, like Table 6, the REs of the proposed estimators and some existing estimators are given for both ρ_{yxN}=[0.90, 0.90] and ρ_{yxN}=[-0.90, -0.90]. In Table 7 like Table 6, the highest REs of the estimators $\bar{y}_{SErpN}$, $\bar{y}_{SrpeN}$, $\bar{y}_{R_1rN}$, $\bar{y}_{R_2rN}$ are in bold font. The propounded neutrosophic Searls estimators $\bar{y}_{R_1rN}$, and $\bar{y}_{R_2rN}$ are superior to other proposed and existing estimators under NeSRS. We also seethe REsof theproposed estimators and other existing estimators under NeSRS are increasing proportionally to sample size and correlation coefficient.

Like Table 6, in Table 8, the REs of the proposed estimators and some existing estimators are given under classical simple random sampling through the Monte-Carlo simulation method for both ρ_{yx}=0.70 and ρ_{yx}=-0.70. In Table 8 like Table 6, The highest REs of the estimators $\bar{y}_{SErp}$, $\bar{y}_{Srpe}$, $\bar{y}_{R_1r}$, $\bar{y}_{R_2r}$ are in bold font. The proposed Searls estimators $\bar{y}_{R_1r}$, and $\bar{y}_{R_2r}$

Table 7: REs of the proposed estimators under NeSRS over $\bar{y}_{nN}$

$\rho_{xyN} = [0.90, 0.90]$			
Estimators	REs	Estimators	REs
	$n = 42$		$n = 72$
$\bar{y}_{nN}$	$[1.000, 1.000]$	$\bar{y}_{nN}$	$[1.000, 1.000]$
$\bar{y}_{rN}$	$[1.459, 1.615]$	$\bar{y}_{rN}$	$[1.466, 1.940]$
$\bar{y}_{pN}$	$[4.604, 5.090]$	$\bar{y}_{pN}$	$[4.933, 5.755]$
$\bar{y}_{BTrN}$	$[3.025, 2.919]$	$\bar{y}_{BTrN}$	$[2.313, 2.427]$
$\bar{y}_{BTpN}$	$[0.449, 0.461]$	$\bar{y}_{BTpN}$	$[0.525, 0.516]$
$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**5.185, 5.414**]	$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**5.069, 5.950**]
$\bar{y}_{R_1rN}$	[**5.193, 5.421**]	$\bar{y}_{R_1rN}$	[**5.084, 5.962**]
$\bar{y}_{R_2rN}$	[**5.186, 5.415**]	$\bar{y}_{R_2rN}$	[**5.070, 5.952**]
$\rho_{xyN} = [-0.90, -0.90]$			
Estimators	REs	Estimators	REs
	$n = 42$		$n = 72$
$\bar{y}_{nN}$	$[1.000, 1.000]$	$\bar{y}_{nN}$	$[1.000, 1.000]$
$\bar{y}_{rN}$	$[0.252, 0.278]$	$\bar{y}_{rN}$	$[0.264, 0.274]$
$\bar{y}_{pN}$	$[3.980, 5.851]$	$\bar{y}_{pN}$	$[4.919, 6.233]$
$\bar{y}_{BTrN}$	$[0.453, 0.482]$	$\bar{y}_{BTrN}$	$[0.466, 0.477]$
$\bar{y}_{BTpN}$	$[2.869, 2.741]$	$\bar{y}_{BTpN}$	$[2.837, 2.819]$
$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**4.502, 5.879**]	$\bar{y}_{SErpN} = \bar{y}_{SrpeN}$	[**5.174, 6.275**]
$\bar{y}_{R_1rN}$	[**4.503, 5.879**]	$\bar{y}_{R_1rN}$	[**5.174,6.277**]
$\bar{y}_{R_2rN}$	[**4.508, 5.890**]	$\bar{y}_{R_2rN}$	[**5.197,6.306**]

The bold highlighted fonts are the estimators which performed better over other estimators

are superior to proposed and other existing estimators under SRS. We also see that the REsof theproposed estimators and other existing estimators under SRS are increasing with the increasing value of sample size.We also observe that for interval types of data, neutrosophic proposed estimators are better than corresponding estimators as there is some loss of information due to their crisp nature, and overall neutrosophic results are better than corresponding classical values.

7 Conclusion

In this article, following Tahir et al. (2021), Vishwakarma and Singh (2022a, b), and by employing neutrosophic ancillary information, we have proposed neutrosophic generalized estimators with and without (Searls

Table 8: REs of the proposed estimators under SRS over $\bar{y}_{nN}$

$\rho_{xyN} = 0.70$			
Estimators	REs	Estimators	REs
	$n = 42$		$n = 72$
$\bar{y}_n$	1.000	$\bar{y}_n$	1.000
$\bar{y}_r$	1.149	$\bar{y}_r$	1.521
$\bar{y}_p$	0.325	$\bar{y}_p$	0.295
$\bar{y}_{BTr}$	1.447	$\bar{y}_{BTr}$	1.744
$\bar{y}_{BTp}$	0.557	$\bar{y}_{BTp}$	0.516
$\bar{y}_{SErp} = \bar{y}_{Srpe}$	**1.456**	$\bar{y}_{SErp} = \bar{y}_{Srpe}$	**1.836**
$\bar{y}_{R_1r}$	**1.458**	$\bar{y}_{R_1r}$	**1.837**
$\bar{y}_{R_2r}$	**1.457**	$\bar{y}_{R_2r}$	**1.837**
$\rho_{xyN} = -0.70$			
Estimators	REs	Estimators	REs
	$n = 42$		$n = 72$
$\bar{y}_n$	1.000	$\bar{y}_n$	1.000
$\bar{y}_r$	0.313	$\bar{y}_r$	0.293
$\bar{y}_p$	1.527	$\bar{y}_p$	1.559
$\bar{y}_{BTr}$	0.536	$\bar{y}_{BTr}$	0.513
$\bar{y}_{BTp}$	1.675	$\bar{y}_{BTp}$	1.772
$\bar{y}_{SErp} = \bar{y}_{Srpe}$	**1.770**	$\bar{y}_{SErp} = \bar{y}_{Srpe}$	**1.875**
$\bar{y}_{R_1r}$	**1.772**	$\bar{y}_{R_1r}$	**1.876**
$\bar{y}_{R_2r}$	**1.772**	$\bar{y}_{R_2r}$	**1.877**

The bold highlighted fonts are the estimators which performed better over other estimators

1964) technique. We haveshown the mathematical expressions of approximated Biases and MSEs for the neutrosophic proposed estimators up to 1st. Further, real data application and simulation studies have been carried out to highlight the properties of the proposed neutrosophic estimators. The propounded neutrosophic estimators with highREs have gained supremacy over the other existing estimators under NeSRS. Also, proposed neutrosophic generalized estimators using the Searls technique have shown supremacy over proposed neutrosophic generalized estimators without the Searls technique. We have also delineatedand concludedthat the neutrosophic method of estimation is more reliable and efficient than the classical method of estimationfor interval/uncertain types of data.

The work in sampling theory under a neutrosophic environment is initiated by Tahir et al. (2021) for estimatingthe population mean and later by

Vishwakarma and Singh (2022a, b), and motivated by them, we have also given generalized estimators with and without the Searls technique under NeSRS.The limitation of this study is that it is applicable to uncertain/fuzzy data.

The future study can also be done in other neutrosophic sampling methods like neutrosophic systematic sampling, stratified sampling, cluster sampling, successive sampling, neutrosophic ranked set sampling,and under its variantsor by applying some efficient estimators better than the estimators in this manuscript. Also, this same study can be done utilizing the Searls technique under RSS as by Singh and Vishwakarma (2021).

Funding The authors declare that there is no funding for this research.

Data Availability Statement We, the author confirm that the data helping the findings of this manuscript are available within the manuscript.

Declarations

Conflict of interest The authors declare that there is no conflict or competing interests.

Compliance with Ethical Standards The authors declare that there is no research involving Human Participants and/or Animals.

References

Abbasi, H., Shahzad, U., Emam, W., Hanif, M., Ali, N., & Mukhtar, M. (2024). Calibrated empirical neutrosophic cumulative distribution function estimation for both symmetric and asymmetric data. Symmetry, 16(5), 633.

AlAita, A., & Aslam, M. (2023). Analysis of covariance under neutrosophic statistics. *Journal of Statistical Computation and Simulation*, **93(3)**, 397-415.

Albassam, M., & Aslam, M. (2021). Testing Internal Quality Control of Clinical Laboratory Data Using Paired-Test under Uncertainty. *BioMed Research International*. https://doi.org/10.1155/2021/5527845

Alhabib, R., Ranna, M. M., Farah, H., & Salama, A. A. (2018). Some neutrosophic probability distributions. *Neutrosophic Sets and Systems*, **22**, 30-38.

Alomair, A. M., & Shahzad, U. (2023). Neutrosophic Mean Estimation of Sensitive and Non-Sensitive Variables with Robust Hartley?Ross-Type Estimators. *Axioms*, **12(6)**, 578. https://doi.org/10.3390/axioms12060578

Al-Omari, A. I., Jemain, A. A., & Ibrahim, K. (2009). New ratio estimators of the mean using simple random sampling and ranked set sampling methods. *Investigacio′nn Operacional*, **30(2)**, 97-108.

Aslam M (2019). Neutrosophic analysis of variance: application to university students. *Complex & Intelligent System*, **5**:403?407.https://doi.org/10.1007/s40747-019-0107-2

Aslam, M., & Algarni, A. (2020). Analyzing the Solar Energy Data Using a New Anderson-Darling Test under Indeterminacy. *International Journal of Photoenergy*, 2020. https://doi.org/10.1155/2020/6662389

Aslam, M., Arif, O. H., & Sherwani, R. A. K. (2020). New diagnosis test under the neutrosophic statistics: an application to diabetic patients. *BioMed Research International*, 00:7. https://doi.org/10.1155/2020/2086185

Atanassov K. 1986. Intuitionistic fuzzy sets. *Fuzzy Sets Syst*; **20(1)**:87-96.

Atanassov K. 1999. New operations defined over the intuitionistic fuzzy sets. *Fuzzy Sets Syst*;**61(2)**:137-42.

Bahl, S., & Tuteja, R. (1991). Ratio and product type exponential estimators. *Journal of information and optimization sciences*, **12(1)**, 159-164.

Chakraborty, A., Mondal, S. P., Alam, S., & Dey, A. (2021). Classification of trapezoidal bipolar neutrosophic number, de-bipolarization technique and its execution in cloud service-based MCGDM problem. *Complex & Intelligent Systems*, **7(1)**, 145-162.

Chen, C. T. (2000). Extensions of the TOPSIS for group decision-making under fuzzy environment. *Fuzzy sets and systems*, **114(1)**, 1-9.

Chen, J., Ye, J., & Du, S. (2017a). Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. *Symmetry*, **9(10)**, 208.

Chen, J., Ye, J., Du, S., & Yong, R. (2017b). Expressions of rock joint roughness coefficient using neutrosophic interval statistical numbers. *Symmetry*, **9(7)**, 123.

Cochran, G. (1940). Some properties of estimators based on sampling scheme with varying probabilities. *Aust. J. Stat.*, **17**,22-28.

Jana, C., & Pal, M. (2019). A robust single-valued neutrosophic soft aggregation operators in multi-criteria decision making. *Symmetry*, **11(1)**, 110.

Kadilar, C. and Cingi, H. (2004). Ratio estimators in simple random sampling, *Applied Mathematics and Computation*,**151**, 893-902

Kadilar, C. and Cingi, H. (2006). An Improvement in Estimating the Population mean by using the Correlation Co-efficient, *Hacettepe Journal of Mathematics and Statistics*, Volume **35 (1)**, 103-109

Kadilar, C., Unyazici, Y. And Cingi, H. (2009) Ratio estimator for the population mean using ranked set sampling, *Statistical Papers*, **50**, 301-309.

Khan, H., Sanaullah, A., Khan, M. A., & Siddiqi, A. F. (2014). Improved exponential ratio type estimators for estimating population mean regarding full information in survey sampling. *World Applied Science journal*, **26(5)**, 1897-1902.

Khanna, B. (2024). Generalized Regressed Exponential Estimator for Estimation of Mean Under Neutrosophic Ranked Set Sampling. *National Academy Science Letters*, 1-7.

Kumar, S. (2015). An estimator of the mean estimation of study variable using median of auxiliary variable. *Sri Lankan Journal of Applied Statistics*, **16(2)**, 107-115.

Kumari, A., Singh, R., & Smarandache, F. (2024). New Modification of Ranked Set Sampling for Estimating Population Mean: Neutrosophic Median Ranked Set Sampling with an Application to Demographic Data. *International Journal of Computational Intelligence Systems*, **17(1)**, 1-15.

Murthy, M. N. (1964). Product method of estimation. Sankhya: *The Indian Journal of Statistics, Series A*, 69-74.

Nabeeh, N. A., Smarandache, F., Abdel-Basset, M., El-Ghareeb, H. A., & Aboelfetouh, A. (2019). An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. *IEEE Access*, **7**, 29734-29744.

Raghav, Y. S. (2023). Neutrosophic generalized exponential robust ratio type estimators. *International Journal of Analysis and Applications*, **21**, 41-41.

Rao, G. S., & Aslam, M. (2021). Inspection plan for COVID-19 patients for Weibull distribution using repetitive sampling under indeterminacy. *BMC Medical Research Methodology*, **21(1)**, 1-15.

Rehman, S. A., & Asif, M. (2020). A Class of Estimator for Population Mean Under SRS. *STATISTICS, COMPUTING AND INTERDISCIPLINARY RESEARCH*, **2(1)**, 9-26.

Searls, D. T. (1964). The utilization of a known coefficient of variation in the estimation procedure. *Journal of the American Statistical Association*, **59 (308)**:1225–1226. https://doi.org/10.1080/01621459.1964.10480765.

Shahzad, U., Zhu, H., Albalawi, O., & Arslan, M. (2025). Neutrosophic developments in Horvitz-Thompson type estimators. *Mathematical Population Studies*, 1-14.

Shukla, S., Singh, A., & Vishwakarma, G. K. (2023). Predictive estimation for mean under median ranked set sampling: an application to COVID-19 data. *Indian Journal of Pure and Applied Mathematics*, 1-12.

Singh, A., Aslam, M., Vishwakarma, G. K., Dhital, A., & Patrascu, I. (2023a). Neutrosophic regression cum ratio estimators for the population mean: an application in medical science. *In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics*,313-333. Academic Press.

Singh, A., Kulkarni, H., Smarandache, F., & Vishwakarma, G. K. (2024a). Computation of Separate Ratio and Regression Estimator Under Neutrosophic Stratified Sampling: An Application to Climate Data. *Journal of Fuzzy Extension and Applications*.https://doi.org/10.22105/jfea.2024.422211.1313

Singh, R., Kumari, A., Smarandache, F., & Tiwari, S. N. (2024b). Construction Of Almost Unbiased Estimator for Population Mean Using Neutrosophic Information. *Infinite Study*.

Singh, A., Singh, P., Sharma, P., & Aloraini, B. (2025a). Estimation of Population Mean using Neutrosophic Exponential Estimators with Application to Real Data. *International Journal of Neutrosophic Science (IJNS)*, **25(3)**.

Singh, R., Smarandache, F., & Mishra, R. (2023b). Generalized robust-type neutrosophic ratio estimators of pharmaceutical daily stock prices. *In Cognitive intelligence with neutrosophic statistics in bioinformatics*, 417-429. Academic Press.

Singh, R., Smarandache, F., & Mishra, R. (2023c). Generalized robust-type neutrosophic ratio estimators of pharmaceutical daily stock prices. *In Cognitive Intelligence with Neutrosophic Statistics in Bioinformatics* (pp. 417-429). Academic Press.

Singh, H. P., & Espejo, M. R. (2003). On linear regression and ratio?product estimation of a finite population mean. *Journal of the Royal Statistical Society: Series D (The Statistician)*, **52(1)**, 59-67.

Singh, P., & Gupta, S. (2025). Combining Two Auxiliary Variables for Elevated Estimation of Finite Population Mean under Neutrosophic Framework. *Neutrosophic Sets and Systems*, **76**, 275-287.

Singh, R., & Kumari, A. (2024). Neutrosophic Ranked Set Sampling Scheme for Estimating Population Mean: An Application to Demographic Data. *Neutrosophic Sets and Systems*, **68**, 246-270.

Singh, R., & Tiwari, S. N. (2025). Improved Estimator for Population Mean Utilizing Known Medians of Two Auxiliary Variables under Neutrosophic Framework. Neutrosophic Systems with Applications, 25(1), 38-52.

Singh, A., & Vishwakarma, G. K. (2021). Improved predictive estimation for mean using the Searls technique under ranked set sampling. *Communications in Statistics-Theory and Methods*, **50(9)**, 2015-2038.

Singh, H.P., Tailor, R., Tailor, R. and Kakran, M.S. (2004). An Improved Estimator of population mean using Power transformation. Journal of the Indian Society of Agricultural Statistics 58(2), 223-230.

Singh, R., Chauhan, P., & Sawan, N. (2008). On linear combination of ratio and product type exponential estimator for estimating the finite population mean. *Statistics in Transition*, **9(1)**, 105-115.

Singh, P., Sharma, P., & Singh, A. (2025b). Estimation of population mean using generalized neutrosophic exponential estimators. *Research in Mathematics*, **12(1)**, 2474774.

Sisodia, B.V.S. and Dwivedi, V.K. (1981). A modified ratio estimator using co-efficient of variation of auxiliary variable. *Journal of the Inndian Society of Agricultural Statistics*,**33(1)**, 13-18

Smarandache, F. (1999). A unifying field in Logics: Neutrosophic Logic. In Philosophy (pp. 1-141). *American Research Press*.

Smarandache, F. (2001). A unifying field in logics: Neutrosophic logic, neutrosophic set, neutrosophic probability, and statistics. arXiv preprint math/0101228.

Smarandache, F. (2010). Neutrosophic logic-a generalization of the intuitionistic fuzzy logic. Multispace & multi structure. *Neutrosophic transdisciplinarity(100 collected papers of science)*, **4**, 396.

Smarandache, F. (2013). Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability. Infinite Study.

Smarandache, F. (2014). Introduction to neutrosophic statistics: Infinite Study.

Smarandache, F, Aslam, M. (2023). *Cognitive intelligence with neutrosophic statistics in bioinformatics*, Elsevier. https://doi.org/10.1016/C2021-0-02227-6

Smarandache, F. (1998). Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. *ProQuest Information & Learning*, Ann Arbor, MI, USA, vol. **105**, pp. 118-123.

Smarandache, F. (2005). Neutrosophic set a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, **24(3)**, 287.

Smarandache, F. (2019). Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). *Journal of New Theory,***(29)**, 1-31.

Tahir, Z., Khan, H., Alamri, F. S., & Aslam, M. (2023). Neutrosophic ratio-type exponential estimators for estimation of population mean. *Journal of Intelligent & Fuzzy Systems*, (Preprint), 1-25. https://content.iospress.com/articles/journal-of-intelligent-and-fuzzy-systems/ifs223539

Tahir, Z., Khan, H., Aslam, M., Shabbir, J., Mahmood, Y., & Smarandache, F. (2021). Neutrosophic ratio-type estimators for estimating the population mean. *Complex & Intelligent Systems*, 1-11.

Tailor, R. and Sharma. B. (2009). A modified ratio-cum-product product estimator of finite population mean using known coefficient of variation and coefficient of kurtosis. *Statistics in Transition*, **10**: 15-24.

Upadhyaya, L.N. and Singh, H.P. (1999). Use of transformed auxiliary variable in estimating the finite population mean, *Biometrical Journal*, **41 (5)**, 627-636.

Upadhyaya, L. N., Singh, H. P., Chatterjee, S., & Yadav, R. (2011). Improved ratio and product exponential type estimators. *Journal of statistical theory and practice*, **5**, 285-302.

Vishwakarma, G. K., & Singh, A. (2022a). Generalized estimator for computation of population mean under neutrosophic ranked set technique: An application to solar energy data. *Computational and Applied Mathematics*, **41(4)**, 144.

Vishwakarma, G., & Singh, A. (2022b). Computing the effect of measurement errors on ranked set sampling estimators of the population mean. *Concurrency and Computation: Practice and Experience*, **34(27)**, e7333.

Wang, H., Smarandache, F., Sunderraman, R., & Zhang, Y. Q. (2005). Interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing (Vol. **5**). *Infinite Study*

Yadav, S. K., & Smarandache, F. (2023). Generalized Neutrosophic Sampling Strategy for Elevated estimation of Population Mean. *Neutrosophic Sets and Systems*, **53(1)**, 14. https://digitalrepository.unm.edu/nssjournal/vol53/iss1/14/

Yan, Z. and Tian, B. (2010). Ratio method to the mean estimation using co-efficient of skewness of auxiliary variable, ICICA 2010, Part II, CCIS **106**, pp. 103-110.

Zadeh, L. A. (1996). Fuzzy sets. In Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A. Zadeh (pp. 394-432).

Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, **8(3)**, 338-353.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix A

Table 9: Some particulars of the suggested Searl estimators

Case 1		
k	α	**Estimators**
k	α	$\bar{y}_{R_1 rN} = k\bar{y}_N\left[\alpha\left(\frac{\bar{X}_N}{\bar{x}_N}\right) + (1-\alpha)\left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right]$ Proposed estimator
1	α	$\bar{y}_{SErpN} = \bar{y}_N\left[\alpha\left(\frac{\bar{X}_N}{\bar{x}_N}\right) + (1-\alpha)\left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right]$, Proposed estimator
k	1	$\bar{y}_{kaur\ rN} = k\bar{y}_N\left(\frac{\bar{X}_N}{\bar{x}_N}\right)$
k	0	$\bar{y}_{kaur\ pN} = k\bar{y}_N\left(\frac{\bar{x}_N}{\bar{X}_N}\right)$
1	1	$\bar{y}_{kaur\ rN} = k\bar{y}_N\left(\frac{\bar{X}_N}{\bar{x}_N}\right)$, Tahir et al.(2021)

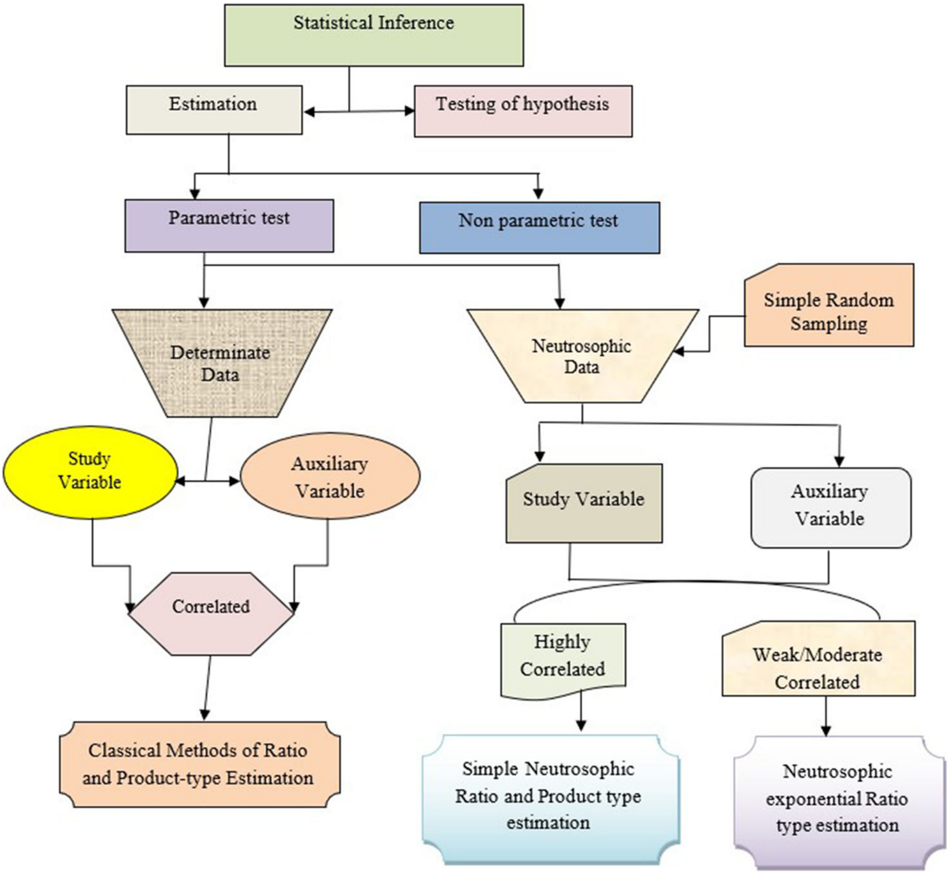
Table 9: continued

Case 2			
k	α	**Estimators**	
k	α	$\bar{y}_{R_2rN} = \bar{y}_N\left[\alpha\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right)+(1-\alpha)\left(\frac{\bar{x}_N-\bar{X}_N}{\bar{X}_N+\bar{x}_N}\right)\right]$	Proposed estimator
1	α	$\bar{y}_{SrpeN} = k\bar{y}_N\left[\alpha\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right)+(1-\alpha)\left(\frac{\bar{x}_N-\bar{X}_N}{\bar{X}_N+\bar{x}_N}\right)\right]$,	Proposed estimator
k	1	$\bar{y}_{SreN} = k\bar{y}_N exp\left[\alpha\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right)\right]$,	
k	1	$\bar{y}_{SpeN} = k\bar{y}_N exp\left[\alpha\left(\frac{\bar{x}_N-\bar{X}_N}{\bar{x}_N+\bar{X}_N}\right)\right]$,	
1	1	$\bar{y}_{BTrN} = \bar{y}_N exp\left[\alpha\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right)\right]$,	Tahir et al.(2021)
k	1	$\bar{y}_{BTpN} = \bar{y}_N exp\left[\alpha\left(\frac{\bar{x}_N-\bar{X}_N}{\bar{x}_N+\bar{X}_N}\right)\right]$,	Tahir et al.(2021)

Appendix B

Flow Chart

Here, Flow diagram is given below, which explians the path of using the proposed methods under neutrosophic numbers.



Bavita Singh
Department of Statistics, Amity Institute of Applied Sciences, Amity University, Sector 125, Noida 201313 U.P., India
E-mail: bavita43@gmail.com

Abhishek Singh
Department of Mathematics and Statistics, Dr. Vishwanath Karad MIT World Peace University, Paud Road, Kothrud, Pune 411038 Maharashtra, India
E-mail: abhisuryak@gmail.com

Florentin Smarandache
Mathematics, Physics and Natural Science Division, University of New Mexico, Gallup, 705 Gurley Ave., Gallup NM 87301, USA
E-mail: smarand@unm.edu

Paper received: 3 December 2024; accepted 22 May 2025.