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A Generalization of an Inequality of Tchebychev

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Statement: If $a_i^{(k)} \geq a_{i+1}^{(k)}$, $i \in \{1, 2, ..., n-1\}$, $k \in \{1, 2, ..., m\}$, then:

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} \geq \frac{1}{n^m}\prod_{k=1}^{m}\sum_{i=1}^{n} a_i^{(k)} .$$

Demonstration by recurrence on m .

Case $m = 1$ is obvious: $\frac{1}{n}\sum_{i=1}^{n} a_i^{(1)} \geq \frac{1}{n}\sum_{i=1}^{n} a_i^{(1)}$.

In the case $m = 2$, this is the inequality of Tchebychev itself:

If $a_1^{(1)} \geq a_2^{(1)} \geq ... \geq a_n^{(1)}$ and $a_1^{(2)} \geq a_2^{(2)} \geq ... \geq a_n^{(2)}$, then:

$$\frac{a_1^{(1)}a_1^{(2)} + a_2^{(1)}a_2^{(2)} + ... + a_n^{(1)}a_n^{(2)}}{n} \geq \frac{a_1^{(1)} + a_2^{(1)} + ... + a_n^{(1)}}{n} \times \frac{a_1^{(2)} + ... + a_n^{(2)}}{n}$$

One supposes that the inequality is true for all the values smaller or equal to m . It is necessary to prove for the rang $m+1$:

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m+1} a_i^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\left(\prod_{k=1}^{m} a_i^{(k)}\right) \cdot a_i^{(m+1)} .$$

This is $\geq \left(\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)}\right) \cdot \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{(m+1)}\right) \geq \left(\frac{1}{n^m}\prod_{k=1}^{m}\sum_{i=1}^{n} a_i^{(k)}\right) \cdot \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{(m+1)}\right)$

and this is exactly $\frac{1}{n^{m+1}}\prod_{k=1}^{m+1}\sum_{i=1}^{n} a_i^{(k)}$ (Quod Erat Demonstrandum).