

A NEW VIEW INTO COMPLEX PLANE CURVES

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Abstract. In this study, we give some characterizations of complex curves with isotropic principal normal in the complex plane \mathbb{C}^2 . First, we obtain a differential equation that characterizes the position vector of the curve and give some results about the Sturmian spirals for complex curves. Also, we take this curve into consideration as a curve of constant breadth. Finally, we introduce the notion of involute-evolute curve pair for complex curves in the complex plane.

1. Introduction

The concept of isotropic curve was first introduced by Poncelet [13]. Then the notion of complex curve and its moving frame were pioneered in the studies of Cartan [3]. Subsequently, Blaschke and Şemin elaborated the subject in both the complex plane and space by taking one of the frame vector fields of a complex curve as an isotropic vector. In particular, we are interested in the frame fields of complex plane curves with isotropic principal normal given in their works [2, 15].

In complex space (or plane), a curve is called an isotropic curve if the inner product of the tangent vector field with itself is zero. The notion of isotropic curve had been firstly mentioned about two hundred years ago [13], from that now on, a lot of work has been done on the subject. To give examples from recent history, some characterizations have been given on isotropic curves in 3- and 4-dimensional complex spaces [4, 14, 17-20].

Curves have many different fields of study in physics and differential geometry. Some studies on position vectors and constant breadth curves in Minkowski and complex spaces are noteworthy. These studies have also detailed special curves such as Smarandache curves and circular indicatrices on these spaces [5, 10]. Also, the studies on Sturmian spirals in the Euclidean and Minkowski planes have guided our researches [8, 11, 21]. On the other hand, some special curve pairs such as Bertrand, Mannheim and involute-evolute ones have been

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studied for different kinds of frames and in various spaces. The Bertrand and involute – evolute curve pairs have also been studied for isotropic curves in complex spaces [12, 17]. We focused on the involute-evolute curve pair and noticed that these curve pairs have been considered due to the frames such as Bishop, Frenet, Darboux ones in Euclidean space [1, 7, 9].

The motivation of our study is to find how to characterize the position vector, curves of constant breadth and involute-evolute curve pair for complex curves with isotropic principal normal in the complex plane \mathbb{C}^2 . First, we obtain the position vector of our special plane curve and give the Sturmian spiral characterization through this vector, and then investigate the complex curves as curves of constant breadth. Finally, due to the E. Cartan frames of the isotropic curve and the curve with isotropic principal normal, we characterize involute-evolute curve pair of these ones.

2. Preliminaries

The complex plane \mathbb{C}^2 is the set of ordered pairs of complex numbers with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2,$$

where $\mathbf{x} = (x_1, x_2) = a + ib \in \mathbb{C}^2$, $\mathbf{y} = (y_1, y_2) = c + id \in \mathbb{C}^2$, and $a, b, c, d \in \mathbb{R}^2$ [2, 15].

Definition 1. A function of a complex variable is said to be analytic if it is differentiable everywhere in its domain of definition [16].

Let x_p be a complex analytic function of a complex variable t . Then the vector function

$$\mathbf{x}(t) = \sum_{p=1}^2 x_p(t) \mathbf{k}_p$$

is called a complex plane curve where $t = t_1 + it_2$, $t_1, t_2 \in \mathbb{R}$, $\mathbf{x} : U \subset \mathbb{C} \rightarrow \mathbb{C}^2$, and the vectors \mathbf{k}_p are the standard basis unit vectors of \mathbb{E}^2 .

In this plane, a vector which has a minimal direction is called an isotropic vector or minimal vector, that is, a vector \mathbf{u} is a minimal vector if and only if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ [12]. The curves whose distance square between the two points is equal to zero, are called minimal or isotropic curves. Let s denote pseudo-arc length, a curve is an isotropic (minimal) curve if and only if $ds^2 = 0$ [19].

A complex curve is a smooth mapping $\mathbf{x} : U \subset \mathbb{C} \rightarrow \mathbb{C}^2$ from an open interval I onto \mathbb{C}^2 . The coordinate representation of x is denoted as $\mathbf{x} = (x_1(s), x_2(s))$ and also the vector

$$\frac{d\mathbf{x}}{ds} = \left(\frac{dx_1}{ds}, \frac{dx_2}{ds} \right) = \mathbf{b}_1$$

is called the tangent vector field of the curve $\mathbf{x} = \mathbf{x}(s)$, and \mathbf{b}_2 is the normal vector field of the curve [2, 15]. In the rest of the paper, we shall consider the complex curves with the isotropic normal vector field. So, we have the

derivative formula with two dimensional version of E. Cartan frame $\{\mathbf{b}_1, \mathbf{b}_2\}$ as follows;

$$(1) \quad \begin{bmatrix} \mathbf{b}_1' \\ \mathbf{b}_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix},$$

where $\rho = \rho(s)$ is the pseudo curvature of the complex curve $\mathbf{x} = \mathbf{x}(s)$. Here, $\langle \mathbf{b}_1, \mathbf{b}_1 \rangle = 1$, $\langle \mathbf{b}_2, \mathbf{b}_2 \rangle = 0$, $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle = 0$. In the complex plane, we have two cases for the vector product as follows;

$$\mathbf{b}_1 \wedge \mathbf{b}_2 = -i\mathbf{b}_2,$$

$$\mathbf{b}_2 \wedge \mathbf{b}_1 = i\mathbf{b}_1.$$

The basic idea, in the cross product of one real vector with an isotropic vector results an isotropic one. Therefore, shortly we present this product mentioned above as $\mathbf{b}_2 = i\mathbf{b}_1 \wedge \mathbf{b}_2$ [2, 15].

Definition 2. The Sturmiian spirals are planar curves whose pseudo curvature functions $\rho(s)$ satisfy the relations

$$(2) \quad \rho(s) = \frac{\sigma}{r}, \quad r = \sqrt{|\langle \alpha(s), \alpha(s) \rangle|},$$

where $r \in \mathbb{R}^+$ is the distance from the origin and σ is a positive real constant in Euclidean plane \mathbb{E}^2 [8,11].

3. Position vector of a complex plane curve with isotropic principal normal in \mathbb{C}^2

Let $\alpha = \alpha(s)$ be a unit speed complex curve with isotropic principal normal in the complex plane \mathbb{C}^2 . Then the position vector of this curve with respect to the frame given by the Eq. (1) can be written as follows;

$$(3) \quad \alpha = \alpha(s) = \lambda(s)\mathbf{b}_1(s) + \mu(s)\mathbf{b}_2(s),$$

where λ, μ are the arbitrary functions of s . By using the Eq. (1) in differentiation of the Eq. (3), we obtain

$$(4) \quad \begin{cases} \lambda'(s) - 1 = 0, \\ \mu'(s) + \mu(s)\rho(s) + \lambda(s) = 0. \end{cases}$$

The solution of the system given in Eq. (4) gives the differential equation

$$(5) \quad \mu'(s) + \rho(s)\mu(s) = c - s.$$

This equation characterizes the curve $\mathbf{x} = \mathbf{x}(s)$ with isotropic principal normal due to μ . Solving the Eq. (5) presents

$$(6) \quad \mu = e^{-\int \rho ds} \int e^{\int \rho ds} (c - s) ds.$$

Then, this result can be enclosed as follows;

Theorem 1. Let $\alpha = \alpha(s)$ be a unit speed complex curve with isotropic principal normal in the complex plane \mathbb{C}^2 . The position vector of the curve $\alpha = \alpha(s)$ is given by following equation

$$(7) \quad \alpha(s) = s\mathbf{b}_1 + e^{-\int \rho ds} \int e^{\int \rho ds} (c - s) ds \mathbf{b}_2,$$

where $\rho(s)$ is the pseudo curvature function.

Using the Eq. (7), we characterize the Sturmian spirals for complex curves in the complex space by using the Eq. (2) and the following corollary can be given:

Corollary 1: Let $\alpha = \alpha(s)$ be a complex curve with isotropic principal normal in the complex plane. The condition for the curve $\alpha(s)$ to be a Sturmian spiral is

$$(8) \quad \alpha(s) = s\mathbf{b}_1 - \frac{s^2}{a+2}\mathbf{b}_2, \quad a \in \mathbb{R}.$$

Proof. Substituting the Eq. (7) into the Eq. (2), we have

$$r = s,$$

and

$$s.\rho = \sigma = const.$$

In this case, we have $\rho = 0$ or $\rho = \frac{a}{s}$, $a \in \mathbb{R}$. It is not possible $\rho = 0$, because of the Eq. (2) and $\sigma > 0$. Then, substituting the equation $\rho = \frac{a}{s}$ into the Eq. (7) gives us the Eq. (8), straightforwardly.

4. Complex plane curves of constant breadth

Let $\alpha = \alpha(s)$ and $\alpha^* = \alpha^*(s)$ be simple closed curves with isotropic principal normal in the complex plane \mathbb{C}^2 . These curves will be denoted by C and C^* . At every point p on the curve, the isotropic normal plane meets the curve at a single point q (opposite point of p) other than p . In the class Γ as in [6], we consider curves with parallel tangents \mathbf{b}_1 and \mathbf{b}_1^* in opposite directions at opposite points α and α^* of the curve.

A simple closed curve of constant breadth with an isotropic principal normal having parallel tangents in opposite directions at opposite points can be represented by the following equation with respect to the frame given by the Eq. (1):

$$(9) \quad \alpha^* = \alpha + \gamma\mathbf{b}_1 + \delta\mathbf{b}_2,$$

where $\gamma(s)$ and $\delta(s)$ are the arbitrary functions of s , and α and α^* are opposite points of curves. Here, the distance vector between the opposite points of C and C^* is defined as follows;

$$(10) \quad \mathbf{d} = \alpha^* - \alpha.$$

If we differentiate the Eq. (9) and substitute the derivative formulas given by the Eq. (1), we reach

$$(11) \quad \mathbf{b}_1^* \frac{ds^*}{ds} = \left(1 + \frac{d\gamma}{ds}\right) \mathbf{b}_1 + \left(\gamma + \rho\delta + \frac{d\delta}{ds}\right) \mathbf{b}_2.$$

It is known that, the tangents \mathbf{b}_1 and \mathbf{b}_1^* are parallel to each other in opposite directions at the opposite points α and α^* of the curve from the definition of curve of constant breadth. Then, there is the relation

$$(12) \quad \mathbf{b}_1^* = -\mathbf{b}_1,$$

and also using the Eq. (12) in the Eq. (11), we get

$$(13) \quad \begin{cases} \frac{d\gamma}{ds} + 1 = -\frac{ds^*}{ds}, \\ \frac{d\delta}{ds} + \rho\delta + \gamma = 0. \end{cases}$$

Let θ be the angle between the tangent vector \mathbf{b}_1 at the point $\alpha(s)$ of an oval and a fixed direction, then it can be written

$$(14) \quad \frac{ds}{d\theta} = \omega = \frac{1}{\rho}, \quad \text{and} \quad \frac{ds^*}{d\theta} = \omega^* = \frac{1}{\rho^*}.$$

Thus, the Eq. (13) takes the following form:

$$(15) \quad \begin{cases} -\frac{d\gamma}{d\theta} = \omega^* + \omega = f(\theta), \\ \frac{d\delta}{d\theta} + \delta = -\gamma\omega. \end{cases}$$

Here, if we take the integration of both side of the Eq. (15)₁ with respect to θ , it is obtained that

$$(16) \quad \gamma = -\int (\omega^* + \omega) d\theta = -\int f(\theta)d\theta.$$

Substituting the Eq. (16) into the Eq. (15)₂, we have the following differential equation according to δ

$$(17) \quad \delta' + \delta = -\omega \int f(\theta)d\theta.$$

Then, from the solution of the Eq. (17), we find

$$(18) \quad \delta = -e^{-\theta} \int [\omega e^{\theta} \int f(\theta)d\theta] d\theta.$$

Therefore, substituting the Eqs. (16) and (18) into the Eq. (9) give the equation characterizing the curves of constant breadth.

Theorem 2. The position vector of a simple closed complex plane curve of constant breadth with an isotropic principal normal having parallel tangents in opposite directions at opposite points is expressed as follows;

$$(19) \quad \alpha^* = \alpha - \int \left(\frac{1}{\rho^*} + \frac{1}{\rho}\right) d\theta \mathbf{b}_1 - e^{-\theta} \int \left(\frac{1}{\rho^*} + \frac{1}{\rho}\right) d\theta \int \frac{e^{\theta}}{\rho} d\theta \mathbf{b}_2,$$

where θ is the angle between the tangent vector \mathbf{b}_1 at the point $\alpha(s)$ of an oval and a fixed direction, ρ^* and ρ are pseudo curvatures of the curves α^* and α , respectively.

On the other hand, if the distance function between the opposite points of C^* and C is constant, then it can be written that

$$(20) \quad \|\alpha^* - \alpha\| = \gamma^2 = \text{const.}$$

Differentiating the Eq. (20) gives

$$(21) \quad \gamma \frac{d\gamma}{d\theta} = 0.$$

From the Eq. (21), we obtain

$$(22) \quad \gamma = 0 \quad \text{or} \quad \gamma = \text{const.}$$

In this case, considering the Eq. (15)₁, we reach the following equation:

$$(23) \quad \rho^* = -\rho.$$

Thus the Eq. (23) shows the pseudo curvatures belonging to the curves α^* and α which are parallel in the opposite directions. Therefore, substituting the Eq. (23) into the Eq. (19), we can reach the following result:

$$(24) \quad \alpha^* = \alpha.$$

Hence, the following corollary is given:

Corollary 2. Let $\alpha = \alpha(s)$ and $\alpha^* = \alpha^*(s)$ be simple closed curves with isotropic principal normal denoted by C and C^* in the complex plane \mathbb{C}^2 . If the distance function between the opposite points α and α^* of the curve C and C^* are constant, the position vectors of the curves coincide for every point on the curves.

5. Involute-evolute curve pair of complex curves in \mathbb{C}^2

Definition 3. Let $\tilde{\alpha}$ and α be two regular complex curves in the complex plane \mathbb{C}^2 . If the tangent vector of the complex curve α at the point $\alpha(s)$ passes through the tangent vector of the curve $\tilde{\alpha}$ at the point $\tilde{\alpha}(s)$ and the following condition holds

$$(25) \quad \langle \tilde{\mathbf{b}}_1, \mathbf{b}_1 \rangle = 0,$$

the complex curve $\tilde{\alpha}$ is called the involute of the curve α and also the curve α is called the evolute of the curve $\tilde{\alpha}$. Here, the sets of the vector fields $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$ represent the complex curves $\tilde{\alpha}$ and α , respectively. In the further of the study, \tilde{s} and s are pseudo-arclength parameters of the complex curves $\tilde{\alpha}$ and α , respectively [7]. Then, the following corollary can be given:

Corollary 3. Let $\tilde{\alpha}$ be the involute complex curve of the complex curve with isotropic principal normal α and the sets of vector fields $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$ represent the curves $\tilde{\alpha}$ and α , respectively. There exists the following relation between these curve pair:

$$(26) \quad \tilde{\alpha}(\tilde{s}) = \alpha(s) + \xi(s)\mathbf{b}_1(s),$$

where $\xi(s)$ is an arbitrary function of s .

Corollary 4. Let $\tilde{\alpha}$ be the involute complex curve of the curve with isotropic principal normal α and the sets of vector fields $\{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$ represent the curves $\tilde{\alpha}$ and α , respectively. If the evolute α is a complex curve with isotropic principal normal, then the involute $\tilde{\alpha}$ has to be an isotropic curve lying in the complex 3-space since $\tilde{\mathbf{b}}_1(\tilde{s}) = \mathbf{b}_2(s)$. Then, the E. Cartan frame of the curve $\tilde{\alpha}$ is given as follows;

$$(27) \quad \begin{bmatrix} \tilde{\mathbf{b}}_1' \\ \tilde{\mathbf{b}}_2' \\ \tilde{\mathbf{b}}_3' \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 \\ i\tilde{\rho} & 0 & i \\ 0 & -i\tilde{\rho} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_1 \\ \tilde{\mathbf{b}}_2 \\ \tilde{\mathbf{b}}_3 \end{bmatrix},$$

where, $i^2 = -1$ and $\tilde{\rho}(s)$ is the pseudo curvature of the curve $\tilde{\alpha}$ [2, 15, 19].

Remark 1. The involute curve of a curve with isotropic principal normal has an isotropic tangent vector. This leads to the conclusion that the involute curve lies in complex 3-space [2, 15].

In this case, we can give the following theorems for involute-evolute curve pair:

Theorem 3. Let $\tilde{\alpha}$ be the involute isotropic curve of the curve α in the complex space \mathbb{C}^3 . The distance between the complex curves $\tilde{\alpha}$ and α is as follows;

$$(28) \quad d(\tilde{\alpha}(\tilde{s}) - \alpha(s)) = |c - s|,$$

where c is constant.

Proof. Differentiating both sides of the Eq. (26) with respect to s , and using the Eqs. (1) and (27), we reach

$$\tilde{\mathbf{b}}_1(\tilde{s}) \frac{d\tilde{s}}{ds} = \left(\frac{d\xi(s)}{ds} + 1 \right) \mathbf{b}_1(s) + \xi(s) \mathbf{b}_2(s).$$

From the condition of the involute-evolute curve pair $\langle \tilde{\mathbf{b}}_1, \mathbf{b}_1 \rangle = 0$. Taking the inner product of both sides by \mathbf{b}_1 gives

$$\frac{d\xi}{ds} + 1 = 0$$

whose solution is

$$(29) \quad \xi(s) = c - s,$$

where c is a constant. In this case, the Eq. (26) can be written as follows;

$$(30) \quad \tilde{\alpha}(\tilde{s}) - \alpha(s) = (c - s) \mathbf{b}_1(s).$$

Taking the norm of the Eq. (30) completes the proof.

Theorem 4. Let $\tilde{\alpha}$ be the involute isotropic curve in the complex space \mathbb{C}^3 . $\tilde{\alpha}$ is always a point in complex space \mathbb{C}^3 for the curve α .

Proof. Differentiating the Eq. (30) two times with respect to s , and using the Eqs. (1) and (27), we get

$$(31) \quad -i\tilde{\mathbf{b}}_2(\tilde{s}) \left(\frac{d\tilde{s}}{ds}\right)^2 + \tilde{\mathbf{b}}_1(\tilde{s}) \frac{d^2\tilde{s}}{ds^2} = [(c-s)\rho - 1] \mathbf{b}_2(s).$$

Taking the norm of the last equation gives

$$(32) \quad \frac{d\tilde{s}}{ds} = 0,$$

and \tilde{s} is a constant. Multiplying the both side of the Eq. (31) with the vector $\tilde{\mathbf{b}}_1$, we have

$$(33) \quad \rho = \frac{1}{c-s}.$$

From the Eqs. (32) and (33), it is seen that the involute isotropic curve $\tilde{\alpha}$ is a point in the complex 3-space while the evolute α is a curve with isotropic principal normal lying in the complex plane.

On the other hand, from the definition of the involute - evolute curve pair, we know that

$$(34) \quad \tilde{\mathbf{b}}_1(\tilde{s}) = \mathbf{b}_2(s).$$

If we differentiate the Eq. (34), then we reach the following equation:

$$(35) \quad -i\tilde{\mathbf{b}}_2(\tilde{s}) \frac{d\tilde{s}}{ds} = \rho(s)\mathbf{b}_2(s).$$

Taking the norm of the Eq. (35), we get again

$$(36) \quad \frac{d\tilde{s}}{ds} = 0,$$

and \tilde{s} is a constant. Multiplying the both side of the Eq. (35) with the vector $\tilde{\mathbf{b}}_1$, we have

$$(37) \quad \rho = 0.$$

From the Eqs. (36) and (37), it can be seen that the involute isotropic curve $\tilde{\alpha}$ is a point in the complex 3-space while the evolute curve α is a straight line with isotropic principal normal lying in the complex plane.

It is clear that when the evolute α is a curve or a straight line in the complex plane \mathbb{C}^2 , the involute $\tilde{\alpha}$ is always a point in complex space \mathbb{C}^3 . Then the proof is completed.

6. Conclusion

In this research, the complex curves with the isotropic principal normal vector fields have been investigated in the complex plane \mathbb{C}^2 . Firstly, the position vectors of these curves have been characterized due to the corresponding frame. We have analyzed the complex curves as the Sturmian spirals in the complex plane. Secondly, we have examined complex planar curves with an isotropic principal normal vector to be curve of constant breadth in the complex plane. Finally, the notion of the involute-evolute curve pair has been considered for complex curves (the involute-evolute curve pair corresponds to the isotropic

curve, and the curve with isotropic principal normal, respectively) in the complex plane \mathbb{C}^2 . Based upon on the distance function between these curve pair, a certain result about the pseudo-arclength parameter of involute curve has been reached in the complex plane.

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