Almost Unbiased Estimator for Estimating Population Mean Using Known Value of Some Population Parameter(s)

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Abstract

In this paper we have proposed an almost unbiased estimator using known value of some population parameter(s). Various existing estimators are shown particular members of the proposed estimator. Under simple random sampling without replacement (SRSWOR) scheme the expressions for bias and mean square error (MSE) are derived. The study is extended to the two phase sampling. Empirical study is carried out to demonstrate the superiority of the proposed estimator.

Key words: Auxiliary information, bias, mean square error, unbiased estimator, two phase sampling.

1. Introduction

Consider a finite population U = $U_1, U_2, \ldots U_N$ of N units. Let y and x stand for the variable under study and auxiliary variable respectively. Let (y_i, x_i), i=1, 2,.., n denote the values of the units included in a sample s_n of size n drawn by simple random sampling without replacement (SRSWOR). The auxiliary information has been used in improving the precision of the estimate of a parameter (See Cochran (1977), Sukhatme et. al. (1984) and the references cited there in). Out of many methods, ratio and product methods of estimation are good illustrations in this context.

In order to have a survey estimate of the population mean $\bar{Y}$ of the study character y, assuming the knowledge of the population mean $\bar{X}$ of the auxiliary character x, the well-known ratio estimator is

$$t_r = \bar{y}\,\frac{\bar{X}}{\bar{x}} \tag{1.1}$$

Bahl and Tuteja (1991) suggested an exponential ratio type estimator as –

$$t_{re} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{1.2}$$

Several authors have used prior value of certain population parameter(s) to find more precise estimates. Sisodiya and Dwivedi (1981), Sen (1978) and Upadhyaya and Singh (1984) used the known coefficient of variation (CV) of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of coefficient of kurtosis in estimating the population variance of study character y was first made by Singh et. al. (1973). Later used by Singh and Kakaran (1993) in the estimation of population mean of study character. Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient. Kadilar and Cingi (2006), Khosnevisan et. al. (2007), Singh et. al. (2007) Singh and Kumar (2009) and Singh et. al. (2009) have suggested modified ratio estimators by using different pairs of known value of population parameter(s).

In this paper under SRSWOR, we have proposed almost unbiased estimator for estimating $\overline{Y}$.

2. Almost unbiased ratio type estimator

Suppose

$$t_0 = \bar{y}, \qquad t_{rs} = \bar{y}\left(\frac{a\bar{X}+b}{\bar{x}+b}\right), \qquad t_{rse} = \bar{y}\exp\left\{\frac{(a\bar{X}+b)-(a\bar{x}+b)}{(a\bar{X}+b)+(a\bar{x}+b)}\right\}$$

Such that $t_0, t_{rs}, t_{rse} \in w_r$ where w_r denotes the set of all possible ratio type estimators for estimating the population mean $\bar{Y}$. By definition the set w_r is a linear variety, if

$$t_{wr} = \omega_0\bar{y} + \omega_1 t_{rs} + \omega_2 t_{rse} \in w \qquad (2.1)$$

for $\sum_{i=0}^{2}\omega_i = 1, \quad \omega_i \in R$ (2.2)

where ω_i (i=0, 1, 2) denotes the statistical constants and R denotes the set of real numbers.

To obtain the bias and MSE of t_w, we write

$$\bar{y} = \bar{Y}(1+e_0), \quad \bar{x} = \bar{X}(1+e_1),$$

such that

$$E(e_0) = E(e_1) = 0.$$

$$E(e_0^2) = f_1 C_y^2, \qquad E(e_1^2) = f_1 C_x^2, \qquad E(e_0 e_1) = f_1 \rho C_y C_x.$$

where $f_1 = (\frac{1}{n} - \frac{1}{N})$, $\quad S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})^2$, $\quad S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \bar{X})^2$,

$$C_y = \frac{S_y}{\bar{Y}}, \; C_x = \frac{S_x}{\bar{X}}, \qquad K = \rho\left(\frac{C_y}{C_x}\right), \; \rho = \frac{S_{yx}}{(S_y S_x)},$$

$$S_{yx} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}).$$

Expressing t_w in terms of e's, we have

$$t_w = \bar{Y}(1+e_0)\left[\omega_0 + \omega_1(1+\theta e_1)^{-1} + \omega_2 \exp\left\{-\frac{\theta e_1}{2}(1+\theta e_1)^{-1}\right\}\right] \tag{2.3}$$

where $\theta = \frac{a\bar{X}}{a\bar{X}+b}$.

Expanding the right hand side of (2.3) and retaining terms up to second order of e's, we have

$$t_w \cong \bar{Y}\left[1 + e_0 - \omega\theta e_1 + \theta^2\left(\omega_1 + \frac{3}{8}\right)e_1^2 - \theta\omega e_0 e_1\right] \tag{2.4}$$

where $\omega = (\omega_1 + \frac{\omega_2}{2})$. (2.5)

Taking expectations of both side of (2.4) and then subtracting $\bar{Y}$ from both side, we get the bias of the estimator t_w, up to the first order of approximation as

$$B(t_w) = f_1\bar{Y}\left[\theta^2 C_x^2\left(\omega_1 + \frac{3\omega_2}{8}\right) - \theta\omega\rho C_y C_x\right] \tag{2.6}$$

$$B(t_{rs}) = f_1\bar{Y}\left[\theta^2 C_x^2 - \theta\rho C_y C_x\right] \tag{2.7}$$

$$B(t_{rse}) = f_1\bar{Y}\left[\frac{3\theta^2 C_x^2}{8} - \frac{\theta\rho C_y C_x}{2}\right] \tag{2.8}$$

From (2.4), we have

$$(t_w - \bar{Y}) \cong \bar{Y}\left[e_0 - \theta\omega e_1\right] \tag{2.9}$$

Squaring both sides of (2.9) and then taking expectations, we get MSE of the estimator t_w, up to the first order of approximation, as

$$MSE(t_w) = f_1\bar{Y}\left[C_y^2 + \theta^2\omega^2 C_x^2 - 2\theta\omega\rho C_y C_x\right] \tag{2.10}$$

This is minimum when

$$\omega = k\left(= \rho\frac{C_y}{C_x}\right). \tag{2.11}$$

Putting this value of $\omega(=k)$ in (2.10), we get the minimum MSE of t_w as

min. $MSE(t_w) = f_1\bar{Y}^2 C_y^2(1-\rho^2)$ (2.12)

which is same as that of traditional linear regression estimator

from (2.5) and (2.11), we have

$$\omega_1 + \frac{\omega_2}{2} = k. \tag{2.13}$$

From (2.2) and (2.13), we have only two equations in three unknowns. It is not possible to find the unique values for $\omega_{i's}, i = 0,1,2.$ In order to get unique values for $\omega_{i's}$, we shall impose the linear restriction

$$\omega_0 B(\bar{y}) + \omega_1 B(t_{rs}) + \omega_2 B(t_{rse}) = 0 \tag{2.14}$$

Equations (2.2), (2.11) and (2.14) can be written as in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & B(t_{rs}) & B(t_{rse}) \end{bmatrix}\begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix} \tag{2.15}$$

Using (2.15) ,we get unique values of $\omega_{i's}$ (i=0,1,2) as

$$\left.\begin{array}{l}\omega_0 = \frac{\Delta_0}{\Delta_r}\\ \omega_1 = \frac{\Delta_1}{\Delta_r}\\ \omega_2 = \frac{\Delta_2}{\Delta_r}\end{array}\right\} \tag{2.16}$$

where

$$\left.\begin{array}{c}\Delta_r = B(t_{rse}) - \frac{1}{2}B(t_{rs})\\ \Delta_{r0} = B(t_{rse})\{1-k\} + \frac{1}{2}B(t_{rs})\left\{k-\frac{1}{2}\right\}\\ \Delta_r = k.B(t_{rse})\\ \Delta_{r2} = -k\,B(t_{rs})\end{array}\right\} \tag{2.17}$$

Use of these $\omega_{i's}$ (i=0,1,2) remove the bias up to terms of order (n^{-1}) at (2.1).

3. Product –type estimators

Suppose $t_0 = \bar{y}$, $\quad t_{ps} = \bar{y}\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)$, $\quad t_{pse} = \bar{y}\exp\left\{\frac{(a\bar{x}+b)-(a\bar{X}+b)}{(a\bar{x}+b)+(a\bar{X}+b)}\right\}$

such that t_0, t_{ps}, $t_{pse} \in Q$, where Q denotes the set of all possible product –type estimators for estimating the population mean $\bar{Y}$. By definition, the set Q is linear variety if

$$t_q = q_0\bar{y} + q_1 t_{ps} + q_2 t_{pse} \quad \in Q \tag{3.1}$$

for $\sum_{i=0}^{2} q_i = 1, \quad q_i \in R$ (3.2)

where q_i(i=0,1,2) denotes the statistical constants.

Expressing t_q in terms of e's, we have

$$t_q = \bar{y}(1+e_0)\left[\omega_0 + \omega_1(1+\theta e_1) + \omega_2 \exp\left\{\frac{\theta e_1}{2}(1+\theta e_1)^{-1}\right\}\right] \tag{3.3}$$

where $\theta = \frac{a\bar{X}}{a\bar{X}+b}$.

Expanding the right hand side of (3.3) and retaining terms up to second power of e's, we have

$$t_q \cong \bar{Y}\left[1 + e_0 + \theta q e_1 - \frac{q_2}{8}e_1^2 + q\theta e_0 e_1\right] \tag{3.4}$$

where $q = q_1 + \frac{q_2}{2}$ (3.5)

Taking expectations of both sides of (3.4) and then subtracting $\bar{Y}$ from both sides, we get the bias of the estimator t_q, up to the first order of approximation as

$$B(t_q) = f_1\bar{Y}\left[-\frac{q_2}{8}\theta^2 C_x^2 + q\theta\rho C_y C_x\right] \tag{3.6}$$

Bias expression for the estimators t_{ps} and t_{pse} is given by

$$B(t_{ps}) = f_1\bar{Y}\left[\theta\rho C_y C_x\right] \quad (3.7)$$

$$B(t_{pse}) = f_1\bar{Y}\left[-\frac{1}{8}\theta^2 C_x^2 + \frac{\theta\rho C_y C_x}{2}\right] \quad (3.8)$$

From (3.4), we have

$$(t_q - \bar{Y}) \cong \bar{Y}[e_0 + \theta q e_1] \quad (3.9)$$

Squaring both the sides of (3.9) and then taking expectations, we get MSE of the estimator t_q, up to the first order of approximation, as

$$MSE(t_q) = f_1\bar{Y}^2\left[C_y^2 + \theta^2 q^2 C_x^2 + 2\theta q\rho C_y C_x\right] \quad (3.10)$$

which is minimum for

$$q = -k = -\rho\frac{C_y}{C_x} \quad (3.11)$$

Putting this value of q(=-k) in (3.10),we get the minimum MSE of t_q as

$$\text{min. } MSE(t_q) = f_1\bar{Y}^2 C_y^2(1-\rho^2) \quad (3.12)$$

which is same as that of traditional linear regression estimator.

From (3.5) and (3.11), we have

$$q_1 + \frac{q_2}{2} = -k \quad (3.13)$$

From (3.2) and (3.13), we have only two equations in three unknowns. It is not possible to find the unique values for q_i's, i=0,1,2. In order to get unique values of q_i's, we shall impose the linear restriction

$$q_0 B(\bar{y}) + q_1 B(t_{ps}) + q_2 B(t_{pse}) = 0 \quad (3.14)$$

Equations (3.2),(3.13) and (3.14) can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & B(t_{ps}) & B(t_{pse}) \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -k \\ 0 \end{bmatrix} \quad (3.15)$$

Solving (3.15), we get the unique values of q_i's (i=0,1,2) as-

$$\left.\begin{aligned} q_0 &= \frac{\Delta_{p0}}{\Delta_p} \\ q_1 &= \frac{\Delta_{p1}}{\Delta_p} \\ q_2 &= \frac{\Delta_{p2}}{\Delta_p} \end{aligned}\right\} \quad (3.16)$$

where

$$\left.\begin{array}{c}\Delta_p = B(t_{pse}) - \frac{1}{2}B(t_{ps}) \\ \Delta_{p0} = B(t_{pse})\{1+k\} + B(t_{ps})\left\{-k-\frac{1}{2}\right\} \\ \Delta_{p1} = -k.B(t_{pse}) \\ \Delta_{p2} = k\,B(t_{ps})\end{array}\right\} \qquad (3.17)$$

Use of these q_i's (i=0,1,2) remove the bias up to terms of order $o(n^{-1})$ at (3.1).

In Appendix A we have listed some of the important known estimators of the population mean, which can be obtained by suitable choice of constants $\omega_i, i = 0,1,2$, $q_i, i = 0,1,2$ and a and b.

4. Proposed estimators in two phase sampling

When $\bar{X}$ is unknown, it is sometimes estimated from a preliminary large sample of size n' on which only the characteristic x is measured (for details see Singh et. al. (2007)). Then a second phase sample of size n (n < n') is drawn on which both y and x characteristics are measured. Let $\bar{x} = \frac{1}{n'}\sum_{i=1}^{n'} x_i$ denote the sample mean of x based on first phase sample of size n', $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, be the sample means of y and x respectively based on second phase of size n.

In two phase sampling the estimator t_{wr} will take the following form

$$t_{wd} = \omega_{0d}\bar{y} + \omega_{1d}t_{rsd} + \omega_{2d}t_{rsed} \in w_d \qquad (4.1)$$

for $\sum_{i=0}^{2}\omega_{id} = 1,$ $\qquad \omega_{id} \in R$ (4.2)

where $t_{rs} = \bar{y}\left(\frac{a\bar{X}' + b}{a\bar{x} + b}\right)$ and $t_{rse} = \bar{y}\exp\left\{\frac{(a\bar{X}' + b) - (a\bar{x} + b)}{(a\bar{X}' + b) + (a\bar{x} + b)}\right\}$

To obtain the bias and MSE of t_{wd}, we write

$\bar{y} = \bar{Y}(1 + e_0)$, $\bar{x} = \bar{X}(1 + e_1)$, $\bar{x}' = \bar{X}(1 + e_1')$

such that

$E(e_0) = E(e_1) = E(e_1') = 0.$

$E(e_0^2) = f_1C_y^2,$ $\qquad E(e_1^2) = f_1C_x^2,$ $\qquad E(e_1'^2) = f_2C_x^2$ $\qquad E(e_0e_1) = f_1\rho C_yC_x$

$E(e_0e_1') = f_2\rho C_yC_x$ $\qquad E(e_1e_1') = f_2C_x^2$

where $\qquad f_1 = (\frac{1}{n} - \frac{1}{N})$, $\qquad f_2 = (\frac{1}{n'} - \frac{1}{N})$

Following the procedure mentioned in section 2 and 3, we get bias and MSE of t_{wd} as

$$B(t_{\omega d}) = \bar{y}\left[\theta^2 C_x^2 f_3\left(\omega_{1d} + \frac{3\omega_{2d}}{8}\right) - \theta\rho C_y C_x f_3\left(\omega_{1d} + \frac{\omega_{2d}}{2}\right)\right] \qquad (4.3)$$

$$\mathrm{MSE}(t_{\omega d}) = \bar{Y}^2\left[f_1 C_y^2 + f_3 C_x^2 \omega_d^2 - 2f_3 \rho C_y C_x \omega_d\right] \tag{4.4}$$

where $f_3 = \frac{1}{n} - \frac{1}{n'} = f_1 - f_2$.

MSE ($t_{\omega d}$) is minimum, when

$$\omega_{1d} + \frac{\omega_2 d}{2} = \omega_d = k. \tag{4.5}$$

Putting this value of ω_d in (4.4), we get the minimum MSE of $t_{\omega d}$ as

$$\text{min. } \mathrm{MSE}(t_{\omega d}) = \bar{Y}^2 C_y^2 [f_1 - f_3 \rho^2] \tag{4.6}$$

This is same as that of traditional two phase linear regression estimator.

The bias expression for the estimators t_{rsd} and t_{rsde} is respectively given by

$$B(t_{rsd}) = \bar{Y}\left[\theta^2 C_x^2 f_3 \omega_{1d} - \theta\rho C_y C_x f_3 \omega_{1d}\right] \tag{4.7}$$

$$B(t_{rsde}) = \bar{Y}\left[\theta^2 C_x^2 f_3 \frac{3\omega_{2d}}{8} - \theta\rho C_y C_x f_3 \frac{\omega_{2d}}{2}\right] \tag{4.8}$$

From (4.2) and (4.5), we have only two equations in three unknowns. It is not possible to find the unique values for w_{id}'s i=0,1,2.

In order to get unique values of w_{id}, we shall impose linear restriction

$$\omega_{0d} B(\bar{y}) + \omega_{1d} B(t_{rsd}) + \omega_{2d} B(t_{rsed}) = 0 \tag{4.9}$$

Equations (4.2), (4.5) and (4.9) can be written in matrix form, as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & B(t_{rsd}) & B(t_{rsed}) \end{bmatrix} \begin{bmatrix} \omega_{0d} \\ \omega_{1d} \\ \omega_{2d} \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix} \tag{4.10}$$

Solving (4.10), we get the unique values of $\omega'_{id}s,\ (i = 0,1,2)$ as

$$\left.\begin{aligned} \omega_{0d} &= \frac{\Delta_{0d}}{\Delta_{rd}} \\ \omega_{1d} &= \frac{\Delta_{1d}}{\Delta_{rd}} \\ \omega_{2d} &= \frac{\Delta_{2d}}{\Delta_{rd}} \end{aligned}\right\} \tag{4.11}$$

where

$$\left.\begin{aligned} \Delta_{rd} &= B(t_{rsed}) - \frac{1}{2} B(t_{rsd}) \\ \Delta_{r0d} &= B(t_{rsed})\{1 - k\} + \frac{1}{2} B(t_{rsd})\left\{k - \frac{1}{2}\right\} \\ \Delta_{r1d} &= k.B(t_{rsed}) \\ \Delta_{r2d} &= -k.B(t_{rsd}) \end{aligned}\right\} \tag{4.12}$$

Use of these w_{id}'s (i=0,1,2) will remove the bias up to terms of order $O(n^{-1})$ at (4.1).

The estimator t_q written in (3.1), in two phase sampling, will take following form

$$t_{qd} = q_{0d}\bar{y} + q_{1d}t_{psd} + q_{2d}t_{psed} \quad \in Q_d \tag{4.13}$$

For $\sum_{i=0}^{2} q_{id} = 1, \ q_{id} \in R.$ (4.14)

where q_{id}(i=0,1,2) denotes the statistical constants .

The estimators t_{psd} and t_{psde} are

$t_{psd} = \bar{y}\left(\frac{a\bar{x}+b}{a\bar{X}'+b}\right)$ and

$$t_{psed} = \bar{y}\exp\left\{\frac{(a\bar{x}+b)-(a\bar{X}'+b)}{(a\bar{x}+b)+(a\bar{X}'+b)}\right\}$$

Following the procedure of section 4, we get the unique values of q_{id}'s (i=0,1,2) as

$$\left.\begin{aligned} q_{0d} &= \frac{\Delta_{p0d}}{\Delta_{pd}} \\ q_{1d} &= \frac{\Delta_{p1d}}{\Delta_{pd}} \\ q_{2d} &= \frac{\Delta_{p2d}}{\Delta_{pd}} \end{aligned}\right\} \tag{4.15}$$

where

$$\left.\begin{aligned} \Delta_{pd} &= B(t_{rsed}) - \frac{1}{2}B(t_{rsd}) \\ \Delta_{p0d} &= B(t_{rsed})\{1+k\} + \frac{1}{2}B(t_{rsd})\left\{-k-\frac{1}{2}\right\} \\ \Delta_{p1d} &= -k.B(t_{psed}) \\ \Delta_{p2d} &= k.B(t_{psd}) \end{aligned}\right\} \tag{4.16}$$

where

$$B(t_{psd}) = \bar{Y}[\theta q_{1d} f_3 \rho C_y C_x] \tag{4.17}$$

$$B(t_{psed}) = \bar{Y}\left[\theta\frac{q_{2d}}{2}f_3\rho C_y C_x - \frac{\theta^2}{8}q_{2d}f_3C_x^2\right] \tag{4.18}$$

The minimum MSE of t_{qd} is given by

$$MSE(t_{qd}) = \bar{Y}^2C_y^2[f_1 - f_3\rho^2].$$

5. Empirical study

For empirical study we use the data sets earlier used by Kadilar and Cingi (2006) (population 1) and Khosnevisan et. al. (2007) (population 2) to verify the theoretical results.

Data statistics

Population	N	n	$\bar{Y}$	$\bar{X}$	C_y	C_x	ρ	$\beta_2(x)$
Population 1	106	20	2212.59	27421.7	5.22	2.10	0.86	34.57
Population 2	20	8	19.55	18.8	0.355	0.394	-0.92	3.06

Table 5.1: **Values of ω_i's and q_i's**

ω_i's	Population 1	Population 2
ω_0 (q_0)	8.590718 (21.417)	7.892148 (2.919085)
ω_1 (q_1)	11.86615 (16.14158)	5.234461 (3.576773)
ω_2 (q_2)	-19.4569 (-36.5586)	-12.1266 (-5.49586)

The percent relative efficiencies (PRE) of various estimators of $\bar{Y}$ are computed and presented in Table 5.2 below.

Table 5.2: PRE of different estimators of $\bar{Y}$

Estimator	PRE (Pop I)	Estimator	PRE (Pop II)
t_0	100	q_0	100
t_1	212.816	q_1	526
t_2	212.803	q_2	550.261
t_3	212.606	q_3	645.256
t_4	212.815	q_4	534.592
t_5	212.716	q_5	581.732
t_6	212.810	q_6	465.501
t_7	143.992	q_7	384.447
t_8	143.923	q_8	285.920
t_9	143.988	q_9	338.487
t_{10}	143.990	q_{10}	374.584
t_{11}	143.991	q_{11}	345.118
t_{12}	143.959	q_{12}	231.602
t_{13}	143.991	q_{13}	424.194
t_{14}	143.987	q_{14}	360.086
t_{15}	143.992	q_{15}	356.520
t_{16}	143.911	q_{16}	467.051
t_w(opt)	384.025	t_q(opt)	650.263

In order to see the performance of the suggested estimators in two phase sampling we use the data set of Murthy (1967) (Population III) and Steel and Torrie (1960) (Population IV).

Population	C_y	C_x	ρ	N	n'	n
Population 1	0.3542	0.9484	0.9150	80	30	10
Population2	0.4803	0.7493	-0.4996	30	12	4

Table 5.3 : The values of $\omega_{id}{'s}$ and $q_{id}{'s}$

$\omega_{id}{'s}$ ($q_{id}{'s}$)	**Population III**	**Population IV**
ω_{0d} (q_{0d})	-0.241523 (1.808833)	3.011435 (1.089979)
ω_{1d} (q_{1d})	-0.558071 (0.125381)	1.370950 (0.730464)
ω_{2d} (q_{2d})	1.799595 (-0.934214)	-3.382385 (-0.820443)

The percent relative efficiencies of various estimators of $\bar{Y}$ in two phase sampling are computed and presented in Table 5.4 below.

Table 5.4 : PRE of different estimators of $\bar{Y}$ in two phase sampling

Estimator	PRE (Population I)	PRE (Population II)
$\bar{y}$	100	100
t_{rsd}	36.642	24.562
t_{psd}	4.849	59.770
t_{rsde}	200.420	48.365
t_{psed}	23.628	115.142
$t_{\omega d}$	276.156	63.452
t_{qd}	34.321	123.762
t_{opt}	276.156	123.762

6. Conclusion

From theoretical discussion and empirical study we conclude that the proposed estimators under optimum conditions perform better than other estimators considered in the article. The relative efficiencies of various estimators are listed in Table 5.2 and 5.4.

Appendix A

Table A.1: Some members of the proposed family of estimators -

a	b	ω_0 (q_0)	ω_1 (q_1)	ω_2 (q_2)	Ratio Estimator (corresponding to ω_i, i=0,1,2)	Product Estimator (corresponding to q_i, i=0,1,2)
0	0	1	0	0	$t_0 = \bar{y}$ The mean per unit estimator	$q_0 = \bar{y}$ The mean per unit estimator
1	o	0	1	0	$t_1 = \bar{y}\frac{\bar{X}}{\bar{x}}$ The usual ratio estimator	$q_1 = \bar{y}\frac{\bar{x}}{\bar{X}}$ The usual product estimator
1	C_x	0	1	0	$t_2 = \bar{y}\frac{\bar{X}+C_x}{\bar{x}+C_x}$ Sisodia and Dwivedi (1981) estimator	$q_2 = \bar{y}\frac{\bar{x}+C_x}{\bar{X}+C_x}$ Pandey and Dubey (1988) estimator
1	$\beta_2(x)$	0	1	0	$t_3 = \bar{y}\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}$ Singh et. al. (2004) estimator	$q_3 = \bar{y}\frac{\bar{x}+\beta_2(x)}{\bar{X}+\beta_2(x)}$ Singh et. al. (2004) estimator
$\beta_2(x)$	C_x	0	1	0	$t_4 = \bar{y}\frac{\bar{X}\beta_2(x)+C_x}{\bar{x}\beta_2(x)+C_x}$ Upadhyaya and Singh (1999) estimator	$q_4 = \bar{y}\frac{\bar{x}\beta_2(x)+C_x}{\bar{X}\beta_2(x)+C_x}$ Upadhyaya and Singh (1999) estimator
C_x	$\beta_2(x)$	0	1	0	$t_5 = \bar{y}\frac{\bar{X}C_x+\beta_2(x)}{\bar{x}C_x+\beta_2(x)}$ Upadhyaya and Singh (1999) estimator	$q_5 = \bar{y}\frac{\bar{x}C_x+\beta_2(x)}{\bar{X}C_x+\beta_2(x)}$ Upadhyaya and Singh (1999) estimator
1	ρ	0	1	0	$t_6 = \bar{y}\left[\frac{\bar{X}+\rho}{\bar{x}+\rho}\right]$ Singh and Tailor (2003) estimator	$q_6 = \bar{y}\left[\frac{\bar{x}+\rho}{\bar{X}+\rho}\right]$ Singh and Tailor (2003) estimator
1	0	0	0	1	$t_7 = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]$ Bahl and Tuteja (1991) estimator	$q_7 = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right]$ Bahl and Tuteja (1991) estimator

a	**b**	ω_0 (q_0)	ω_1 (q_1)	ω_2 (q_2)	**Ratio Estimator (corresponding to ω_i,i=0,1,2)**	**Product Estimator (corresponding to q_i,i=0,1,2)**
1	$\beta_2(x)$	0	0	1	$t_8 = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator	$q_8 = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator
1	C_x	0	0	1	$t_9 = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2C_x}\right]$ Singh et.al. (2007) estimator	$q_9 = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}+2C_x}\right]$ Singh et.al. (2007) estimator
1	ρ	0	0	1	$t_{10} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho}\right]$ Singh et. al.(2007) estimator	$q_{10} = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}+2\rho}\right]$ Singh et. al.(2007) estimator
$\beta_2(x)$	C_x	0	0	1	$t_{11} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator	$q_{11} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator
C_x	$\beta_2(x)$	0	0	1	$t_{12} = \bar{y}\exp\left[\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator	$q_{12} = \bar{y}\exp\left[\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator
C_x	ρ	0	0	1	$t_{13} = \bar{y}\exp\left[\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator	$q_{13} = \bar{y}\exp\left[\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator
ρ	C_x	0	0	1	$t_{14} = \bar{y}\exp\left[\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator	$q_{14} = \bar{y}\exp\left[\frac{\rho(\bar{x}-\bar{X})}{\rho(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator
$\beta_2(x)$	ρ	0	0	1	$t_{15} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator	$q_{15} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator
ρ	$\beta_2(x)$	0	0	1	$t_{16} = \bar{y}\exp\left[\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator	$q_{16} = \bar{y}\exp\left[\frac{\rho(\bar{x}-\bar{X})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator

In addition to above estimators a large number of estimators can also be generated from the proposed estimators just by putting different values of constants $\omega_i, i = 0,1,2$, $q_i, i = 0,1,2$ and a and b.

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References

1. Bahl, S. and Tuteja, R.K. (1991): Ratio and product type exponential estimator. Infrm. and Optim. Sci., XII, I, 159-163.
2. Kadilar, C. and Cingi, H. (2006): New ratio estimators using correlation coefficient. InterStat, 1-11.
3. Khoshnevisan,M., Singh, R., Chauhan, P., Sawan, N. and Smarandache, F.(2007): A general family of estimators for estimating population means using known value of some population parameter(s). Far east journal of statistics, 22(2), 181-191.
4. Pandey, B.N. and Dubey, V. (1988): Modified product estimator using coefficient of variation of auxiliary variate. Assam Statistical Rev. 2(2), 64-66.
5. Sen, A.R. (1978): Estimation of the population mean when the coefficient of variation is known. Commun. Stat. Theo. Meth. A (7), 657-672.
6. Singh, H.P. and Kakran, M.S. (1993): A modified ratio estimator using coefficient of variation of auxiliary character. Unpublished.
7. Singh, H.P. and Tailor, R. (2003): Use of known correlation coefficient in estimating the finite population mean. Statistics in Transition, 6, 4, 555-560.
8. Singh, H.P., Tailor, R, Tailor, R. and Kakran, M.S. (2004): An improved estimator of population mean using power transformation. Jor. Ind. Soc. Agri. Statist. 58(2), 223-230.
9. Singh, J., Pandey, B.N. and Hirano, K. (1973): On the utilization of a known coefficient of kurtosis in the estimation procedure of variance. Ann. Inst. Stat. Math., 25, 51-55.
10. Singh, R. and Kumar, M. (2009): A note on transformations on auxiliary variable in survey sampling. (to appear in MASA)
11. Singh, R., Cauhan, P., Sawan, N. and Smarandache, F. (2007): Auxiliary information and a priory values in construction of improved estimators. Renaissance High press, USA.

12. Singh, R., Singh, J. and Smarandache, F. (2009): Studies in statistical inference, sampling techniques and demography. ProQuest Information & Learning, USA.

13. Sisodiya, B. V. S. and Dwivedi, V.K. (1981): A modified ratio estimator using coefficient of variation of auxiliary variable. Jour. Ind. Soc. Agri. Stat., 33, 13-18.

14. Sukhatme, P.V., Sukhatme, B.V., Sukhatme, S. and Ashok, C. (1984): Sampling theory of surveys with applications. Iowa State University Press, USA.

15. Upadhyaya, L.N. and Singh, H.P. (1984): On the estimation of the population mean with known coefficient of variation. Biom. Jour., 26(8), 915-922.