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An Introduction to Single-Valued Neutrosophic Primal Theory

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Abstract: This article explores the interconnections among the single-valued neutrosophic grill, single-valued neutrosophic primal and their stratification, uncovering their fundamental characteristics and correlated findings. By introducing the notion of a single-valued neutrosophic primal, a broader framework including the fuzzy primal and intuitionistic fuzzy primal is established. Additionally, the concept of a single-valued neutrosophic open local function for a single-valued neutrosophic topological space is presented. We introduce an operator based on a single-valued neutrosophic primal, illustrating that the single-valued neutrosophic primal topology is finer than the single-valued neutrosophic topology. Lastly, the concept of single-valued neutrosophic open compatibility between the single-valued neutrosophic primal and single-valued neutrosophic topologies is introduced, along with the establishment of several equivalent conditions related to this notion.

Keywords: single-valued neutrosophic primal; single-valued neutrosophic primal topology; single-valued neutrosophic open compatibility

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1. Introduction

Topology, a highly versatile field of mathematics [1], finds extensive application across both the scientific and social science domains, prompting the emergence of numerous innovative concepts within its standard frameworks. Kuratowski [2] examined the notion of ideals derived from filters, which can be seen as dual to filters. The notion of the fuzzy grill was given by [3]. Chattopadhyay and Thron [4] utilized grills to establish various topics, including closure spaces, while Thron [5] defined proximity structures within grills. Roy et al. [6] introduced novel definitions related to grills, with Roy and Mukherjee [7–9] subsequently exploring diverse topological properties associated with grills. Numerous applications stemming from these studies are documented in various works [10–16]. Given the dual nature of primal concerning grills, we draw inspiration from Janković and Hamlett [17] to introduce a new topology based on ideal structures.

The concept of neutrosophic sets, introduced as a generalization of intuitionistic fuzzy sets, was initially proposed in [18]. Salama et al. [19] and Wang et al. [20] have extensively investigated neutrosophic sets and their single-valued neutrosophic, abbreviated as *svn*, counterparts. Numerous applications stemming from these studies are documented in various works [21–24]. Stratified single-valued soft topogenous structures have been studied by Alsharari et al. [25].

Saber et al. have conducted extensive research on single-valued neutrosophic soft uniform spaces, single-valued neutrosophic ideals, abbreviated as *svnis*, and the connectedness and stratification of single-valued neutrosophic topological spaces expanded with an ideal [26–28]. The neutrosophic compound orthogonal neural network (NCONN), for the first time, contained the NsN weight values, *NsN* input and output, and hidden layer neutrosophic neuron functions; to approximate neutrosophic functions, *NsN* data have been studied by Ye et al. [29]. Shao et al. [30] introduced the concept of the probabilistic single-valued (interval) neutrosophic hesitant fuzzy set, extensively investigating the operational relations of *PINHFS* and the comparison method of probabilistic interval neutrosophic hesitant fuzzy numbers (*PINHFNs*). Rıdvan et al. [31] examined the notion of the neutrosophic subsethood measure for single-valued neutrosophic sets. The neutrosophic fuzzy set and its application in decision-making was defined by Das et al. [32].

The objective of this paper is to explore the inter-relations between the single-valued neutrosophic grill (*svn-grill*) and single-valued neutrosophic primal (*svn-primal*), along with their stratification, while showcasing some of their inherent properties. Additionally, we investigate the quantum behaviors within a novel structure denoted as $\Xi_r^\star(\mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$, as defined in Definition 11. Furthermore, we introduce and analyze both the *svn primal* and its associated topology. We also derive several preservation properties and characterizations regarding svn-primal open compatibility.

2. Preliminaries

This section presents the fundamental definitions and results necessary for our study. Initially, we define a neutrosophic set (for short, *n-set*) and a single-valued neutrosophic set (for short, *svn-set*). For a more comprehensive understanding of *n-set* theory and *svn-set* theory, readers are directed to [18,20,33]. Conventionally, $\xi^{\pounds}$ denotes the family encompassing all *svn-sets*, defined on £. Here, $\xi = [0,1]$, $\xi_0 = (0,1]$ and for any $\alpha \in \xi$ and $z \in \pounds$, $\overline{\alpha}(z) = \alpha$.

We begin with the definition of a neutrosophic set as follows:

Definition 1 ([18]). *Let £ be a non-empty set. An n-set on £ is defined as*

$$\Xi = \{\langle z, \tau_{\Xi}(z), \pi_{\Xi}(z), \sigma_{\Xi}(z)\rangle \mid x \in \pounds, \tau_{\Xi}(z), \pi_{\Xi}(Z), \sigma_{\Xi}(z) \in \rfloor^{-}0, 1^{+}\lfloor\},$$

representing the degree of membership where ($\tau_{\Xi}(z)$), the degree of indeterminacy ($\pi_{\Xi}(z)$) and degree of nonmembership ($\sigma_{\Xi}(z)$); $\forall\, z \in \pounds$ to the set Ξ.

We now discuss the concept of the *svn-set*, which is a more specific type of neutrosophic set.

Definition 2 ([20]). *Let £ be a space of points (objects) with a generic element in £ denoted by z. Then Ξ is called an svn-set in £ if Ξ has the form $\Xi = \{\langle z, \tau_{\Xi}(z), \pi_{\Xi}(z), \sigma_{\Xi}(z)\rangle \mid x \in \pounds\}$ where $\tau_{\Xi}, \pi_{\Xi}, \sigma_{\Xi} : \pounds \to \xi = [0,1]$. In this case, σ_{Ξ}, π_{Ξ} and τ_{Ξ} are called the falsity membership function, indeterminancy membership function and truth membership function, respectively.*

An svn-set Ξ *on £ is named as a null* svn-set *(for short, $\overline{0}$), if $\tau_{\Xi}(z) = 0$, $\pi_{\Xi}(z) = 1$ and $\sigma_{\Xi}(x) = 1$, for all $z \in \pounds$.*

An svn-set Ξ *on £ is named as an absolute* svn-set *(for short, $\overline{1}$), if $\tau_{\Xi}(z) = 1$, $\pi_{\Xi}(z) = 0$ and $\sigma_{\Xi}(z) = 0$, for all $z \in \pounds$.*

Example 1. *Suppose that $\pounds = \{l_1, l_2, l_3\}$, l_1 is capability, l_2 is trustworthiness and l_3 is price. The values of l_1, l_2 and l_3 are in $\xi = [0,1]$. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". Ξ is an svn-set of £ defined by*

$$\Xi = \langle 0.2, 0.3, 0.8\rangle / l_1 + \langle 0.8, 02, 0.3\rangle / l_2 + \langle 0.7, 0.1, 0.2\rangle / l_3,$$

Θ is an svn-set of £ defined by

$$\Theta = \langle 0.6, 0.1, 0.2\rangle / l_1 + \langle 0.3, 02, 0.6\rangle / l_2 + \langle 0.4, 0.1, 0.5\rangle / l_3.$$

To better understand the properties of *svn-sets*, we will now discuss the complement of an *svn-set*.

Definition 3 ([20])**.** *Let* $\Xi = \{\langle x, \tau_{\Xi}(z), \pi_{\Xi}(z), \sigma_{\Xi}(z)\rangle\}e : x \in £\}$ *be an svn-set on £. The complement of the set* Ξ (Ξ^c) *is defined as follows:*

$$\tau_{\Xi^c}(z) = \sigma_{\Xi}(z),\ \pi_{\Xi^c}(z) = [\pi_{\Xi}]^c(z),\ \sigma_{\Xi^c}(z) = \tau_{\Xi}(z).$$

The following definition provides more insight into the relationships between *svn-sets*, introducing the notions of subsets, equality and special sets.

Definition 4 ([34])**.** *Let* $\Xi, \Theta \in \zeta^{£}$*, then,*

(1) Ξ *is said to be contained in* Θ*, denoted by* $\Xi \subseteq \Theta$*, if, for each* $z \in £$*,*

$$\tau_{\Xi}(z) \leq \tau_{\Theta}(z),\ \pi_{\Xi}(z) \geq \pi_{\Theta}(z),\ \sigma_{\Xi}(z) \geq \sigma_{\Theta}(z).$$

(2) Ξ *is said to be equal to* Θ*, denoted by* $\Xi = \Theta$*, iff* $\Xi \subseteq \Theta$ *and* $\Theta \subseteq \Xi$*.*

In the context of *svn-sets*, we introduce definitions related to the intersection and union of *svn-sets*. Further, we discuss the concept of a single-valued neutrosophic topological space (*svnts*) and the properties it entails.

Definition 5 ([33])**.** *Let* $\Xi, \Theta \in \zeta^{£}$*. Then,*

(a) $\Xi \wedge \Theta$ *is an (svn-set), if* $\forall\, z \in £$*,*

$$\Xi \wedge \Theta = \langle (\tau_{\Xi} \wedge \tau_{\Theta})(z),\ (\pi_{\Xi} \vee \pi_{\Theta})(z),\ (\sigma_{\Xi} \vee \sigma_{\Theta})(z)\rangle$$

where $(\tau_{\Xi} \wedge \tau_{\Theta})(z) = \tau_{\Xi}(z) \wedge \tau_{\Theta}(z)$*,* $(\pi_{\Xi} \vee \pi_{\Theta})(z) = \pi_{\Xi}(z) \vee \pi_{\Theta}(z)$ *and* $(\sigma_{\Xi} \vee \sigma_{\Theta})(z) = \sigma_{\Xi}(z) \vee \sigma_{\Theta}(z), \forall\, z \in £$*,*

(b) $\Xi \vee \Theta$ *is an (svn-set), if* $\forall\, z \in £$*,*

$$\Xi \vee \Theta = \langle (\tau_{\Xi} \vee \tau_{\Theta})(z),\ (\pi_{\Xi} \wedge \pi_{\Theta})(z),\ (\sigma_{\Xi} \wedge \sigma_{\Theta})(z)\rangle.$$

Now, we discuss the concept of svnts, which consists of a set £ and three mappings $\mathcal{T}^{\tau}, \mathcal{T}^{\pi}, \mathcal{T}^{\sigma} : \zeta^{£} \to \zeta$ that satisfy specific axioms.

Definition 6 ([27])**.** *An svnts is an ordered* $(£, \mathcal{T}^{\tau}, \mathcal{T}^{\pi}, \mathcal{T}^{\sigma})$ *where* $\mathcal{T}^{\tau}, \mathcal{T}^{\pi}, \mathcal{T}^{\sigma} : \zeta^{£} \to \zeta$ *is a mapping satisfying the following axioms:*

(SVNT1) $\mathcal{T}^{\tau}(\bar{0}) = \mathcal{T}^{\tau}(\bar{1}) = 1$ *and* $\mathcal{T}^{\pi}(\bar{0}) = \mathcal{T}^{\pi}(\bar{1}) = \mathcal{T}^{\sigma}(\bar{0}) = \mathcal{T}^{\sigma}(\bar{1}) = 0$
(SVNT2) $\mathcal{T}^{\tau}(\Xi \wedge \Theta) \geq \mathcal{T}^{\tau}(\Xi) \wedge \mathcal{T}^{\tau}(\Theta)$, $\quad \mathcal{T}^{\pi}(\Xi \wedge \Theta) \leq \mathcal{T}^{\pi}(\xi) \vee \mathcal{T}^{\pi}(\Theta)$,
$\mathcal{T}^{\sigma}(\Xi \wedge \Theta) \leq \mathcal{T}^{\sigma}(\Xi) \vee \mathcal{T}^{\sigma}(\Theta)$*, for every* $\Xi, \Theta \in \zeta^{£}$ *,*
(SVNT3) $\mathcal{T}^{\tau}(\bigvee_{j \in J}(\Xi_j) \geq \bigwedge_{j \in J} \mathcal{T}^{\tau}(\Xi_j)$, $\quad \mathcal{T}^{\pi}(\bigvee_{j \in J}(\Xi_j) \leq \bigvee_{j \in J} \mathcal{T}^{\pi}(\Xi_j)$,
$\mathcal{T}^{\sigma}(\bigvee_{j \in J}(\Xi_j) \leq \bigvee_{j \in J} \mathcal{T}^{\sigma}(\Xi_j)$*, for every* $\Xi_j \in \zeta^{£}$*.*

For the sake of brevity and clarity, we sometimes denote $(\mathcal{T}^{\tau}, \mathcal{T}^{\pi}, \mathcal{T}^{\sigma})$ as $\mathcal{T}^{\tau\pi\sigma}$ without causing any ambiguity.

The following theorem establishes an operator that satisfies specific conditions, which further clarifies the properties of *svnts*.

Theorem 1 ([27])**.** *Let* $(£, \mathcal{T}^{\tau\pi\sigma})$ *be an svnts. Then, for all* $r \in \zeta_0$ *and* $\Xi \in \zeta^{£}$*, we define the operator* $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}} : \zeta^{£} \times \zeta_0 \to \zeta^{£}$ *as follows:*

$$\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \bigwedge\{\Theta \in \zeta^{£} | \Xi \leq \Theta,\ \mathcal{T}^{\tau}(\Theta^c)) \geq r,\quad \mathcal{T}^{\pi}(\Theta^c)) \leq 1 - r,\quad \mathcal{T}^{\sigma}(\Theta^c)) \leq 1 - r\}.$$

For any $\Xi, \Theta \in \zeta^{£}$ and $r, s \in \zeta_0$, the operator $\mathcal{C}_{\mathcal{T}\tau\pi\sigma}$ satisfies the following conditions:

(C1) $\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\overline{0}, r) = \overline{0}$.
(C2) $\Xi \leq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi, r)$.
(C3) $\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi, r) \vee \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Theta, r) = \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi \vee \Theta, r)$.
(C4) $\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Xi, r) \leq \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\Theta, s)$ if $r \leq s$.
(C5) $\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\xi, r), r) = \mathcal{C}_{\mathcal{T}\tau\pi\sigma}(\xi, r)$.

3. Stratified Single-Valued Neutrosophic Grill with Single-Valued Neutrosophic Primal

In this section, we explores the interconnections between single-valued neutrosophic grills (*svn-grills*) and single-valued neutrosophic primals (*svn-primals*), along with their stratification. We also present a new structure within the context of single-valued neutrosophic topology, referred to as an svn-primal. This novel structure is the dual counterpart to the *svn-grill*.

We start with the following definition:

Definition 7. *A map $\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma} : \zeta^{£} \to \zeta$ is called an svn-grill on £, if it meets the following criteria:*

(G_1) $\mathcal{G}^{\tau}(\overline{0}) = 0, \mathcal{G}^{\pi}(\overline{0}) = 1, \mathcal{G}^{\sigma}(\overline{0}) = 1$ *and* $\mathcal{G}^{\tau}(\overline{1}) = 1, \mathcal{G}^{\pi}(\overline{1}) = 0, \mathcal{G}^{\sigma}(\overline{1}) = 0$.
(G_2) *If* $\Xi \leq \Theta$, *then* $\mathcal{G}^{\tau}(\Xi) \leq \mathcal{G}^{\tau}(\Theta)$, $\mathcal{G}^{\pi}(\Xi) \geq \mathcal{G}^{\pi}(\Theta)$, $\mathcal{G}^{\sigma}(\Xi) \geq \mathcal{G}^{\sigma}(\Theta), \forall\, \Xi, \Theta \in \zeta^{£}$.
(G_3) $\mathcal{G}^{\tau}(\Xi \vee \Theta) \leq \mathcal{G}^{\tau}(\Xi) \vee \mathcal{G}^{\tau}(\Theta)$, $\mathcal{G}^{\pi}(\Xi \vee \Theta) \geq \mathcal{G}^{\pi}(\Xi) \wedge \mathcal{G}^{\pi}(\Theta)$, $\mathcal{G}^{\sigma}(\Xi \vee \Theta) \geq \mathcal{G}^{\sigma}(\Xi) \wedge \mathcal{G}^{\sigma}(\Theta), \forall\, \Xi, \Theta \in \zeta^{£}$.

An svn-grill *is called stratified iff $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$ satisfies the following condition:*
(G_{st}) $\mathcal{G}^{\tau}(\Xi \vee \overline{\alpha}) \leq \mathcal{G}^{\tau}(\Xi) \vee \alpha$, $\mathcal{G}^{\pi}(\Xi \vee \overline{\alpha}) \geq \mathcal{G}^{\pi}(\Xi) \wedge \alpha$, $\mathcal{G}^{\sigma}(\Xi \vee \overline{\alpha}) \geq \mathcal{G}^{\sigma}(\Xi) \wedge \alpha, \forall\, \Xi \in \zeta^{£}$ *and* $\alpha \in \zeta$.

For the sake of brevity and clarity, we will occasionally denote $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$ as $\mathcal{G}^{\tau\pi\sigma}$.

A single-valued neutrosophic grill topological space (svngts) is defined as the triple (£, $\mathcal{T}^{\tau\pi\sigma}$, $\mathcal{G}^{\tau\pi\sigma}$).

Theorem 2. *Let $(\mathcal{G}^{\tau\pi\sigma})$ be an svn-grill on £. Define $\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st} : \zeta^{£} \to \zeta$*

$$\mathcal{G}^{\tau}_{st}(\Xi) = \bigwedge_{\{(\Xi_i, \overline{\alpha_i}) | i \in J\} \in \mathcal{K}(\Xi)} \left\{ \bigvee_{(\Xi_l, \overline{\alpha_l}) \in \{(\Xi_i, \overline{\alpha_i}) | i \in J\}} \mathcal{G}^{\tau}(\Xi_l) \vee \alpha_l \right\},$$

$$\mathcal{G}^{\pi}_{st}(\Xi) = \bigvee_{\{(\Xi_i, \overline{\alpha_i}) | i \in J\} \in \mathcal{K}(\Xi)} \left\{ \bigwedge_{(\Xi_l, \overline{\alpha_l}) \in \{(\Xi_i, \overline{\alpha_i}) | i \in J\}} \mathcal{G}^{\pi}(\Xi_l) \vee \alpha_l \right\},$$

$$\mathcal{G}^{\sigma}_{st}(\Xi) = \bigvee_{\{(\Xi_i, \overline{\alpha_i}) | i \in J\} \in \mathcal{K}(\Xi)} \left\{ \bigwedge_{(\Xi_l, \overline{\alpha_l}) \in \{(\Xi_i, \overline{\alpha_i}) | i \in J\}} \mathcal{G}^{\sigma}(\Xi_l) \vee \alpha_l \right\},$$

where $\mathcal{K}(\xi) = \{\{(\Xi_i, \overline{\alpha_i}) \mid i \in J, J \text{ is finite index set}\} \text{ and } \Xi \leq \bigwedge_{i \in J}(\Xi_i \vee \overline{\alpha_i})\}$. Then, $(\mathcal{G}^{\tau}_{st}$, $\mathcal{G}^{\pi}_{st}$, $\mathcal{G}^{\sigma}_{st})$ is the coarsest stratified svn-grill on £ which is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$.

Proof. First, we will prove that $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is a stratified *svn-grill* on £.

(G_1) and (G_2) are straightforward.

(G_3) Assume that there exist $\Xi, \Theta \in \zeta^{£}$ such that

$$\mathcal{G}^{\tau}(\Xi \vee \Theta) \not\leq \mathcal{G}^{\tau}(\Xi) \vee \mathcal{G}^{\tau}(\Theta), \qquad \mathcal{G}^{\pi}(\Xi \vee \Theta) \not\geq \mathcal{G}^{\pi}(\Xi) \wedge \mathcal{G}^{\pi}(\Theta)$$

$$\mathcal{G}^{\sigma}(\Xi \vee \Theta) \not\geq \mathcal{G}^{\sigma}(\Xi) \wedge \mathcal{G}^{\sigma}(\Theta).$$

By the definition of $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$, there exist $\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\} \in \mathcal{K}(\Xi)$ and $\{(\Theta_\iota, \overline{\omega_\iota}) \mid \iota \in \Gamma\} \in \mathcal{K}(\Theta)$ such that

$$\mathcal{G}^{\tau}(\Xi \vee \Theta) \not\geq \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_t) \vee \alpha_t \right) \vee \left(\bigvee_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) | \iota \in \Gamma\}} \mathcal{G}^{\tau}(\Theta_j) \vee \omega_j \right).$$

$$\mathcal{G}^{\pi}(\Xi \vee \Theta) \not\leq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_t) \vee \alpha_t \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) | \iota \in \Gamma\}} \mathcal{G}^{\pi}(\Theta_j) \vee \omega_j \right).$$

$$\mathcal{G}^{\sigma}(\Xi \vee \Theta) \not\leq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_t) \vee \alpha_t \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) | \iota \in \Gamma\}} \mathcal{G}^{\sigma}(\Theta_j) \vee \omega_j \right).$$

Set $k \in J \cup \Gamma$ such that

$$\Pi_k \vee \overline{\varrho_k} = \begin{cases} \Xi_k \vee \alpha_k, & \text{if } k \in J - (J \cap \Gamma), \\ \Theta_k \vee \omega_k, & \text{if } k \in \Gamma - (J \cap \Gamma), \\ (\Xi_k \vee \alpha_k) \vee (\Theta_k \vee \omega_k), & \text{if } k \in (J \cap \Gamma). \end{cases}$$

On the other hand,

$$\Xi \vee \Theta \leq \left(\bigwedge_{t \in J} (\Xi_t \vee \alpha_t) \right) \vee \left(\bigwedge_{\iota \in \Gamma} (\Theta_\iota \vee \omega_\iota) \right) = \bigwedge_{k \in (J \cap \Gamma)} (\Pi_k \vee \overline{\varrho_k}) \textit{ and } \{(\Pi_k, \overline{\varrho_k}) \mid k \in (J \cup \Gamma)\} \in \mathcal{K}(\Xi \vee \Theta).$$

Then, we have

$$\begin{aligned} \mathcal{G}^{\tau}_{st}(\Xi \vee \Theta) &\leq \bigvee_{(\Pi_\nu, \overline{\varrho_\nu}) \in \{(\Pi_k, \overline{\varrho_k}) | k \in (J \cup \Gamma)\}} \mathcal{G}^{\tau}(\Pi_\nu) \vee \varrho_\nu \\ &= \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) \vee \left(\bigvee_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) | \iota \in \Gamma\}} \mathcal{G}^{\tau}(\Theta_j) \vee \omega_j \right). \end{aligned}$$

$$\begin{aligned} \mathcal{G}^{\pi}_{st}(\Xi \vee \Theta) &\geq \bigwedge_{(\Pi_\nu, \overline{\varrho_\nu}) \in \{(\Pi_k, \overline{\varrho_k}) | k \in (J \cup \Gamma)\}} \mathcal{G}^{\tau}(\Pi_\nu) \vee \varrho_\nu \\ &= \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) | \iota \in \Gamma\}} \mathcal{G}^{\pi}(\Theta_j) \vee \omega_j \right). \end{aligned}$$

$$\begin{aligned} \mathcal{G}^{\sigma}_{st}(\Xi \vee \Theta) &\geq \bigwedge_{(\Pi_\nu, \overline{\varrho_\nu}) \in \{(\Pi_k, \overline{\varrho_k}) | k \in (J \cup \Gamma)\}} \mathcal{G}^{\tau}(\Pi_\nu) \vee \varrho_\nu \\ &= \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) \wedge \left(\bigwedge_{(\Theta_j, \overline{\omega_j}) \in \{(\Theta_\iota, \overline{\omega_\iota}) | \iota \in \Gamma\}} \mathcal{G}^{\sigma}(\Theta_j) \vee \omega_j \right). \end{aligned}$$

This is a contradiction. Thus, (G_3) holds.

(G_{st}) Assume that there exist $\Xi \in \zeta^{£}$ and $\alpha \in \zeta$ such that

$$\mathcal{G}^{\tau}_{st}(\Xi \vee \overline{\alpha}) \not\leq \mathcal{G}^{\tau}_{st}(\Xi) \vee \alpha, \qquad \mathcal{G}^{\pi}_{st}(\Xi \vee \overline{\alpha}) \not\geq \mathcal{G}^{\pi}_{st}(\Xi) \wedge \alpha,$$

$$\mathcal{G}^{\sigma}_{st}(\Xi \vee \overline{\alpha}) \ngeq \mathcal{G}^{\sigma}_{st}(\Xi) \wedge \alpha.$$

By the definition of $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$, there exist $\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\} \in \mathcal{K}(\Xi)$ such that

$$\mathcal{G}^{\tau}_{st}(\Xi \vee \overline{\alpha}) \nleq \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) \vee \alpha$$

$$\mathcal{G}^{\pi}_{st}(\Xi \vee \overline{\alpha}) \ngeq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) \wedge \alpha$$

$$\mathcal{G}^{\sigma}_{st}(\Xi \vee \overline{\alpha}) \ngeq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) \wedge \alpha$$

On the other hand, $\Xi \vee \overline{\alpha} \leq \bigwedge_{t \in (J \cap \Gamma)} (\Xi_t \vee \overline{\varrho_t})$ where $\varrho_t = \alpha_i \vee \alpha$; then, $\{(\Xi_t, \overline{\varrho_t}) \mid t \in J\} \in \mathcal{K}(\Xi \vee \overline{\alpha})$. Then, we have

$$\mathcal{G}^{\tau}_{st}(\Xi \vee \overline{\alpha}) \leq \left(\bigvee_{(\Xi_i, \overline{\varrho_i}) \in \{(\Xi_t, \overline{\varrho_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) = \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right) \vee \alpha,$$

$$\mathcal{G}^{\pi}_{st}(\Xi \vee \overline{\alpha}) \geq \left(\bigwedge_{(\Xi_i, \overline{\varrho_i}) \in \{(\Xi_t, \overline{\varrho_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) = \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right) \wedge \alpha,$$

$$\mathcal{G}^{\sigma}_{st}(\Xi \vee \overline{\alpha}) \geq \left(\bigwedge_{(\Xi_i, \overline{\varrho_i}) \in \{(\Xi_t, \overline{\varrho_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) = \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right) \wedge \alpha.$$

This is a contradiction. Thus, $\mathcal{G}_{st}$ holds. Hence, $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is stratified.

Second, for any $\Xi \in \zeta^{\pounds}$, there exists a collection $\{\overline{\alpha}\}$ with $\Xi \leq \Xi \vee \overline{\alpha}$ such that $\mathcal{G}^{\tau}_{st}(\Xi) \leq \mathcal{G}^{\tau}(\Xi)$, $\mathcal{G}^{\pi}_{st}(\Xi) \geq \mathcal{G}^{\pi}(\Xi)$ and $\mathcal{G}^{\sigma}_{st}(\Xi) \geq \mathcal{G}^{\sigma}(\Xi)$. Thus, $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$.

Finally, consider $(\mathcal{G}^{\star\tau}, \mathcal{G}^{\star\pi}, \mathcal{G}^{\star\sigma})$, a stratified *svn-grill* on £ which is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$. And we will show that $\mathcal{G}^{\tau}_{st}(\Xi) \geq \mathcal{G}^{\star\tau}(\Xi)$, $\mathcal{G}^{\pi}_{st}(\Xi) \leq \mathcal{G}^{\star\pi}(\Xi)$ and $\mathcal{G}^{\sigma}_{st}(\Xi) \leq \mathcal{G}^{\star\sigma}(\Xi)$. Assume that $\Xi \in \zeta^{\pounds}$ such that $\mathcal{G}^{\tau}_{st}(\Xi) \ngeq \mathcal{G}^{\star\tau}(\Xi)$, $\mathcal{G}^{\pi}_{st}(\Xi) \nleq \mathcal{G}^{\star\pi}(\Xi)$ and $\mathcal{G}^{\sigma}_{st}(\Xi) \nleq \mathcal{G}^{\star\sigma}(\Xi)$. . By the definition of $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$, there exist $\{(\Xi_t, \overline{\alpha_t}) \mid t \in J\} \in \mathcal{K}(\Xi)$ such that

$$\mathcal{G}^{\star\tau}(\Xi) \nleq \left(\bigvee_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\tau}(\Xi_i) \vee \alpha_i \right),$$

$$\mathcal{G}^{\star\pi}(\Xi) \ngeq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\pi}(\Xi_i) \vee \alpha_i \right),$$

$$\mathcal{G}^{\star\sigma}(\Xi) \ngeq \left(\bigwedge_{(\Xi_i, \overline{\alpha_i}) \in \{(\Xi_t, \overline{\alpha_t}) | t \in J\}} \mathcal{G}^{\sigma}(\Xi_i) \vee \alpha_i \right).$$

On the other hand, $(\mathcal{G}^{\star\tau}, \mathcal{G}^{\star\pi}, \mathcal{G}^{\star\sigma})$ is stratified; then, we have

$$\begin{aligned}\mathcal{G}^{\star\tau}(\Xi) &\leq \mathcal{G}^{\star\tau}\left(\bigwedge_{t\in J}(\Xi_t \vee \overline{\alpha_t})\right)\\ &\leq \bigvee_{t\in J}\mathcal{G}^{\star\tau}(\Xi_t \vee \overline{\alpha_t})\\ &\leq \bigvee_{t\in J}(\mathcal{G}^{\star\tau}(\Xi_t) \vee \overline{\alpha_t})\\ &\leq \bigvee_{(\Xi_i,\overline{\alpha_i})\in\left\{(\Xi_t,\overline{\alpha_t})|t\in J\right\}}\mathcal{G}^{\tau}(\Xi_i)\vee\alpha_i\end{aligned}$$

Likewise, we can establish through a similar line of reasoning that

$$\mathcal{G}^{\star\pi}(\Xi) \geq \bigwedge_{(\Xi_i,\overline{\alpha_i})\in\left\{(\Xi_t,\overline{\alpha_t})|t\in J\right\}}\mathcal{G}^{\pi}(\Xi_i)\vee\alpha_i \quad \mathcal{G}^{\star\sigma}(\Xi) \geq \bigwedge_{(\Xi_i,\overline{\alpha_i})\in\left\{(\Xi_t,\overline{\alpha_t})|t\in J\right\}}\mathcal{G}^{\sigma}(\Xi_i)\vee\alpha_i.$$

This is a contradiction. Hence, $(\mathcal{G}^{\tau}_{st}, \mathcal{G}^{\pi}_{st}, \mathcal{G}^{\sigma}_{st})$ is the coarsest stratified *svn-grill* on £ which is finer than $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$. $\square$

In order to better understand the notion of *svn-primal* mappings, let us provide some context for the following definition. Consider a non-empty set £ and a mapping $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \zeta^{£} \to \zeta$. We will now introduce certain conditions that, when satisfied, characterize the mapping as an *svn-primal* on £.

Definition 8. *Let £ be a non-empty set. A mapping $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \zeta^{£} \to \zeta$ is said to be svn-primal on £, if it meets the following conditions:*

(P1) $\mathcal{P}^{\tau}(\overline{1}) = 0, \mathcal{P}^{\pi}(\overline{1}) = 1, \mathcal{P}^{\sigma}(\overline{1}) = 1$ *and* $\mathcal{P}^{\tau}(\overline{0}) = 1, \mathcal{P}^{\pi}(\overline{0}) = 0, \mathcal{P}^{\sigma}(\overline{0}) = 0.$

(P2) *If* $\Xi \leq \Theta$, *then* $\mathcal{P}^{\tau}(\Theta) \leq \mathcal{P}^{\tau}(\Xi), \mathcal{P}^{\pi}(\Theta) \geq \mathcal{P}^{\pi}(\Xi),\ \mathcal{P}^{\sigma}(\Theta) \geq \mathcal{P}^{\sigma}(\Xi), \forall\, \Xi, \Theta \in \zeta^{£}.$

(P3) $\mathcal{P}^{\tau}(\Xi \wedge \Theta) \leq \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta),\ \mathcal{P}^{\pi}(\Xi \wedge \Theta) \geq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta),\ \mathcal{P}^{\sigma}(\Xi \wedge \Theta) \geq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta), \forall\, \Xi, \Theta \in \zeta^{£}$. *Sometimes, we will write* $\mathcal{P}^{\tau\pi\sigma}$ *for* $(\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma})$.

If $\mathcal{P}^{\tau\pi\sigma}$ and $\mathcal{P}^{\star\tau\pi\sigma}$ are svn-primals on £, then "$\mathcal{P}^{\tau\pi\sigma}$ is finer than $\mathcal{P}^{\star\tau\pi\sigma}$ or ($\mathcal{P}^{\star\tau\pi\sigma}$ is coarser than $\mathcal{P}^{\tau\pi\sigma}$)" denoted by $\mathcal{P}^{\tau\pi\sigma} \leq \mathcal{P}^{\star\tau\pi\sigma}$ if and only if

$$\mathcal{P}^{\tau}(\Xi) \leq \mathcal{P}^{\star\tau}(\Theta),\ \mathcal{P}^{\pi}(\Xi) \geq \mathcal{P}^{\star\pi}(\Theta),\ \mathcal{P}^{\sigma}(\Xi) \geq \mathcal{P}^{\star\sigma}(\Theta), \forall, \Xi, \Theta \in \zeta^{£}.$$

The triable $(£, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ is called a single-valued neutrosophic primal topological space (svnpts). For $\alpha \in \zeta_0$, $(£, \mathcal{T}^{\tau\pi\sigma}_{\alpha}, \mathcal{P}^{\tau\pi\sigma}_{\alpha})$ is primal topological space (pts) in [35].

Remark 1. *The terms (P_2) and (P_3), in Definition 10, correspond to the next condition*

$$\mathcal{P}^{\tau}(\Xi \wedge \Theta) = \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta), \quad \mathcal{P}^{\pi}(\Xi \wedge \Theta) \neq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta), \quad \mathcal{P}^{\sigma}(\Xi \wedge \Theta) \neq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta)$$

Example 2. *Suppose that $£ = \{l_1, l_2, l_3\}$; define the svn-set $\Theta \in \xi^{£}$ as follows*

$$\Theta = \langle (0.4, 0.4, 0.4), (0, 0, 0), (0, 0, 0) \rangle$$

Define the mappings $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \xi^{£} \to \xi$ as follows:

$$\mathcal{P}^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ \frac{1}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{P}^{\pi}(\Xi) = \begin{cases} 0, & if\ \Xi = \overline{0}, \\ \frac{3}{4}, & if\ \Xi = \Theta, \\ \frac{1}{2}, & if\ \overline{0} < \Xi < \Theta, \\ 1, & otherwise, \end{cases}$$

$$\mathcal{P}^{\sigma}(\Xi) = \begin{cases} 0, & if\ \Xi = \overline{0}, \\ \frac{1}{2}, & if\ \Xi = \Theta, \\ \frac{1}{4}, & if\ \overline{0} < \Xi < \Theta, \\ 1, & otherwise, \end{cases}$$

Then, $\mathcal{P}^{\tau\pi\sigma}$ is svn-primal on £.

Theorem 3. *Let $(\mathcal{G}^{\tau}, \mathcal{G}^{\pi}, \mathcal{G}^{\sigma})$ be an svng on £. Then, the collection $\{\Theta \in \xi^{£} : \mathcal{G}^{\tau}(\Theta^c) \geq r$, $\mathcal{G}^{\pi}(\Theta^c) \leq 1 - r$, $\mathcal{G}^{\sigma}(\Theta^c) \leq 1 - r$, $r \in \xi_0\}$ is an svn-primal on £.*

Proof. (P_1) Since $\mathcal{G}^{\tau}(\overline{0}) = 0, \mathcal{G}^{\pi}(\overline{0}) = 1, \mathcal{G}^{\sigma}(\overline{0}) = 1$ and $\mathcal{G}^{\tau}(\overline{1}) = 1, \mathcal{G}^{\pi}(\overline{1}) = 0, \mathcal{G}^{\sigma}(\overline{1}) = 0$ implies that $\mathcal{P}^{\tau}(\overline{1}) = 0, \mathcal{P}^{\pi}(\overline{1}) = 1, \mathcal{P}^{\sigma}(\overline{1}) = 1$ and $\mathcal{P}^{\tau}(\overline{0}) = 1, \mathcal{G}^{\pi}(\overline{0}) = 0, \mathcal{G}^{\sigma}(\overline{0}) = 0$.

(P_2) Let $\Xi \leq \Theta$; then, $\Theta^c \leq \Xi^c$ and, thus,

$$r \leq \mathcal{G}^{\tau}(\Theta^c) \leq \mathcal{G}^{\tau}(\Xi^c),\ \ 1 - r \geq \mathcal{G}^{\pi}(\Theta^c) \geq \mathcal{G}^{\pi}(\Xi^c),\ \ 1 - r \geq \mathcal{G}^{\sigma}(\Theta^c) \geq \mathcal{G}^{\sigma}(\Xi^c).$$

Hence, $\mathcal{P}^{\tau}(\Theta) \leq \mathcal{P}^{\tau}(\Xi), \mathcal{P}^{\pi}(\Theta) \geq \mathcal{P}^{\pi}(\Xi),\ \mathcal{P}^{\sigma}(\Theta) \geq \mathcal{P}^{\sigma}(\Xi)$.

(P_3) Suppose

$$\mathcal{P}^{\tau}(\Xi \wedge \Theta) \not\leq \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta), \qquad \mathcal{P}^{\pi}(\Xi \wedge \Theta) \not\geq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta),$$

$$\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \not\geq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta).$$

Then, there exists $r \in \xi_0$ such that

$$\mathcal{P}^{\tau}(\Xi \wedge \Theta) \geq r \geq \mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta), \qquad \mathcal{P}^{\pi}(\Xi \wedge \Theta) \leq 1 - r \leq \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta),$$

$$\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \leq 1 - r \leq \mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta).$$

Since $\mathcal{P}^{\tau}(\Xi \wedge \Theta) \geq r$, $\mathcal{P}^{\pi}(\Xi \wedge \Theta) \leq 1 - r$ and $\mathcal{P}^{\sigma}(\Xi \wedge \Theta) \leq 1 - r$, we have $\mathcal{G}^{\tau}([\Xi \wedge \Theta]^c) \geq r$, $\mathcal{P}^{\pi}([\Xi \wedge \Theta]^c) \leq 1 - r$ and $\mathcal{G}^{\sigma}([\Xi \wedge \Theta]^c) \leq 1 - r$ implies that

$$\mathcal{G}^{\tau}(\Xi^c \vee \Theta^c) \geq r,\ \ \mathcal{G}^{\pi}(\Xi^c \vee \Theta^c) \leq 1 - r,\ \ \mathcal{G}^{\sigma}(\Xi^c \vee \Theta^c) \leq 1 - r.$$

From the definition of $\mathcal{G}^{\tau\pi\sigma}$, we have

$$\mathcal{G}^{\tau}(\Xi^c) \vee \mathcal{G}^{\tau}(\Theta^c) \geq \mathcal{G}^{\tau}(\Xi^c \vee \Theta^c) \geq r, \qquad \mathcal{G}^{\pi}(\Xi^c) \wedge \mathcal{G}^{\pi}(\Theta^c) \leq \mathcal{G}^{\pi}(\Xi^c \vee \Theta^c) \leq 1 - r,$$

$$\mathcal{G}^{\sigma}(\Xi^c) \wedge \mathcal{G}^{\sigma}(\Theta^c) \leq \mathcal{G}^{\sigma}(\Xi^c \vee \Theta^c) \leq 1 - r.$$

Since $\mathcal{G}^{\tau}(\Xi^c) \vee \mathcal{G}^{\tau}(\Theta^c) \geq r$, $\mathcal{G}^{\pi}(\Xi^c) \wedge \mathcal{G}^{\pi}(\Theta^c) \leq 1 - r$ and $\mathcal{G}^{\sigma}(\Xi^c \vee \Theta^c) \leq 1 - r$, we obtain $\mathcal{G}^{\tau}(\Theta^c) \geq r$, $\mathcal{G}^{\pi}(\Theta^c) \leq 1 - r$, $\mathcal{G}^{\sigma}(\Theta^c) \leq 1 - r$ and $\mathcal{G}^{\tau}(\Xi^c) \geq r$, $\mathcal{G}^{\pi}(\Xi^c) \leq 1 - r$, $\mathcal{G}^{\sigma}(\Xi^c) \leq 1 - r$ implies that $\mathcal{P}^{\tau}(\Theta) \geq r$, $\mathcal{P}^{\pi}(\Theta) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Theta) \leq 1 - r$ and $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$. Hence,

$$\mathcal{P}^{\tau}(\Xi) \vee \mathcal{P}^{\tau}(\Theta) \geq r, \qquad \mathcal{P}^{\pi}(\Xi) \wedge \mathcal{P}^{\pi}(\Theta) \leq 1 - r,$$

$$\mathcal{P}^{\sigma}(\Xi) \wedge \mathcal{P}^{\sigma}(\Theta) \leq 1 - r.$$

This is a contradiction. Consequently, (P_3) holds. □

Proposition 1. *Let $\{\mathcal{P}_j^{\tau\pi\sigma}\}_{j \in J}$ be a collection of svn-primals on £. Then, their union $\bigvee_{i \in J} \mathcal{P}_j^{\tau\pi\sigma}$ is also an* svn-primal *on £.*

Proof. Directly from Definition 7. □

4. Single-Valued Neutrosophic Primal Open Local Function in Šostak Sense

In this section, we investigate the concept of svn-primal open local functions within the context of Šostak's sense. Our primary focus is to explore the properties of these functions and their relationship with svnts and neutrosophic primals. Through a series of definitions, theorems and discussions, we aim to provide a comprehensive understanding of this unique concept and its implications in the domain of neutrosophic topology. The introductory results presented here lay the foundation for further exploration of this fascinating topic, shedding light on the details of svn structures in the sense of Šostak.

Definition 9. *Let $n, m, v \in \zeta_0$ and $n + m + v \leq 3$. A single-valued neutrosophic point (svn-point) $y_{n,m,v}$ is an svn-set in $\zeta^{£}$ for each $z \in \Theta$, defined by*

$$y_{n,m,v}(z) = \begin{cases} (n, m, v), & \text{if } y = z, \\ (0, 1, 1), & \text{if } y \neq z, \end{cases}$$

An svn-point $y_{n,m,v}$ *is said to belong to an* svn-set $\Xi = \langle z, \tau_{\Xi}(z), \pi_{\Xi}(z), \sigma_{\Xi}(z) \rangle \in \zeta^{£}$ *, denoted by $y_{n,m,v} \in \Xi$ iff $n \leq \tau_{\Xi}(z)$, $m \geq \pi_{\Xi}(z)$ and $v \geq \sigma_{\Xi}(z)$. We indicate the set of all* svn-points *in £ as (svn-point (£)).*

For each $y_{n,m,v} \in$ svn-point(£) *and $\Theta \in \zeta^{\omega}$ we shall write $y_{n,m,v}$ quasi-coincident with Ξ, denoted by $y_{n,m,v} q \Xi$, if*

$$n + \tau_{\Xi}(z) > 1, \quad n + \pi_{\Xi}(z) \leq 1, \quad v + \sigma_{\Xi}(z) \leq 1.$$

For all $\Theta, \Xi \in \zeta^{£}$ we shall write $\Xi \; q \; \theta$ to mean that Ξ is quasi-coincident with Θ if there exists $z \in £$ such that

$$\tau_{\Xi}(z) + \tau_{\Theta}(z) > 1, \quad \pi_{\Xi}(z) + \pi_{\Theta}(z) \leq 1, \quad \sigma_{\Xi}(z) + \sigma_{\Theta}(z) \leq 1.$$

Definition 10. *Let $(£, \mathcal{T}^{\tau\pi\sigma})$ be an svnts, for all $\Xi \in \zeta^{£}$, $y_{n,m,v} \in$svn-point (£) and $r \in \zeta_0$. Then, Ξ is said to be an r-open $\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}$-neighborhood (r-OQN) of $y_{n,m,v}$, defined as follows*

$$\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r) = \{\Xi \in \zeta^{£} | y_{n,m,v} q \Xi, \quad \mathcal{T}^{\tau}(\Xi)) \geq r, \quad \mathcal{T}^{\pi}(\Xi)) \leq 1 - r, \quad \mathcal{T}^{\sigma}(\Xi)) \leq 1 - r\}.$$

Lemma 1. *An* svn-point $y_{n.m.v} \in \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r)$ *iff every* r-O$\mathcal{Q}$N *of $y_{n,n,v}$ is quasi-coincident with Ξ.*

Definition 11. *Let $(£, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts, for each $r \in \zeta_0$ and $\Xi \in \zeta^{£}$. Then, the single-valued neutrosophic primal open local function $\Xi^{\star}_r(\mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ of Ξ is the union of all svn-points $y_{n,m,v}$ such that if $\Theta \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, then there is at least one $z \in £$, for which $\tau_{\Xi}(z) + \tau_{\Theta}(z) - 1 > \tau_{\Pi}(z)$, $\pi_{\Xi}(z) + \pi_{\Theta}(z) - 1 \leq \pi_{\Pi}(z)$, $\sigma_{\Xi}(z) + \sigma_{\Theta}(z) - 1 \leq \sigma_{\Pi}(z)$.*

In this article, we will write $\Xi^{\star}_r$ for $\Xi^{\star}_r(\mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ without any ambiguity.

Example 3. *Let $(£, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ be an svnpts. The simplest svn-primal on £ is $\mathcal{P}^{\tau}_{0}, \mathcal{P}^{\pi}_{0}, \mathcal{P}^{\sigma}_{0} : \zeta^{£} \to \zeta$ where*

$$\mathcal{P}^{\tau}_{0}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathcal{P}^{\pi}_{0}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\mathcal{P}^{\sigma}_{0}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 1, & \text{otherwise}, \end{cases}$$

If $\mathcal{P}^{\tau\pi\sigma} = \mathcal{P}^{\tau\pi\sigma}_{0}$, then, for each $\Xi \in \zeta^{\pounds}$, $r \in \zeta_0$, we have $\Xi^{\star}_{r} = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.

Theorem 4. *Let $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ be an svnts and $\mathcal{P}^{\tau\pi\sigma}_1$, $\mathcal{P}^{\tau\pi\sigma}_2$ be two svn-primals on £. Then, $\forall$ $\Xi, \Theta \in \zeta^{\pounds}$ and $r \in \zeta_0$, we have*

(1) *If $\Xi \leq \Theta$, then $\Xi^{\star}_r \leq \Theta^{\star}_r$.*
(2) *If $\mathcal{P}^{\tau}_1 \leq \mathcal{P}^{\tau}_2$, $\mathcal{P}^{\pi}_1 \geq \mathcal{P}^{\pi}_2$ and $\mathcal{P}^{\sigma}_1 \geq \mathcal{P}^{\sigma}_2$, then $\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma}) \geq \Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})$.*
(3) *$\Xi^{\star}_r = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^{\star}_r, r) \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.*
(4) *If $(\Xi^{\star}_r)^{\star}_r \leq \Xi^{\star}_r$.*
(5) *If $\mathcal{P}^{\tau}(\Theta) \geq r$, $\mathcal{P}^{\pi}(\Theta) \leq 1 - r$ and $\mathcal{P}^{\sigma}(\Theta) \leq 1 - r$, then $(\Xi \vee \Theta)^{\star}_r = \Xi^{\star}_r \vee \Theta^{\star}_r = \Xi^{\star}_r$.*
(6) *$(\Xi^{\star}_r \vee \Theta^{\star}_r) = (\Xi \vee \Theta)^{\star}_r$.*
(7) *$\mathcal{T}^{\tau}(\Theta) \geq r$, $\mathcal{T}^{\pi}(\Theta) \leq 1 - r$ and $\mathcal{T}^{\sigma}(\Theta) \leq 1 - r$, then , $(\Theta \wedge \Xi^{\star}_r) \leq (\Theta \wedge \Xi)^{\star}_r$.*
(8) *$(\Xi^{\star}_r \wedge \Theta^{\star}_r) \geq (\Xi \wedge \Theta)^{\star}_r$.*

Proof. (1) Let $\Xi \in \zeta^{\pounds}$ and $r \in \zeta_0$ such that $\Xi^{\star}_r \not\leq \Theta^{\star}_r$. Then, there exists $z \in \pounds$ and $n, m, v \in \zeta_0$ such that

$$\tau_{\Xi^{\star}_r}(z) \geq n > \tau_{\Theta^{\star}_r}(z), \qquad \pi_{\Xi^{\star}_r}(z) < m \leq \pi_{\Theta^{\star}_r}(z), \qquad \sigma_{\Xi^{\star}_r}(z) < v \leq \sigma_{\Theta^{\star}_r}(z). \tag{1}$$

Since $\tau_{\Theta^{\star}_r}(z) < n$, $\pi_{\Theta^{\star}_r}(z) \geq m$ and $\sigma_{\Theta^{\star}_r}(z) \geq v$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, such that for every $x \in \pounds$, we have,

$$\tau_{\mathcal{A}}(x) + \tau_{\Theta}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Theta}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Theta}(x) - 1 > \sigma_{\Pi}(x).$$

Since $\Xi \leq \Theta$, we have

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

So, $\tau_{\Xi^{\star}_r}(z) < n$, $\pi_{\Xi^{\star}_r}(z) \geq m$ and $\sigma_{\Xi^{\star}_r}(z) \geq v$, and this is a contradiction for Equation (1). Hence, $\Xi^{\star}_r \leq \Theta^{\star}_r$.

(2) Suppose that $\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma}) \not\geq \Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})$. Then, there exist $z \in \pounds$ and $n, m, v \in \zeta_0$ such that

$$\begin{gathered} \tau_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma})}(z) < n \leq \tau_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})}(z), \\ \pi_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma})}(z) \geq m > \pi_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})}(z), \\ \sigma_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma})}(z) \geq v > \sigma_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})}(z). \end{gathered} \tag{2}$$

Since $\tau_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma})}(z) < n$, $\pi_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma})}(z) \geq m$ and $\sigma_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma})}(z) \geq v$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}^{\tau}_1(\Pi) \geq r$, $\mathcal{P}^{\pi}_1(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}_1(\Pi) \leq 1 - r$, such that for every $x \in \pounds$, we have

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

Since, $\mathcal{P}^{\tau}_2(\Pi) \geq \mathcal{P}^{\tau}_1(\Pi) \geq r$, $\mathcal{P}^{\pi}_2(\Pi) \leq \mathcal{P}^{\pi}_1(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}_2(\Pi) \leq \mathcal{P}^{\sigma}_1(\Pi) \leq 1 - r$, we have

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

Hence, $\tau_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})}(z) < s$, $\pi_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})}(z) \geq m$ and $\sigma_{\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})}(z) \geq v$, and this is a contradiction for Equation (2). Thus, $\Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_1, \mathcal{T}^{\tau\pi\sigma}) \geq \Xi^{\star}_r(\mathcal{P}^{\tau\pi\sigma}_2, \mathcal{T}^{\tau\pi\sigma})$.

(3) Assume that $\Xi_r^\star \not\leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$; then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^\star}(z) \geq n > \tau_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi,r)}(z), \quad \pi_{\Xi_r^\star}(z) < m \leq \pi_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi,r)}, \quad \sigma_{\Xi_r^\star}(z) < v \leq \sigma_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi,r)}(z). \quad (3)$$

Since $\tau_{\Xi_r^\star}(z) \geq n$, $\pi_{\Xi_r^\star}(z) < m$ and $\pi_{\Xi_r^\star}(z) < v$, we have $y_{n,m,v} \in \Xi_r^\star$. So, there is at least one $x \in \pounds$, for each $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1-r$, $\mathcal{P}^\sigma(\Pi) \leq 1-r$ such that

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(x) > \tau_{\Pi}(z) + 1, \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) \leq \pi_{\Pi}(z) + 1, \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) \leq \sigma_{\Pi}(x) + 1.$$

From Lemma 1, we have $y_{n,m,v} \in \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$. This is a contradiction for Equation (3). Thus, $\Xi_r^\star \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.

We will now prove this relationship $\Xi_r^\star \geq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star, r)$. Assume that $\Xi_r^\star \not\geq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star, r)$; then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi_r^\star}(z) < n \leq \tau_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star,r)}(z), \quad \pi_{\Xi_r^\star}(z) \geq m > \pi_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star,r)}(z), \quad \sigma_{\Xi_r^\star}(z) \geq v > \sigma_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star,r)}(z). \quad (4)$$

Since $\tau_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star,r)}(z) \geq n$, $\pi_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star,r)}(z) < m$ and $\sigma_{\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star,r)}(z) < v$, we have $y_{n,m,v} \in \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star, r)$. So, there exists at least one $x \in \pounds$ with $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi_r^\star}(x) > 1, \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi_r^\star}(x) \leq 1, \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi_r^\star}(x) \leq 1.$$

Thus, $\Xi_r^\star(x) \neq 0$. Suppose that $n_1 = \tau_{\Xi_r^\star}(x)$, $m_1 = \pi_{\Xi^\star}(x)$ and $v_1 = \sigma_{\Xi_r^\star}(x)$. Then, $x_{n_1,m_1,v_1} \in \Xi_r^*$ and $n_1 + \tau_{\mathcal{A}}(x) > 1$, $m_1 + \pi_{\mathcal{A}} \leq 1$, $v_1 + \sigma_{\mathcal{A}}(x) \leq 1$, so that $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(x_{n_1,m_1,v_1}, r)$. Now, $x_{n_1,m_1,v_1} \in \Xi_r^*$ implies there is at least one $z' \in \pounds$ such that $\tau_{\mathcal{B}}(z') + \tau_{\Xi}(z') - 1 > \tau_{\Pi}(z')$, $\pi_{\mathcal{B}}(z') + \pi_{\Xi}(z') - 1 \leq \pi_{\Pi}(z')$, $\sigma_{\mathcal{B}}(z') + \sigma_{\Xi}(z') - 1 \leq \sigma_{\Pi}(z')$ $\forall$, $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1-r$, $\mathcal{P}^\sigma(\Pi) \leq 1-r$ and $\mathcal{B} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(x_{n_1,m_2,v_1}, r)$. This is also true for $\mathcal{A}$. So, there is at least one $z'' \in \pounds$ such that $\tau_{\mathcal{A}}(z'') + \tau_{\Xi}(z'') - 1 > \tau_{\Pi}(z')$, $\pi_{\mathcal{A}}(z'') + \pi_{\Xi}(z') - 1 \leq \pi_{\Pi}(z'')$, $\sigma_{\mathcal{A}}(z'') + \sigma_{\Xi}(z'') - 1 \leq \sigma_{\Pi}(z';)$. Since $\mathcal{A}$ is an arbitrary and $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, then $\tau_{\Xi_r^\star}(z) > n$, $\pi_{\Xi_r^\star}(z) \leq m$, $\sigma_{\Xi_r^\star}(z) \leq v$. This contradicts Equation (4). Thus, $\Xi_r^\star \geq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi_r^\star, r)$.

(4) Using (3) we obtain that $(\Xi_r^\star)_r^\star = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}((\Xi_r^\star)_r^\star, r) \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}((\Xi_r^\star, r) \leq \Xi_r^\star$.

(5) Straightforward.

(6) ($\Rightarrow$) Since $\Xi, \Theta \leq \Xi \vee \Theta$. By (1), we have $\Xi_r^\star \leq (\Xi \vee \Theta)_r^\star$ and $\Theta_r^\star \leq (\Xi \vee \Theta)_r^\star$. Thus, $\Xi_r^\star \vee \Theta_r^\star \leq (\Xi \vee \Theta)_r^\star$.

($\Leftarrow$) Let $\Xi_r^\star \vee \Theta_r^\star \not\geq (\Xi \vee \Theta)_r^\star$; then, there exist $z \in \pounds$ and $n.m.v \in \xi_0$ such that

$$\tau_{(\Xi_r^\star \vee \Theta_r^\star)}(z) < n \leq \tau_{(\Xi \cup \Theta)_r^\star}(z),$$

$$\pi_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq m > \pi_{(\Xi \cup \Theta)_r^\star}(z), \quad (5)$$

$$\sigma_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq v > \sigma_{(\Xi \cup \Theta)_r^\star}(z).$$

Since $\tau_{(\Xi_r^\star \vee \Theta_r^\star)}(z) < n$, $\pi_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq m$, $\sigma_{(\Xi_r^\star \vee \Theta_r^\star)}(z) \geq v$, we obtain $\tau_{\Xi_r^\star}(z) < n$, $\pi_{\Xi_r^\star}(z) \geq m$, $\sigma_{\Xi_r^\star}(z) \geq v$ or $\tau_{\Theta_r^\star}(z) < n$, $\pi_{\Theta_r^\star}(z) \geq m$, $\sigma_{\Theta_r^\star)}(z) \geq v$. So, there exists $\mathcal{A}_1 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that for each $x \in \pounds$ and for some $\mathcal{P}^\tau(\Pi_1) \geq r$, $\mathcal{P}^\pi(\Pi_1) \leq 1-r$, $\mathcal{P}^\sigma(\Pi_1) \leq 1-r$ we have,

$$\tau_{\mathcal{A}_1}(x) + \tau_{\Xi}(x) - 1 \leq \tau_{\Pi_1}(x),$$

$$\pi_{\mathcal{A}_1}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi_1}(x),$$

$$\sigma_{\mathcal{A}_1}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi_1}(x).$$

Similarly, there exists $\mathcal{A}_2 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that for each $x \in \pounds$ and for some $\mathcal{P}^{\tau}(\Pi_2) \geq r$, $\mathcal{P}^{\pi}(\Pi_2) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi_2) \leq 1-r$ we have,

$$\tau_{\mathcal{A}_2}(x) + \tau_{\Xi}(x) - 1 \leq \tau_{\Pi_2}(x),$$

$$\pi_{\mathcal{A}_2}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi_2}(x),$$

$$\sigma_{\mathcal{A}_2}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi_2}(x).$$

Since $\mathcal{A} = \mathcal{A}_1 \wedge \mathcal{A}_2 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and by (P_3), we obtain $\mathcal{P}^{\tau}(\Pi_1 \wedge \Pi_2) \geq r$, $\mathcal{P}^{\pi}(\Pi_1 \wedge \Pi_2) \leq 1-r$, $\mathcal{P}^{\sigma}(\Pi_1 \wedge \Pi_2) \leq 1-r$. Thus, for each $x \in \xi$,

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi \vee \Theta}(x) - 1 \leq \tau_{\Pi_1 \wedge \Pi_2}(x),$$

$$\pi_{\mathcal{A}}(x) + \pi_{\Xi \vee \Theta}(x) - 1 > \pi_{\Pi_1 \wedge \Pi_2}(x),$$

$$\sigma_{\mathcal{A}}(x) + \sigma_{\Xi \vee \Theta}(x) - 1 > \sigma_{\Pi_1 \wedge \Pi_2}(x).$$

Hence, $\tau_{(\Xi \cup \Theta)^{\star}_r}(z) < n$, $\pi_{(\Xi \cup \Theta)^{\star}_r}(z) \geq m$, $\sigma_{(\Xi \cup \Theta)^{\star}_r}(z) \geq v$. This is a conflict with Equation (5). Thus, $\Xi^{\star}_r \vee \Theta^{\star}_r \geq (\Xi \vee \Theta)^{\star}_r$.

(7) and (8) are obvious. □

Example 4. *Suppose that* $\pounds = \{l_1, l_2, l_3\}$*; define the* svns $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \Theta \in \xi^{\pounds}$ *as follows*

$$\mathcal{A}_1 = \langle (0.8, 08, 0.8), (0.8, 0.8, 0.8), (0.8, 0.8, 0.8) \rangle,$$

$$\mathcal{A}_2 = \langle (0.6, 0.6, 0.6), (0.6, 0.6, 0.6), (0.6, 0.6, 0.6) \rangle,$$

$$\mathcal{A}_3 = \langle (0.5, 0.5, 0.5), (0.5, 0.5, 0.5), (0.5, 0.5, 0.5) \rangle,$$

$$\Theta = \langle (0.2, 0.2, 0.2), (0, 0, 0), (0, 0, 0) \rangle.$$

Define the mappings $\mathcal{T}^{\tau}, \mathcal{T}^{\pi}, \mathcal{T}^{\sigma} : \xi^{\pounds} \to \xi$ *and* $\mathcal{P}^{\tau}, \mathcal{P}^{\pi}, \mathcal{P}^{\sigma} : \xi^{\pounds} \to \xi$ *as follows:*

$$\mathcal{T}^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ 1, & \text{if } \Xi = \overline{1}, \\ \frac{1}{2}, & \text{if } \Xi = \mathcal{A}_1, \\ 0, & \text{otherwise,} \end{cases} \quad \mathcal{P}^{\tau}(\Xi) = \begin{cases} 1, & \text{if } \Xi = \overline{0}, \\ \frac{1}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^{\pi}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 0, & \text{if } \Xi = \overline{1}, \\ \frac{1}{2}, & \text{if } \Xi = \mathcal{A}_2, \\ 1, & \text{otherwise,} \end{cases} \quad \mathcal{P}^{\pi}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ \frac{3}{4}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 1, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^{\sigma}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ 0, & \text{if } \Xi = \overline{1}, \\ \frac{1}{2}, & \text{if } \Xi = \mathcal{A}_3, \\ 1, & \text{otherwise,} \end{cases} \quad \mathcal{P}^{\sigma}(\Xi) = \begin{cases} 0, & \text{if } \Xi = \overline{0}, \\ \frac{2}{3}, & \text{if } \Xi = \Theta, \\ \frac{1}{2}, & \text{if } \overline{0} < \Xi < \Theta, \\ 1, & \text{otherwise,} \end{cases}$$

Let $\mathcal{B} = \langle (0.4, 0.4, 0.4), (0.4, 0.4, 0.4), (0.4, 0.4, 0.4) \rangle$*. Then,* $\overline{0} = (\mathcal{B}^{\star}_{\frac{1}{2}})^{\star}_{\frac{1}{2}} \neq \mathcal{B}^{\star}_{\frac{1}{2}} = \overline{0.2}$.

Theorem 5. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *be an svnpts and* $\{\Xi_i : i \in J\} \subset \xi^{\pounds}$*. Then:*

(1) $(\bigvee(\Xi_i)^\star_r : i \in J) \leq (\bigvee \Xi_i : i \in J)^\star_r$.
(2) $(\bigwedge \Xi_i : i \in J)^\star_r \leq (\bigwedge(\Xi_i)^\star_r : i \in J)$.

Proof. (1) Since $\Xi_i \leq \bigvee \Xi_i$, for all $i \in J$, by Theorem 4(1), we obtain $(\Xi_i)^\star_r \leq (\bigvee \Xi_i)^\star_r$, for every $i \in J$. Thus,

$$(\bigvee(\Xi_i)^\star_r : i \in J) \leq (\bigvee \Xi_i : i \in J)^\star_r.$$

(2) Since $\bigwedge \Xi_i \leq \mathcal{A}_i$, and by (1) in Theorem 4, we have $(\bigwedge \Xi_i)^\star_r \leq (\Xi_i)^\star_r, \forall\, i \in J$. Hence,

$$(\bigwedge(\Xi_i)^\star_r : i \in J) \geq (\bigwedge \Xi_i : i \in J)^\star_r.$$

□

Remark 2. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *be an svnpts and* $\Xi \in \xi^{\pounds}$*; we can define*

$$\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \Xi \cup \Xi^\star_r, \quad \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \Xi \wedge ((\Xi^c)^\star_r)^c.$$

Clearly, $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}$ *is an svn-closure operator and* $(\mathcal{T}^\tau(\mathcal{P}^\tau), \mathcal{T}^\pi(\mathcal{P}^\pi), \mathcal{T}^\sigma(\mathcal{P}^\sigma))$ *is the svnt generated by* $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}$*, i.e.,*

$$\mathcal{T}^\star(\mathcal{I})(\Xi) = \bigvee\{r \mid \ \mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = \Xi^c\}.$$

Now, if $\mathcal{P}^{\tau\pi\sigma} = \mathcal{P}_0^{\tau\pi\sigma}, \forall\, \Xi \in \xi^{\pounds}$*, then* $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \Xi \cup \Xi^\star_r = \Xi \vee \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) = \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$*. So,* $\mathcal{T}^{\tau\star}(\mathcal{P}_0^\tau) = \mathcal{T}^\tau,\ \mathcal{T}^{\pi\star}(\mathcal{P}_0^\pi) = \mathcal{T}^\pi$ *and* $\mathcal{T}^{\sigma\star}(\mathcal{P}_0^\sigma) = \mathcal{T}^\sigma$*.*

Theorem 6. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *be an svnpts and* $\Xi \in \xi^{\pounds}$ *and* $r \in \xi_0$ *and* $\Xi \in \xi^{\pounds}$*. Then*

(1) $\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi \vee \Theta, r) \leq \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \vee \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r)$.
(2) $\mathcal{I}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \Xi \leq \mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \leq \mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)$.
(3) $\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi \wedge \Theta, r) = \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r) \wedge \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Theta, r)$.
(4) $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = [\mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)]^c$ *and* $[\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi, r)]^c = \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r)$.

Proof. Straightforward from $\mathcal{C}^\star_{\mathcal{T}^{\tau\pi\sigma}}, \mathcal{I}^\star_{\mathcal{T}^{\tau\pi\sigma}}$ and $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}$. □

Theorem 7. *Let* $(\pounds, \mathcal{T}_1^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *and* $(\pounds, \mathcal{T}_2^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *be svnptss and* $\mathcal{T}_1^\tau \leq \mathcal{T}_2^\tau$, $\mathcal{T}_1^\pi \geq \mathcal{T}_2^\pi$, $\mathcal{T}_1^\sigma \geq \mathcal{T}_2^\sigma$*. Then,*

(1) $\Xi^\star_r(\mathcal{T}_2^\tau, \mathcal{P}^\tau) \leq \Xi^\star_r(\mathcal{T}_1^\tau, \mathcal{P}^\tau), \Xi^\star_r(\mathcal{T}_2^\pi, \mathcal{P}^\pi) \geq \Xi^\star_r(\mathcal{T}_1^\pi, \mathcal{P}^\pi)$ *and* $\Xi^\star_r(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma) \geq \Xi^\star_r(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma)$.
(2) $\mathcal{T}_1^{\star\tau}(\mathcal{P}^\tau) \leq \mathcal{T}_2^{\star\tau}(\mathcal{P}^\tau)$, $\mathcal{T}_1^{\star\pi}(\mathcal{P}^\pi) \geq \mathcal{T}_2^{\star\pi}(\mathcal{P}^\pi)$, $\mathcal{T}_1^{\star\sigma}(\mathcal{P}^\sigma) \geq \mathcal{T}_2^{\star\sigma}(\mathcal{P}^\sigma)$.

Proof. (1) Let

$$\Xi^\star_r(\mathcal{T}_2^\tau, \mathcal{P}^\tau) \not\leq \Xi^\star_r(\mathcal{T}_1^\tau, \mathcal{P}^\tau), \quad \Xi^\star_r(\mathcal{T}_2^\pi, \mathcal{P}^\pi) \not\geq \Xi^\star_r(\mathcal{T}_1^\pi, \mathcal{P}^\pi),$$

$$\Xi^\star_r(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma) \not\geq \Xi^\star_r(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma),$$

then, there exist $z \in \pounds$ and $n, m, v \in \xi_0$ such that

$$\tau_{\Xi^\star_r(\mathcal{T}_2^\tau, \mathcal{P}^\tau)}(z) \geq n > \tau_{\Xi^\star_r(\mathcal{T}_1^\tau, \mathcal{P}^\tau)}(z),$$

$$\pi_{\Xi^\star_r(\mathcal{T}_2^\pi, \mathcal{P}^\pi)}(z) < m \leq \pi_{\Xi^\star_r(\mathcal{T}_1^\pi, \mathcal{P}^\pi)}(z), \tag{6}$$

$$\sigma_{\Xi^\star_r(\mathcal{T}_2^\sigma, \mathcal{P}^\sigma)}(z) < v \leq \sigma_{\Xi^\star_r(\mathcal{T}_1^\sigma, \mathcal{P}^\sigma)}(z).$$

Since $\tau_{\Xi^\star_r(\mathcal{T}_1^\tau,\mathcal{P}^\tau)}(z) < n$, $\pi_{\Xi^\star_r(\mathcal{T}_1^\pi,\mathcal{P}^\pi)}(z) \geq m$, $\sigma_{\Xi^\star_r(\mathcal{T}_1^\sigma,\mathcal{P}^\sigma)}(z) \geq v$, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}_1^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1-r$, $\mathcal{P}^\sigma(\Pi) \leq 1-r$, such that for each $x \in £$,

$$\tau_{\mathcal{A}}(x) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(x), \quad \pi_{\mathcal{A}}(x) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(x), \quad \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(x) - 1 > \sigma_{\Pi}(x).$$

Since $\mathcal{T}_1^\tau \leq \mathcal{T}_2^\tau$, $\mathcal{T}_1^\pi \geq \mathcal{T}_2^\pi$, $\mathcal{T}_1^\sigma \geq \mathcal{T}_2^\sigma$, we have $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}_2^{\tau\pi\sigma}}(y_{n,m,v}, r)$. Hence, $\tau_{\Xi^\star_r(\mathcal{T}_2^\tau,\mathcal{P}^\tau)}(z) < n$, $\pi_{\Xi^\star_r(\mathcal{T}_2^\pi,\mathcal{P}^\pi)}(z) \geq m$, $\sigma_{\Xi^\star_r(\mathcal{T}_2^\sigma,\mathcal{P}^\sigma)}(z) \geq v$. This is a contradiction of Equation (6).

(2) Similar to part (1). □

Theorem 8. *Let $(£, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}_1^{\tau\pi\sigma})$ and $(£, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}_2^{\tau\pi\sigma})$ be svnptss and $\mathcal{P}_1^\tau \leq \mathcal{P}_2^\tau$, $\mathcal{P}_1^\pi \geq \mathcal{P}_2^\pi$, $\mathcal{P}_1^\sigma \geq \mathcal{P}_2^\sigma$. Then,*

(1) $\Xi^\star_r(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \geq \Xi^\star_r(\mathcal{P}_2^\tau, \mathcal{T}^\pi)$, $\Xi^\star_r(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \leq \Xi^\star_r(\mathcal{P}_2^\pi, \mathcal{T}^\pi)$ *and* $\Xi^\star_r(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \leq \Xi^\star_r(\mathcal{P}_2^\tau, \mathcal{T}^\tau)$.

(2) $\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau) \leq \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau)$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi) \geq \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi)$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma) \geq \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)$.

Proof. The same method as the proof of Theorem 7. □

Theorem 9. *Let $(£, \mathcal{T}^{\tau\pi\sigma})$ be an svnts and $\mathcal{P}_1^{\tau\pi\sigma}$, $\mathcal{P}_2^{\tau\pi\sigma}$ be two svn-primals on £. Then, $\forall\ \Xi \in \xi^£$ and $r \in \xi_0$,*

(1)

$$\Xi^\star_r(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau) = \Xi^\star_r(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi^\star_r(\mathcal{P}_2^\tau, \mathcal{T}^\tau),$$
$$\Xi^\star_r(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) = \Xi^\star_r(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi^\star_r(\mathcal{P}_2^\pi, \mathcal{T}^\pi),$$
$$\Xi^\star_r(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) = \Xi^\star_r(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi^\star_r(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma).$$

(2)

$$\Xi^\star_r(\mathcal{P}_1^\tau \vee \mathcal{P}_2^\tau, \mathcal{T}^\tau) = \Xi^\star_r(\mathcal{P}_1^\tau, \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau)) \wedge \Xi^\star_r(\mathcal{P}_2^\tau, \mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau))),$$
$$\Xi^\star_r(\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi, \mathcal{T}^\pi) = \Xi^\star_r(\mathcal{P}_1^\pi, \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi)) \vee \Xi^\star_r(\mathcal{P}_2^\pi, \mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi)),$$
$$\Xi^\star_r(\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) = \Xi^\star_r(\mathcal{P}_1^\sigma, \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)) \vee \Xi^\star_r(\mathcal{P}_2^\sigma, \mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma)).$$

Proof. (1) Let

$$\Xi^\star_r(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau)) \not\leq \Xi^\star_r(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi^\star_r(\mathcal{P}_2^\tau, \mathcal{T}^\tau),$$

$$\Xi^\star_r(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) \not\geq \Xi^\star_r(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi^\star_r(\mathcal{P}_2^\pi, \mathcal{T}^\pi),$$

$$\Xi^\star_r(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) \not\geq \Xi^\star_r(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi^\star_r(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma).$$

Then there exist $z \in £$ and $n, m, v \in \xi_0$ s.t.

$$\tau_{\Xi^\star_r(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau))}(z) \geq n > \tau_{\Xi^\star_r(\mathcal{P}_1^\tau,\mathcal{T}^\tau)}(z) \vee \tau_{\Xi^\star_r(\mathcal{P}_2^\tau,\mathcal{T}^\tau)}(z),$$

$$\pi_{\Xi^\star_r(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi)}(z) < m \leq \pi_{\Xi^\star_r(\mathcal{P}_1^\pi,\mathcal{T}^\pi)}(z) \wedge \pi_{\Xi^\star_r(\mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z), \tag{7}$$

$$\sigma_{\Xi^\star_r(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma)}(z) < v \leq \sigma_{\Xi^\star_r(\mathcal{P}_1^\sigma,\mathcal{T}^\sigma)}(z) \wedge \sigma_{\Xi^\star_r(\mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z).$$

Since $[\tau_{\Xi^\star_r(\mathcal{P}_1^\tau,\mathcal{T}^\tau)}(z) \vee \tau_{\Xi^\star_r(\mathcal{P}_2^\tau,\mathcal{T}^\tau)}(z)] < n$, $[\pi_{\Xi^\star_r(\mathcal{P}_1^\pi,\mathcal{T}^\pi)}(z) \wedge \pi_{\Xi^\star_r(\mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z)] \geq m$, $[\sigma_{\Xi^\star_r(\mathcal{P}_1^\sigma,\mathcal{T}^\sigma)}(z) \wedge \sigma_{\Xi^\star_r(\mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z)] \geq v$, we obtain $\tau_{\Xi^\star_r(\mathcal{P}_1^\tau,\mathcal{T}^\tau)}(z) < n$, $\pi_{\Xi^\star_r(\mathcal{P}_1^\pi,\mathcal{T}^\pi)}(z) \geq m$, $\sigma_{\Xi^\star_r(\mathcal{P}_1^\sigma,\mathcal{T}^\sigma)}(z) \geq v$ and $\tau_{\Xi^\star_r(\mathcal{P}_2^\tau,\mathcal{T}^\tau)}(z) < n$, $\pi_{\Xi^\star_r(\mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z) \geq m$, $\sigma_{\Xi^\star_r(\mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z) \geq v$.

First, by taking the first part, $\tau_{\Xi_r^\star(\mathcal{P}_1^\tau,\mathcal{T}^\tau)}(z) < n,\ \pi_{\Xi_r^\star(\mathcal{P}_1^\pi,\mathcal{T}^\pi)}(z) \geq m,\ \sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma,\mathcal{T}^\sigma)}(z) \geq v$, there will be $\mathcal{A}_1 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}_1^\tau(\Pi_1) \geq r$, $\mathcal{P}_1^\pi(\Pi_1) \leq 1-r$, $\mathcal{P}_1^\sigma(\Pi_1) \leq 1-r$, such that $\forall$, $x \in \pounds$,

$$\tau_{\mathcal{A}_1}(x) + \tau_\Xi(x) - 1 \leq \tau_{\Pi_1}(x),$$

$$\pi_{\mathcal{A}_1}(x) + \pi_\Xi(x) - 1 > \pi_{\Pi_1}(x),$$

$$\sigma_{\mathcal{A}_1}(x) + \sigma_\Xi(x) - 1 > \sigma_{\Pi_1}(x).$$

Secondly, by taking the second part, $\tau_{\Xi_r^\star(\mathcal{P}_2^\tau,\mathcal{T}^\tau)}(z) < n,\ \pi_{\Xi_r^\star(\mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z) \geq m,\ \sigma_{\Xi_r^\star(\mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z) \geq v$, there will be $\mathcal{A}_2 \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ with $\mathcal{P}_2^\tau(\Pi_2) \geq r$, $\mathcal{P}_2^\pi(\Pi_2) \leq 1-r$, $\mathcal{P}_2^\sigma(\Pi_2) \leq 1-r$, such that $\forall$, $x \in \pounds$,

$$\tau_{\mathcal{A}_2}(x) + \tau_\Xi(x) - 1 \leq \tau_{\Pi_2}(x),$$

$$\pi_{\mathcal{A}_2}(x) + \pi_\Xi(x) - 1 > \pi_{\Pi_2}(x),$$

$$\sigma_{\mathcal{A}_2}(x) + \sigma_\Xi(x) - 1 > \sigma_{\Pi_2}(x).$$

Thus, $\forall\ y \in \pounds$, we obtain

$$\tau_{(\mathcal{A}_1 \wedge \mathcal{A}_2)}(x) + \tau_\Xi(x) - 1 > \tau_{(\Pi_1 \wedge \Pi_2)}(x),$$

$$\pi_{(\mathcal{A}_1 \vee \mathcal{A}_2)}(x) + \pi_\Xi(x) - 1 \leq \pi_{(\Pi_1 \vee \Pi_2)}(x),$$

$$\sigma_{(\mathcal{A}_1 \vee \mathcal{A}_2)}(x) + \sigma_\Xi(x) - 1 \leq \sigma_{(\Pi_1 \vee \Pi_2)}(x).$$

Since $(\mathcal{A}_1 \wedge \mathcal{A}_2) \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and

$$(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau)(\Pi_1 \wedge \Pi_2) \geq r, \quad (\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi)(\Pi_1 \vee \Pi_2) \leq 1-r,$$
$$(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi)(\Pi_1 \vee \Pi_2) \leq 1-r,$$

we obtain that $\tau_{\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau,\mathcal{T}^\tau))}(z) \leq n$, $\pi_{\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi,\mathcal{T}^\pi)}(z) > m$, $\sigma_{\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma,\mathcal{T}^\sigma)}(z) > v$ and this is a contradiction of Equation (7). So,

$$\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau)) \leq \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau),$$

$$\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) \geq \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi),$$

$$\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) \geq \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma).$$

On the other side $\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau \leq \mathcal{P}_1^\tau$, $\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi \geq \mathcal{P}_1^\pi$, $\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma \geq \mathcal{P}_1^\sigma$ and $\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau \leq \mathcal{P}_2^\tau$, $\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi \geq \mathcal{P}_2^\pi$, $\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma \geq \mathcal{P}_2^\sigma$, so by Theorem 4(2),

$$\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau) \geq \Xi_r^\star(\mathcal{P}_1^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau), \quad \Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi) \leq \Xi_r^\star(\mathcal{P}_1^\tau) \wedge \Xi_r^\star(\mathcal{P}_2^\pi),$$

$$\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma) \leq \Xi_r^\star(\mathcal{P}_1^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma).$$

Hence, $\Xi_r^\star(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau, \mathcal{T}^\tau) = \Xi_r^\star(\mathcal{P}_1^\tau, \mathcal{T}^\tau) \vee \Xi_r^\star(\mathcal{P}_2^\tau, \mathcal{T}^\tau)$, $\Xi_r^\star(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi, \mathcal{T}^\pi) = \Xi_r^\star(\mathcal{P}_1^\pi, \mathcal{T}^\pi) \wedge \Xi_r^\star(\mathcal{P}_2^\pi, \mathcal{T}^\pi)$ and $\Xi_r^\star(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma, \mathcal{T}^\sigma) = \Xi_r^\star(\mathcal{P}_1^\sigma, \mathcal{T}^\sigma) \wedge \Xi_r^\star(\mathcal{P}_2^\sigma, \mathcal{T}^\sigma)$.

(2) Similar to part (1). □

From the previous theory, we can see the important result that $(\mathcal{T}^{\star\tau}(\mathcal{P}^\tau), \mathcal{T}^{\star\pi}(\mathcal{P}^\pi), \mathcal{T}^{\star\sigma}(\mathcal{P}^\star))$ (for short $\mathcal{T}^\star(\mathcal{P})$) and $([\mathcal{T}^{\star\tau}(\mathcal{P}^\tau)]^\star(\mathcal{P}^\tau), [\mathcal{T}^{\star\pi}(\mathcal{P}^\pi)]^\star((\mathcal{P}^\pi)), [\mathcal{T}^{\star\sigma}(\mathcal{P}^\star)]^\star(\mathcal{P}^\sigma))$ (for short $\mathcal{T}^{\star\star}$) are equal for any *svn-primals* on £.

Theorem 10. *Let* $(£, \mathcal{T}^{\tau\pi\sigma}, \mathcal{P}^{\tau\pi\sigma})$ *be an svnpts. Then, for each* $\Xi \in \zeta^£$ *and* $r \in \zeta_0$*,*

(1) $\Xi_r^\star(\mathcal{P}^\tau) = \Xi_r^\star(\mathcal{P}^\tau, \mathcal{T}^{\star\tau})$, $\Xi_r^\star(\mathcal{P}^\pi) = \Xi_r^\star(\mathcal{P}^\pi, \mathcal{T}^{\star\pi})$ *and* $\Xi_r^\star(\mathcal{P}^\sigma) = \Xi_r^\star(\mathcal{P}^\sigma, \mathcal{T}^{\star\sigma}))$.
(2) $(\mathcal{T}^{\star\tau}(\mathcal{P}^\tau) = \mathcal{T}^{\star\star\tau}, \mathcal{T}^{\star\pi}(\mathcal{P}^\pi) = \mathcal{T}^{\star\star\pi}, \mathcal{T}^{\star\sigma}(\mathcal{P}^\pi)) = \mathcal{T}^{\star\star\sigma}$.

Proof. By putting $\mathcal{P}_1^\tau = \mathcal{P}_2^\tau$, $\mathcal{P}_1^\pi = \mathcal{P}_2^\pi$ and $\mathcal{P}_1^\sigma = \mathcal{P}_2^\sigma$ in the second part of the above theorem, we obtain the required proof. □

Theorem 11. *Let* $(£, \mathcal{T}^{\tau\pi\sigma})$ *be an svnts and* $\mathcal{P}_1^{\tau\pi\sigma}$, $\mathcal{P}_2^{\tau\pi\sigma}$ *be two svn-primals on £. Then,* $\forall$ $r \in \zeta_0$ *and* $\Xi \in \zeta^£$, $\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau) = \mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau) \wedge \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau)$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi) = \mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi) \vee \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi)$ *and* $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma) = \mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma) \vee \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)$.

Proof. ($\Rightarrow$): Since $\mathcal{P}_1^\tau \geq \mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau$, $\mathcal{P}_1^\pi \leq \mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi$, $\mathcal{P}_1^\sigma \leq \mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma$ and $\mathcal{P}_2^\tau \geq \mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau$, $\mathcal{P}_2^\pi \leq \mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi$, $\mathcal{P}_2^\sigma \leq \mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma$, by Theorem 8(2) we obtain $\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau) \geq \mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau)$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi) \leq \mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi)$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma) \leq \mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma)$ and $\mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau) \geq \mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau)$, $\mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi) \leq \mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi)$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma) \leq \mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma)$. Hence,

$$\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau) \leq \mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau) \wedge \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau),$$

$$\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi \wedge \mathcal{P}_2^\pi) \geq \mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi) \vee \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi),$$

$$\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma \wedge \mathcal{P}_2^\sigma) \geq \mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma) \vee \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma).$$

($\Leftarrow$): Suppose that $(\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau) \wedge \mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau))(\Xi) \geq r$, $(\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi) \vee \mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi))(\Xi) \leq 1 - r$, $(\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma) \vee \mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)(\Xi) \leq 1 - r$. Then, $\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau)(\Xi) \geq r$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi)(\Xi) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma)(\Xi) \leq 1 - r$ and $\mathcal{T}^{\star\tau}(\mathcal{P}_2^\tau)(\Xi) \geq r$, $\mathcal{T}^{\star\pi}(\mathcal{P}_2^\pi)(\Xi) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_2^\sigma)(\Xi) \leq 1 - r$; that means $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = \Xi^c \vee [\Xi^c]_r^\star(\mathcal{P}_1^{\tau\pi\sigma}) = \Xi^c$ and $\mathcal{C}_{\mathcal{T}^{\tau\pi\sigma}}(\Xi^c, r) = \Xi^c \vee [\Xi^c]_r^\star(\mathcal{P}_2^{\tau\pi\sigma}) = \Xi^c$. Thus, $[\Xi^c]_r^\star(\mathcal{P}_1^{\tau\pi\sigma}) \leq \Xi^c$ and $[\Xi^c]_r^\star(\mathcal{P}_2^{\tau\pi\sigma}) \leq \Xi^c$. So, $[\Xi^c]_r^\star(\mathcal{P}_1^{\tau\pi\sigma}) \vee [\Xi^c]_r^\star(\mathcal{P}_2^{\tau\pi\sigma}) \leq \Xi^c$. From Theorem 9(1), we have $[\Xi^c]_r^\star(\mathcal{P}_1^{\tau\pi\sigma} \wedge \mathcal{P}_2^{\tau\pi\sigma}) \leq \Xi^c$. Therefore, $\mathcal{T}^{\star\tau}(\mathcal{P}_1^\tau \wedge \mathcal{P}_2^\tau)(\Xi) \geq r$, $\mathcal{T}^{\star\pi}(\mathcal{P}_1^\pi \vee \mathcal{P}_2^\pi)(\Xi) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\mathcal{P}_1^\sigma \vee \mathcal{P}_2^\sigma)(\Xi) \leq 1 - r$. This completes the proof. □

Definition 12. *Let* $(£, \mathcal{T}^{\tau\pi\sigma})$ *be an svnts with* $\mathcal{P}^{\tau\pi\sigma}$ *an svn-primal on £. Then,* $\mathcal{T}^{\tau\pi\sigma}$ *is called single-valued neutrosophic primal open compatible with* $\mathcal{P}^{\tau\pi\sigma}$*, indicated by* $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$*, if* $\forall$ $\Xi \in \zeta^£$, $y_{n,m,v} \in \Xi$ *and* $\Pi \in \zeta^£$ *with* $\mathcal{P}^\tau(\Pi) \geq r$, $\mathcal{P}^\pi(\Pi) \leq 1 - r$, $\mathcal{P}^\sigma(\Pi) \leq 1 - r$*, there exists* $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ *such that*

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\Xi}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\xi}(z) - 1 > \sigma_{\Pi}(z),$$

holds for every $z \in £$*, then* $\mathcal{P}^\tau(\Xi) \geq r$, $\mathcal{P}^\pi(\Xi) \leq 1 - r$, $\mathcal{P}^\sigma(\Xi) \leq 1 - r$.

Definition 13. *Let* $\{\Theta_i : i \in \Gamma\} \sqsubseteq \zeta^£$ *such that* $\Theta_i q \Xi$, $\forall$ $i \in \Gamma$ *and* $\Xi \in \zeta^£$*. Then,* $\{\Theta_i : i \in \Gamma\}$ *is called an r-single-valued neutrosophic quasi-cover (for short, r-svnq-cover) of* Ξ *iff,* $\forall$ $z \in £$,

$$\tau_{\Xi}(z) + \tau_{\bigvee_{i\in\Gamma}(\Theta_i)}(z) \geq 1, \quad \pi_{\Xi}(z) + \pi_{\bigvee_{i\in\Gamma}(\Theta_i)}(z) < 1, \quad \sigma_{\Xi}(z) + \sigma_{\bigvee_{i\in\Gamma}(\Theta_i)}(z) < 1.$$

Further, let $(£, \mathcal{T}^{\tau\pi\sigma})$ be an *svnts*, for any $\mathcal{T}^\tau(\Theta_i) \geq r$, $\mathcal{T}^\pi(\Theta_i) \leq 1 - r$, $\mathcal{T}^\sigma(\Theta_i) \leq 1 - r$. Then, this *r-svnq-cover* will be called a single-valued neutrosophic quasi open-cover (for short, *svnqo-cover*) of Ξ.

Theorem 12. *Let* $(\pounds, \mathcal{T}^{\tau\pi\sigma})$ *be an* svnts *with* svn-primal $\mathcal{P}^{\tau\pi\sigma}$ *on* £. *Then, the following conditions are equivalent*

(1) $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$.

(2) *If for each* $\Xi \in \zeta^{\pounds}$ *has a svnqo-cover of* $\{\Theta_i : i \in \Gamma\}$ *such that* $\tau_{\mathcal{A}}(z) + \tau_{\Theta_i}(z) - 1 \leq \tau_{\Pi}(z)$ $\pi_{\mathcal{A}}(z) + \pi_{\Theta_i}(z) - 1 > \pi_{\Pi}(z)$, $\sigma_{\mathcal{A}}(z) + \sigma_{\Theta_i}(z) - 1 > \sigma_{\Pi}(z)$, *for every* $z \in \pounds$ *and for some* $\mathcal{T}^{\tau}(\Pi) \geq r$, $\mathcal{T}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{T}^{\sigma}(\Pi) \leq 1 - r$, *then* $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.

(3) *For any* $\Xi \in \zeta^{\pounds}$, $\Xi \wedge \Xi^{\star}_r = \overline{0}$ *implies* $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.

(4) *For any* $\Xi \in \zeta^{\pounds}$, $\mathcal{P}^{\tau}(\overbrace{\ \Xi\ }) \geq r$, $\mathcal{P}^{\pi}(\overbrace{\ \Xi\ }) \leq 1 - r$, $\mathcal{P}^{\sigma}(\overbrace{\ \Xi\ }) \leq 1 - r$, *where* $\overbrace{\ \Xi\ } = \bigvee y_{n.m,v}$ *such that* $y_{n,m,v} \in \Xi$ *but* $y_{n,m,v} \notin \Xi^{\star}_r$.

(5) *For all* $\mathcal{T}^{\star\tau}(\Xi^c) \geq r$, $\mathcal{T}^{\star\pi}(\Xi^c) \leq 1 - r$, $\mathcal{T}^{\star\sigma}(\Xi^c) \leq 1 - r$, *we have* $\mathcal{P}^{\tau}(\overbrace{\ \Xi\ }) \geq r$, $\mathcal{P}^{\pi}(\overbrace{\ \Xi\ }) \leq 1 - r$, $\mathcal{P}^{\sigma}(\overbrace{\ \Xi\ }) \leq 1 - r$.

(6) *For each* $\Xi \in \zeta^{\pounds}$, *if* Ξ *contains no* $\Theta \neq \overline{0}$ *with* $\Theta \leq \Theta^{\star}_r$, *then* $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.

Proof. $1 \Rightarrow 2$: Let $\{\Theta_i : i \in \Gamma\}$ be an *svnqo-cover* of $\Xi \in \zeta^{\pounds}$ such that, $\forall\, i \in \Gamma$, $\tau_{\mathcal{A}}(z) + \tau_{\Theta_i}(z) - 1 \leq \tau_{\Pi}(z)$, $\pi_{\mathcal{A}}(z) + \pi_{\Theta_i}(z) - 1 > \pi_{\Pi}(z)$, $\sigma_{\mathcal{A}}(z) + \sigma_{\Theta_i}(z) - 1 > \sigma_{\Pi}(z)$, for any $z \in \pounds$ and for some $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$. Thus, as $\{\Theta_i : i \in \Gamma\}$ is an *svnqo-cover* of Ξ, for each $y_{n,m,v} \in \Xi$, there exists at least one $\Theta_{i\circ}$ such that $y_{n,m,v} q \Theta_{i\circ}$ and for each $z \in \pounds$, $\tau_{\mathcal{A}}(z) + \tau_{\Theta_{i\circ}}(z) - 1 \leq \tau_{\Pi}(z)$, $\pi_{\mathcal{A}}(z) + \pi_{\Theta_{i\circ}}(z) - 1 > \pi_{\Pi}(z)$, $\sigma_{\mathcal{A}}(z) + \sigma_{\Theta_{i\circ}}(z) - 1 > \sigma_{\Pi}(z)$ for some $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$, $\mathcal{T}^{\tau}(\Pi) \geq r$, $\mathcal{T}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{T}^{\sigma}(\Pi) \leq 1 - r$. Significantly, $\Theta_{i\circ} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$. From (1), we obtain $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.

$2 \Rightarrow 1$: It is clear from this that the family of $\{\Theta_i : i \in \Gamma\}$ contains at least one $\Theta_{i\circ} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, such that every *svn-point* of Ξ constitutes a *svnqo-cover* of Ξ.

$1 \Rightarrow 3$: Let $\Xi \wedge \Xi^{\star}_r = \overline{0}$, for each $z \in \zeta$, $y_{n,m,v} \in \Xi$ implies $y_{n,m,v} \notin \Xi^{\star}_r$. Then, there exists $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ and $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, such that for all $z \in \pounds$

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \;\; \pi_{\mathcal{A}}(z) + \pi_{\Xi}(x) - 1 > \pi_{\Pi}(z), \;\; \sigma_{\mathcal{A}}(x) + \sigma_{\Xi}(z) - 1 > \sigma_{\Pi}(x).$$

Since $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, from (1), we have $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$.

$3 \Rightarrow 1$: For each $y_{n,m,v} \in \Xi$, there exists an $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that, for any $z \in \pounds$,

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\Xi}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\Xi}(z) - 1 > \sigma_{\Pi}(z),$$

for some $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, implies $y_{n,m,v} \notin \Xi^{\star}_r$. Firstly, there are two cases: either $\Xi^{\star}_r = \overline{0}$ or $\Xi^{\star}_r \neq \overline{0}$ but $n > \tau_{\Xi^{\star}_r}(z) \neq \overline{0}$, $m \leq \pi_{\Xi^{\star}_r}(z) \neq \overline{0}$, $v \leq \sigma_{\Xi^{\star}_r}(z) \neq \overline{0}$. Assume, if possible, $y_{n.m.v} \in \Xi$ such that $n > \tau_{\Xi^{\star}_r}(z) \neq \overline{0}$, $m \leq \pi_{\Xi^{\star}_r}(z) \neq \overline{0}$, $v \leq \sigma_{\Xi^{\star}_r}(z) \neq \overline{0}$. Let $n_1 = \tau_{\Xi^{\star}_r}(z)$, $m_1 = \pi_{\Xi^{\star}_r}(z)$, $v_1 = \sigma_{\Xi^{\star}_r}(z)$, . Then, $y_{n_1,m_1,v_1} \in \Xi^{\star}_r(z)$. Secondly, $y_{n_1,m_1,v_1} \in \Xi$. Thus, for all $\mathcal{B} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$, $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$, there is at least one $z \in \pounds$ such that

$$\tau_{\mathcal{B}}(z) + \tau_{\Xi}(z) - 1 > \tau_{\Pi}(z), \quad \pi_{\mathcal{B}}(z) + \pi_{\Xi}(z) - 1 \leq \pi_{\Pi}(z), \quad \sigma_{\mathcal{B}}(z) + \sigma_{\Xi}(z) - 1 \leq \sigma_{\Pi}(z).$$

Since $y_{n_1,m_1,v_1} \in \Xi$, this contradicts the assumption for each *svn-point* of Ξ. So, $\Xi^{\star}_r = \overline{0}$. That means $y_{n,m,v} \in \Xi$ implies $y_{n,m,v} \notin \Xi^{\star}_r$. Now, this is true for each $\Xi \in \zeta^{\pounds}$. So, $\Xi \vee \Xi^{\star}_r = \overline{0}$. Thus, by (3), we obtain $\mathcal{P}^{\tau}(\Xi) \geq r$, $\mathcal{P}^{\pi}(\Xi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Xi) \leq 1 - r$ implies $\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$.

$3 \Rightarrow 4$: Let $y_{n,m,v} \in \overbrace{\ \Xi\ }$. Then, $y_{n,m,v} \in \Xi$ but $y_{n,m,v} \notin \Xi^{\star}_r$. So, there exists an $\mathcal{A} \in \mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v}, r)$ such that $\forall\, z \in \pounds$,

$$\tau_{\mathcal{A}}(z) + \tau_{\Xi}(z) - 1 \leq \tau_{\Pi}(z), \quad \pi_{\mathcal{A}}(z) + \pi_{\Xi}(z) - 1 > \pi_{\Pi}(z), \quad \sigma_{\mathcal{A}}(z) + \sigma_{\Xi}(z) - 1 > \sigma_{\Pi}(z),$$

for some $\mathcal{P}^{\tau}(\Pi) \geq r$, $\mathcal{P}^{\pi}(\Pi) \leq 1 - r$, $\mathcal{P}^{\sigma}(\Pi) \leq 1 - r$. Since $\overbrace{\ \Xi\ } \leq \Xi$,

$$\tau_{\mathcal{A}}(z)+\tau_{\overbrace{\Xi}}(z)-1\leq\tau_{\Pi}(z),\quad \pi_{\mathcal{A}}(z)+\pi_{\overbrace{\Xi}}(z)-1>\pi_{\Pi}(z),\quad \sigma_{\mathcal{A}}(z)+\sigma_{\overbrace{\Xi}}(z)-1>\sigma_{\Pi}(z),$$

for some $\mathcal{P}^{\tau}(\Pi)\geq r$, $\mathcal{P}^{\pi}(\Pi)\leq 1-r$, $\mathcal{P}^{\sigma}(\Pi)\leq 1-r$. Thus, $y_{n,m,v}\notin\overbrace{\Xi^{\star}_r}$ implies that $\overbrace{\Xi^{\star}_r}=\overline{0}$ or $\overbrace{\Xi^{\star}_r}\neq\overline{0}$ but $n>\tau_{\overbrace{\Xi^{\star}_r}}(z)$, $m\leq\pi_{\overbrace{\Xi^{\star}_r}}(z)$, $v\leq\sigma_{\overbrace{\Xi^{\star}_r}}(z)$. Let $y_{s_1,m_1,v_1}\in$ *svn-point*$(\pounds)$ such that $n_1\leq\tau_{\overbrace{\Xi^{\star}_r}}(z)<n$, $m_1>\pi_{\overbrace{\Xi^{\star}_r}}(z)\geq m$, $v_1>\sigma_{\overbrace{\Xi^{\star}_r}}(z)\geq v$; this means $y_{n_1,m_1,v_1}\in\overbrace{\Xi^{\star}_r}$. Then, $\forall\ \mathcal{B}\in\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n_1,m_1,v_1},r)$ and for every $\mathcal{P}^{\tau}(\Pi)\geq r$, $\mathcal{P}^{\pi}(\Pi)\leq 1-r$, $\mathcal{P}^{\sigma}(\Pi)\leq 1-r$, there is at least one $z\in\pounds$ such that

$$\tau_{\mathcal{B}}(z)+\tau_{\overbrace{\Xi}}(z)-1>\tau_{\Pi}(z),\quad \pi_{\mathcal{B}}(z)+\pi_{\overbrace{\Xi}}(z)-1\leq\pi_{\Pi}(z),\quad \sigma_{\mathcal{B}}(z)+\sigma_{\overbrace{\Xi}}(z)-1\leq\sigma_{\Pi}(z).$$

Since $\overbrace{\Xi}\leq\Xi$, then for any $\mathcal{B}\in\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n_1,m_1,v_1},r)$ and $\mathcal{P}^{\tau}(\Pi)\geq r$, $\mathcal{P}^{\pi}(\Pi)\leq 1-r$, $\mathcal{P}^{\sigma}(\Pi)\leq 1-r$, there is at least one $z\in\pounds$ such that

$$\tau_{\mathcal{B}}(z)+\tau_{\Xi}(z)-1>\tau_{\Pi}(z),\quad \pi_{\mathcal{B}}(z)+\pi_{\Xi}(z)-1\leq\pi_{\Pi}(z),\quad \sigma_{\mathcal{B}}(z)+\sigma_{\Xi}(z)-1\leq\sigma_{\Pi}(z).$$

This implies $y_{n_1,m_1,v_1}\in\Xi^{\star}_r$. But $n_1<n$, $m_1\geq m$, $v_1\geq v$ and $y_{n,m,v}\in\overbrace{\Xi}$ implies $y_{n_1,m_1,v_1}\in\overbrace{\Xi}$ and hence $y_{n_1,m_1,v_1}\notin\overbrace{\Xi^{\star}_r}$. This is a contradiction. Thus, $\Xi^{\star}_r=\overline{0}$, so that $y_{n,m,v}\in\overbrace{\Xi}$ implies $y_{n,m,v}\notin\overbrace{\Xi^{\star}_r}$ with $\overbrace{\Xi^{\star}_r}=\overline{0}$. Thus, $\overbrace{\Xi}\wedge\overbrace{\Xi^{\star}_r}=\overline{0}$ for every $\Xi\in\zeta^{\pounds}$. Hence, by (3), $\mathcal{P}^{\tau}(\overbrace{\Xi})\geq r$, $\mathcal{P}^{\pi}(\overbrace{\Xi})\leq 1-r$, $\mathcal{P}^{\sigma}(\overbrace{\Xi})\leq 1-r$.

$4\Rightarrow 5$: The same method as the proof of $3\Rightarrow 4$.

$4\Rightarrow 6$: Let $\Xi\in\zeta^{\pounds}$, Ξ contains no $\Theta\neq\overline{0}$ with $\Theta\leq\Theta^{\star}_r$. Then, $\forall\ \Xi\in\zeta^{\pounds}$, $\Xi=\overbrace{\Xi}\vee(\Xi\wedge\Xi^{\star}_r)$. By Theorem 4(5), we have $\Xi^{\star}_r=[\overbrace{\Xi}\vee(\Xi\wedge\Xi^{\star}_r)]^{\star}_r=\overbrace{\Xi^{\star}_r}\vee[\Xi\wedge\Xi^{\star}_r]^{\star}_r$. Now, by (4), we obtain $\mathcal{P}^{\tau}(\overbrace{\Xi})\geq r$, $\mathcal{P}^{\pi}(\overbrace{\Xi})\leq 1-r$, $\mathcal{P}^{\sigma}(\overbrace{\Xi})\leq 1-r$; then, $\overbrace{\Xi^{\star}_r}=\overline{0}$. Thus, $[\Xi\wedge\Xi^{\star}_r]^{\star}_r=\Xi^{\star}_r$ but $\Xi\wedge\Xi^{\star}_r\leq\Xi^{\star}_r$, then $\Xi\wedge\Xi^{\star}_r\leq(\Xi\wedge\Xi^{\star}_r)^{\star}_r$. This contradicts the hypothesis if $\overline{0}\neq\Theta\leq\Xi$ with $\Theta\leq\Theta^{\star}_r$. Hence, $\Xi\wedge\Xi^{\star}_r=\overline{0}$, so, $\Xi=\overbrace{\Xi}$ by (4), we obtain that $\mathcal{P}^{\tau}(\Xi)\geq r$, $\mathcal{P}^{\pi}(\Xi)\leq 1-r$, $\mathcal{P}^{\sigma}(\Xi)\leq 1-r$.

$6\Rightarrow 4$: Since $\forall\ \Xi\in\zeta^{\pounds}$, $\Xi\wedge\Xi^{\star}_r=\overline{0}$. Hence, by (6), as Ξ contains no non-empty single-valued neutrosophic subset Θ with $\Theta\leq\Theta^{\star}_r$, $\mathcal{P}^{\tau}(\Xi)\geq r$, $\mathcal{P}^{\pi}(\Xi)\leq 1-r$, $\mathcal{P}^{\sigma}(\Xi)\leq 1-r$.

$5\Rightarrow 1$: For every $\Xi\in\zeta^{\pounds}$, $y_{n,m,v}\in\Xi$, there exists $\mathcal{A}\in\mathcal{Q}_{\mathcal{T}^{\tau\pi\sigma}}(y_{n,m,v},r)$ such that

$$\tau_{\mathcal{A}}(z)+\tau_{\Xi}(z)-1\leq\tau_{\Pi}(z),\quad \pi_{\mathcal{A}}(z)+\pi_{\Xi}(z)-1>\pi_{\Pi}(z),\quad \sigma_{\mathcal{A}}(z)+\sigma_{\xi}(z)-1>\sigma_{\Pi}(z),$$

holds for every $z\in\pounds$ and for some $\mathcal{P}^{\tau}(\Xi)\geq r$, $\mathcal{P}^{\pi}(\Xi)\leq 1-r$, $\mathcal{P}^{\sigma}(\Xi)\leq 1-r$. This implies $y_{n,m,v}\notin\Xi^{\star}_r$. Let $\Theta=\Xi\vee\Xi^{\star}_r$. Then, $\Theta^{*}_r=(\Xi\vee\Xi^{\star}_r)^{\star}_r=\Xi^{\star}_r\vee(\Xi^{\star}_r)^{\star}_r=\Xi^{\star}_r$ by Theorem 4(4). So, $\mathcal{C}^{\star}_{\mathcal{T}^{\tau\pi\sigma}}(\Theta,r)=\Theta\vee\Theta^{\star}_r=\Theta$. That means $\mathcal{T}^{\star\tau}(\Theta^c)\geq r$, $\mathcal{T}^{\star\pi}(\Theta^c)\leq 1-r$, $\mathcal{T}^{\star\sigma}(\Theta^c)\leq 1-r$. Thus, by (5), we obtain that $\mathcal{P}^{\tau}(\Theta)\geq r$, $\mathcal{P}^{\pi}(\Theta)\leq 1-r$, $\mathcal{P}^{\sigma}(\Theta)\leq 1-r$.

Again, For any $y_{s,m,v}\in$ *svn-point*$(\pounds)$, $y_{s,m,v}\notin\overbrace{\Theta^{\star}_r}$ implies $y_{s,m,v}\in\Theta$ but $y_{s,m,v}\notin\Theta^{\star}_r=\Xi^{\star}_r$. So, $\Theta=\Xi\vee\Xi^{\star}_r$, $y_{s,m,v}\in\Xi$. Now, by the hypothesis about Ξ, we have for every $y_{s,m,v}\in\Xi^{\star}_r$. So, $\overbrace{\Theta}=\Xi$. Hence, $\mathcal{P}^{\tau}(\Theta)\geq r$, $\mathcal{P}^{\pi}(\Theta)\leq 1-r$, $\mathcal{P}^{\sigma}(\Theta)\leq 1-r$. Thus, $\mathcal{T}^{\tau\pi\sigma}\models\mathcal{P}^{\tau\pi\sigma}$. □

Theorem 13. *Let* $(\pounds,\mathcal{T}^{\tau\pi\sigma})$ *be an* svnts *with* svn-primal $\mathcal{P}^{\tau\pi\sigma}$ *on* £*. Then, the following cases are equivalent and implied* $\mathcal{T}^{\tau\pi\sigma}\models\mathcal{P}^{\tau\pi\sigma}$*.*

(1) *For each* $\Xi \in \zeta^{\pounds}$, $\Xi \wedge \Xi^{\star}_{r} = \overline{0}$ *implies* $\Xi^{*}_{r} = \overline{0}$.
(2) *For each* $\Xi \in \zeta^{\pounds}$, $\overbrace{\zeta^{\star}_{r}} = \overline{0}$.
(3) *For each* $\Xi \in \zeta^{\pounds}$, $\Xi \wedge \Xi^{\star}_{r} = \Xi^{\star}_{r}$.

Proof. The proof follows a similar line of reasoning to that of Theorem 12. □

5. Conclusions

In this paper, first, we investigated the complex area of stratification of svn grills and determined some of their fundamental features. The links between svn grills and svn-primals were investigated. We also introduced and explored the concept of svn-primal open local functions in the context of Šostak's sense. By extending the notions of svn sets and related topological structures, we have presented a novel approach to understanding the properties and relationships within this unique framework.

Our investigation began by defining svn-sets and their corresponding notions. Building upon these fundamental definitions, we introduced svnts and explored various operations and properties within these spaces, such as the neutrosophic closure and interior. A central contribution of this work has been the introduction and exploration of svn-primals and their associated operators. We have provided several equivalent conditions characterizing the compatibility of svnts with neutrosophic primals. Additionally, we discussed the properties of primal open local functions, their relationship with svnts, and the induced operators.

In conclusion, the results presented in this paper contribute to the growing field of neutrosophic topology by offering a deeper understanding of svn structures in Šostak's sense. The properties and correlations investigated here establish the framework for further studies in this field, opening options for future investigations and applications of these innovative notions in various fields.

In terms of related and future research directions, it is of great interest to investigate the connections between our findings and the advancements in the field of neural networks (Gu and Sheng [36]; Gu et al. [37]; Deng et al. [38]; Gu et al. [39]). Furthermore, the possible connections to multidimensional systems and signal processing need further investigation, as illustrated by Wang et al. [40] and Xiong et al. [41].

By bridging the gap between svn structures and related domains, we might promote collaboration across disciplines and discover new applications for our findings. This not only enriches the discipline of neutrosophic topology, but also benefits the larger scientific community by providing new insights and fresh approaches to difficult problems.

For forthcoming papers

The theory can be extended in the following normal methods.

1—Basic concepts of neutrosophic metric topological spaces can be studied using the notion of svn-primal present in this article;

2—Examine the connected, separation axioms and soft closure spaces in the context of svn-primals.

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Abbreviations

The following abbreviations are used in this manuscript:

n-set	neutrosophic set
svn-set	single-valued neutrosophic set
svnits	single-valued neutrosophic ideals
svnts	single-valued neutrosophic topological spaces
svngts	single-valued neutrosophic grill topological spaces
svnpts	single-valued neutrosophic primal topological spaces
svnqo-cover	single-valued neutrosophic quasi open-cover
svns-primal	single-valued neutrosophic primal
$\mathcal{T}^{\tau\pi\sigma} \models \mathcal{P}^{\tau\pi\sigma}$	single-valued neutrosophic primal open compatible
svn-point	single-valued neutrosophic point

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