An Application of a Theorem of Orthology

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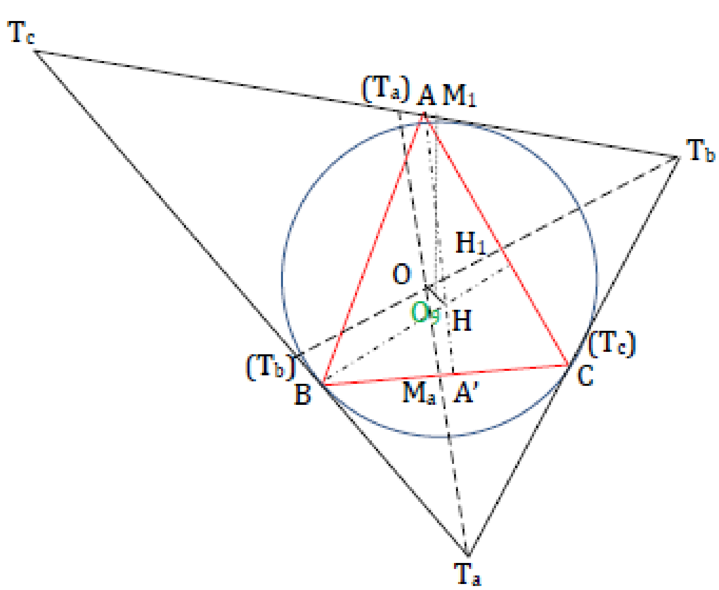
In this note, we make connections between Problem 21 of [1] and the theory of orthological triangles. The enunciation of the competitional problem is:

Let (T_A), (T_B), (T_C) be the tangents in the peaks A, B, C of the triangle ABC to the circle circumscribed to the triangle. Prove that the perpendiculars drawn from the means of the opposite sides on (T_A), (T_B), (T_C) are concurrent and determine their concurrent point.

We formulate and we demonstrate below a sentence containing in its proof the solution of the competitional problem in this case.



Proposition. The tangential triangle and the median triangle of a given triangle ABC are orthological. The orthological centers are O – the center of the circle circumscribed to the triangle ABC, and O_9 – the center of the circle of ABC triangle's 9 points.

Proof. Let $M_aM_bM_c$ be the median triangle of triangle ABC and $T_aT_bT_c$ the tangential triangle of the triangle ABC. It is obvious that the triangle $T_aT_bT_c$ and the triangle ABC are orthological and that O is the orthological center. Verily, the perpendiculars taken from T_a, T_b, T_c on BC; CA; AB respectively are internal bisectors in the triangle $T_aT_bT_c$ and consequently passing through O, which is center of the circle inscribed in triangle $T_aT_bT_c$. Moreover, T_aO is the mediator of (BC) and accordingly passing through M_a, and T_aM_c is perpendicular on BC, being a mediator, but also on M_bM_c which is a median line.

From orthological triangles theorem, it follows that the perpendiculars taken from M_a, M_b, M_c on T_bT_c, T_cT_a, T_aT_b respectively, are concurrent. The point of concurrency is the second orthological center of triangles $T_aT_bT_c$ and $M_aM_bM_c$. We prove that this point is O_9 – the center of Euler circle of triangle ABC. We take $M_aM_1 \perp T_bT_c$ and denote by $\{H_1\} = M_aM_1 \cap AH$, H being the orthocenter of the triangle ABC. We know that $AH = 2OM_a$; we prove this relation vectorially.

From Sylvester's relation, we have that: $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, but $\overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OM_a}$; it follows that $\overrightarrow{OH} - \overrightarrow{OA} = 2\overrightarrow{OM_a}$, so $\overrightarrow{AH} = 2\overrightarrow{OM_a}$; changing to module, we have $AH = 2OM_a$. Uniting O to A, we have $OA \perp T_bT_c$, and because $M_aM_1 \perp T_bT_c$ and $AH \parallel OM_a$, it follows that the quadrilateral OM_aH_1A is a parallelogram.

From $AH_1 = OM_a$ and $AH = 2OM_a$ we get that H_1 is the middle of (AH), so H_1 is situated on the circle of the 9 points of triangle ABC. On this circle, we find as well the points A' - the height foot from A and M_a; since $\sphericalangle AA'M_a = 90^0$, it follows that M_aH_1 is the diameter of Euler circle, therefore the middle of (M_aH_1), which we denote by O_9, is the center of Euler's circle; we observe that the quadrilateral H_1HM_aO is as well a parallelogram; it follows that O_9 is the middle of segment $[OH]$. In conclusion, the perpendicular from M_a on T_bT_c pass through O_9.

Analogously, we show that the perpendicular taken from M_b on T_aT_c pass through O_9 and consequently O_9 is the enunciated orthological center.

Remark. The triangles $M_aM_bM_c$ and $T_aT_bT_c$ are homological as well, since T_aM_a, T_bM_b, T_cM_c are concurrent in O, as we already observed, therefore the triangles $T_aT_bT_c$ and $M_aM_bM_c$ are orthohomological of rank I (see [2]).

From P. Sondat theorem (see [4]), it follows that the Euler line OO_9 is perpendicular on the homological axis of the median triangle and of the tangential triangle corresponding to the given triangle ABC.

A Note (regarding the triangles that are simultaneously orthological and homological).

In the article *A Theorem about Simultaneous Orthological and Homological Triangles*, by Ion Patrascu and Florentin Smarandache, we stated and proved a theorem which was called by Mihai Dicu in [2] and [3] *The Smarandache-Patrascu Theorem of Orthohomological Triangle*; then, in [4], we used the term *ortohomological triangles* for the triangles that are simultaneously orthological and homological.

The term *ortohomological triangles* was used by J. Neuberg in *Nouvelles Annales de Mathematiques* (1885) to name the triangles that are simultaneously orthogonal (one has the sides perpendicular to the sides of the other) and homological.

We suggest that the triangles that are simultaneously orthogonal and homological to be called *ortohomological triangles of rank I*, and triangles that are simultaneously orthological and homological to be called *ortohomological triangles of rank II*.

Bibliography.

[1] Titu Andreescu and Dorin Andrica: *360 de probleme de matematică pentru concursuri/360 Competitional Math Problem*, Editura GIL, Zalau, Romania, 2002.

[2] Mihai Dicu, *The Smarandache-Patrascu Theorem of Orthohomological Triangle*, http://www.scrib.com/doc/28311880/Smarandache-Patrascu-Theorem-of-Orthohomological-Triangles, 2010

[3] Mihai Dicu, *The Smarandache-Patrascu Theorem of Orthohomological Triangle*, in *Revista de matematică ALPHA*, Year XVI, no.1/2010, Craiova, Romania.

[4] Ion Patrascu, Florentin Smarandache: *Variance on topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, 2013.

[5] *Multispace and multistructure neutrosophic transdisciplinarity (100 collected papers of sciences),* Vol IV, Edited by prof. Florentin Smarandache, Dept. of Mathematics and Sciences, University of New Mexico, USA - North European Scientific Publishers, Hanco, Finland, 2010.