About an Identity and its Applications

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Theorem 1. If $x, y \in C$ then $2\left(x^2+y^2\right)^3-(x+y)^3\left(x^3+y^3\right)=(x-y)^4\left(x^2+xy+y^2\right)$.

Proof. With elementary calculus.

Application 1.1. If $x, y \in C$ then

$$\left(2\left(x^2+y^2\right)^3-(x+y)^3\left(x^3+y^3\right)\right)\left(2\left(x^2+y^2\right)^3-(x-y)^3\left(x^3-y^3\right)\right)=\left(x^2-y^2\right)^4\left(x^4+x^2y^2+y^4\right)$$

Proof. In Theorem 1 we replace $y \to -y$, etc.

Application 1.2. If $x \in R$ then

$$(\sin x-\cos x)^4(1+\sin x\cos x)+(\sin x+\cos x)^3\left(\sin^3 x+\cos^3 x\right)=2$$

Proof. In Theorem 1 we replace $x \to \sin x,\ y \to \cos x$

Application 1.3. If $x \in R$ then $2ch^6 x-(1+shx)^3\left(1+sh^3 x\right)=(1+shx)^4\left(shx+ch^2 x\right)$.

Proof. In Theorem 1 we replace $x \to 1,\ y \to shx$

Application 1.4. If $x, y \in C$ $(x \neq \pm y)$ then

$$\frac{2\left(x^2+y^2\right)^3-(x+y)^3\left(x^3+y^3\right)}{(x-y)^4}+\frac{2\left(x^2+y^2\right)^3-(x-y)^3\left(x^3-y^3\right)}{(x+y)^4}=2\left(x^2+y^2\right)$$

Application 1.5. If $x, y \in C$ then

$$\frac{2\left(x^2+y^2\right)^3-(x+y)^3\left(x^3+y^3\right)}{x^2+xy+y^2}+\frac{2\left(x^2+y^2\right)^3-(x-y)^3\left(x^3-y^3\right)}{x^2-xy+y^2}=2\left(x^4+6x^2y^2+y^4\right)$$

Application 1.6. If $x, y \in R$ then $2\left(x^2+y^2\right)^3 \geq (x+y)^3\left(x^3+y^3\right)$.

(See Jószef Sándor, Problem L.667, Matlap, Kolozsvar, 9/2001.)

Proof. See Theorem 1.

Theorem 2. If $x, y, z \in R$ then $3\left(x^2+y^2+z^2\right)^3 \geq (x+y+z)^3\left(x^3+y^3+z^3\right)$.

Proof. With elementary calculus.

Application 2.1. Let $ABCDA_1B_1C_1D_1$ be a rectangle parallelepiped with sides a, b, c and diagonal d. Prove that $3d^6 \geq (a+b+c)^3\left(a^3+b^3+c^3\right)$.

Application 2.2. In any triangle ABC the followings hold:

1) $3\left(p^2-r^2-4Rr\right)^3 \geq 2p^4\left(p^2-3r^2-6Rr\right)$

2) $3\left(p^2-2r^2-8Rr\right)^3 \geq p^4\left(p^2-12Rr\right)$

3) $3\left((4R+r)^2-2p^2\right)^3 \geq (4R+r)^3\left((4R+r)^3-12p^2R\right)$

4) $3\left(8R^2+r^2-p^2\right)^3 \ge (2R-r)^3\left((2R-r)\left((4R+r)^2-3p^2\right)+6Rr^2\right)$

5) $3\left((4R+r)^2-p^2\right)^3 \ge (4R+r)^3\left((4R+r)^3-3p^2(2R+r)\right)$

Proof. In Theorem 2 we take:

$\{x,y,z\}\in$

$$\in\left\{\{a,b,c\};\{p-a,p-b,p-c\};\{r_a,r_b,r_c\};\left\{\sin^2\frac{A}{2},\sin^2\frac{B}{2},\sin^2\frac{C}{2}\right\};\left\{\cos^2\frac{A}{2},\cos^2\frac{B}{2},\cos^2\frac{C}{2}\right\}\right\}.$$

Application 2.3. Let ABC be a rectangle triangle, with sides $a>b>c$ then

$$24a^6 \ge (a+b+c)^3\left(a^3+b^3+c^3\right)$$

Theorem 3. If $x_k>0,\ k=1,2,...,n$, then $n\left(\sum_{k=1}^{n}x_k^2\right)^3 \ge \left(\sum_{k=1}^{n}x_k\right)^3\sum_{k=1}^{n}x_k^3$.

Application 3.1 The following inequality is true: $\sum_{k=0}^{n}\left(C_n^k\right)^3 \le (n+1)\left(\frac{C_{2n}^n}{2}\right)^3$.

Proof. In Theorem 3 we take $x_k=C_n^k,\ k=0,1,2,...,n.$.

Application 3.2. In all tetrahedron $ABCD$ holds:

1) $\dfrac{\left(\sum\frac{1}{h_a^2}\right)^3}{\sum\frac{1}{h_a^3}} \ge \dfrac{4}{r^3}$ 2) $\dfrac{\left(\sum\frac{1}{r_a^2}\right)^3}{\sum\frac{1}{r_a^3}} \ge \dfrac{2}{r^3}$

Proof. In Theorem 3 we take $x_1=\frac{1}{h_a},x_2=\frac{1}{h_b},x_3=\frac{1}{h_c},x_4=\frac{1}{h_d}$ and

$$x_1=\frac{1}{r_a},x_2=\frac{1}{r_b},x_3=\frac{1}{r_c},x_4=\frac{1}{r_d}.$$

Application 3.3. If $S_n^\alpha=\sum_{k=1}^{n}k^\alpha$ then $n\left(S_n^{2\alpha}\right)^3 \ge \left(S_n^\alpha\right)^3 S_n^{3\alpha}$.

Proof. In Theorem 3 we take $x_k=k^\alpha,\ k=0,1,2,...,n.$.

Application 3.4. If F_k denote Fibonacci numbers, then $\sum_{k=1}^{n}F_k^3 \le n\left(\frac{F_nF_{n+1}}{F_{n+2}-1}\right)^3$.

Proof. In Theorem 3 we take $x_k=F_k,\ k=1,2,...,n.$.

References:

[1] Mihály Bencze, *Inequalities* (manuscript), 1982.

[2] Collection of "Octogon Mathematical Magazine", 1993-2004.

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