Another proof of the I. Pătraşcu′s theorem

Florentin SMARANDACHE [1]

Abstract. In this note the author presents a new proof for the theorem of **I. Pătraşcu**.
Keywords: median, symmedian, Brocard′s points.
MSC 2010: 97G40.

In [**1**], **Ion Pătraşcu** proves the following

Theorem. *The Brocard′s point of an isosceles triangle is the intersection of a median and the symmedian constructed from the another vertex of the triangle′s base, and reciprocal.*

We′ll provide below a different proof of this theorem than the proof given in [**1**] and [**2**].

We′ll recall the following definitions:

Definition 1. The symmetric cevian of the triangle′s median in rapport to the bisector constructed from the same vertex is called the triangle′s symmedian.

Definition 2. The points Ω, Ω' from the plane of the triangle ABC with the property $\widehat{\Omega BA} \equiv \widehat{\Omega AC} \equiv \widehat{\Omega CB}$, respectively $\widehat{\Omega' AB} \equiv \widehat{\Omega' BC} \equiv \widehat{\Omega' CA}$, are called the Brocard′s points of the given triangle.

Remark. In an arbitrary triangle there exist two Brocard′s points.

Proof of the Theorem. Let ABC an isosceles triangle, $AB = AC$, and Ω the Brocard′s point, therefore $\widehat{\Omega BA} \equiv \widehat{\Omega AC} \equiv \widehat{\Omega CB} = \omega$.

We′ll construct the circumscribed circle to the triangle $B\Omega C$. Having $\widehat{\Omega BA} \equiv \widehat{\Omega CB}$ and $\widehat{\Omega CA} \equiv \widehat{\Omega BC}$, it results that this circle is tangent in B, respectively in C to the sides AB, respectively AC.

We note M the intersection point of the line $B\Omega$ with AC and with N the intersection point of the lines $C\Omega$ and AB. From the similarity of the triangles ABM, ΩAM, we obtain

$$MB \cdot M\Omega = AM^2. \tag{1}$$

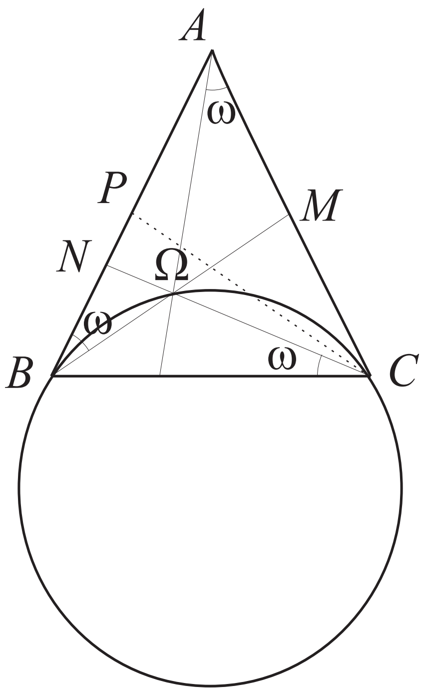
Considering the power of the point M in rapport to the constructed circle, we obtain

$$MB \cdot M\Omega = MC^2. \tag{2}$$

From the relations (1) and (2) it results that $AM = MC$, therefore, BM is a median.

If CP is the median from C of the triangle, then from the congruency of the triangles ABM, ACP we find that $\widehat{ACP} \equiv \widehat{ABM} = \omega$. It results that the cevian CN is a symmedian and the direct theorem is proved.

[1] University of New Mexico, Gallup Campus, U.S.A.



We′ll prove the reciprocal of this theorem. In the triangle ABC is known that the median BM and the symmedian CN intersect in the Brocard′s point Ω. We′ll construct the circumscribed circle to the triangle $B\Omega C$. We observe that because

$$\widehat{\Omega BA} \equiv \widehat{\Omega CB}, \tag{3}$$

this circle is tangent in B to the side AB. From the similarity of the triangles ABM, ΩAM it results $AM^2 = MB \cdot M\Omega$. But $AM = MC$, it results that $MC^2 = MB \cdot M\Omega$. This relation shows that the line AC is tangent in C to the circumscribed circle to the triangle $B\Omega C$, therefore

$$\widehat{\Omega BC} \equiv \widehat{\Omega CA}. \tag{4}$$

By adding up relations (3) and (4) side by side, we obtain $\widehat{ABC} \equiv \widehat{ACB}$, consequently, the triangle ABC is an isosceles triangle.

References

1. **I. Pătraşcu** – *O teoremă relativă la punctual lui Brocard*, Gazeta Matematică, nr. 9/1984, LXXXIX, 328-329.
2. **I. Pătraşcu** – *Asupra unei teoreme relative la punctual lui Brocard*, Revista Gamma, nr. 1-2/1988, Braşov.