1. INTRODUCTION: CENTRIC CARDINAL SINE FUNCTION

According to any standard dictionary, the word "cardinal" is synonymous with "principal", "essential", "fundamental".

In centric mathematics (CM), or ordinary mathematics, cardinal is, on the one hand, a number equal to a number of finite aggregate, called the power of the aggregate, and on the other hand, known as the sine cardinal $sinc(x)$ or cosine cardinal $cosc(x)$, is a special function defined by the centric circular function (CCF). $sin(x)$ and $cos(x)$ are commonly used in undulatory physics (see Figure 1) and whose graph, the graph of cardinal sine, which is called as "Mexican hat" (sombrero) because of its shape (see Figure 2).

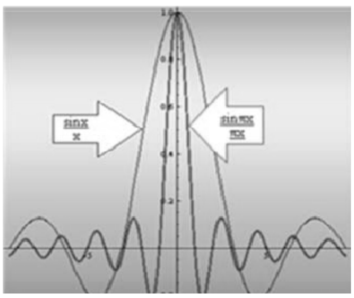
Note that $sinc(x)$ cardinal sine function is given in the speciality literature, in three variants

(1)
$$sinc(x) = \begin{cases} 1, & \text{for } x = 0 \\ \dfrac{sin(x)}{x}, & \text{for } x \in [-\infty, +\infty] \setminus 0 \end{cases}$$
$$= \frac{sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^{76}}{5040} + \frac{x^8}{362880} + 0[x]^{11}$$
$$= \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} \rightarrow sinc(\frac{\pi}{2}) = \frac{2}{\pi},$$
$$\frac{d(sinc(x))}{dx} = \frac{cos(x)}{x} - \frac{sin(x)}{x^2} = cosc(x) - \frac{sinc(x)}{x},$$

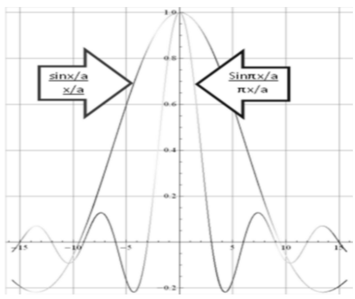
(2)
$$sinc(x) = \frac{sin(\pi x)}{\pi x}$$

(3)
$$sinc_a(x) = \frac{sin(\frac{\pi x}{a})}{\frac{\pi x}{a}}$$

It is a special function because its primitive, called sine integral and denoted $Si(x)$



Centric circular cardinal sine functions



Modified centric circular cardinal sine functions

Figure 1: The graphs of centric circular functions cardinal sine, in 2D, as known in literature

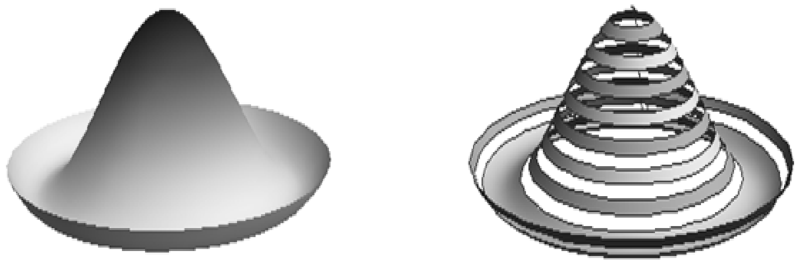


Figure 2: Cardinal sine function in 3D Mexican hat (sombrero)

$$(4)\qquad \begin{aligned} Si(x) &= \int_0^x \frac{sin(t)}{t}dt = \int_0^x sinc(t)\cdot dt \\ &= x - \frac{x^3}{18} + \frac{x^5}{600}\frac{x^7}{35280} + \frac{x^9}{3265920} + 0[x]^{11} \\ &= x - \frac{x^8}{3.3!} + \frac{x^5}{5.5!} - \frac{x7}{7.7!} + \cdots - \ldots \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n+1)^2(2n)!},\quad \forall x \in \mathbb{R} \end{aligned}$$

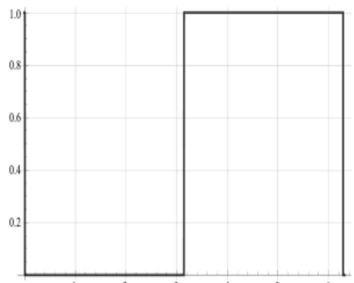
can not be expressed exactly by elementary functions, but only by expansion of power series, as shown in equation (4). Therefore, its derivative is

$$(5)\qquad \forall x \in \mathbb{R},\ Si^{'}(x) = \frac{d(Si(x))}{dx} = \frac{sin(x)}{x} = sinc(x),$$

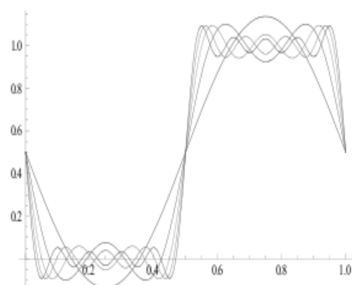
an integral sine function $Si(x)$, that satisfies the differential equation

$$(6)\qquad x\cdot f^{'''}(x) + 2f^{''}(x) + x\cdot f^{\,'}(x) = 0 \rightarrow f(x) = Si(x).$$

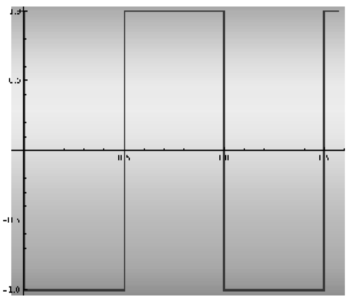
The Gibbs phenomenon appears at the approximation of the square with a continuous and differentiable Fourier series (Figure 3 right ►). This operation could be substitute with the circular eccentric supermathematics functions (CE-SMF), because the eccentric derivative function of eccentric variable θ can express exactly this rectangular function (Figure 3 ▲ top) or square (Figure 3 ▼ below) as shown on their graphs (Figure 3 ◄ left).



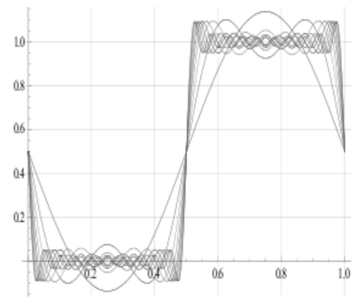
$1 - \cos\frac{x-\pi/2}{\sqrt{1-\sin(x-\pi/2)^2}}$,
$\{x, -\pi, 2.01\pi\}$



$\frac{1}{2} - 4x\sum \mathrm{Si}\,nc[2\pi(2k-1)x]$,
$\{k,n\}\{n,5\}\{x,0,1\}$



$\frac{1}{2}dex[(\theta, \frac{\pi}{2}), S(1,0)]$



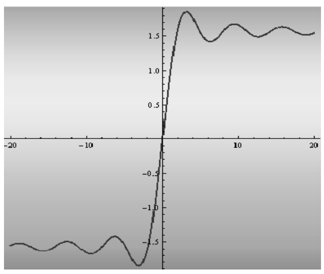
Gibbs phenomenon for a square wave with $n = 5$ and $n = 10$

Figure 3: Comparison between the square function, eccentric derivative and its approximation by **Fourier** serial expansion

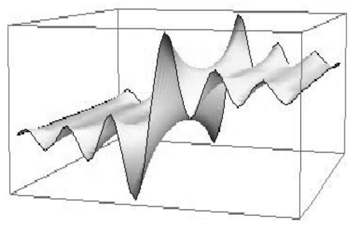
Integral sine function (4) can be approximated with sufficient accuracy. The maximum difference is less than 1%, except the area near the origin. By the CE-SMF eccentric amplitude of eccentric variable θ

(7) $$F(\theta) = 1.57 \; aex[\theta, S(0.6, 0)],$$

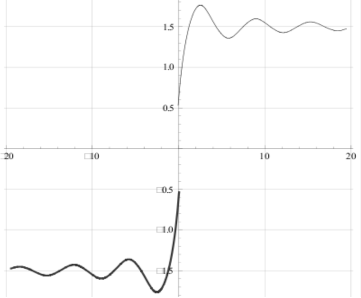
as shown on the graph on Figure 5.



$Sin\int x, \{x, -20, 20\}$



$\text{Sin}\int(x + Iy)$
$\{x, -20, 20\}, \{y, -3, 3\}$



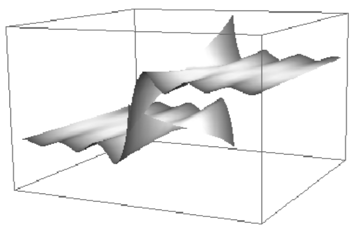


Figure 4: The graph of integral sine function $\text{Si}(x)$ ▲ compared with the graph CE-SMF Eccentric amplitude $1,57aex[\theta, S(0,6;0)]$ of eccentric variable θ▼

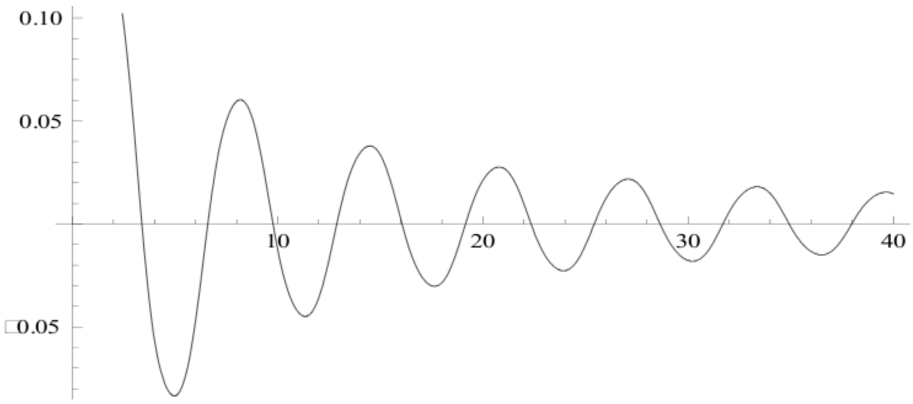


Figure 5: The difference between integral sine and CE-SMF eccentric amplitude $F(\theta) = 1,5aex[\theta, S(0,6;0]$ of eccentric variable θ

2. ECCENTRIC CIRCULAR SUPERMATHEMATICS CARDINAL FUNCTIONS, CARDINAL ECCENTRIC SINE (ECC-SMF)

Like all other supermathematics functions (SMF),they may be eccentric (ECC-SMF), elevated (ELC-SMF) and exotic (CEX-SMF), of eccentric variable θ, of centric variable $\alpha_{1,2}$ of main determination, of index 1, or secondary determination of index 2. At the passage from centric circular domain to the eccentric one, by positioning of the eccenter $S(s, \varepsilon)$ in any point in the plane of the unit circle, all supermathematics functions multiply from one to infinity. It means that in CM there exists each unique function for a certain type. In EM there are infinitely many such functions, and for $s = 0$ one will get the centric function. In other words, any supermathematics function contains both the eccentric and the centric ones.

Notations $sexc(x)$ and respectively, $Sexc(x)$ are not standard in the literature and thus will be defined in three variants by the relations:

(8) $$sexc(x) = \frac{sex(x)}{x} = \frac{sex[\theta, S(s,s)]}{\theta}$$

of eccentric variable θ and

(8') $$Sexc(x) = \frac{Sex(x)}{x} = \frac{Sex[\alpha, S(s,s)]}{\alpha}$$

of eccentric variable α.

(9) $$sexc(x) = \frac{sex(\pi x)}{\pi x},$$

of eccentric variable θ, noted also by $sexc_{\pi}(x)$ and

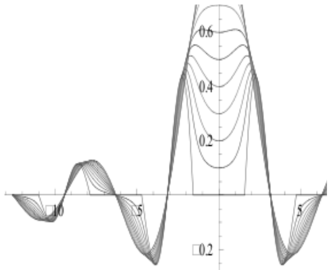
(9') $$Sexc(x) = \frac{sex(\pi x)}{\pi x} = \frac{Sex[\alpha, S(s,s)]}{\alpha},$$

of eccentric variable α, noted also by $Sexc_{\pi}(x)$

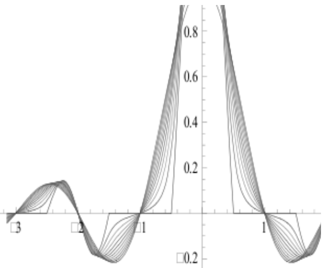
(10) $$sexc_a(x) = \frac{sex\frac{\pi x}{a}}{\frac{\pi x}{a}} = \frac{sex\frac{\pi\theta}{\theta}}{\frac{\pi\theta}{\theta}},$$

of eccentric variable θ, with the graphs from Figure 6 and Figure 7.

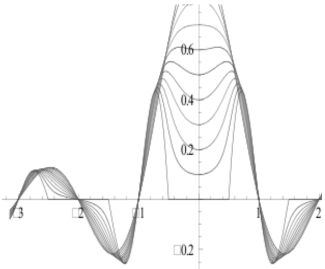
(10')
$$Sex_a(x) = \frac{Sex\frac{\pi x}{a}}{\frac{\pi x}{a}} = \frac{Sex\frac{\pi a}{a}}{\frac{\pi a}{a}}$$



$Sin\frac{\theta - ArcSin[s\ Sin(\theta)]}{\theta}$
$\{s, 0, +1\}, \{\theta, \pi, 4\pi\}$

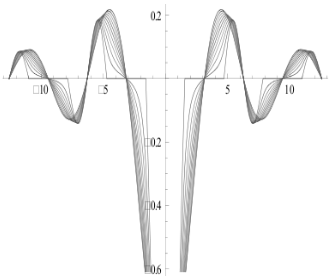


$Sin\frac{\pi\theta - ArcSin[s\ Sin(\pi\theta)]}{\theta}$
$\{s, -1, 0\}, \{\theta, -\pi, \pi\}$

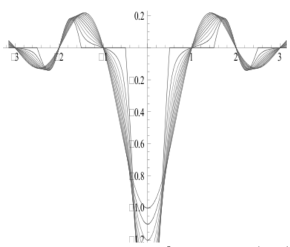


$Sin\frac{\pi\theta - ArcSin[s\ Sin(\pi\theta)]}{\theta}$
$\{s, 0, 1\}, \{\theta, -\pi, \pi\}$

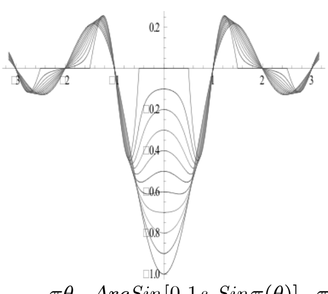
Figure 6: The ECCC-SMF graphs $sexc_1[\theta, S(s, \varepsilon)]$ of eccentric variable θ



$Sin\frac{\theta + ArcSin[s\ Sin(\theta)] - \pi}{\theta}$
$\{s, 0, 1\}, \{\theta, -4\pi, 4\pi\}$



$Sin\frac{\pi\theta - ArcSin[0.1s\ Sin(\pi\theta)] - \pi}{\theta}$
$\{s, -10, 0\}, \{\theta, -\pi, \pi\}$



$Sin\frac{\pi\theta - ArcSin[0.1s\ Sin\pi(\theta)] - \pi}{\theta}$
$\{s, 0.1, 0\}, \{\theta, -\pi, \pi\}$

Figure 7: Graphs ECCC-SMF $sexc_2[\theta, S(s, \varepsilon)]$, eccentric variable θ

3. ECCENTRIC CIRCULAR SUPERMATHEMATICS FUNCTIONS CARDINAL ELEVATED SINE AND COSINE (ECC- SMF-CEL)

Supermathematical elevated circular functions (ELC-SMF), elevated sine $sel(\theta)$ and elevated cosine $cel(\theta)$, is the projection of the fazor/vector

$$\vec{r} = rex(\theta) \cdot rad(\theta) = rex[\theta, S(s, \varepsilon)] \cdot rad(\theta)$$

on the two coordinate axis X_S and Y_S respectively, with the origin in the eccenter $S(s, \varepsilon)$, the axis parallel with the axis x and y which originate in $O(0, 0)$.

If the eccentric cosine and sine are the coordinates of the point $W(x, y)$, by the origin $O(0, 0)$ of the intersection of the straight line $d = d + \cup\ d\hat{a}e$“, revolving around the point $S(s, \varepsilon)$, the elevated cosine and sine are the same coordinates to the eccenter $S(s, \varepsilon)$; ie, considering the origin of the coordinate straight rectangular axes XSY /as landmark in $S(s, \varepsilon)$. Therefore, the relations between these functions are as follows:

$$(11) \qquad \begin{cases} x = cex(\theta) = X + s \cdot cos(\varepsilon) = cel(\theta) + s \cdot cos(\varepsilon) \\ y = Y + s \cdot sin(\varepsilon) = sex(\theta) = sel(\theta) + s \cdot sin(\varepsilon) \end{cases}$$

Thus, for $\varepsilon = 0$, ie S eccenter S located on the axis $x > 0, sex(\theta) = sel(\theta)$, and for $\varepsilon = \frac{\pi}{2}, cex(\theta) = cel(\theta)$, as shown on Figure 8.

On Figure 8 were represented simultaneously the elevated $cel(\theta)$ and the $sel(\theta)$ graphics functions, but also graphs of $cex(\theta)$ functions, respectively, for comparison and revealing $sex(\theta)$ elevation Eccentricity of the functions is the same, of $s = 0.4$, with the attached drawing and $sel(\theta)$ are $\varepsilon = \frac{\pi}{2}$, and $cel(\theta)$ has $\varepsilon = 0$.

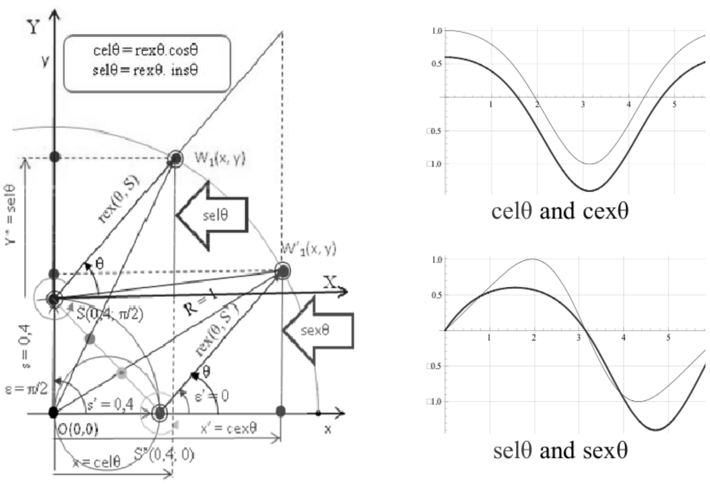


Figure 8: Comparison between elevated supermathematics function and eccentric functions

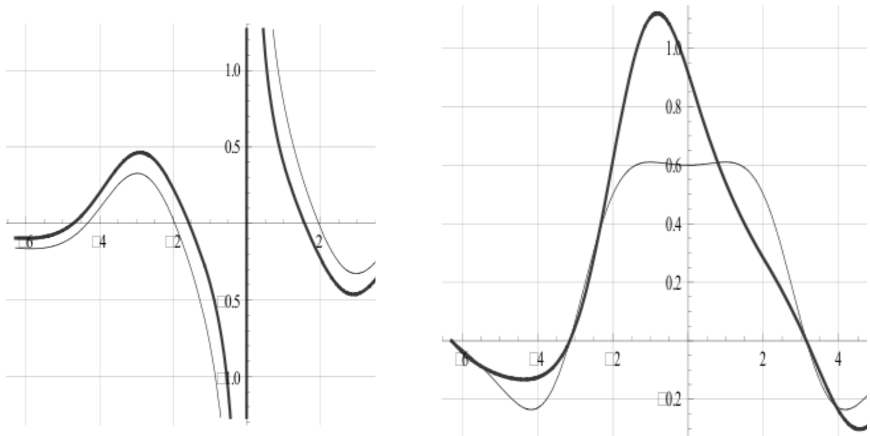


Figure 9: Elevated supermathematics function and cardinal eccentric functions $celc(x)$ ◀ and $selc(x)$ ▶ of $s = 0.4$

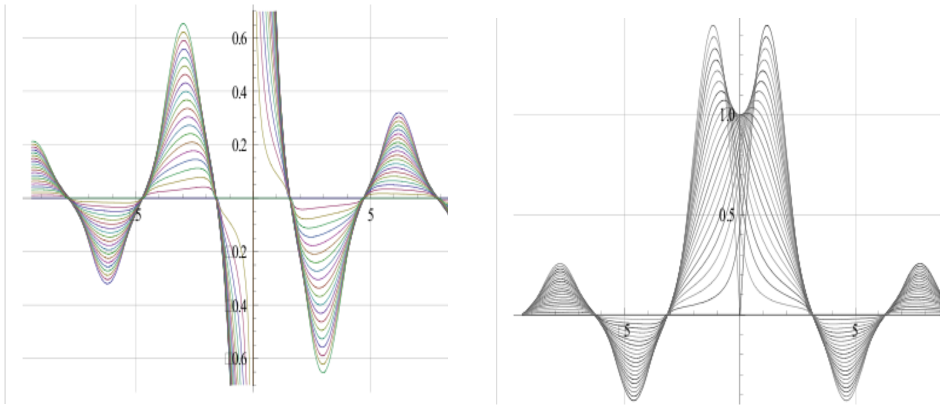


Figure 10: Cardinal eccentric elevated supermathematics function $celc(x)$ ◀ and $selc(x)$ ▶

Elevate functions (11) divided by θ become cosine functions and cardinal elevated sine, denoted $celc(\theta) = [\theta, S]$ and $selc(\theta) = [\theta, S]$, given by the equations

$$(12) \qquad \begin{cases} X = celc(\theta) = celc[\theta, S(s,\varepsilon)] = cexc(\theta) - \dfrac{s \cdot cos(s)}{\theta} \\ Y = selc(\theta) = selc[\theta, S(s,\varepsilon)] = sexc(\theta) - \dfrac{s \cdot sin(s)}{\theta} \end{cases}$$

with the graphs on Figure 9 and Figure 10.

4. NEW SUPERMATHEMATICS CARDINAL ECCENTRIC CIRCULAR FUNCTIONS (ECCC-SMF)

The functions that will be introduced in this section are unknown in mathematics literature. These functions are centrics and cardinal functions or integrals. They are supermathematics eccentric functions amplitude, beta, radial, eccentric derivative of eccentric variable [1], [2], [3], [4], [6], [7] cardinals and cardinal cvadrilobe functions [5].

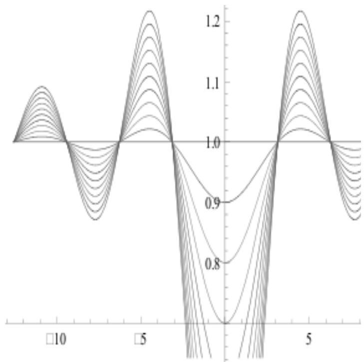
Eccentric amplitude function cardinal $aex(\theta)$, denoted as

$$(x) = aex[\theta, S(s, \varepsilon)], x \equiv \theta,$$

is expressed in

$$(13) \qquad aexc(\theta) = \frac{aex(\theta}{\theta} = \frac{aex[\theta, S(s,s)]}{\theta} = \frac{\theta - arcsin[s\ sin(\theta - s]}{\theta}$$

and the graphs from Figure 11.



$\frac{\theta - Sin(\theta)}{\theta}, \{s, 0, 1\}, \{\theta, -4\pi, +4\pi\}$

Figure 11: The graph of cardinal eccentric circular supermathematics function $aexc(\theta)$

The beta cardinal eccentric function will be

$$(14) \qquad bexc(\theta) = \frac{bex(\theta)}{\theta} = \frac{bex[\theta S(s,s)]}{\theta} = \frac{arcsin[s\ sin(\theta - s)]}{\theta},$$

with the graphs from Figure 12.

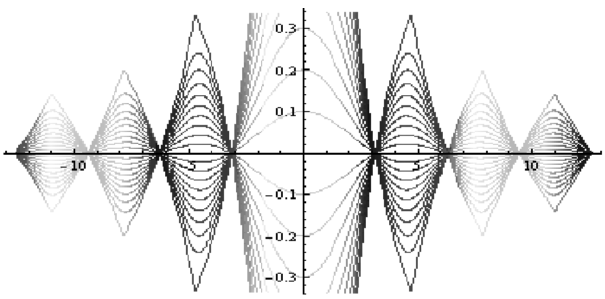


Figure 12: The graph of cardinal eccentric circular supermathematics function $bexc(\theta)$ $(\{s, -1, 1\}, \{\theta, -4\pi, 4\pi\})$

The cardinal eccentric function of eccentric variable θ is expressed by

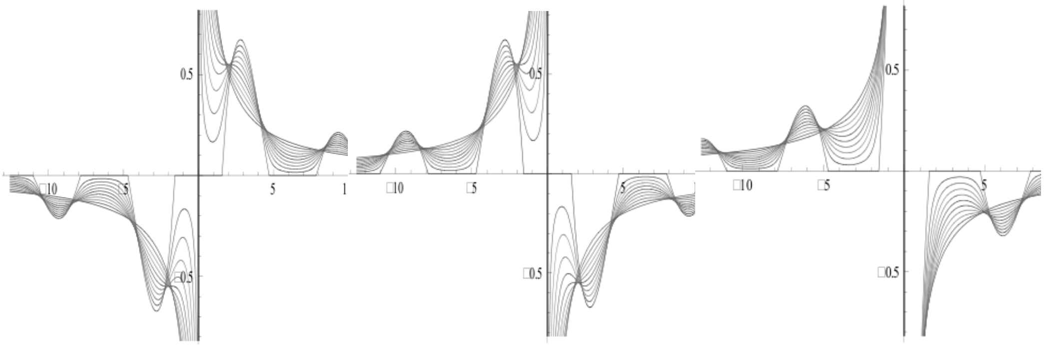
$$(15) \qquad rexc_{1,2}(\theta) = \frac{rex(\theta)}{\theta}$$

$$= \frac{rex[\theta, S(s,s)]}{\theta} = \frac{-s\ cos(\theta - s) \pm \sqrt{1 - s^2 sin(\theta - s)}}{\theta}$$

and the graphs from Figure 13, and the same function, but of centric variable α is expressed by

$$(16) \qquad Rexc(\alpha_{1,2}) = \frac{Rex(\alpha_{1,2})}{\alpha_{1,2}}$$

$$= \frac{Rex[\alpha_{1,2} S(s,s)]}{\alpha_{1,2}} = \frac{\pm\sqrt{1 + s^2 - 2s\ cos(\alpha_{1,2} - s)}}{\alpha_{1,2}}$$



$\frac{-sCos(\theta)+\sqrt{1-[s\,Sin(\theta)]^2}}{\theta}$, $\{s, 0, 1\}, \{\theta, -4\pi, 4\pi\}$ $\frac{-sCos(\theta)-\sqrt{1-[s\,Sin(\theta)]^2}}{\theta}$, $\{s, -1, 0\}, \{\theta, -4\pi, 4\pi\}$ $\frac{-sCos(\theta)-\sqrt{1-[s\,Sin(\theta)]^2}}{\theta}$, $\{s, 0, 1\}, \{\theta, -4\pi, 4\pi\}$

Figure 13: The graph of cardinal eccentric circular supermathematical function $rexc_{1,2}(\theta)$

And the graphs for $Rexc(\alpha 1)$, from Figure 14.

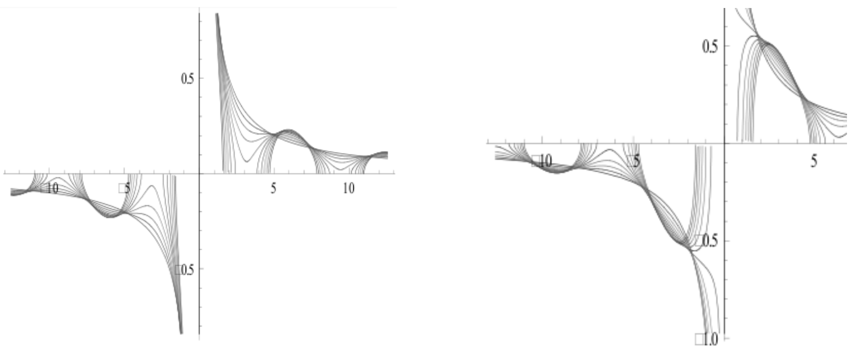


Figure 14: The graph of cardinal eccentric radial circular supermathematics function $Rexc(\theta)$

An eccentric circular supermathematics function with large applications, representing the function of transmitting speeds and/or the turning speeds of all known planar mechanisms is the derived eccentric $dex_{1,2}(\theta)$ and $Dex(\alpha_{1,2})$, functions that by dividing/reporting with arguments θ and, respectively, α

lead to corresponding cardinal functions, denoted $dexc_{1,2}(\theta)$, respectively $Dexc(\alpha_{1,2})$ and expressions

$$(17) \qquad dexc_{1,2}(\theta) = \frac{dex_{1,2}(\theta)}{\theta} = \frac{dex_{1,2}[\theta, S(s,s)]}{\theta} = \frac{1 - \frac{s \cdot cos(\theta - \varepsilon)}{1 - s^2 sin^2(\theta - \varepsilon)}}{\theta}$$

(18)

$$Dexc(\alpha_{1,2}) = \frac{Dex(\alpha_{1,2})}{\alpha_{1,2}} = \frac{Dex\{\alpha[\alpha_{1,2}S(s,s)]\}}{\alpha_{1,2}} = \frac{\sqrt{1 + s^2 - 2s \cdot cos(\alpha_{1,2} - s)}}{\alpha_{1,2}}$$

the graphs on Figure 15.

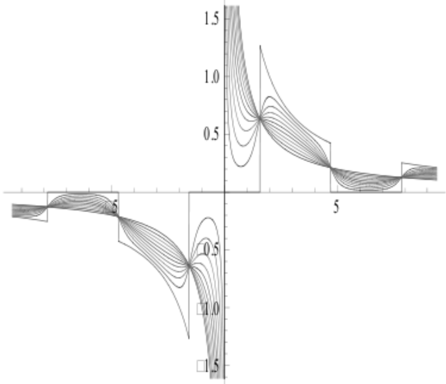


Figure 15: The graph of supermathematical cardinal eccentric radial circular function $dexc_1(\theta)$

Because $Dex(\alpha_{1,2}) = \frac{1}{dex_{1,2}(\theta)}$ results that $Dexc(\alpha_{1,2}) = \frac{1}{dexc_{1,2}(\theta)} siq(\theta)$ and $coq(\theta)$ are also cvadrilobe functions, dividing by their arguments lead to cardinal cvadrilobe functions $siqc(\theta)$ and $coqc(\theta)$ obtaining with the expressions

$$(19) \qquad coqc(\theta) = \frac{coq(\theta)}{\theta} = \frac{coq[\theta S(s,s)]}{\theta} = \frac{cos(\theta - s)}{\theta\sqrt{1 - s^2 sin^2(\theta - s)}}$$

$$(20) \qquad siqc(\theta) = \frac{siq(\theta)}{\theta} = \frac{siq[\theta S(s,s)]}{\theta} = \frac{sin(\theta - s)}{\theta\sqrt{1 - s^2 cos^2(\theta - s)}}$$

the graphs on Figure 16.

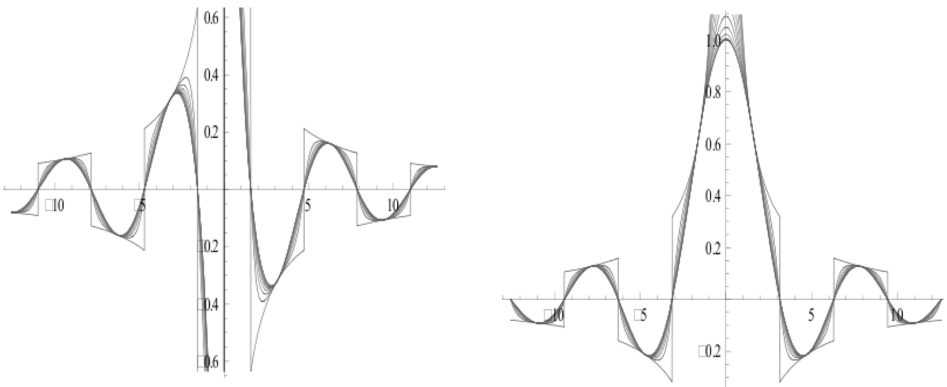


Figure 16: The graph of supermathematics cardinal cvadrilobe function $ceqc(\theta)$ ◄ and $siqc(\theta)$ ►

It is known that, by definite integrating of cardinal centric and eccentric functions in the field of supermathematics, we obtain the corresponding integral functions.

Such integral supermathematics functions are presented below. For zero eccentricity, they degenerate into the centric integral functions. Otherwise they belong to the new eccentric mathematics.

5. ECCENTRIC SINE INTEGRAL FUNCTIONS

Are obtained by integrating eccentric cardinal sine functions (13) and are

$$sie(x) = \int_0^x sexc(\theta) \cdot d\theta \tag{21}$$

with the graphs on Figure 17 for the ones with the eccentric variable $x \equiv \theta$.

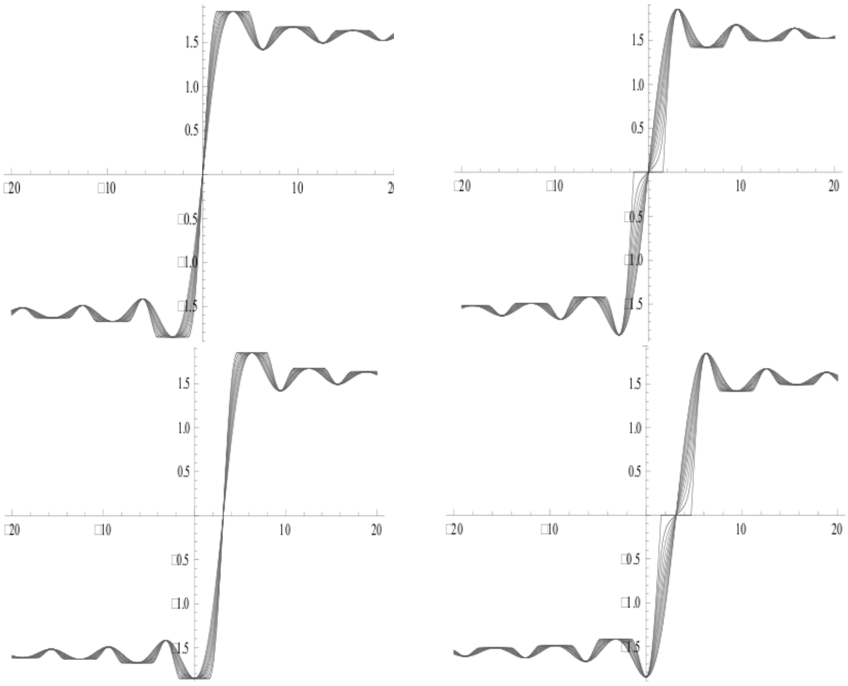


Figure 17: The graph of eccentric integral sine function $sie_1(x)$▲ and $sie_2(x)$▼

Unlike the corresponding centric functions, which is denoted $Sie(x)$, the eccentric integral sine of eccentric variable was noted $sie(x)$, without the capital S, which will be assigned according to the convention only for the ECCC-SMF of centric variable. The eccentric integral sine function of centric variable, noted $Sie(x)$ is obtained by integrating the cardinal eccentric sine of the eccentric circular supermathematics function, with centric variable

$$Sexc(x) = Sexc[\alpha, S(s, \varepsilon)], \tag{22}$$

thus

$$Sie(x) = \int_0^x \frac{Sex[\alpha, S(s,\varepsilon]}{\alpha}, \tag{23}$$

with the graphs from Figure 18.

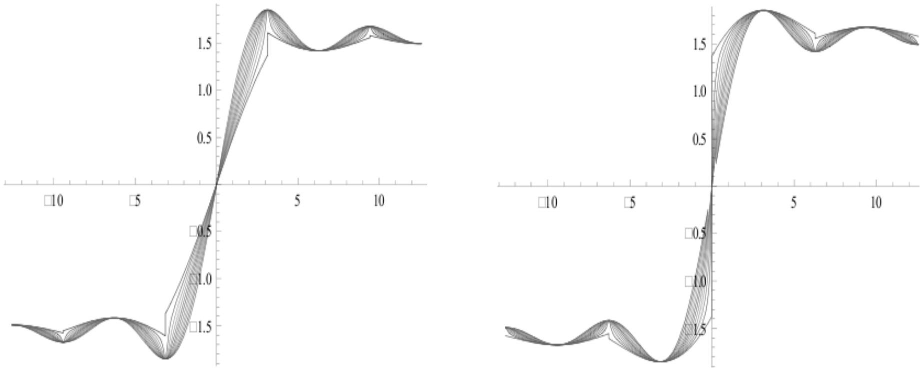


Figure 18: The graph of eccentric integral sine function $sie_2(x)$

6. CONCLUSION

The paper highlighted the possibility of indefinite multiplication of cardinal and integral functions from the centric mathematics domain in the eccentric mathematics's or of supermathematics's which is a reunion of the two mathematics. Supermathematics, cardinal and integral functions were also introduced with correspondences in centric mathematics, a series new cardinal functions that have no corresponding centric mathematics.

The applications of the new supermathematics cardinal and eccentric functions certainly will not leave themselves too much expected.

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