

ABOUT THE CHARACTERISTIC FUNCTION OF A SET

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Abstract:

In this paper we give a method, based on the characteristic function of a set, to solve some difficult problems of set theory found in undergraduate studies.

Definition: Let's consider $A \subset E \neq \emptyset$ (a universal set), then $f_A : E \rightarrow \{0, 1\}$, where the function $f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$ is called the characteristic function of the set A .

Theorem 1: Let's consider $A, B \subset E$. In this case $f_A = f_B$ if and only if $A = B$.

Proof.

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A = B \\ 0, & \text{if } x \notin A = B \end{cases} = f_B(x)$$

Reciprocally: For any $x \in A$, $f_A(x) = 1$, but $f_A = f_B$, therefore $f_B(x) = 1$, namely $x \in B$ from where $A \subset B$. The same way we prove that $B \subset A$, namely $A = B$.

Theorem 2: $f_{\tilde{A}} = 1 - f_A$, $\tilde{A} = C_E A$.

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$$f_{\tilde{A}}(x) = \begin{cases} 1, & \text{if } x \in \tilde{A} \\ 0, & \text{if } x \notin \tilde{A} \end{cases} = \begin{cases} 1, & \text{if } x \notin A \\ 0, & \text{if } x \in A \end{cases} = \begin{cases} 1-0, & \text{if } x \notin A \\ 1-1, & \text{if } x \in A \end{cases} = 1 - \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases} = 1 - f_A(x)$$

Theorem 3: $f_{A \cap B} = f_A * f_B$.

Proof.

$$f_{A \cap B}(x) = \begin{cases} 1, & \text{if } x \in A \cap B \\ 0, & \text{if } x \notin A \cap B \end{cases} = \begin{cases} 1, & \text{if } x \in A \text{ and } x \in B \\ 0, & \text{if } x \notin A \text{ or } x \notin B \end{cases} = \begin{cases} 1, & \text{if } x \in A, x \in B \\ 0, & \text{if } x \in A, x \notin B \\ 0, & \text{if } x \notin A, x \in B \\ 0, & \text{if } x \notin A, x \notin B \end{cases} = \\ = \left(\begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \right) \left(\begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases} \right) = f_A(x) f_B(x).$$

The theorem can be generalized by induction:

Theorem 4: $f_{\bigcap_{k=1}^n A_k} = \prod_{k=1}^n f_{A_k}$

Consequence. For any $n \in \mathbb{N}^*$, $f_M^n = f_M$.

Proof. In the previous theorem we chose $A_1 = A_2 = \dots = A_n = M$.

Theorem 5: $f_{A \cup B} = f_A + f_B - f_A f_B$.

Proof.

$$f_{A \cup B} = f_{\overline{A \cap B}} = f_{\overline{A \cap B}} = 1 - f_{\overline{A \cap B}} = 1 - f_{\overline{A}} f_{\overline{B}} = 1 - (1 - f_A)(1 - f_B) = f_A + f_B - f_A f_B$$

It can be generalized by induction:

Theorem 6: $f_{\bigcup_{k=1}^n A_k} = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} (-1)^{k-1} f_{A_{i_1}} f_{A_{i_2}} \dots f_{A_{i_k}}$

Theorem 7: $f_{A-B} = f_A (1 - f_B)$

Proof. $f_{A-B} = f_{A \cap \overline{B}} = f_A f_{\overline{B}} = f_A (1 - f_B)$.

It can be generalized by induction:

Theorem 8: $f_{A_1 - A_2 - \dots - A_n} = \sum_{k=1}^n (-1)^{k-1} f_{A_{i_1}} f_{A_{i_2}} \dots f_{A_{i_k}}$.

Theorem 9: $f_{A \Delta B} = f_A + f_B - 2 f_A f_B$

Proof.

$$f_{A \Delta B} = f_{A \cup B - A \cap B} = f_{A \cup B} (1 - f_{A \cap B}) = (f_A + f_B - f_A f_B)(1 - f_A f_B) = f_A + f_B - 2 f_A f_B.$$

It can be generalized by induction:

Theorem 10: $F_{\Delta_{k=1}^n A_k} = \sum_{k=1}^n (-2)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1} A_{i_2} \dots A_{i_k}}$.

Theorem 11: $f_{A \times B}(x, y) = f_A(x) f_B(y)$.

Proof. If $(x, y) \in A \times B$, then $f_{A \times B}(x, y) = 1$ and $x \in A$, namely $f_A(x) = 1$ and $y \in B$, namely $f_B(y) = 1$, therefore $f_A(x)f_B(y) = 1$. If $(x, y) \notin A \times B$, then $f_{A \times B}(x, y) = 0$ and $x \notin A$, namely $f_A(x) = 0$ or $y \notin B$, namely $f_B(y) = 0$, therefore $f_A(x)f_B(y) = 0$.

This theorem can be generalized by induction.

Theorem 12: $f_{\times_{k=1}^n A_k}(x_1, x_2, \dots, x_n) = \prod_{k=1}^n f_{A_k}(x_k).$

Theorem 13: (De Morgan) $\overline{\bigcup_{k=1}^n A_k} = \bigcap_{k=1}^n \overline{A_k}$

Proof.

$$f_{\overline{\bigcup_{k=1}^n A_k}} = 1 - f_{\bigcup_{k=1}^n A_k} = 1 - \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1}} f_{A_{i_2}} \dots f_{A_{i_k}} = \prod_{k=1}^n (1 - f_{A_k}) = \prod_{k=1}^n f_{\overline{A_k}} = f_{\bigcap_{k=1}^n \overline{A_k}}.$$

We prove in the same way the following theorem:

Theorem 14: (De Morgan) $\overline{\bigcap_{k=1}^n A_k} = \bigcup_{k=1}^n \overline{A_k}.$

Theorem 15: $\left(\bigcup_{k=1}^n A_k \right) \cap M = \bigcup_{k=1}^n (A_k \cap M).$

Proof.

$$\begin{aligned} f_{\left(\bigcup_{k=1}^n A_k \right) \cap M} &= f_{\bigcup_{k=1}^n A_k} f_M = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1}} f_{A_{i_2}} \dots f_{A_{i_k}} f_M = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1}} f_{A_{i_2}} \dots f_{A_{i_k}} f_M^k = \\ &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1} \cap M} f_{A_{i_2} \cap M} \dots f_{A_{i_k} \cap M} = f_{\bigcup_{k=1}^n (A_k \cap M)} \end{aligned}$$

In the same way we prove that:

Theorem 16: $\left(\bigcap_{k=1}^n A_k \right) \cup M = \bigcap_{k=1}^n (A_k \cup M).$

Theorem 17: $\left(\Delta_{k=1}^n A_k \right) \cap M = \Delta_{k=1}^n (A_k \cap M)$

Application.

$\left(\Delta_{k=1}^n A_k \right) \cup M = \Delta_{k=1}^n (A_k \cup M)$ if and only if $M = \Phi.$

Theorem 18: $M \times \left(\bigcup_{k=1}^n A_k \right) = \bigcup_{k=1}^n (M \times A_k)$

Proof.

$$\begin{aligned}
f_{M \times \left(\bigcup_{k=1}^n A_k\right)}(x, y) &= f_M(y) f_{\bigcup_{k=1}^n A_k}(x) = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1}}(x) f_{A_{i_2}}(x) \dots f_{A_{i_k}}(x) f_M(y) = \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1}}(x) f_{A_{i_2}}(x) \dots f_{A_{i_k}}(x) f_M^k(y) = \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1} \times M}(x, y) \dots f_{A_{i_k} \times M}(x, y) = f_{\bigcup_{k=1}^n (M \times A_k)}
\end{aligned}$$

In the same way we prove that:

$$\mathbf{Theorem 19:} \quad M \times \left(\bigcap_{k=1}^n A_k\right) = \bigcap_{k=1}^n (M \times A_k).$$

$$\mathbf{Theorem 20:} \quad M \times (A_1 - A_2 - \dots - A_n) = (M \times A_1) - (M \times A_2) - \dots - (M \times A_n).$$

$$\mathbf{Theorem 21:} \quad (A_1 - A_2) \cup (A_2 - A_3) \cup \dots \cup (A_{n-1} - A_n) \cup (A_n - A_1) = \bigcup_{k=1}^n A_k - \bigcap_{k=1}^n A_k$$

Proof 1.

$$\begin{aligned}
f_{(A_1 - A_2) \cup \dots \cup (A_n - A_1)} &= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1} - A_{i_2}} \dots f_{A_{i_k} - A_{i_1}} = \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} (f_{A_{i_1}} - f_{A_{i_2}} - f_{A_{i_1}} f_{A_{i_2}}) \dots (f_{A_{i_k}} - f_{A_{i_1}} - f_{A_{i_k}} f_{A_{i_1}}) = \\
&= \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{A_{i_1}} \dots f_{A_{i_k}} \left(1 - \prod_{p=1}^n f_{A_p}\right) = f_{\bigcup_{k=1}^n A_k} \left(1 - f_{\bigcap_{k=1}^n A_k}\right) = f_{\bigcup_{k=1}^n A_k - \bigcap_{k=1}^n A_k}.
\end{aligned}$$

Proof 2. Let's consider $x \in \bigcup_{i=1}^n (A_i - A_{i+1})$, (where $A_{n+1} = A_1$), then there exists k such that $x \in (A_k - A_{k+1})$, namely $x \notin (A_k \cap A_{k+1}) \subset A_1 \cap A_2 \cap \dots \cap A_n$, namely $x \notin A_1 \cap A_2 \cap \dots \cap A_n$, and $x \in \bigcup_{k=1}^n A_k - \bigcap_{k=1}^n A_k$.

Now we prove the inverse statement:

Let's consider $x \in \bigcup_{k=1}^n A_k - \bigcap_{k=1}^n A_k$, we show that there exists k such that $x \in A_k$ and $x \notin A_{k+1}$. On the contrary, it would result that for any $k \in \{1, 2, \dots, n\}$, $x \in A_k$ and $x \in A_{k+1}$ namely $x \in \bigcup_{k=1}^n A_k$, it results that there exists p such that $x \in A_p$, but from the previous reasoning it results that $x \in A_{p+1}$, and using this we consequently obtain that $x \in A_k$ for $k = \overline{p, n}$. But from $x \in A_n$ we obtain that $x \in A_1$, therefore, it results that $x \in A_k$, $k = \overline{1, p}$, from where $x \in A_k$, $k = \overline{1, n}$, namely $x \in A_1 \cap \dots \cap A_n$, that is a contradiction. Thus there exists r such that $x \in A_r$ and $x \notin A_{r+1}$, namely $x \in (A_r - A_{r+1})$ and therefore $x \in \bigcup_{k=1}^n (A_k - A_{k+1})$.

In the same way we prove the following theorem:

$$\textbf{Theorem 22: } (A_1 \Delta A_2) \cup (A_2 \Delta A_3) \cup \dots \cup (A_{n-1} \Delta A_n) = \bigcup_{k=1}^n A_k - \bigcap_{k=1}^n A_k.$$

Theorem 23:

$$(A_1 \times A_2 \times \dots \times A_k) \cap (A_{k+1} \times A_{k+2} \times \dots \times A_{2k}) \cap (A_n \times A_1 \times \dots \times A_{k-1}) = (A_1 \cap A_2 \cap \dots \cap A_n)^k.$$

$$\begin{aligned} \textbf{Proof. } f_{(A_1 \times \dots \times A_k) \cap \dots \cap (A_n \times A_1 \times \dots \times A_{k-1})}(x_1, \dots, x_n) &= \\ &= f_{A_1 \times \dots \times A_k}(x_1, \dots, x_n) \dots f_{A_n \times \dots \times A_{k-1}}(x_1, \dots, x_n) = \\ &= (f_{A_1}(x_1) \dots f_{A_k}(x_k)) \dots (f_{A_n}(x_n) \dots f_{A_{k-1}}(x_{k-1})) = \\ &= f_{A_1}^k(x_1) \dots f_{A_n}^k(x_n) = f_{A_1 \cap \dots \cap A_n}^k(x_1, \dots, x_n) = \\ &= f_{(A_1 \cap \dots \cap A_n)^k}(x_1, \dots, x_n). \end{aligned}$$

Theorem 24. $(P(E), \cup)$ is a commutative monoid.

Proof. For any $A, B \in P(E)$; $A \cup B \in P(E)$, namely the intern operation. Because $(A \cup B) \cup C = A \cup (B \cup C)$ is associative, $A \cup B = B \cup A$ commutative, and because $A \cup \emptyset = A$ then \emptyset is the neutral element.

Theorem 25: $(P(E), \cap)$ is a commutative monoid.

Proof. For any $A, B \in P(E)$; $A \cap B \in P(E)$ namely intern operation. $(A \cap B) \cap C = A \cap (B \cap C)$ associative, $A \cap B = B \cap A$, commutative $A \cap E = A$, E is the neutral element.

Theorem 26: $(P(E), \Delta)$ is an abelian group.

Proof. For any $A, B \in P(E)$; $A \Delta B \in P(E)$, namely the intern operation. $A \Delta B = B \Delta A$ commutative. The proof of associativity is in the XIIth grade manual as a problem. We'll prove it using the characteristic function of the set.

$$f_{(A \Delta B) \Delta C} = 4f_A f_B f_C - 2f_A f_B + f_B f_C + f_C f_A + f_A + f_B + f_C = f_{A \Delta (B \Delta C)}.$$

Because $A \Delta \emptyset = A$, \emptyset is the neutral element and because $A \Delta A = \emptyset$; the symmetric element of A is A itself.

Theorem 27: $(P(E), \Delta, \cap)$ is a commutative Boole ring with a divisor of zero.

Proof. Because the previous theorem satisfies the commutative ring axioms, the first part of the theorem is proved. Now we prove that it has a divisor of zero. If $A \neq \emptyset$ and $B \neq \emptyset$ are two disjoint sets, then $A \cap B = \emptyset$, thus it has divisor of zero. From Theorem 17 we get that it is distributive for $n=2$. Because for any $A \in P(E)$; $A \cap A = A$ and $A \Delta A = \emptyset$ it also satisfies the Boole-type axioms.

Theorem 28: Let's consider $H = \{f \mid f : E \rightarrow \{0,1\}\}$, then (H, \oplus) is an abelian group, where $f_A \oplus f_B = f_A + f_B - 2f_A f_B$ and $(P(E), \Delta) \cong (H, \oplus)$.

Proof. Let's consider $F : P(E) \rightarrow H$, where $f(A) = f_A$, then, from the previous theorem we get that it is bijective and because $F(A \Delta B) = f_{A \Delta B} = F(A) \oplus F(B)$ it is compatible.

Theorem 29: $card(A_1 \Delta A_n) \leq card(A_1 \Delta A_2) + card(A_2 \Delta A_3) + \dots + card(A_{n-1} \Delta A_n)$.

Proof. By induction. If $n = 2$, then it is true, we show that for $n = 3$ it is also true. Because $(A_1 \cap A_2) \cup (A_2 \cap A_3) \subseteq A_2 \cup (A_1 \cap A_3)$;

$$card((A_1 \cap A_2) \cup (A_2 \cap A_3)) \leq card(A_2 \cup (A_1 \cap A_3)) \text{ but}$$

$$card(M \cup N) = cardM + cardN - card(M \cap N), \text{ and thus}$$

$$cardA_2 + card(A_1 \cap A_3) - card(A_1 \cap A_2) - card(A_2 \cap A_3) \geq 0, \text{ can be}$$

written as

$$cardA_1 + cardA_3 - 2card(A_1 \cap A_3) \leq$$

$$\leq (cardA_1 + cardA_2 - 2card(A_1 \cap A_2)) + (cardA_2 + cardA_3 - 2card(A_2 \cap A_3)).$$

But because of

$$(M \Delta N) = cardM + cardN - 2card(M \cap N)$$

then $card(A_1 \Delta A_3) \leq card(A_1 \Delta A_2) + card(A_2 \Delta A_3)$. The proof of this step of the induction relies on the above method.

Theorem 30: $(P^2(E), card(A \Delta B))$ is a metric space.

Proof. Let $d(A, B) = card(A \Delta B) : P(E) \times P(E) \rightarrow \square$

1. $d(A, B) = 0 \Leftrightarrow card(A \Delta B) = 0 \Leftrightarrow card((A - B) \cup (B - A)) = 0$ but

because $(A - B) \cap (B - A) = \emptyset$ we obtain $(A - B) + card(B - A) = 0$ and because $(A - B) = 0$ and $card(B - A) = 0$, then $A - B = \emptyset$, $B - A = \emptyset$, and $A = B$.

2. $d(A, B) = d(B, A)$ results from $A \Delta B = B \Delta A$.

3. As a consequence of the previous theorem $d(A, C) \leq d(A, B) + d(B, C)$.

As a result of the above three properties it is a metric space.

PROBLEMS

Problem 1.

Let's consider $A = B \cup C$ and $f : P(A) \rightarrow P(A) \times P(A)$, where $f(x) = (X \cup B, X \cup C)$. Prove that f is injective if and only if $B \cap C = \emptyset$.

Solution 1. If f is injective. Then

$$f(\emptyset) = (\emptyset \cup B, \emptyset \cup C) = (B, C) = ((B \cap C) \cup B, (B \cap C) \cup C) = f(B \cap C) \text{ from}$$

which we obtain $B \cap C = \emptyset$. Now reciprocally: Let's consider $B \cap C = \emptyset$, then $f(X) = f(Y)$; it results that $X \cup B = Y \cup B$ and $X \cup C = Y \cup C$ or

$X = X \cup \emptyset = X \cup (B \cap C) = (X \cup B) \cap (X \cup C) = (Y \cup B) \cap (Y \cup C) = Y \cup (B \cap C) = Y \cup \emptyset = Y$
namely it is injective.

Solution 2. Let's consider $B \cap C = \emptyset$ passing over the set function $f(X) = f(Y)$ if and only if $X \cup B = Y \cup B$ and $X \cup C = Y \cup C$, namely $f_{X \cup B} = f_{Y \cup B}$ and $f_{X \cup C} = f_{Y \cup C}$ or $f_X + f_B - f_X f_B = f_Y + f_B - f_Y f_B$ and $f_X + f_C - f_X f_C = f_Y + f_C - f_Y f_C$ from which we obtain $(f_X - f_Y)(f_B - f_C) = 0$.

Because $A = B \cup C$ and $B \cap C = \emptyset$, we have

$$(f_B - f_C)(u) = \begin{cases} 1, & \text{if } u \in B \\ -1, & \text{if } u \in C \end{cases} \neq 0$$

therefore $f_X - f_Y = 0$, namely $X = Y$ and thus it is injective.

Generalization. Let $M = \bigcup_{k=1}^n A_k$ and $f : P(A) \rightarrow P^n(A)$, where

$$f(X) = (X \cup A_1, X \cup A_2, \dots, X \cup A_n).$$

Prove that f is injective if and only if $A_1 \cap A_2 \cap \dots \cap A_n = \emptyset$.

Problem 2. Let $E \neq \emptyset$, $A \in P(E)$, and $f : P(E) \rightarrow P(E) \times P(E)$, where $f(X) = (X \cap A, X \cup A)$.

- Prove that f is injective
- Prove that $\{f(x), x \in P(E)\} = \{(M, N) \mid M \subset A \subset N \subset E\} = K$.
- Let $g : P(E) \rightarrow K$, where $g(X) = f(X)$. Prove that g is bijective and compute its inverse.

Solution.

a. $f(X) = f(Y)$, namely $(X \cap A, X \cup A) = (Y \cap A, Y \cup A)$ and then $X \cap A = Y \cap A$, $X \cup A = Y \cup A$, from where $X \Delta A = Y \Delta A$ or $(X \Delta A) \Delta A = (Y \Delta A) \Delta A$, $X \Delta (A \Delta A) = Y \Delta (A \Delta A)$, $X \Delta \emptyset = Y \Delta \emptyset$ and thus $X = Y$, namely f is injective.

b. $\{f(X), X \in P(E)\} = f(P(E))$. We'll show that $f(P(E)) \subset K$. For any $(M, N) \in f(P(E))$, $\exists X \in P(E) : f(X) = (M, N)$; $(X \cap A, X \cup A) = (M, N)$.

From here $X \cap A = M$, $X \cup A = N$, namely $M \subset A$ and $A \subset N$

thus $M \subset A \subset N$, and, therefore $(M, N) \in K$.

Now, we'll show that $K \subset f(P(E))$, for any $(M, N) \in K$, $\exists X \in P(E)$ such that $f(X) = (M, N)$. $f(X) = (M, N)$, namely $(X \cap A, X \cup A) = (M, N)$ from where $X \cap A = M$ and $X \cup A = N$, namely $X \Delta A = N - M$, $(X \Delta A) \Delta A = (N - M) \Delta A$, $X \Delta \emptyset = (N - M) \Delta A$,

$$X = (N - M) \Delta A, \quad X = (N \cap \overline{M}) \Delta A,$$

$$X = ((N \cap \overline{M}) - A) \cup (A - (N \cap \overline{M})) = ((N \cap \overline{M}) \cap A) \cup (A \cap \overline{(N \cap \overline{M})}) =$$

$$= (N \cap (\overline{M} \cap \overline{A})) \cup (A \cap (N \cap \overline{M})) = (N \cap \overline{A}) \cup ((A \cap \overline{N}) \cup (A \cap M)) =$$

$$= (N \cap \bar{A}) \cup (\emptyset \cup M) = (N - A) \cup M.$$

From here we get the unique solution: $X = (N - A) \cup M$.

$$\text{We test } f((N - A) \cup M) = (((N - A) \cup M) \cap A, ((N - A) \cup M) \cup A)$$

but

$$\begin{aligned} ((N - A) \cup M) \cap A &= ((N \cap \bar{A}) \cup M) \cap A = ((N \cap \bar{A}) \cap A) \cup (M \cap A) = \\ &= ((N \cap (\bar{A} \cap A)) \cup M) = (N \cap \emptyset) \cup M = \emptyset \cup M = M \end{aligned}$$

and

$$\begin{aligned} ((N - A) \cup M) \cup A &= (N - A) \cup (M \cup A) = (N - A) \cup A = (N \cap \bar{A}) \cup A = \\ &= (N \cup A) \cap (\bar{A} \cup A) = N \cap E = N, \quad f((N - A) \cup M) = (M, N). \end{aligned}$$

Thus $f(P(E)) = K$.

c. From point a. we have that g is injective, from point b. we have that g surjective, thus g is bijective. The inverse function is:

$$g^{-1}(M, N) = (N - A) \cup M.$$

Problem 3. Let $E \neq \emptyset$, $A, B \in P(E)$ and $f : P(E) \rightarrow P(E) \times P(E)$, where $f(X) = (X \cap A, X \cap B)$.

- Give the necessary and sufficient condition such that f is injective.
- Give the necessary and sufficient condition such that f is surjective.
- Supposing that f is bijective, compute its inverse.

Solution.

a. Suppose that f is injective. Then:

$$f(A \cup B) = ((A \cup B) \cap A, (A \cup B) \cap B) = (A, B) = (E \cap A, E \cap B) = f(E),$$

from where $A \cup B = E$.

Now we suppose that $A \cup B = E$, it results that:

$$X = X \cap E = X \cap (A \cup B) = (X \cap A) \cup (X \cap B) = (Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y \cap E = Y$$

namely from $f(X) = f(Y)$ we obtain that $X = Y$, namely f is injective.

b. Suppose that f is surjective, for any $M, N \in P(A) \times P(B)$, there exists

$$X \in P(E), f(X) = (M, N), (X \cap A, X \cap B) = (M, N), X \cap A = M, X \cap B = N.$$

In special cases $(M, N) = (A, \emptyset)$, there exists $X \in P(E)$, from

$$X \supset A, \emptyset = X \cap B \supset A \cap B, A \cap B = \emptyset.$$

Now we suppose that $A \cap B = \emptyset$ and show that it is surjective.

Let $(M, N) \in P(A) \times P(B)$, then $M \subset A, N \subset B$, $M \cap B \subset A \cap B = \emptyset$, and $N \cap A \subset B \cap A = \emptyset$, namely $M \cap B = \emptyset$, $N \cap A = \emptyset$ and

$$\begin{aligned} f(M \cup N) &= ((M \cup N) \cap A, (M \cup N) \cap B) = \\ &= ((M \cap A) \cup (N \cap A), (M \cap B) \cup (N \cap B)) = (M \cup \emptyset, \emptyset \cup N) = (M, N), \end{aligned}$$

for any (M, N) there exists $X = M \cup N$ such that $f(X) = (M, N)$, namely f is surjective.

c. We'll show that $f^{-1}((M, N)) = M \cup N$.

Remark. In the previous two problems we can use the characteristic function of the set as in the first problem. We leave this method for the readers.

Application. Let $E \neq \emptyset$, $A_k \in P(E)$ ($k = 1, \dots, n$) and $f : P(E) \rightarrow P^n(E)$, where $f(X) = (X \cap A_1, X \cap A_2, \dots, X \cap A_n)$.

Prove that f is injective if and only if $\bigcup_{k=1}^n A_k = E$.

Application. Let $E \neq \emptyset$, $A_k \in P(E)$, ($k = 1, \dots, n$) and $f : P(E) \rightarrow P^n(E)$, where $f(X) = (X \cap A_1, X \cap A_2, \dots, X \cap A_n)$.

Prove that f is surjective if and only if $\bigcap_{k=1}^n \bar{A}_k = \emptyset$.

Problem 4. We name the set M convex if for any $x, y \in M$ $tx + (1-t)y \in M$, for any $t \in [0, 1]$.

Prove that if A_k , ($k = 1, \dots, n$) are convex sets, then $\bigcap_{k=1}^n A_k$ is also convex.

Problem 5. If A_k , ($k = 1, \dots, n$) are convex sets, then $\bigcap_{k=1}^n A_k$ is also convex.

Problem 6. Give the necessary and sufficient condition such that if A, B are convex/concave sets, then $A \cup B$ is also convex/concave. Generalization for the \mathbb{N} set.

Problem 7. Give the necessary and sufficient condition such that if A, B are convex/concave sets then $A \Delta B$ is also convex/concave. Generalization for the \mathbb{N} set.

Problem 8. Let $f, g : P(E) \rightarrow P(E)$, where $f(x) = A - X$, and $g(x) = A \Delta X$, $A \in P(E)$.

Prove that f, g are bijective and compute their inverse functions.

Problem 9. Let $A \circ B = \{(x, y) \in \square \times \square \mid \exists z \in \square : (x, z) \in A \text{ and } (z, y) \in B\}$. In a particular case let $A = \{(x, \{x\}) \mid x \in \square\}$ and $B = \{(\{y\}, y) \mid y \in \square\}$.

Represent the $A \circ A$, $B \circ A$, $B \circ B$ cases.

Problem 10.

i. If $A \cup B \cup C = D$, $A \cup B \cup D = C$, $A \cup C \cup D = B$, $B \cup C \cup D = A$, then $A = B = C = D$

- ii. Are there different A, B, C, D sets such that
 $A \cup B \cup C = A \cup B \cup D = A \cup C \cup D = B \cup C \cup D$?

Problem 11. Prove that $A \Delta B = A \cup B$ if and only if $A \cap B = \emptyset$.

Problem 12. Prove the following identity.

$$\bigcap_{i,j=1,i < j}^n A_k \cup A_j = \bigcup_{i=1}^n \left(\bigcap_{j=1, j \neq i}^n A_j \right)$$

Problem 13. Prove the following identities.

$$(A \cup B) - (B \cap C) = (A - (B \cap C)) \cup (B - C) = (A - B) \cup (A - C) \cup (B - C)$$

and

$$A - [(A \cap C) - (A \cap B)] = (A - \bar{B}) \cup (A - C).$$

Problem 14. Prove that $A \cup (B \cap C) = (A \cup B) \cap C = (A \cup C) \cap B$ if and only if $A \subset B$ and $A \subset C$.

Problem 15. Prove the following identities:

$$(A - B) - C = (A - B) - (C - B),$$

$$(A \cup B) - (A \cup C) = B - (A \cap C),$$

$$(A \cap B) - (A \cap C) = (A \cap B) - C.$$

Problem 16. Solve the following system of equations:

$$\begin{cases} A \cup X \cup Y = (A \cup X) \cap (A \cup Y) \\ A \cap X \cap Y = (A \cap X) \cup (A \cap Y) \end{cases}$$

Problem 17. Solve the following system of equations:

$$\begin{cases} A \Delta X \Delta B = A \\ A \Delta Y \Delta B = B \end{cases}$$

Problem 18. Let $X, Y, Z \subseteq A$. Prove that:

$$Z = (X \cap \bar{Z}) \cup (Y \cap \bar{Z}) \cup (\bar{X} \cap Z \cap \bar{Y}) \text{ if and only if } X = Y = \emptyset.$$

Problem 19. Prove the following identity:

$$\bigcup_{k=1}^n [A_k \cup (B_k - C)] = \left(\bigcup_{k=1}^n A_k \right) \cup \left[\left(\bigcup_{k=1}^n A_k \right) - C \right].$$

Problem 20. Prove that: $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

Problem 21. Prove that:

$$(A\Delta B)\Delta C = (A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)\cup(A\cap B\cap C).$$

REFERENCES:

- [1] Mihály Bencze, F. Popovici – Permutaciok - Matematikai Lapok, Kolozsvár, pp. 7-8, 1991.
- [2] Pellegrini Miklós – Egy újabb kísérlet, a retegezett halmaz. – M.L., Kolozsvár, 6, 1978.
- [3] Halmazokra vonatkozó egyenletekrol – Matematikai Lapok, Kolozsvár, 6, 1970.
- [4] Alkalmazások a halmazokkal kapcsolatban - Matematikai Lapok, Kolozsvár, 3, 1970.
- [5] Ion Savu – Produsul elementelor într-un grup finit comutativ – Gazeta Matematică Perf., 1, 1989.
- [6] Nicolae Negoescu – Principiul includerii-excluderii – RMT 2, 1987.
- [7] F. C. Gheorghe, T. Spiru – Teorema de prelungire a unei probabilități, dedusă din teorema de completare metrică – Gazeta Matematică, Seria A, 2, 1974.
- [8] C. P. Popovici – Funcții Boolene – Gazeta Matematică, Seria A, 1, 1973.
- [9] Algebra tankönyv IX oszt., Romania.
- [10] Năstăsescu stb. – Exerciții și probleme de algebră pentru clasele IX-XII – Romania.

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