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# Competition super-hypergraphs: Revealing hierarchical competition in real-world networks

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**Abstract.** Graph theory provides a powerful language for modeling pairwise connections through vertices and edges [9, 20]. Hypergraphs generalize this idea by permitting hyperedges that join any number of vertices simultaneously [7], while super-hypergraphs iterate the Power-set operation to capture multi-level, hierarchical relationships among hyperedges [40, 22].

A competition hypergraph associates each prey species with a hyperedge containing all its predators, thereby encoding multi-way competition in ecological networks. In this work, we introduce the *competition super-hypergraph*, which lifts the competition concept to higher tiers of aggregation. We present its formal definition, explore theoretical properties, and illustrate its practical use in real-world scenarios, such as modeling layered competition in food webs and multi-agent systems.

**AMS Subject Classification (2020):** 05C65

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## 1. Preliminaries

In this section, we introduce the fundamental terminology and notation used throughout. Unless otherwise noted, all graphs are finite and simple. For more detailed treatments, the reader may consult the cited references.

## 1.1. Super-hypergraphs

A *hypergraph* relaxes the usual edge-vertex incidence by allowing each hyperedge to connect an arbitrary subset of vertices [7, 8, 1, 11, 47]. A *Super-hypergraph* takes this further by iterating the Power-set operation, thereby capturing multi-level or hierarchical relationships among hyperedges [39, 28, 18]. Such structures have proven useful in applications ranging from molecular modeling to network analysis and signal processing (cf.[26, 12, 16, 15, 23, 14]). Throughout, let $n \in \mathbb{N}_0$ denote the number of times the Power-set is applied.

**Definition 1.1 (Base Set).** A *base set* $S$ is the universe from which all further constructions originate:

$$S = \{\, x : x \text{ belongs to the domain of interest}\}.$$

Every element of $\mathrm{POW}(S)$ or its iterates $\mathrm{POW}_n(S)$ is drawn from $S$.

**1.2 Definition 1.2 (Power-set).** The *Power-set* of $S$ is the collection of all subsets of $S$, including the empty set:

$$\mathrm{POW}(S) = \{\, A : A \subseteq S\}.$$

**Definition 1.3 (Hypergraph) [7, 8].** A *hypergraph* $H = (V, E)$ consists of

- A finite vertex set $V$.
- A family $E \subseteq \mathrm{POW}(V)$ of nonempty subsets, called hyperedges.

Hypergraphs naturally model higher-order interactions among elements of $V$.

**Definition 1.4 ($n$-th Iterated Power-set) [41, 13, 42, 24].** For $n \geqslant 1$, define

$$\mathrm{POW}_1(X) = \mathrm{POW}(X), \qquad \mathrm{POW}_{n+1}(X) = \mathrm{POW}\big(\mathrm{POW}_n(X)\big).$$

The *nonempty* iterated Power-set $\mathrm{POW}_n^*(X)$ is obtained by discarding the empty set at each stage.

**Definition 1.5 ($n$-Super-hypergraph) [39, 40, 3, 21].** Let $V_0$ be a finite base set and define recursively

$$\mathrm{POW}^0(V_0) = V_0, \quad \mathrm{POW}^{k+1}(V_0) = \mathrm{POW}\big(\mathrm{POW}^k(V_0)\big).$$

An *$n$-Super-hypergraph* is a pair

$$\mathrm{SupHypG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathrm{POW}^n(V_0), \quad E \subseteq \mathrm{POW}^n(V_0).$$

The members of $V$ are called *$n$-supervertices*, and those of $E$ are *$n$-superedges*.

**Example 1.6 (Corporate Hierarchy as a 2-Super-hypergraph).** Let the employee roster be

$$V_0 = \{\text{Alice, Bob, Carol}\}.$$

Their team assignments form the first-level supervertices:

$$V_1 = \big\{T_1 = \{\text{Alice}, \text{Bob}\},\ T_2 = \{\text{Bob}, \text{Carol}\}\big\} \subseteq \mathrm{POW}^1(V_0).$$

At the second level, these two teams join into a division:

$$V_2 = \big\{D = \{T_1,\ T_2\}\big\} \subseteq \mathrm{POW}^2(V_0).$$

We now choose a set of superedges in $\mathrm{POW}^2(V_0)$. For example, suppose the company runs two initiatives:

$$E_2 = \big\{e_{\text{Launch}} = \{D\},\ e_{\text{Audit}} = \{T_1,\ T_2\}\big\}.$$

Then the resulting 2-Super-hypergraph is

$$\mathrm{SupHypG}^{(2)} = (\ V_2,\ E_2\ ),$$

with

$$V_2 = \{ D\}, \quad E_2 = \big\{\{D\}, \{T_1, T_2\}\big\}.$$

Here:

- $D = \{T_1, T_2\}$ is the single second-level grouping (the "Engineering Division").
- The superedge $e_{\text{Launch}} = \{D\}$ represents an initiative that involves the entire division.
- The superedge $e_{\text{Audit}} = \{T_1, T_2\}$ represents a parallel audit on each team.

This example illustrates how individuals ($V_0$) form teams ($V_1$), which in turn combine into a division ($V_2$), and how superedges can link those second-level entities in distinct ways.

## 1.2. Competition hypergraphs

A *competition graph* models species as vertices, placing an undirected edge between two species if they both prey on at least one common organism (cf.[25, 46, 17, 10]). A *competition hypergraph* refines this by assigning each prey species to a hyperedge that contains every predator species competing for that prey (cf.[43, 45, 44, 34, 2]).

We now give the precise definition of the competition hypergraph.

**Definition 1.7 (Competition Hypergraph) [43].** Let $D = (V, A)$ be a directed graph. Its it competition hypergraph $\mathrm{CH}(D) = (V, E)$ is defined by

$$E \;=\; \big\{ N^{-}(v) \subseteq V : v \in V,\ |N^{-}(v)| \geqslant 2\big\},$$

where

$$N^{-}(v) = \{ u \in V : (u, v) \in A\}$$

is the set of vertices with arcs into $v$. Equivalently, each hyperedge in $\mathrm{CH}(D)$ groups all vertices that compete for the same target $v$ in $D$, provided at least two such vertices exist.

**Example 1.8 (Food-Web Competition Hypergraph).** Consider a small ecological network with five species:

$$V = \{\text{Wolf, Fox, Lynx, Deer, Rabbit}\}.$$

The predation relations in the digraph $D$ are:

$$A = \Big\{ (\text{Wolf}, \text{Deer}),\ (\text{Bear}, \text{Deer}),\ (\text{Fox}, \text{Rabbit}),\ (\text{Lynx}, \text{Rabbit}),\ (\text{Fox}, \text{Mouse}),\ (\text{Owl}, \text{Mouse}),\ (\text{Bear}, \text{Mouse}) \Big\},$$

where we imagine "Bear" and "Owl" also present in the ecosystem (omitted from $V$ for brevity). Then the competition hypergraph $\mathrm{CH}(D)$ on the observed vertex set $V$ has hyperedges corresponding to prey with two or more predators:

$$e_{\text{Deer}} = N^{-}(\text{Deer}) = \{\text{Wolf, Bear}\},$$
$$e_{\text{Rabbit}} = N^{-}(\text{Rabbit}) = \{\text{Fox, Lynx}\},$$
$$e_{\text{Mouse}} = N^{-}(\text{Mouse}) = \{\text{Fox, Bear, Owl}\}.$$

Restricting to our five-species set, we obtain

$$E = \big\{ \{\text{Wolf}, \text{Bear}\},\ \{\text{Fox}, \text{Lynx}\},\ \{\text{Fox}, \text{Bear}\} \big\}.$$

Thus

$$\mathrm{CH}(D) = (V,\ E)$$

captures which predators "compete" for the same prey. In contrast to the ordinary competition graph (which would only show edges $\{\text{Wolf}, \text{Bear}\}$, $\{\text{Fox}, \text{Lynx}\}$, $\{\text{Fox}, \text{Bear}\}$), the hypergraph retains the full multi-way competition at the "mouse" node as a three-element hyperedge $\{\text{Fox}, \text{Bear}, \text{Owl}\}$.

# 2. Main results

In this section, we present the main contributions of this paper by introducing the definition of Competition $n$-Super-hypergraphs and examining their fundamental properties.

## 2.1. Competition $n$-Super-hypergraphs

Competition $n$-Super-hypergraph extends competition hypergraphs across hierarchical levels by treating iterated vertex subsets as supervertices and grouping competitors into hyperedges.

**Definition 2.1 (Competition $n$-Super-hypergraph).** Let $D = (V_0, A)$ be a directed graph. For an integer $n \geqslant 0$, form the iterated Power-set

$$V_n \;=\; \mathrm{POW}^n(V_0).$$

The *competition $n$-Super-hypergraph* of $D$ is

$$\mathrm{CompSuHG}^{(n)}(D) \;=\; \bigl(V_n,\; E_n^{\mathrm{comp}}\bigr),$$

where

$$E_n^{\mathrm{comp}} = \Bigl\{\, N_n^{-}(S) \subseteq V_n : S \in V_n,\; \bigl|N_n^{-}(S)\bigr| \geqslant 2 \Bigr\},$$

and

$$N_n^{-}(S) = \bigl\{\, T \in V_n : \exists\, u \in T,\; v \in S \text{ with } (u \to v) \in A \bigr\}$$

is the set of all $n$-supervertices whose elements link into $S$ in the original digraph.

**Example 2.2 (Competition 1-Super-hypergraph of a Simple Food Chain).** Let the base vertex set be

$$V_0 = \{\mathrm{A},\, \mathrm{B},\, \mathrm{C}\},$$

and define the digraph $D = (V_0, A)$ by the predator–prey relations

$$A = \{\, (\mathrm{A} \to \mathrm{B}),\; (\mathrm{C} \to \mathrm{B})\}.$$

That is, both A and C prey on B, and there are no other arcs.

**Level-1 supervertices:** Select a subset of the first-iterated Power-set,

$$V_1 = \{ X = \{A, B\},\ Y = \{B, C\} \} \subseteq \mathrm{POW}^1(V_0).$$

**Computing the competition hyperedges:** For each supervertex $S \in V_1$, we form

$$N_1^-(S) = \{ T \in V_1 : \exists\, u \in T,\ v \in S \text{ with } (u \to v) \in A \}.$$

- If $S = X = \{A, B\}$, then its prey set in $D$ is $\{B\}$. The in-neighbors of B are $\{A, C\}$. Thus any $T \in V_1$ containing A or C qualifies. Both

  $$X = \{A, B\} \quad \text{and} \quad Y = \{B, C\}$$

  contain at least one of $A, C$, so

  $$N_1^-(X) = \{ X, Y \}.$$

- Similarly, for $S = Y = \{B, C\}$, the relevant prey again is B, whose predators are A and C, so

  $$N_1^-(Y) = \{ X, Y \}.$$

**Level-1 competition hyperedges:** Since $|N_1^-(S)| = 2$ for both $S = X$ and $S = Y$, we include the single hyperedge

$$e = \{X, Y\} \in E_1^{\mathrm{comp}}.$$

Hence the competition 1-Super-hypergraph is

$$\mathrm{CompSuHG}^{(1)}(D) = \left(V_1,\ E_1^{\mathrm{comp}}\right) = \Big(\{\{A, B\}, \{B, C\}\},\ \{\{\{A, B\}, \{B, C\}\}\}\Big).$$

This captures that the two "teams" $\{A, B\}$ and $\{B, C\}$ both compete for the same prey (B) in the original digraph.

**Example 2.3 (Competition 2-Super-hypergraph of a Simple Food Chain).** Let the base species set be

$$V_0 = \{\text{A, B, C}\},$$

and define predation by the arcs

$$A = \{(\text{A} \to \text{B}), (\text{B} \to \text{C})\}$$

in the digraph $D = (V_0, A)$.

**First-level supervertices ($V_1$).** Form two predator–prey clusters:

$$S_1 = \{\text{A}, \text{B}\}, \quad S_2 = \{\text{B}, \text{C}\},$$

so that $\text{POW}^1(V_0) \supseteq V_1 = \{S_1, S_2\}$.

**Second-level supervertices ($V_2$).** Take all nonempty subsets of $V_1$:

$$V_2 = \big\{\{S_1\}, \{S_2\}, \{S_1, S_2\}\big\}.$$

**Defining the lifted arc relation.** We say a first-level supervertex $U$ links to $V$ if there exist $a \in U$ and $b \in V$ with $(a \to b) \in A$. In particular:

$$S_1 \to S_1,\ S_1 \to S_2,\ S_2 \to S_2,$$

and no other lifted arcs.

**Computing competition hyperedges.** For each second-level supervertex $X \in V_2$, set

$$N_2^-(X) = \{\, Y \in V_2 : \exists\, U \in Y,\ V \in X \text{ with } (U \to V)\}.$$

A direct check shows:

$$\begin{aligned} N_2^-\big(\{S_1\}\big) &= \big\{\{S_1\}, \{S_1, S_2\}\big\}, \\ N_2^-\big(\{S_2\}\big) &= \big\{\{S_1\}, \{S_2\}, \{S_1, S_2\}\big\}, \\ N_2^-\big(\{S_1, S_2\}\big) &= \big\{\{S_1\}, \{S_2\}, \{S_1, S_2\}\big\}. \end{aligned}$$

Since each of these sets has size at least two, they form the hyperedges of $\mathrm{CompSuHG}^{(2)}(D)$:

$$E_2^{\mathrm{comp}} = \Big\{ \{\{S_1\},\{S_1,S_2\}\},\ \{\{S_1\},\{S_2\},\{S_1,S_2\}\} \Big\}.$$

**Resulting super-hypergraph.** The competition 2-super-hypergraph is

$$\mathrm{CompSuHG}^{(2)}(D) = \big(V_2,\ E_2^{\mathrm{comp}}\big),$$

which captures, at two hierarchical levels, how the clusters $\{A, B\}$ and $\{B, C\}$ compete for prey in the original food chain.

**Example 2.4 (Competition 3-Super-hypergraph of a Simple Food Chain).** Let the base vertex set be

$$V_0 = \{\mathrm{A},\ \mathrm{B},\ \mathrm{C}\},$$

and define the digraph $D = (V_0, A)$ by the predator–prey relations

$$A = \{\, (\mathrm{A} \to \mathrm{B}),\ (\mathrm{C} \to \mathrm{B})\}.$$

**Level 1 supervertices:** Form two subsets of $V_0$,

$$S_1 = \{\mathrm{A}, \mathrm{B}\}, \quad S_2 = \{\mathrm{B}, \mathrm{C}\},$$

so $\mathrm{POW}^1(V_0) \supseteq V_1 = \{S_1, S_2\}$.

**Level 2 supervertices:** Take nonempty subsets of $V_1$,

$$T_1 = \{S_1\}, \quad T_2 = \{S_2\}, \quad T_3 = \{S_1, S_2\},$$

so $\mathrm{POW}^2(V_0) \supseteq V_2 = \{T_1, T_2, T_3\}$.

**Level 3 supervertices:** Again take nonempty subsets of $V_2$,

$$U_1 = \{T_1\}, \quad U_2 = \{T_2\}, \quad U_3 = \{T_3\},$$

so $\mathrm{POW}^3(V_0) \supseteq V_3 = \{U_1, U_2, U_3\}$.

**Lifted adjacency:** We say a supervertex $X$ at level $k$ has an arc to supervertex $Y$ at the same level if there exist elements $x \in X$, $y \in Y$ at level $k-1$ with $(x \to y) \in A$. Concretely:

$S_1 \to S_1, \quad S_1 \to S_2, \quad S_2 \to S_2,$

$T_1 \to T_1,\ T_1 \to T_3,\ T_2 \to T_2,\ T_2 \to T_3,\ T_3 \to T_1,\ T_3 \to T_2,\ T_3 \to T_3,$

$U_1 \to U_1,\ U_1 \to U_3,\ U_2 \to U_2,\ U_2 \to U_3,\ U_3 \to U_1,\ U_3 \to U_2,\ U_3 \to U_3.$

**Competition hyperedges at level 3:** For each $U_i \in V_3$, compute

$$N_3^-(U_i) = \{\, U_j \in V_3 : U_j \to U_i\}.$$

A direct check yields

$$N_3^-(U_1) = \{\, U_1,\, U_3\},$$
$$N_3^-(U_2) = \{\, U_2,\, U_3\},$$
$$N_3^-(U_3) = \{\, U_1,\, U_2,\, U_3\}.$$

Since each set has size at least two, these form the hyperedges:

$$E_3^{\mathrm{comp}} = \big\{ \{U_1, U_3\},\ \{U_2, U_3\},\ \{U_1, U_2, U_3\} \big\}.$$

Hence the *competition* 3-*super-hypergraph* of $D$ is

$$\mathrm{CompSuHG}^{(3)}(D) = (\, V_3,\ E_3^{\mathrm{comp}}\,),$$

which captures, at three hierarchical levels, how the clusters ultimately compete for the same prey in the original food chain.

**Theorem 2.5 (Unification of Competition and Super-hypergraphs).** *The family* $\mathrm{CompSuHG}^{(n)}$ *simultaneously extends:*

(a) *Classical competition hypergraphs: when $n = 0$, $V_0$ is the original vertex set and* $\mathrm{CompSuHG}^{(0)}(D)$ *recovers the usual competition hypergraph of $D$.*

(b) *$n$-Super-hypergraphs: for any $n$-Super-hypergraph* $\mathrm{SuHG}^{(n)} = (V_n, E_n)$, *there exists a digraph $D$ on base $V_n$ such that* $\mathrm{CompSuHG}^{(0)}(D) = \mathrm{SuHG}^{(n)}$.

*Hence* $\mathrm{CompSuHG}^{(n)}$ *unifies both constructions in a single parameterized framework.*

**Proof. (a)** $n = 0$**.** By definition $\mathrm{POW}^0(V_0) = V_0$. Then

$$E_0^{\mathrm{comp}} = \left\{ N_0^-(v) \subseteq V_0 : |N_0^-(v)| \geqslant 2 \right\}, \quad N_0^-(v) = \{u \in V_0 : (u \to v) \in A\},$$

which is exactly the classical competition hypergraph of $D$.

**(b) Realizing any** $\mathrm{SuHG}^{(n)}$**.** Let $\mathrm{SuHG}^{(n)} = (V_n, E_n)$ be given, where $V_n \subseteq \mathrm{POW}^n(V_0)$ and $E_n \subseteq \mathrm{POW}^n(V_0)$. We define a directed graph

$$D = \left(V_n,\ A\right)$$

on base $V_n$ by introducing an arc $T \to S$ whenever $S \in E_n$ and $T \in S$. Concretely:

$$A = \{\, (T, S) : S \in E_n,\ T \in S\}.$$

Then for each $S \in V_n$,

$$N_0^-(S) \;=\; \{T \in V_n : (T \to S) \in A\} = \{T \in V_n : T \in S\} = S.$$

Hence

$$E_0^{\mathrm{comp}} = \left\{ N_0^-(S) : S \in V_n,\ |S| \geqslant 2 \right\} = \{\, S \in E_n : |S| \geqslant 2\}.$$

If some superedges in $E_n$ are singletons or empty, one can either include them by relaxing the size-threshold, or disregard them to match the usual requirement $|E| \geqslant 2$. Thus up to this minor convention, $\mathrm{CompSuHG}^{(0)}(D)$ reproduces exactly the given $\mathrm{SuHG}^{(n)}$.

Combining (a) and (b), we see that the competition $n$-Super-hypergraph construction recovers both classical competition hypergraphs (for $n = 0$)

and arbitrary $n$-Super-hypergraphs (via an appropriate choice of digraph). This completes the proof. □

**Theorem 2.6 (Isomorphism Invariance).** *If two digraphs $D = (V_0, A)$ and $D' = (V_0', A')$ are isomorphic via $\varphi : V_0 \to V_0'$ (i.e. $(u \to v) \in A \iff (\varphi(u) \to \varphi(v)) \in A'$), then for each $n \geqslant 0$, their competition $n$-Super-hypergraphs $\mathrm{CompSuHG}^{(n)}(D)$ and $\mathrm{CompSuHG}^{(n)}(D')$ are isomorphic.*

**Proof.** The isomorphism $\varphi$ induces bijections $\varphi_n : \mathrm{POW}^n(V_0) \to \mathrm{POW}^n(V_0')$ by

$$\varphi_0 = \varphi, \quad \varphi_{k+1}(S) = \{\varphi_k(T) \mid T \in S\} \quad (k \geqslant 0).$$

We claim $\varphi_n$ carries hyperedges of
$\mathrm{CompSuHG}^{(n)}(D)$ to those of $\mathrm{CompSuHG}^{(n)}(D')$.

Indeed, for any $S \in \mathrm{POW}^n(V_0)$,

$$N_n^D(S) = \{T \in \mathrm{POW}^n(V_0) \mid \exists\, u \in T,\ v \in S : (u \to v) \in A\},$$

and similarly $N_n^{D'}(\varphi_n(S))$ is defined with $A'$. By the edge-preserving property of $\varphi$,

$$(u \to v) \in A \iff (\varphi(u) \to \varphi(v)) \in A',$$

one checks $\varphi_n\big(N_n^D(S)\big) = N_n^{D'}\big(\varphi_n(S)\big)$. Since $\varphi_n$ is a bijection, it maps hyperedges $\{N_n^D(S) : |N_n^D(S)| \geqslant 2\}$ exactly onto $\{N_n^{D'}(S') : |N_n^{D'}(S')| \geqslant 2\}$.

Hence $\varphi_n$ defines an isomorphism of competition $n$-Super-hypergraphs. □

**Theorem 2.7 (Recovery of Classical Competition Hypergraph).** *For $n = 0$, the competition 0-Super-hypergraph $\mathrm{CompSuHG}^{(0)}(D)$ coincides with the standard competition hypergraph of $D$.*

**Proof.** By definition, $\mathrm{POW}^0(V_0) = V_0$. Then for each $v \in V_0$,

$$N_0^-(v) \ = \ \{u \in V_0 : (u \to v) \in A\},$$

and

$$E_0^{\text{comp}} = \{ N_0^-(v) \subseteq V_0 : |N_0^-(v)| \geqslant 2\},$$

which is exactly the collection of hyperedges in the classical competition hypergraph of $D$. Therefore $\text{CompSuHG}^{(0)}(D)$ recovers the usual construction. □

**Theorem 2.8 (Hierarchical Consistency).** *For each $n \geqslant 0$, there is a natural projection morphism $\pi_n : \text{CompSuHG}^{(n+1)}(D) \to \text{CompSuHG}^{(n)}(D)$ sending each $(n+1)$-supervertex $S \subseteq \text{POW}^n(V_0)$ to itself viewed as an element of $\text{POW}^n(V_0)$. This map preserves hyperedges in the sense that*

$$\pi_n\big(N_{n+1}^-(S)\big) = N_n^-(S) \quad \forall\, S \in \text{POW}^{n+1}(V_0).$$

**Proof.** Define $\pi_n : \text{POW}^{n+1}(V_0) \to \text{POW}^n(V_0)$ by $\pi_n(S) = S$ (viewing a set of $n$-supervertices as an $n$-supervertex itself). For each $S \in \text{POW}^{n+1}(V_0)$,

$$N_{n+1}^-(S) = \{T \in \text{POW}^{n+1}(V_0) : \exists\, u \in T,\, v \in S : (u \to v) \in A\}.$$

Applying $\pi_n$ to each $T$ yields the set of all $n$-supervertices that link into $S$, i.e. $N_n^-(S)$. Since $\pi_n$ is surjective onto $\text{POW}^n(V_0)$ and respects set-inclusion, it maps hyperedges of $\text{CompSuHG}^{(n+1)}(D)$ onto those of $\text{CompSuHG}^{(n)}(D)$. This establishes hierarchical consistency. □

**Theorem 2.9 (Monotonicity of Hyperedge Sizes).** *For any $n \geqslant 0$ and any $S \in \text{POW}^{n+1}(V_0)$,*

$$\big|N_{n+1}^-(S)\big| \;\geqslant\; \big|N_n^-(S)\big|.$$

**Proof.** By definition,

$$N_{n+1}^-(S) = \{T \subseteq \text{POW}^n(V_0) : \exists\, u \in T, v \in S : (u \to v) \in A\},$$

while

$$N_n^-(S) = \{\tau \in \mathrm{POW}^n(V_0) : \exists\, u \in \tau, v \in S : (u \to v) \in A\}.$$

Every $n$-supervertex $\tau \in N_n^-(S)$ gives at least one $(n+1)$-supervertex $T \supseteq \{\tau\}$ in $N_{n+1}^-(S)$. Since there are $2^{|\mathrm{POW}^n(V_0)|-1}$ such supersets of $\{\tau\}$, one concludes $\left|N_{n+1}^-(S)\right| \geqslant |\{\tau\}| = |N_n^-(S)|$. Hence hyperedge sizes cannot decrease as $n$ increases. □

## 3. Conclusion and future work

In this paper, we introduced the *competition super-hypergraph*, a novel extension of classical competition hypergraphs across multiple hierarchical levels. For future research, we plan to explore several avenues:

- **Graph-based extensions:** Integrate related structures such as *bidirected graphs* [30, 31].
- **Fuzzy and uncertainty models:** Develop versions using *fuzzy sets* [32-33].
- **Neutrosophic and plithogenic generalizations:** Incorporate *neutrosophic sets* [35, 36, 37] and *plithogenic sets* [6, 27, 38] to capture higher-order uncertainty and conflicting attributes.

These directions promise to enrich the theoretical framework and broaden the applicability of competition super-hypergraphs in real-world scenarios.

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