ON AN ERDÖS'S OPEN PROBLEM

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In one of his books ("Analysis…") Mr. Paul Erdös proposed the following problem:

"The integer n is called a barrier for an arithmetic function f if $m + f(m) \leq n$ for all $m < n$.

Question: Are there infinitely many barriers for $\varepsilon v(n)$, for some $\varepsilon > 0$? Here $v(n)$ denotes the number of distinct prime factors of n."

We found some results regarding this question, which results make us to conjecture that there is a finite number of barriers, for all $\varepsilon > 0$.

Let $R(n)$ be the relation: $m + \varepsilon v(m) \leq n, \ \forall m < n$.

Lemma 1. If $\varepsilon > 1$ there are two barriers only: $n = 1$ and $n = 2$ (which we call trivial barriers).

Proof. It is clear for $n = 1$ and $n = 2$ because $v(0) = v(1) = 0$.

Let's consider $n \geq 3$. Then, if $m = n - 1$ we have $m + \varepsilon v(m) \geq n - 1 + \varepsilon > n$, absurd.

Lemma 2. There is an infinity of numbers which cannot be barriers for $\varepsilon v(n)$, $\forall \varepsilon > 0$.

Proof. Let's consider $s, k \in \mathrm{N}^*$ such that $s \cdot \varepsilon > k$. We write n in the form $n = p_{i_1}^{\alpha_{i_1}} \cdots p_{i_s}^{\alpha_{i_s}} + k$, where for all j, $\alpha_{i_j} \in \mathrm{N}^*$ and p_{i_j} are positive distinct primes.

Taking $m = n - k$ we have $m + \varepsilon v(m) = n - k + \varepsilon \cdot s > n$.

But there exists an infinity of n because the parameters $\alpha_{i_1}, ..., \alpha_{i_s}$ are arbitrary in N^* and $p_{i_1}, ..., p_{i_s}$ are arbitrary positive distinct primes, also there is an infinity of couples (s, k) for an $\varepsilon > 0$, fixed , with the property $s \cdot \varepsilon > k$.

Lemma 3. For all $\varepsilon \in (0,1]$ there are nontrivial barriers for $\varepsilon v(n)$.

Proof. Let t be the greatest natural number such that $t\varepsilon \leq 1$ (always there is this t).

Let n be from $[3, ..., p_1 \cdots p_t p_{t+1})$, where p_i is the sequence of the positive primes. Then $1 \leq v(n) \leq t$.

All $n \in [1,..., p_1 \cdots p_t p_{t+1}]$ is a barrier, because: $\forall\ 1 \le k \le n-1$, if $m = n-k$ we have $m + \varepsilon v(m) \le n - k + \varepsilon \cdot t \le n$.

Hence, there are at list $p_1 \cdots p_t p_{t+1}$ barriers.

Corollary. If $\varepsilon \to 0$ then n (the number of barriers) $\to \infty$.

Lemma 4. Let's consider $n \in [1,..., p_1 \cdots p_r p_{r+1}]$ and $\varepsilon \in (0,1]$. Then: n is a barrier if and only if $R(n)$ is verified for $m \in \{ n-1, n-2, ..., n-r+1 \}$.

Proof. It is sufficient to prove that $R(n)$ is always verified for $m \le n - r$.

Let's consider $m = n - r - u$, $u \ge 0$. Then $m + \varepsilon v(m) \le n - r - u + \varepsilon \cdot r \le n$.

Conjecture.

We note $I_r \in [p_1 \cdots p_r, ..., \cdot p_1 \cdots p_r p_{r+1})$. Of course $\bigcup_{r \ge 1} I_r = \mathrm{N} \setminus \{0,1\}$, and $I_{r_1} \cap I_{r_2} = \Phi$ for $r_1 \ne r_2$.

Let $\mathrm{N}_r(1+t)$ be the number of all numbers n from I_r such that $1 \le v(n) \le t$.

We conjecture that there is a finite number of barriers for $\varepsilon v(n)$, $\forall \varepsilon > 0$; because

$$\lim_{r \to \infty} \frac{\mathrm{N}_r(1+t)}{p_1 \cdots p_{r+1} - p_1 \cdots p_r} = 0$$

and the probability (of finding of $r-1$ consecutive values for m, which verify the relation $R(n)$) goes to zero.