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Fusion of qualitative information using imprecise 2 -tuple labels

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Abstract: *In this chapter, Herrera-Martínez' 2-tuple linguistic representation model is extended for combining imprecise qualitative information using fusion rules drawn from Dezert-Smarandache Theory (DSmT) or from Dempster-Shafer Theory (DST) frameworks. The proposed approach preserves the precision and the efficiency of the combination of linguistic information. Some basic operators on imprecise 2-tuple labels are presented. We also give simple examples to show how precise and imprecise qualitative information can be combined for reasoning under uncertainty. It is concluded that DSmT can deal efficiently with both precise and imprecise quantitative and qualitative beliefs, which extends the scope of this theory.*

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8.1 Introduction

Qualitative methods for reasoning under uncertainty have gained more and more attention by Information Fusion community, especially by the researchers and system designers working in the development of modern multi-source systems for information retrieval, fusion and management in defense, in robotics and so on. This is because traditional methods based only on quantitative representation and analysis are not able to adequately satisfy the need of the development of science and technology that integrate at higher fusion levels human beliefs and reports in complex systems. Therefore qualitative knowledge representation and analysis become more and more important and necessary in next generations of decision-making support systems.

In 1954, Polya was one of the pioneers to characterize formally the qualitative human reports [15]. Then Zadeh [26–30] made important contributions in this field in proposing a fuzzy linguistic approach to model and to combine qualitative/vague information expressed in natural language. However, since the combination process highly depends on the fuzzy operators chosen, a possible issue has been pointed out by Yager in [25]. In 1994, Wellman developed Qualitative Probabilistic Networks (QPN) based on a Qualitative Probability Language, which relaxed precision in representation and reasoning within the probabilistic framework [24]. Subrahmanian introduced the annotated logics, which was a powerful formalism for classical (i.e. consistent), as well as paraconsistent reasoning in artificial intelligence [11, 22]. QPN and Annotated Logics belong actually to the family of imprecise probability [23] and probability bounds analysis (PBA) approaches [6]. Parsons proposed a Qualitative Evidence Theory (QET) with new interesting qualitative reasoning techniques but his QET unfortunately cannot deal efficiently with complex problems of qualitative information fusion encountered in real world [12–14]. Dubois and Prade proposed a Qualitative Possibility Theory (QPT) in Decision Analysis (DA) for the representation and the aggregation of preferences. QPT was driven by the principle of minimal specificity [4]. They use refined linguistic quantifiers to represent either the possibility distributions which encode a piece of imprecise knowledge about a situation, or to represent the qualitative belief masses over the elements in 2^Θ. However, the combination process might produce approximate results because of the finite probabilistic scale of the label set [5]. H*á*jek et al. in [7] proposed a Qualitative Fuzzy Possibilistic Logic (QFPL) which was used to deal with both uncertainty (possibility) and vagueness (fuzziness). QFPL is different from our qualitative reasoning in DSmT, though the propositional variables were mapped to a set of values i.e. $\{0, 1/n, 2/n, \cdots, 1\}$ similar to 1-tuple linguistic model, since it built modality-free formulas from propositional variables using connectives, i.e. $\wedge, \vee, \rightarrow, \neg$.

The purpose of this chapter is to propose a model of imprecise qualitative belief structures for solving fusion problems for applications and not to compare all previous theoretical approaches. We adopt here a pragmatic point of view in order to deal with poor and imprecise qualitative sources of information since in reality the requirement that precise labels are assigned to every individual hypotheses is often regarded as too restrictive.

Some research works on quantitative imprecise belief structures have been done at the end of nineties by Denœux who proposed a representation model in DST framework for dealing with imprecise belief and plausibility functions, imprecise pignistic probabilities together with the extension of Dempster's rule [1] for combining imprecise belief masses. Within the DSmT framework, Dezert and Smarandache further proposed new interval-valued beliefs operators and generalized DSm combination rules from precise belief structures fusion to imprecise/sub-unitary intervals fusion, and more generally, to any set of sub-unitary intervals fusion [17]. In [9], Li proposed a revised version of imprecise division operator and the *Min* and *Max* operators for imprecise belief structures, which can be applied to fuzzy-extended reasoning combination rules. Since all the extensions of belief structures proposed so far in the literature concern only *imprecise quantitative belief structures*, we introduce here for the first time a representation for *imprecise qualititative belief structures*. The representation model presented in this chapter is based on the 2-tuple linguistic labels model developed earlier [8] which offers an acceptable computational complexity by working with a finite reduced/coarse granularity set of linguistic labels [3, 19, 20]. The approach adopted here must be viewed as a particular case of the more theoretical approach based on DSm Field and Linear Algebra of Refined Labels (DSm-FLARL) proposed in Chapter 2 in this volume.

The 2-tuple linguistic labels representation allows to take into account some available richer information content (if any), like *less good*, *good enough*, *very good* which is not represented within the 1-tuple linguistic labels representation. It can be interpreted somehow as a remainder technique for linguistic labels. Actually, Herrera and Martínez in [8] were the first to propose a 2-tuple fuzzy linguistic representation model for computing with words (CW) for offering a tractable method for aggregating linguistic information (represented by linguistic variables with equidistant labels) through counting indexes of the corresponding linguistic labels. The advantages of the 2-tuple Linguistic representation of symbolic method over methods based on the extension principle in CW in term of complexity and feasibility have been shown in [8]. In 2007, Li et al. [10] have extended the 1-tuple linguistic representation model to Qualitative Enriched Labels (QEL), denoted $L_i(c_i)$, in the DSmT framework. It must be noted that QEL $L_i(c_i)$ is different from Herrera-Martínez' 2-tuple labels denoted (L_i, σ_i^h). The difference lies in the fact that σ_i^h expresses a kind of refinement correcting term of the standard linguistic label L_i, whereas c_i of QEL expresses a possible confidence factor one may have on the standard linguistic label L_i. In this work, we use Herrera-Martínez' 2-tuple linguistic representation model and introduce new operators for combining imprecise qualitative belief masses based on it.

This chapter is organized as follows: In section 8.2, we remind briefly the basis of DSmT. In section 8.3, we present some 2-tuple linguistic operators and in section 8.4 we present the fusion rules for precise and imprecise qualitative beliefs in DSmT framework. In section 8.5, we provide examples to show how these operators work for combining 2- Tuple qualitative beliefs. Concluding remarks are then given in 8.6.

8.2 DSmT for the fusion of beliefs

8.2.1 Basic belief mass

In Dempster-Shafer Theory (DST) framework [16], one considers a frame of discernment $\Theta = \{\theta_1, \ldots, \theta_n\}$ as a finite set of n exclusive and exhaustive elements (*i.e.* Shafer's model denoted $\mathcal{M}^0(\Theta)$). The power set of Θ is the set of all subsets of Θ. The cardinality of a power set , if the frame of discernment cardinality $|\Theta| = n$ is 2^n. The power set of Θ is denoted 2^Θ. For example, if $\Theta = \{\theta_1, \theta_2\}$, then $2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. In Dezert-Smarandache Theory (DSmT) framework [17, 19], one considers $\Theta = \{\theta_1, \ldots, \theta_n\}$ as a finite set of n exhaustive elements only (*i.e.* free *DSm*-model denoted $\mathcal{M}^f(\Theta)$). Eventually some integrity constraints can be introduced in this free model depending on the nature of problem we have to cope with. The hyper-power set of Θ (*i.e.* the free Dedekind's lattice) denoted D^Θ [17] is defined as:

1. $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.
2. If $A, B \in D^\Theta$, then $A \cap B$ and $A \cup B$ belong to D^Θ.
3. No other elements belong to D^Θ, except those obtained by using rules 1 or 2.

If $|\Theta| = n$, then $|D^\Theta| \leq 2^{2^n}$. Since for any finite set Θ, $|D^\Theta| \geq |2^\Theta|$, we call D^Θ the hyper-power set of Θ. For example, if $\Theta = \{\theta_1, \theta_2\}$, then $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$. The free DSm model $\mathcal{M}^f(\Theta)$ corresponding to D^Θ allows to work with vague concepts which exhibit a continuous and relative intrinsic nature. Such concepts cannot be precisely refined in an absolute interpretation because of the unreachable universal truth. The main differences between DST and DSmT frameworks are (i) the model on which one works with, (ii) the choice of the combination rule and conditioning rules [17, 19], and (iii) aside working with numerical/quantitative beliefs DSmT allows to compute directly with words (more exactly to combine qualitative belief masses as we will show in the sequel). Here we use the generic notation G^Θ for denoting either D^Θ (when working in DSmT with free DSm model) or 2^Θ (when working in DST with Shafer's model) or any other subset of D^Θ (when working with a DSm hybrid model).

From any finite discrete frame Θ, we define a quantitative basic belief assignment (bba) as a mapping $m(.) : G^\Theta \to [0, 1]$ associated to a given body of evidence $\mathcal{B}$ which satisfies

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in G^\Theta} m(A) = 1 \tag{8.1}$$

8.2.2 Fusion of quantitative beliefs

When the free *DSm* model $\mathcal{M}^f(\Theta)$ holds, the pure conjunctive consensus, called *DSm* classic rule (*DSmC*), is performed on $G^\Theta = D^\Theta$. DSmC of two independent[1]

[1] While independence is a difficult concept to define in all theories managing epistemic uncertainty, we consider that two sources of evidence are independent (i.e. distinct and

sources associated with bba's $m_1(.)$ and $m_2(.)$ is thus given by $m_{DSmC}(\emptyset) = 0$ and $\forall X \in D^\Theta$ by [17]:

$$m_{DSmC}(X) = \sum_{\substack{X_1,X_2 \in D^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1)m_2(X_2) \tag{8.2}$$

D^Θ being closed under $\cup$ and $\cap$ operators, $DSmC$ guarantees that $m(.)$ is a proper bba.

When Shafer's model holds, instead of distributing the total conflicting mass onto elements of 2^Θ proportionally with respect to their masses resulted after applying the conjunctive rule as within Demspter's rule (DS) through the normalization step [16], or transferring the partial conflicts onto partial uncertainties as within $DSmH$ rule [17], we propose to use the Proportional Conflict Redistribution rule no.5 (PCR5) [18, 19] which transfers the partial conflicting masses proportionally to non-empty sets involved in the model according to all integrity constraints. PCR5 rule works for any degree of conflict in $[0, 1]$, for any models (Shafer's model, free DSm model or any hybrid DSm model) and both in DST and DSmT frameworks for static or dynamical fusion problems. The PCR5 rule for two sources is defined by: $m_{PCR5}(\emptyset) = 0$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)}] \tag{8.3}$$

where each element X, and Y, is in the disjunctive normal form. $m_{12}(X)$ corresponds to the conjunctive consensus on X between the two sources. All denominators are different from zero. If a denominator is zero, that fraction is discarded. No matter how big or small is the conflicting mass, PCR5 mathematically does a better redistribution of the conflicting mass than Dempster's rule and other rules since PCR5 goes backwards on the tracks of the conjunctive rule and redistributes the partial conflicting masses only to the sets involved in the conflict and proportionally to their masses put in the conflict, considering the conjunctive normal form of the partial conflict. PCR5 is quasi-associative and preserves the neutral impact of the vacuous belief assignment. General PCR5 fusion formula and improvement for the combination of $k \geq 2$ sources of evidence can be found in [19] with many detailed examples.

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noninteracting) if each leaves one totally ignorant about the particular value the other will take.

8.3 Linguistic models of qualitative beliefs

8.3.1 The 1-tuple linguistic model

In order to compute qualitative belief assignments expressed by pure linguistic labels (i.e. 1-tuple linguistic representation model) over G^Θ, Smarandache and Dezert have defined in [19] a qualitative basic belief assignment $q_1m(.)$ as a mapping function from G^Θ into a set of linguistic labels $L = \{L_0, \tilde{L}, L_{n+1}\}$ where $\tilde{L} = \{L_1, \cdots, L_n\}$ is a finite set of linguistic labels and where $n \geq 2$ is an integer. For example, L_1 can take the linguistic value "poor", L_2 the linguistic value "good", etc. $\tilde{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \cdots \prec L_n$,where $\prec$ means inferior to, or less (in quality) than, or smaller than, etc. To work on a true closed linguistic set L under linguistic addition and multiplication operators, Smarandache and Dezert extended naturally $\tilde{L}$ with two extreme values $L_0 = L_{\min}$ and $L_{n+1} = L_{\max}$, where L_0 corresponds to the minimal qualitative value and L_{n+1} corresponds to the maximal qualitative value, in such a way that $L_0 \prec L_1 \prec L_2 \prec \cdots \prec L_n \prec L_{n+1}$. In the sequel $L_i \in L$ are assumed linguistically equidistant labels such that we can make an isomorphism ϕ_L between $L = \{L_0, L_1, L_2, \ldots, L_n, L_{n+1}\}$ and $\{0, 1/(n+1), 2/(n+1), \ldots, n/(n+1), 1\}$, defined as $\phi_L(L_i) = i/(n+1)$ for all $i = 0, 1, 2, \ldots, n, n+1$.

From the extension of the isomorphism between the set of linguistic equidistant labels and a set of numbers in the interval $[0, 1]$, one can built exact operators on linguistic labels which makes possible the extension of all quantitative fusion rules into their qualitative counterparts [10]. We briefly remind the basic (approximate) qualitative operators[2] (or q-operators for short) on (1-tuple) linguistic labels:

- q-addition:
$$L_i + L_j = \begin{cases} L_{i+j} & \text{if } i+j < n+1, \\ L_{n+1} = L_{\max} & \text{if } i+j \geq n+1. \end{cases} \tag{8.4}$$
The q-addition is an extension of the addition operator on equidistant labels which is given by $L_i + L_j = \frac{i}{n+1} + \frac{j}{n+1} = \frac{i+j}{n+1} = L_{i+j}$.

- q-subtraction:
$$L_i - L_j = \begin{cases} L_{i-j} & \text{if} \quad i \geq j, \\ -L_{j-i} & \text{if} \quad i < j. \end{cases} \tag{8.5}$$
where $-L = \{-L_1, -L_2, \ldots, -L_n, -L_{n+1}\}$. The q-subtraction is justified since when $i \geq j$, one has with equidistant labels $L_i - L_j = \frac{i}{n+1} - \frac{j}{n+1} = \frac{i-j}{n+1}$.

- q-multiplication[3]:
$$L_i \cdot L_j = L_{[(i \cdot j)/(n+1)]}. \tag{8.6}$$

[2]more approximate q-operators can be found in [3] and new accurate operators are introduced in Chapter 2 of this volume.

[3]The q-multiplication of two linguistic labels defined here can be extended directly to the multiplication of $n > 2$ linguistic labels. For example the product of three linguistic label will be defined as $L_i \cdot L_j \cdot L_k = L_{[(i \cdot j \cdot k)/(n+1)(n+1)]}$, etc.

where $[x]$ means the closest integer[4] to x (with $[n+0.5] = n+1$, $\forall n \in \mathbb{N}$). This operator is justified by the approximation of the product of equidistant labels given by $L_i \cdot L_j = \frac{i}{n+1} \cdot \frac{j}{n+1} = \frac{(i \cdot j)/(n+1)}{n+1}$. A simpler approximation of the multiplication, but less accurate (as proposed in [19]) is thus

$$L_i \times L_j = L_{\min\{i,j\}} \tag{8.7}$$

- Scalar multiplication of a linguistic label: Let a be a real number. The multiplication of a linguistic label by a scalar is defined by:

$$a \cdot L_i = \frac{a \cdot i}{n+1} \approx \begin{cases} L_{[a \cdot i]} & \text{if } [a \cdot i] \geq 0, \\ L_{-[a \cdot i]} & \text{otherwise.} \end{cases} \tag{8.8}$$

- Division of linguistic labels:

  a) q-division as an internal operator: Let $j \neq 0$, then

$$L_i / L_j = \begin{cases} L_{[(i/j) \cdot (n+1)]} & \text{if} [(i/j) \cdot (n+1)] < n+1, \\ L_{n+1} & \text{otherwise.} \end{cases} \tag{8.9}$$

  The first equality in (8.9) is well justified because with equidistant labels, one gets: $L_i / L_j = \frac{i/(n+1)}{j/(n+1)} = \frac{(i/j) \cdot (n+1)}{n+1} \approx L_{[(i/j) \cdot (n+1)]}$.

  b) Division as an external operator: $\oslash$. Let $j \neq 0$. We define:

$$L_i \oslash L_j = i/j. \tag{8.10}$$

  since for equidistant labels $L_i \oslash L_j = (i/(n+1))/(j/(n+1)) = i/j$.

From the q-operators we now can easily and directly extend all quantitative fusion rules like *DSmC* or PCR5 (8.2) or (8.3) into their qualitative version by replacing classical operators on numbers with linguistic labels defined above. Many detailed examples can be found in [3, 10, 18, 19].

8.3.2 The 1-tuple linguistic enriched model

In order to keep working with a coarse/reduced set of linguistic labels for maintaining a low computational complexity, but for taking into account the confidence one may have on the label value declared by a source, we proposed in [10] a qualitative enriched linguistic representation model denoted by $L_i(c_i)$, where the first component L_i is a classical linguistic label and the second component c_i is an assessment (confidence) value. c_i can be either a numerical supporting degree[5] in

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[4] When working with labels, no matter how many operations we have, the best (most accurate) result is obtained if we do only one approximation, and that one should be just at the very end.

[5] In our previous publication [10], we considered $c_i \in [0, \infty)$ but it seems more natural to take it actually in $[0, 1]$ as in statistics.

$[0,1]$ or a qualitative supporting degree taken its value in a given (ordered) set X of linguistic labels. When $c_i \in [0,1]$, $L_i(c_i)$ is called an enriched label of type 1, whereas when $\alpha_i \in X$, $L_i(c_i)$ is called an enriched label of type 2. The (quantitative or qualitative) value c_i characterizes the confidence weight one has when the source declares label L_i for committing its qualitative belief to a given proposition $A \in G^\Theta$. For example with enriched labels of type 1, if the label $L_1 \triangleq L_1(1)$ represents our full confidence in the linguistic variable *Good* declared by the source, $L_1(0.7)$ means than we are a bit less confident (i.e. 70% confident only) in the declaration *Good* provided by the source, etc. With enriched labels of type 2, if one chooses by example $X = \{SC, MC, HC\}$, where elements of X have the following meaning: $SC \triangleq$ "Small Condidence", $MC \triangleq$ "Medium Confidence" and $FC \triangleq$"Full Confidence", then the enriched label $L_1 \triangleq L_1(FC)$ represents linguistic variable *Good* with the full confidence we grant in this declaration (similarly as $L_1(1)$ for type 1), etc. In [10], we have shown how to work (i.e. how to define new *qe*-operators) and how to combine qualitative beliefs based on this enriched linguistic representation model. The computations are based on an independent derivation mechanism of the 1st and 2nd components of the enriched labels $L_i(c_i)$ because the label L_i and its confidence factor c_i, $i = 1, \dots, n$ do not carry the same intrinsic nature of information.

Herrera-Martínez' approach (i.e. the 2-tuple linguistic model) presented in the next section is totally different as it will be shown. In the 2-tuple linguistic model, one tries to refine the value of the labels in order to deal with a richer/finer information but without regards to the confidence one may have on the (refined/2-tuple) labels. Of course, the enrichment of 2-tuple labels can be easily done following ideas presented in [10].

8.3.3 The precise 2-tuple linguistic model

Herrera and Martínez' (precise) 2-tuple model has been introduced in detail in [8]. Here we denote this model (L_i, σ_i^h) where σ_i^h is chosen in $\mathbf{\Sigma} \triangleq [-0.5/(n+1), 0.5/(n+1)), i \in \{1, \cdots, \infty\}$. The 2-tuple model can be justified since each distance between two equidistant labels is $1/(n+1)$ because of the isomorphism between L and $\{0, 1/(n+1), \dots, n/(n+1), 1\}$, so that $L_i = i/(n+1)$ for all $i = 0, 1, 2, \dots, n, n+1$. Therefore, we take half to the left and half to the right of each label, i.e. $\sigma_i^h \in \mathbf{\Sigma}$. So a 2-tuple equidistant linguistic representation model is used to represent the linguistic information by means of 2-tuple item set $Ł(L, \sigma^h)$ with $L = \{L_0, L_1, L_2, \dots, L_n, L_{n+1}\}$ isomorphic to $\{0, 1/(n+1), 2/(n+1), \dots, n/(n+1), 1\}$ and the set of qualitative assessments isomorphic to $\mathbf{\Sigma}$. This 2-tuple approach is an intricate/hybrid mechanism of derivation using jointly L_i and σ_i^h where σ_i^h is a positive or negative numerical remainder with respect to the labels.

8.3.3.1 Symbolic translation

Let's define the normalized index[6] $i = round((n+1)\times\beta) = [(n+1)\times\beta]$, with $i \in [0,(n+1)]$ and $\beta \in [0,1]$, and the Symbolic Translation $\sigma^h \triangleq \beta - i/(n+1) \in [-0.5/(n+1), 0.5/(n+1))$. Roughly speaking, the Herrera-Martínez symbolic translation of an assessment linguistic value $(n+1)\times\sigma_i^h$ is a numerical value that supports the difference of information between the (normalized) index obtained from the fusion rule and its closest value in $\{0,1,\ldots,n+1\}$.

8.3.3.2 Herrera-Martínez transformations

- $\triangle(.)$: conversion of a numerical value into a 2-tuple

$\triangle(.) : [0,1] \to L \times \mathbf{\Sigma}$ is defined by [8]

$$\triangle(\beta) = (L_i, \sigma^h) \triangleq \begin{cases} L_i, & i = round((n+1)\cdot\beta) \\ \sigma^h = \beta - i/(n+1), & \sigma^h \in \mathbf{\Sigma} \end{cases} \tag{8.11}$$

Thus L_i has the closest index label to β and σ^h is the value of its symbolic translation.

- $\nabla(.)$: conversion of a 2-tuple into a numerical value

The inverse/dual function of $\triangle(.)$ is denoted $\nabla(.)$ and $\nabla(.) : L \times \mathbf{\Sigma} \to [0,1]$ is defined by

$$\nabla((L_i, \sigma_i^h)) = i/(n+1) + \sigma_i^h = \beta_i \tag{8.12}$$

8.3.3.3 Main operators on 2-tuples

Let's consider two 2-tuples (L_i, σ_i^h) and (L_j, σ_j^h), then the following operators are defined as follows.

- Addition of 2-tuples

$$\begin{aligned}(L_i,\sigma_i^h) + (L_j,\sigma_j^h) &\equiv \nabla((L_i,\sigma_i^h) + (L_j,\sigma_j^h)) \\ &= \nabla((L_i,\sigma_i^h)) + \nabla((L_j,\sigma_j^h)) = \beta_i + \beta_j = \beta_z \\ &= \begin{cases} \triangle(\beta_z) & \text{if} \quad \beta_z \in [0,1] \\ L_{n+1} & \text{otherwise} \end{cases}\end{aligned} \tag{8.13}$$

- Product of 2-tuples

$$\begin{aligned}(L_i,\sigma_i^h) \times (L_j,\sigma_j^h) &\equiv \nabla((L_i,\sigma_i^h) \times (L_j,\sigma_j^h)) \\ &= \nabla((L_i,\sigma_i^h)) \times \nabla((L_j,\sigma_j^h)) = \beta_i \times \beta_j = \beta_p \equiv \triangle(\beta_p)\end{aligned} \tag{8.14}$$

with $\beta_p \in [0,1]$. It can be proved that 2-tuple addition and product operators are commutative and associative.

[6]where *round(.)* is the *rounding* operation denoted [.] in our previous q-operators [10].

- Scalar multiplication of a 2-tuple

$$\alpha \cdot (L_i, \sigma_i^h) \equiv \nabla(\alpha \cdot (L_i, \sigma_i^h)) = \alpha \cdot \nabla((L_i, \sigma_i^h))$$
$$= \alpha \cdot \beta_i = \beta_\gamma \equiv \begin{cases} \triangle(\beta_\gamma) & \beta_\gamma \in [0,1] \\ L_{n+1} & \text{otherwise} \end{cases} \quad (8.15)$$

- Division of a 2-tuple by a 2-tuple

Let's consider two 2-tuples (L_i, σ_i^h) and (L_j, σ_j^h) with[7] $(L_i, \sigma_i^h) < (L_j, \sigma_j^h)$, then the division is defined as

$$\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)} \equiv \nabla(\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)}) = \frac{\nabla((L_i, \sigma_i^h))}{\nabla((L_j, \sigma_j^h))}$$
$$= \frac{\beta_i}{\beta_j} = \beta_d \equiv \triangle(\beta_d) \quad \text{with } \beta_d \in [0,1] \quad (8.16)$$

If $(L_i, \sigma_i^h) \geq (L_j, \sigma_j^h)$, then

$$\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)} \equiv \nabla(\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)}) = \frac{\nabla((L_i, \sigma_i^h))}{\nabla((L_j, \sigma_j^h))} = \frac{\beta_i}{\beta_j} \geq 1$$

and in such case $\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)}$ is set to the maximum label, i.e. $\frac{(L_i, \sigma_i^h)}{(L_j, \sigma_j^h)} = (L_{n+1}, 0) \sim L_{n+1}$.

8.3.4 The imprecise 2-tuple linguistic model

Since qualitative belief assignment might be imprecise by expert on some occasions, in order to further combine this imprecise qualitative information, we introduce operators on imprecise 2-tuple labels (i.e. addition, subtraction, product and division, etc.). The definition adopted here is the qualitative extension of the one proposed by Denœux' in [1] for reasoning with (quantitative) Interval-valued Belief Structures (IBS).

Definition 1 (IQBS): Let $\mathcal{L}_{G^\Theta}$ denotes the set of all qualitative belief structures (i.e. precise and imprecise) over G^Θ. An imprecise qualitative belief structure (IQBS) is defined as a non-empty subset $\mathbf{m}$ from $\mathcal{L}_{G^\Theta}$, such that there exist n subsets $F_1, \cdots, F_n$ over G^Θ and n qualitative intervals $[a_i, b_i]$, $1 \leq i \leq n$ (with $L_0 \leq a_i \leq b_i \leq L_{n+1}$) such that

$$\mathbf{m} = \{m \in \mathcal{L}_{G^\Theta} \mid a_i \leq m(F_i) \leq b_i, 1 \leq i \leq n,$$
$$\text{and } m(A) = (L_0, 0), \forall A \notin \{F_1, \ldots, F_n\}\}$$

[7]The comparison operator is defined in [8].

Proposition 1: A necessary and sufficient condition for $\mathbf{m}$ to be non-empty is that $L_0 \leq \sum_{i=1}^{n} a_i \leq L_{n+1} \leq \sum_{i=1}^{n} b_i$ because there should be at least a qualitative value $c_i \in [a_i, b_i]$, for each i, such that $\sum_{i=1}^{n} c_i = L_{n+1}$, i.e. the condition of qualitative normalization of $m(.)$. This is an extension of Denœux' proposition [1].

In order to combine imprecise qualitative belief structures, we use the operations on sets proposed by Dezert and Smarandache in [2].

8.3.4.1 Addition of imprecise 2-tuple labels

The addition operator is very important in most of combination rules for fusing information in most of belief functions theories (in DST framework, in Smets' Transferable Belief Model (TBM) [21] as well as in DSmT framework). The addition operator for imprecise 2-tuple labels (since every imprecise mass of belief is represented here qualitatively by a 2-tuple label) is defined by:

$$\mathbf{m}_1 \boxplus \mathbf{m}_2 = \mathbf{m}_2 \boxplus \mathbf{m}_1 \triangleq \{x \mid x = s_1 + s_2, s_1 \in \mathbf{m}_1, s_2 \in \mathbf{m}_2\} \tag{8.17}$$

where the symbol $+$ means the addition operator on labels and with

$$\begin{cases} \inf(\mathbf{m}_1 + \mathbf{m}_2) = \inf(\mathbf{m}_1) + \inf(\mathbf{m}_2) \\ \sup(\mathbf{m}_1 + \mathbf{m}_2) = \sup(\mathbf{m}_1) + \sup(\mathbf{m}_2) \end{cases}$$

Special case: if a source of evidence supplies precise information, i.e. $\mathbf{m}$ is a precise 2-tuple, say (L_k, α_k^h), then

$$(L_k, \sigma_k^h) \boxplus \mathbf{m}_2 = \mathbf{m}_2 \boxplus (L_k, \sigma_k^h) = \{x \mid x = (L_k, \sigma_k^h) + s_2, s_2 \in \mathbf{m}_2\} \tag{8.18}$$

with

$$\begin{cases} \inf((L_k, \sigma_k^h) + \mathbf{m}_2) = (L_k, \sigma_k^h) + \inf(\mathbf{m}_2) \\ \sup((L_k, \sigma_k^h) + \mathbf{m}_2) = (L_k, \sigma_k^h) + \sup(\mathbf{m}_2) \end{cases}$$

Example: if 9 labels are used, i.e. $n = 9$,

$$[(L_1, 0.01), (L_3, 0.02)] \boxplus [(L_2, 0.02), (L_5, 0.03)] = [(L_3, 0.03), (L_9, -0.05)]$$

$$L_3 \boxplus [(L_2, 0.02), (L_5, 0.03)] = [(L_5, 0.02), (L_8, 0.03))$$

8.3.4.2 Subtraction of imprecise 2-tuple labels

The subtraction operator is defined as follows:

$$\mathbf{m}_1 \boxminus \mathbf{m}_2 \triangleq \{x \mid x = s_1 - s_2, s_1 \in \mathbf{m}_1, s_2 \in \mathbf{m}_2\} \tag{8.19}$$

where the symbol $-$ represents the subtraction operator on labels and with

$$\begin{cases} \inf(\mathbf{m}_1 - \mathbf{m}_2) = \inf(\mathbf{m}_1) - \sup(\mathbf{m}_2) \\ \sup(\mathbf{m}_1 - \mathbf{m}_2) = \sup(\mathbf{m}_1) - \inf(\mathbf{m}_2) \end{cases}$$

When $\sup(\mathbf{m}_1 - \mathbf{m}_2) \leq (L_0, 0)$, one takes $\mathbf{m}_1 \boxminus \mathbf{m}_2 = (L_0, 0)$; If $\inf(\mathbf{m}_1 - \mathbf{m}_2) \leq (L_0, 0), \sup(\mathbf{m}_1 - \mathbf{m}_2) \geq (L_0, 0)$, then $\mathbf{m}_1 \boxminus \mathbf{m}_2 = [(L_0, 0), \sup(\mathbf{m}_1 - \mathbf{m}_2)]$; Otherwise, $\mathbf{m}_1 \boxminus \mathbf{m}_2 = [\inf(\mathbf{m}_1 - \mathbf{m}_2), \sup(\mathbf{m}_1 - \mathbf{m}_2)]$.

Special case: if one of sources of evidence supplies precise information, i.e. **m** is a precise 2-tuple, say (L_k, α_k^h), then

$$(L_k, \sigma_k^h) \boxminus \mathbf{m}_2 = \{x \mid x = (L_k, \sigma_k^h) - s_2, s_2 \in \mathbf{m}_2\} \tag{8.20}$$

with

$$\begin{cases} \inf((L_k, \sigma_k^h) - \mathbf{m}_2) = (L_k, \sigma_k^h) - \sup(\mathbf{m}_2) \\ \sup((L_k, \sigma_k^h) - \mathbf{m}_2) = (L_k, \sigma_k^h) - \inf(\mathbf{m}_2) \end{cases}$$

Similarly,

$$\mathbf{m}_1 \boxminus (L_k, \sigma_k^h) = \{x \mid x = s_1 - (L_k, \sigma_k^h), s_1 \in \mathbf{m}_1\} \tag{8.21}$$

with

$$\begin{cases} \inf(\mathbf{m}_1 - (L_k, \sigma_k^h)) = \inf(\mathbf{m}_1) - (L_k, \sigma_k^h) \\ \sup(\mathbf{m}_1 - (L_k, \sigma_k^h)) = \sup(\mathbf{m}_1) - (L_k, \sigma_k^h) \end{cases}$$

Example: if 9 labels are used, i.e. $n = 9$,

$$[(L_2, 0.02), (L_5, 0.03)] \boxminus [(L_1, 0.01), (L_3, 0.02)] = [(L_0, 0), (L_4, 0.02)]$$

$$[(L_1, 0.01), (L_3, 0.02)] \boxminus (L_5, 0.03) = (L_0, 0)$$

$$L_3 \boxminus [(L_2, 0.02), (L_5, 0.03)] = [(L_0, 0), (L_1, -0.02)]$$

8.3.4.3 Multiplication of imprecise 2-tuple labels

The multiplication operator plays also an important role in most of the rules of combinations. The multiplication of imprecise 2-tuple labels is defined as follows:

$$\mathbf{m}_1 \boxdot \mathbf{m}_2 = \mathbf{m}_2 \boxdot \mathbf{m}_1 \triangleq \{x \mid x = s_1 \times s_2, s_1 \in \mathbf{m}_1, s_2 \in \mathbf{m}_2\} \tag{8.22}$$

where the symbol $\times$ represents the multiplication operator on labels and with

$$\begin{cases} \inf(\mathbf{m}_1 \times \mathbf{m}_2) = \inf(\mathbf{m}_1) \times \inf(\mathbf{m}_2) \\ \sup(\mathbf{m}_1 \times \mathbf{m}_2) = \sup(\mathbf{m}_1) \times \sup(\mathbf{m}_2) \end{cases}$$

Special case: if one of sources of evidence supplies precise information, i.e. **m** is a precise 2-tuple, say (L_k, α_k^h), then

$$(L_k, \sigma_k^h) \boxdot \mathbf{m}_2 = \mathbf{m}_2 \boxdot (L_k, \sigma_k^h) = \{x \mid x = (L_k, \sigma_k^h) \times s_2, s_2 \in \mathbf{m}_2\}$$

with

$$\begin{cases} \inf((L_k, \sigma_k^h) \times \mathbf{m}_2) = (L_k, \sigma_k^h) \times \inf(\mathbf{m}_2) \\ \sup((L_k, \sigma_k^h) \times \mathbf{m}_2) = (L_k, \sigma_k^h) \times \sup(\mathbf{m}_2) \end{cases}$$

Example: if 9 labels are used, i.e. $n = 9$,

$$[(L_1, 0.01), (L_3, 0.02)] \boxdot [(L_2, 0.02), (L_5, 0.03)] = [(L_0, 0.0242), (L_2, -0.0304)]$$

$$L_3 \boxdot [(L_2, 0.02), (L_5, 0.03)] = [(L_1, -0.034), (L_2, -0.041)]$$

8.3.4.4 Division of imprecise 2-tuple labels

The division operator is also necessary in some combinations rules (like in Dempster's rule or PCR5 by example). So we propose the following division operator for imprecise 2-tuple labels based on division of sets introduced in [2]:

If $\mathbf{m}_2 \neq (L_0, 0)$, then

$$\mathbf{m}_1 \boxslash \mathbf{m}_2 \triangleq \{x \mid x = s_1 \div s_2, s_1 \in \mathbf{m}_1, s_2 \in \mathbf{m}_2\} \tag{8.23}$$

where the symbol $\div$ represents the division operator on labels and with

$$\begin{cases} \inf(\mathbf{m}_1 \div \mathbf{m}_2) = \inf(\mathbf{m}_1) \div \sup(\mathbf{m}_2) \\ \sup(\mathbf{m}_1 \div \mathbf{m}_2) = \sup(\mathbf{m}_1) \div \inf(\mathbf{m}_2) \end{cases}$$

when $\sup(\mathbf{m}_1) \div \inf(\mathbf{m}_2) \leq L_{n+1}$. Otherwise we take $\sup(\mathbf{m}_1 \div \mathbf{m}_2) = L_{n+1}$.

Special case: if one of sources of evidence supplies precise information, i.e. $\mathbf{m}$ is a precise 2-tuple, say $(L_k, \alpha_k^h) \neq (L_0, 0)$, then

$$(L_k, \sigma_k^h) \boxslash \mathbf{m}_2 = \{x \mid x = (L_k, \sigma_k^h) \div s_2, s_2 \in \mathbf{m}_2\} \tag{8.24}$$

with

$$\begin{cases} \inf((L_k, \sigma_k^h) \div \mathbf{m}_2) = (L_k, \sigma_k^h) \div \sup(\mathbf{m}_2) \\ \sup((L_k, \sigma_k^h) \div \mathbf{m}_2) = (L_k, \sigma_k^h) \div \inf(\mathbf{m}_2) \end{cases}$$

Similarly,

$$\mathbf{m}_1 \boxslash (L_k, \sigma_k^h) = \{x \mid x = s_1 \div (L_k, \sigma_k^h), s_1 \in \mathbf{m}_1\} \tag{8.25}$$

with

$$\begin{cases} \inf(\mathbf{m}_1 \div (L_k, \sigma_k^h)) = \inf(\mathbf{m}_2) \div (L_k, \sigma_k^h) \\ \sup(\mathbf{m}_1 \div (L_k, \sigma_k^h)) = \sup(\mathbf{m}_2) \div (L_k, \sigma_k^h) \end{cases}$$

Example: if 9 labels are used, i.e. $n = 9$,

$$[(L_1, 0.01), (L_3, 0.02)] \boxslash [(L_2, 0.02), (L_5, 0.03)] = [(L_2, 0.0075), (L_{10}, 0)]$$

$$L_3 \boxslash [(L_2, 0.02), (L_5, 0.03)] = [(L_6, -0.034), (L_{10}, 0)]$$

$$[(L_2, 0.02), (L_5, 0.03)] \boxslash L_3 = [(L_7, 0.033), (L_{10}, 0)]$$

8.4 Fusion of qualitative beliefs

8.4.1 Fusion of precise qualitative beliefs

From the 2-tuple linguistic representation model of qualitative beliefs and the previous operators on 2-tuple labels, we are now able to extend the $DSmC$, $PCR5$ and even Dempster's (DS) fusion rules into the qualitative domain following the track of our previous works [3, 10, 19]. We denote $q_2m(\cdot)$ the qualitative belief mass/assignment (qba) based on 2-tuple representation in order to make a difference with the qba $q_1m(\cdot)$ based on 1-tuple (classical/pure) linguistic labels and $q_em(\cdot)$ based on qualitative enriched linguistic labels [10]. Mathematically, $q_2m(\cdot)$ expressed by a given source/body of evidence S is defined as a mapping function $q_2m(\cdot)$: $G^\Theta \to L \times \alpha$ such that:

$$q_2m(\emptyset) = (L_0, 0) \quad \text{and} \quad \sum_{A \in G^\Theta} q_2m(A) = (L_{n+1}, 0) \tag{8.26}$$

From the expressions of quantitative $DSmC$ (8.2), $PCR5$ (8.3) and Dempster's (DS) [16] fusion rules and from the operators on 2-tuple labels, we can define the classical qualitative combination or proportional redistribution rules (q_2DSmC and q_2PCR5) for dealing with 2-tuple linguistic labels (L_i, α_i). This is done as follows:

- when working with the free DSm model of the frame Θ: $q_2m_{DSmC}(\emptyset) = (L_0, 0)$ and $\forall X \in D^\Theta \setminus \{\emptyset\}$

$$q_2m_{DSmC}(X) = \sum_{\substack{X_1, X_2 \in D^\Theta \\ X_1 \cap X_2 = X}} q_2m_1(X_1)q_2m_2(X_2) \tag{8.27}$$

- when working with Shafer's or hybrid model of the frame Θ: $q_2m_{PCR5}(\emptyset) = (L_0, 0)$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$q_2m_{PCR5}(X) = q_2m_{12}(X) + \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{q_2m_1(X)^2 q_2m_2(Y)}{q_2m_1(X) + q_2m_2(Y)} + \frac{q_2m_2(X)^2 q_2m_1(Y)}{q_2m_2(X) + q_2m_1(Y)}] \tag{8.28}$$

  where $q_2m_{12}(X)$ corresponds to the qualitative conjunctive consensus.

It is important to note that addition, product and division operators involved in formulas (8.27) and (8.28) are 2-tuple operators defined in the previous section. These rules can be easily extended for the qualitative fusion of $k > 2$ sources of evidence. The formulas (8.27) and (8.28) are well justified since every 2-tuple (L_i, α_i) can be mapped into a unique β numerical value corresponding to it which makes the qualitative fusion rules q_2DSmC and q_2PCR5 equivalent to the corresponding numerical fusion rules $DSmC$ and $PCR5$ because of the existence of $\Delta(.)$ transformation.

Theorem 1: (Normalization) If $\sum_{A \in G^\Theta} q_2m(A) = (L_{n+1}, 0)$, then $\sum_{A \in G^\Theta} q_2m_{DSmC}(A) = (L_{n+1}, 0)$, and $\sum_{A \in G^\Theta} q_2m_{PCR5}(A) = (L_{n+1}, 0)$.

Proof: Let's assume that there is a frame of discernment Θ which includes several focal elements. According to DSm model, one defines its hyper-power set

D^Θ, $A_i \in D^\Theta$, $i = \{1, 2, \cdots, n\}$. There exist k evidential sources with qualitative belief mass $a_{ij}, i \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, n\}$. According to the premise, i.e. $\sum_{A \in G^\Theta} q_2m(A) = (L_{n+1}, 0)$, that is, $\sum_{j \in \{1,2,\ldots,n\}} a_{ij} = (L_{n+1}, 0)$. According to (8.14) and the characteristics of Product operator,

$$\prod_{i \in \{1,2,\ldots,k\}} \sum_{j \in \{1,2,\ldots,n\}} a_{ij} = \prod_{i \in \{1,2,\ldots,k\}} (L_{n+1}, 0) = (L_{n+1}, 0)$$

because

$$\begin{aligned} q_2m_{DSmC}(X) &= \sum_{\substack{X_1, X_2, \cdots X_k \in D^\Theta \\ X_1 \cap X_2 \cdots X_k = X}} q_2m_1(X_1) q_2m_2(X_2) \cdots q_2m_k(X_k) \\ &= \prod_{i \in \{1,2,\ldots,k\}} \sum_{j \in \{1,2,\ldots,n\}} a_{ij} = (L_{n+1}, 0). \end{aligned}$$

Moreover, since $qPCR5$ redistributes proportionally the partial conflicting mass to the elements involved in the partial conflict by considering the canonical form of the partial conflict, the total sum of all qualitative belief mass after redistribution doesn't change and therefore it is equal to $(L_{n+1}, 0)$. This completes the proof.

Similarly, Dempster's rule (DS) can be extended for dealing with 2-tuple linguistic labels by taking $q_2m_{DS}(\emptyset) = (L_0, 0)$ and $\forall A \in 2^\Theta \setminus \{\emptyset\}$

$$q_2m_{DS}(A) = \frac{\sum_{\substack{X, Y \in 2^\Theta \\ X \cap Y = A}} q_2m_1(X) q_2m_2(Y)}{(L_{n+1}, 0) - \sum_{\substack{X, Y \in 2^\Theta \\ X \cap Y = \emptyset}} q_2m_1(X) q_2m_2(Y)} \tag{8.29}$$

8.4.2 Fusion of imprecise qualitative beliefs

Let's consider k sources of evidences providing imprecise qualitative belief assignments/masses $\mathbf{m}_{ij}$ defined on G^Θ with $|G^\Theta| = d$. We denote by m_{ij} central value of the label provided by the sourc no. i $(1 \leq i \leq k)$ for the element $X_j \in G^\Theta$, $1 \leq j \leq d$. For example with qualitative interval-valued beliefs, $\mathbf{m}_{ij} = [m_{ij} - \epsilon_{ij}, m_{ij} + \epsilon_{ij}] \in [(L_0, 0), (L_{n+1}, 0)]$, where $(L_0, 0) \leq \epsilon_{ij} \leq L_{n+1}$. More generally, $\mathbf{m}_{ij}$ can be either an union of open intervals, or of closed intervals, or of semi-open intervals.

The set of imprecise qualitative belief masses provided by the sources of evidences can be represented/characterized by the following belief mass matrices with

$$\inf(\mathbf{M}) = \begin{bmatrix} m_{11} - \epsilon_{11} & m_{12} - \epsilon_{12} & \cdots & m_{1d} - \epsilon_{1d} \\ m_{21} - \epsilon_{21} & m_{22} - \epsilon_{22} & \cdots & m_{2d} - \epsilon_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ m_{k1} - \epsilon_{k1} & m_{k2} - \epsilon_{k2} & \cdots & m_{kd} - \epsilon_{kd} \end{bmatrix}$$

$$\sup(\mathbf{M}) = \begin{bmatrix} m_{11}+\epsilon_{11} & m_{12}+\epsilon_{12} & \cdots & m_{1d}+\epsilon_{1d} \\ m_{21}+\epsilon_{21} & m_{22}+\epsilon_{22} & \cdots & m_{2d}+\epsilon_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ m_{k1}+\epsilon_{k1} & m_{k2}+\epsilon_{k2} & \cdots & m_{kd}+\epsilon_{kd} \end{bmatrix}$$

All the previous qualitative fusion rules working with precise 2-tuple labels can be extended directly for dealing with imprecise 2-tuple labels by replacing precise operators on 2-tuple labels by their counterparts for imprecise 2-tuple labels as defined in section 8.5. We just here present the extensions of $DSmC$, $PCR5$ and DS rules of combinations. The extensions of other combination rules (DSmH, Dubois & Prade's, Yager's, etc) can be done easily in a similar way and will not be reported here.

- **The DSmC fusion of imprecise qualitative beliefs**
  The DSm classical rule of combination of $k \geq 2$ imprecise qualitative beliefs is defined for the free DSm model of the frame Θ, i.e. $G^\Theta = D^\Theta$ as follows: $q_2m^I_{DSmC}(\emptyset) = (L_0, 0)$ and $\forall X \in D^\Theta \setminus \{\emptyset\}$

$$q_2\mathbf{m}^I_{DSmC}(X) = \sum_{\substack{X_1,X_2,\cdots,X_k \in D^\Theta \\ X_1 \cap X_2,\cdots,\cap X_k = X}} \prod_{i=1}^{k} q_2\mathbf{m}_i(X_i) \quad (8.30)$$

- **The PCR5 fusion of imprecise qualitative beliefs**
  When working with Shafer's or DSm hybrid models of the frame Θ, the $PCR5$ rule of combination of two imprecise qualitative beliefs is defined by: $q_2m^I_{PCR5}(\emptyset) = (L_0, 0)$ and $\forall X \in G^\Theta \setminus \{\emptyset\}$

$$q_2\mathbf{m}^I_{PCR5}(X) = q_2\mathbf{m}^I_{12}(X) + \sum_{\substack{Y \in G^\Theta \setminus \{X\} \\ X \cap Y = \emptyset}} [\frac{q_2\mathbf{m}_1(X)^2 q_2\mathbf{m}_2(Y)}{q_2\mathbf{m}_1(X) + q_2\mathbf{m}_2(Y)} + \frac{q_2\mathbf{m}_2(X)^2 q_2\mathbf{m}_1(Y)}{q_2\mathbf{m}_2(X) + q_2\mathbf{m}_1(Y)}] \quad (8.31)$$

  where $q_2\mathbf{m}^I_{12}(X)$ corresponds to the imprecise qualitative conjunctive consensus defined by

$$q_2\mathbf{m}^I_{12}(X) = \sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \cap X_2 = X}} q_2\mathbf{m}_1(X_1) q_2\mathbf{m}_2(X_2) \quad (8.32)$$

- **Dempster's fusion of imprecise qualitative beliefs**
  Dempster's rule can also be directly extended for dealing with imprecise qualitative beliefs by taking $q_2\mathbf{m}_{DS}(\emptyset) = (L_0, 0)$ and $\forall A \in 2^\Theta \setminus \{\emptyset\}$

$$q_2\mathbf{m}_{DS}^I(A) = \frac{\sum\limits_{\substack{X,Y\in 2^\Theta\\ X\cap Y=A}} q_2\mathbf{m}_1(X)q_2\mathbf{m}_2(Y)}{(L_{n+1},0) - \sum\limits_{\substack{X,Y\in 2^\Theta\\ X\cap Y=\emptyset}} q_2\mathbf{m}_1(X)q_2\mathbf{m}_2(Y)} \tag{8.33}$$

Theorem 2: The following equality holds

$$q_2\mathbf{m}_{DSmC}^I(X) = [\inf(q_2\mathbf{m}_{DSmC}^I(X)), \sup(q_2\mathbf{m}_{DSmC}^I(X))]$$

with

$$\inf(q_2\mathbf{m}_{DSmC}^I(X)) = \sum_{\substack{X_1,X_2,\cdots,X_k\in D^\Theta\\ X_1\cap X_2,\cdots,\cap X_k=X}} \prod_{i=1}^{k} \inf(q_2\mathbf{m}_i(X_i))$$

$$\sup(q_2\mathbf{m}_{DSmC}^I(X)) = \sum_{\substack{X_1,X_2,\cdots,X_k\in D^\Theta\\ X_1\cap X_2,\cdots,\cap X_k=X}} \prod_{i=1}^{k} \sup(q_2\mathbf{m}_i(X_i))$$

Proof: Let's assume $\inf(q_2\mathbf{m}_i(X_j))$ and $\sup(q_2\mathbf{m}_i(X_j))$ $(1 \leq i \leq k)$ be represented by $a_{ij} \in \inf(\mathbf{M})$ and $b_{ij} \in \sup(\mathbf{M})$ with $a_{ij} \leq b_{ij}$ ($\leq$ represents here a qualitative order). For any label $c_{mj} \in [a_{mj}, b_{mj}]$, one has

$$\sum_{\substack{X_1,X_2,\cdots,X_k\in D^\Theta\\ X_1\cap X_2,\cdots,\cap X_k=X}} \prod_{i=1}^{k} a_{ij} \quad \leq \quad \sum_{\substack{X_1,X_2,\cdots,X_k\in D^\Theta\\ X_1\cap X_2,\cdots,\cap X_k=X}} \prod_{i=1,i\neq m}^{k} a_{ij}c_{mj}$$

and also

$$\sum_{\substack{X_1,X_2,\cdots,X_k\in D^\Theta\\ X_1\cap X_2,\cdots,\cap X_k=X}} \prod_{i=1,i\neq m}^{k} a_{ij}c_{mj} \quad \leq \quad \sum_{\substack{X_1,X_2,\cdots,X_k\in D^\Theta\\ X_1\cap X_2,\cdots,\cap X_k=X}} \prod_{i=1}^{k} b_{ij}$$

Therefore, $q_2\mathbf{m}_{DSmC}^I(X) = [\inf(q_2\mathbf{m}_{DSmC}^I(X)), \sup(q_2\mathbf{m}_{DSmC}^I(X))]$ which completes the proof. Similarly, $q_2\mathbf{m}_{PCR5}^I(X) = [\inf(q_2\mathbf{m}_{PCR5}^I(X)), \sup(q_2\mathbf{m}_{PCR5}^I(X))]$.

This theorem shows that we can compute the upper and lower bounds of imprecise qualitative beliefs by applying the corresponding combination and redistribution rule directly on the bounds.

8.5 Examples of fusion of qualitative beliefs

All examples from this article could easier be calculated using the DSm Field and Algebra of Refined Labels.

8.5.1 Example of fusion of precise qualitative beliefs

Let's consider an investment corporation which has to choose one project among three proposals $\Theta = \{\theta_1, \theta_2, \theta_3\}$ based on two consulting/expert reports. The linguistic labels used by the experts are among the following ones: $I \mapsto$ Impossible, EU $\mapsto$ Extremely-Unlikely, VLC $\mapsto$ Very-Low-Chance, LLC $\mapsto$ Little-Low-Chance, SC $\mapsto$ Small-Chance, IM $\mapsto$ IT-May, MC $\mapsto$ Meanful-Chance, LBC $\mapsto$ Little-Big-Chance, BC $\mapsto$ Big-Chance, ML $\mapsto$ Most-likely, C $\mapsto$ Certain. So, we consider the following ordered set L (with $|L| = n = 9$) of linguistic labels

$$L \triangleq \{L_0 \equiv I, L_1 \equiv EU, L_2 \equiv VLC, L_3 \equiv LLC, L_4 \equiv SC, L_5 \equiv IM,$$
$$L_6 \equiv MC, L_7 \equiv LBC, L_8 \equiv BC, L_9 \equiv ML, L_{10} \equiv C\}$$

The qualitative belief assignments/masses provided by the sources/experts are assumed to be given according to Table 8.1.

	Source 1	Source 2
θ_1	$m_1(\theta_1) = (L_4, 0.03)$	$m_2(\theta_1) = (L_5, 0)$
θ_2	$m_1(\theta_2) = (L_3, -0.03)$	$m_2(\theta_2) = (L_2, 0.01)$
θ_3	$m_1(\theta_3) = (L_3, 0)$	$m_2(\theta_3) = (L_3, -0.01)$

Table 8.1: Precise qualitative belief assignments given by the sources.

When working with the free DSm model and applying the qualitative $DSmC$ rule of combination (8.27), we obtain:

$$q_2m_{DSmC}(\theta_1) = \Delta(0.43 \times 0.50) = (L_2, 0.015)$$
$$q_2m_{DSmC}(\theta_2) = \Delta(0.27 \times 0.21) = (L_1, -0.0433)$$
$$q_2m_{DSmC}(\theta_3) = \Delta(0.30 \times 0.29) = (L_1, -0.013)$$
$$q_2m_{DSmC}(\theta_1 \cap \theta_2) = \Delta(0.43 \times 0.21 + 0.50 \times 0.27) = (L_2, 0.0253)$$
$$q_2m_{DSmC}(\theta_1 \cap \theta_3) = \Delta(0.43 \times 0.29 + 0.50 \times 0.30) = (L_3, -0.0253)$$
$$q_2m_{DSmC}(\theta_2 \cap \theta_3) = \Delta(0.27 \times 0.29 + 0.21 \times 0.30) = (L_1, 0.0413)$$

We can verify the validity of the Theorem 1, i.e. $\sum_{A \in D^\Theta} q_2m(A) = (L_{10}, 0)$, which proves that is $q_2m_{DSmC}(.)$ is normalized.

Now, let's assume that Shafer's model holds for Θ. In this case the sets $\theta_1 \cap \theta_2$,$\theta_1 \cap \theta_3$,$\theta_2 \cap \theta_3$ must be empty and the qualitative conflicting masses $q_2m_{DSmC}(\theta_1 \cap \theta_2)$, $q_2m_{DSmC}(\theta_1 \cap \theta_3)$ and $q_2m_{DSmC}(\theta_2 \cap \theta_3)$ need to be redistributed to the sets involved in these conflicts according to (8.28) if the PCR5 fusion rule is used. So, with PCR5 one gets:

$$\begin{aligned} q_2 m_{PCR5}(\theta_1) &= q_2 m_{DSmC}(\theta_1) + q_2 m_{xA1}(\theta_1) + \\ &\quad q_2 m_{xB1}(\theta_1) + q_2 m_{xA2}(\theta_1) + q_2 m_{xB2}(\theta_1) \\ &= (L_5, 0.03155626126) \end{aligned}$$

$$\begin{aligned} q_2 m_{PCR5}(\theta_2) &= q_2 m_{DSmC}(\theta_2) + q_2 m_{yA1}(\theta_2) + \\ &\quad q_2 m_{yB1}(\theta_2) + q_2 m_{xA3}(\theta_2) + q_2 m_{xB3}(\theta_2) \\ &= (L_2, -0.00263968798) \end{aligned}$$

$$\begin{aligned} q_2 m_{PCR5}(\theta_3) &= q_2 m_{DSmC}(\theta_3) + q_2 m_{yA2}(\theta_3) + \\ &\quad q_2 m_{yB2}(\theta_3) + q_2 m_{yA3}(\theta_3) + q_2 m_{yB3}(\theta_3) \\ &= (L_3, -0.02891657328) \end{aligned}$$

Because $q_2 m_{PCR5}(\theta_1)$ is larger than $q_2 m_{PCR5}(\theta_2)$ and $q_2 m_{PCR5}(\theta_3)$, the investment corporation will choose the first project to invest.

Now, if we prefer to use the extension of Dempter's rule of combination given by the formula (8.33), the total qualitative conflicting mass is $qK_{total} = q_2 m_{DSmC}(\theta_1 \cap \theta_2) + q_2 m_{DSmC}(\theta_1 \cap \theta_3) + q_2 m_{DSmC}(\theta_3 \cap \theta_2) = (L_6, 0.0413)$, and so we obtain:

$$\begin{aligned} q_2 m_{DS}(\emptyset) &\triangleq (L_0, 0) \\ q_2 m_{DS}(\theta_1) &= \frac{q_2 m_{DSmC}(\theta_1)}{L_{10} - qK_{total}} = \frac{(L_2, 0.015)}{L_{10} - (L_6, 0.0413)} = (L_6, -0.0006133) \\ q_2 m_{DS}(\theta_2) &= \frac{q_2 m_{DSmC}(\theta_2)}{L_{10} - qK_{total}} = \frac{(L_1, -0.0433)}{L_{10} - (L_6, 0.0413)} = (L_2, -0.0419292) \\ q_2 m_{DS}(\theta_3) &= \frac{q_2 m_{DSmC}(\theta_3)}{L_{10} - qK_{total}} = \frac{(L_1, -0.013)}{L_{10} - (L_6, 0.0413)} = (L_2, 0.0425425) \end{aligned}$$

We see that $q_2 m_{DS}(\theta_1)$ is larger than $q_2 m_{DS}(\theta_2)$ and $q_2 m_{DS}(\theta_3)$, so the first project is also chosen to invest. The final decision is same to the previous one obtained by $q_2 PCR5$. However, when the total conflict becomes nearer and nearer to L_{10}, then $q_2 DS$ formula will become invalid.

If we adopt the simple arithmetic mean method, the results of the fusion are:

$$\begin{aligned} \theta_1 &: \frac{(L_4, 0.03) + (L_5, 0)}{2} = (L_5, -0.035) \\ \theta_2 &: \frac{(L_3, -0.03) + (L_2, 0.01)}{2} = (L_2, 0.04) \\ \theta_3 &: \frac{(L_3, 0) + (L_3, -0.01)}{2} = (L_3, -0.005) \end{aligned}$$

According to the above results, we easily know which project will be chosen to invest. Though the arithmetic mean is the simplest method among three methods, for some complex problems, it will provide unsatisfactory results since it is not neutral with respect to the introduction of a total ignorant source in the fusion process. This method can also be ill adapted to some particular problems. For example, one also investigates the possibility of investment in two projects together, i.e. $\theta_i \cap \theta_j \neq \emptyset$. However, the corporation only choose one of them to invest. How to do it in this case with simple arithmetic mean method? It is more easy to take decision from $q_2PCR5(.)$.

If all qualitative masses involved in the fusion are normalized, no matter what qualitative fusion rule we use the normalization is kept (i.e. the result will also be a normalized mass).

8.5.2 Example of fusion of imprecise qualitative beliefs

Let's consider again the previous example with imprecise qualitative beliefs providse by the sources according to Table 8.2:

	Source 1	Source 2
θ_1	$m_1(\theta_1) = [(L_4, 0.03), (L_5, 0.03)]$	$m_2(\theta_1) = [(L_5, 0), (L_5, 0.04)]$
θ_2	$m_1(\theta_2) = [(L_3, -0.03), (L_4, -0.03)]$	$m_2(\theta_2) = [(L_2, 0.01), (L_3, -0.03)]$
θ_3	$m_1(\theta_3) = [(L_3, 0), (L_4, 0.03)]$	$m_2(\theta_3) = [(L_3, -0.01), (L_3, 0)]$

Table 8.2: Imprecise qualitative belief assignments given by the sources.

If one works with the free DSm model for the frame Θ, one gets from (8.30) and the theorem 2 the following results:

$$q_2m^I_{DSmC}(\theta_1) = [(L_2, 0.015), (L_3, -0.0138)]$$
$$q_2m^I_{DSmC}(\theta_2) = [(L_1, -0.0433), (L_1, -0.0001)]$$
$$q_2m^I_{DSmC}(\theta_3) = [(L_1, -0.013), (L_1, 0.029)]$$
$$q_2m^I_{DSmC}(\theta_1 \cap \theta_2) = [(L_2, 0.0253), (L_3, 0.0429)]$$
$$q_2m^I_{DSmC}(\theta_1 \cap \theta_3) = [(L_3, -0.0253), (L_4, -0.0088)]$$
$$q_2m^I_{DSmC}(\theta_2 \cap \theta_3) = [(L_1, 0.0413), (L_2, 0.0271)]$$

If one works with Shafer's model for the frame Θ (i.e. all elements of Θ are assumed exclusive), then the imprecise qualitative conflicting masses $q_2m^I_{DSmC}(\theta_1 \cap \theta_2)$, $q_2m^I_{DSmC}(\theta_1 \cap \theta_3)$ and $q_2m^I_{DSmC}(\theta_2 \cap \theta_3)$ need to be redistributed to elements involved in these conflicts if PCR5 is used. In such case and from (8.31) and the

Theorem 2, one gets:

$$q_2 m^I_{PCR5}(\theta_1) = [(L_5, -0.02036), (L_8, 0.01860)]$$
$$q_2 m^I_{PCR5}(\theta_2) = [(L_2, -0.02909), (L_4, -0.0089)]$$
$$q_2 m^I_{PCR5}(\theta_3) = [(L_3, -0.03308), (L_5, 0.01112)]$$

From the values of $q_2 m^I_{PCR5}(.)$, one will choose the project θ_1 as final decision. It is interesting to note that $q_2 DSmC$ and $q_2 PCR5$ can be interpreted as special case (lower bounds) of $q_2^I DSmC$ and $q_2^I PCR5$.

The approach proposed in this work for combining imprecise qualitative beliefs presents the following properties:

1) If one utilizes the q_2-operators on 2-tuples without doing any approximation in the calculations one gets an exact qualitative result, while working on 1-tuples we round the qualitative result so we get approximations. Thus addition and multiplication operators on 2-tuple are truly commutative and associative contraiwise to addition and multiplication operators on 1-tuples. Actually, Herrera-Martínez' representation deals indirectly with exact qualitative (refined) values of the labels. This can be done directly and easier (without 2-tuple representation) from the DSm Field and Linear Algebra of Refined Labels (DSm-FLARL) presented in Chapter 2 of this volume. In DSm-FLARL we get the exact qualitative result.

2) Since the 2-tuples $\{(L_0, \sigma_0^h), \ldots, (L_{n+1}, \sigma_{n+1}^h)\}$ express actually continuous qualitative beliefs, they are equivalent to real numbers. So all quantitative fusion rules (and even the belief conditioning rules) can work directly using this qualitative framework. The imprecise qualitative DSmC and PCR5 fusion rules can deal easily and efficiently with imprecise belief structures, which are usually well adapted in real situations dealing with human reports.

3) The precise qualitative DSmC and PCR5 fusion rules can be seen as special cases of Imprecise qualitative DSmC and PCR5 fusion rules as shown in our examples.

8.6 Conclusion

In this chapter, we have proposed a new approach for combining imprecise qualitative beliefs based on Herrera-Martínez' 2 -Tuple linguistic labels. This approach allows the combination of information in the situations where no precise qualitative information is available. The underlying idea is to work with refined labels expressed as 2-tuples to keep working on the original set of linguistic labels. We have proposed precise and imprecise qualitative operators for 2-tuple labels and we have shown through very simple examples how we can combine precise and/or imprecise qualitative beliefs. The results obtained by this approach are more precise than those based on 1-tuple representation since no rounding approximation is done in operations and all the information is preserved in the fusion process. An enrichment of 2-tuples representation model can be done similarly to the enrichment done for 1-tuple representation in order to take into account the confidence we may commit to each qualitative (precise or imprecise) 2-tuple label given by the sources. The imprecise qualitative DSmC and PCR5 fusion rules are the extensions of precise qualitative DSmC and PCR5 fusion rules.

8.7 References

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