Generating Lemoine Circles

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In this paper, we generalize the theorem relative to the *first circle of Lemoine* and thereby highlight a method to build Lemoine circles. Firstly, we review some notions and results.

Definition 1. It is called a simedian of a triangle the symmetric of a median of the triangle with respect to the internal bisector of the triangle that has in common with the median the peak of the triangle.

Proposition 1. In the triangle ABC, the cevian AS, $S \in (BC)$, is a simedian if and only if $\frac{SB}{SC} = \left(\frac{AB}{AC}\right)^2$.

For *Proof*, see [2].

Definition 2. It is called a simedian center of a triangle (or Lemoine point) the intersection of triangle's simedians.

Theorem 1. The parallels to the sides of a triangle taken through the simedian center intersect the triangle's sides in six concyclic points (the first Lemoine circle - 1873).

A *Proof* of this theorem can be found in [2].

Definition 3. We assert that in a scalene triangle ABC the line MN, where $M \in AB$ and $N \in AC$, is an anti-parallel to BC if $\sphericalangle MNA \equiv \sphericalangle ABC$.

Lemma 1. In the triangle ABC, let AS be a simedian, $S \in (BC)$. If P is the middle of the segment (MN), having $M \in (AB)$ and $N \in (AC)$, belonging to the simedian AS, then MN and BC are anti-parallels.

Proof. We draw through M and N, $MT \parallel AC$ and $NR \parallel AB$, $R, T \in (BC)$, see *Figure 1*. Let $\{Q\} = MT \cap NR$; since $MP = PN$ and $AMQN$ is a parallelogram, it follows that $Q \in AS$.

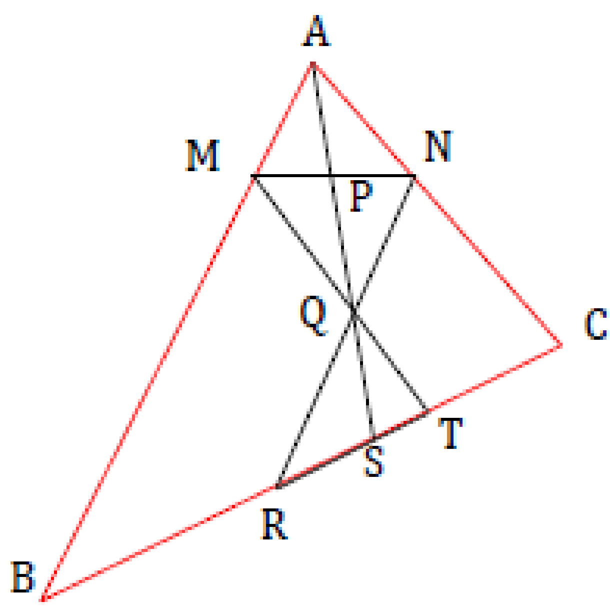

Figure 1.

Thales' Theorem provides the relations:

$$\frac{AN}{AC} = \frac{BR}{BC} \text{ (1)}; \frac{AB}{AM} = \frac{BC}{CT} \text{ (2)}.$$

From (1) and (2), by multiplication, we obtain:

$$\frac{AN}{AM} \cdot \frac{AB}{AC} = \frac{BR}{TC} \text{ (3)}.$$

Using again Thales' Theorem, we obtain:

$$\frac{BR}{BS} = \frac{AQ}{AS} \text{ (4)}, \frac{TC}{SC} = \frac{AQ}{AS} \text{ (5)}.$$

From these relations, we get

$$\frac{BR}{BS} = \frac{TC}{SC} \text{ (6) or } \frac{BS}{SC} = \frac{BR}{TC} \text{ (7).}$$

In view of *Proposition 1*, the relations (7) and (3) drive to $\frac{AN}{AB} = \frac{AB}{AC}$, which shows that $\Delta AMN \sim \Delta ACB$, so $\sphericalangle AMN \equiv \sphericalangle ABC$, therefore MN and BC are anti-parallels in relation to AB and AC.

Remark.

1. The reciprocal of *Lemma 1* is also valid, meaning that if P is the middle of the anti-parallel MN to BC, then P belongs to the simedian from A.

Theorem 2. (Generalization of *Theorem 1*) Let ABC be a scalene triangle and K its simedian center. We take $M \in AK$ and draw $MN \parallel AB$, $MP \parallel AC$, where $N \in BK$, $P \in CK$. Then:

i. $NP \parallel BC$;

ii. MN, NP and MP intersect the sides of triangle ABC in six concyclic points.

Proof. In triangle ABC, let AA_1, BB_1, CC_1 the simedians concurrent in K (see *Figure 2*). We have from Thales' Theorem that: $\frac{AM}{MK} = \frac{BN}{NK}$ (1); $\frac{AM}{MK} = \frac{CP}{PK}$ (2). From relations (1) and (2), it follows that $\frac{BN}{NK} = \frac{CP}{PK}$ (3), which shows that $NP \parallel BC$. Let R, S, V, W, U, T be the intersection points of the parallels MN, MP, NP of the sides of the triangles to the other sides. Obviously, by construction, the quadrilaterals $ASMW$; $CUPV$; $BRNT$ are parallelograms. The middle of the diagonal WS falls on AM, so on the simedian AK, and from *Lemma 1* we get that WS is an anti-parallel to BC. Since $TU \parallel BC$, it follows that WS and TU are anti-parallels, therefore the points W, S, U, T are concyclic (4).

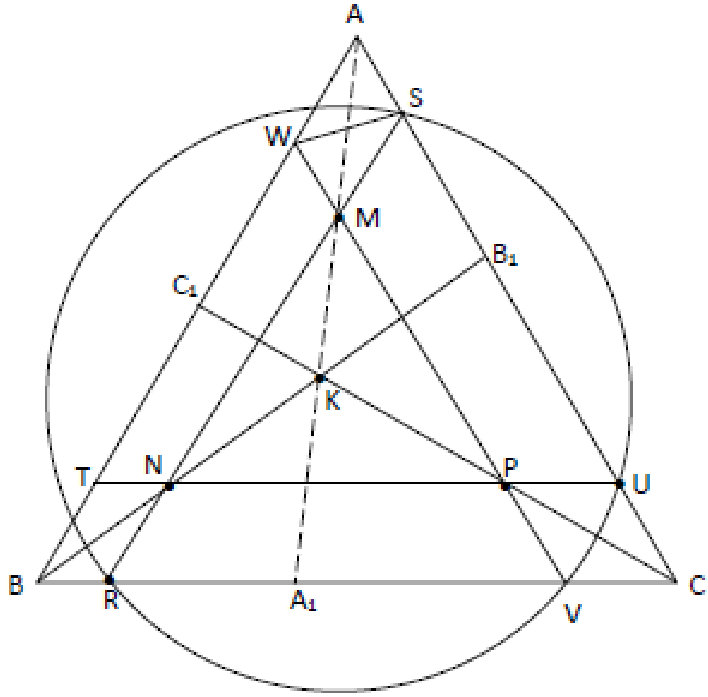

Figure 2.

Analogously, we show that the points U, V, R, S are concyclic (5). From WS and BC anti-parallels, UV and AB anti-parallels, we have that $\sphericalangle WSA \equiv \sphericalangle ABC$ and $\sphericalangle VUC \equiv \sphericalangle ABC$, therefore: $\sphericalangle WSA \equiv \sphericalangle VUC$, and since $VW \parallel AC$, it follows that the trapeze $WSUV$ is isosceles, therefore the points W, S, U, V are concyclic (6). The relations (4), (5), (6) drive to the concyclicality of the points R, U, V, S, W, T, and the theorem is proved.

Remarks.

2. For any point M found on the simedian AA_1, by performing the constructions from hypothesis, we get a circumscribed circle of the 6 points of intersection of the parallels taken to the sides of triangle.

3. The *Theorem 2* generalizes the *Theorem 1* because we get the second in the case the parallels are taken to the sides through the simedian center k.

4. We get a circle built as in *Theorem 2* from the first Lemoine circle by homothety of pole k and of ratio $\lambda \in \mathbb{R}$.

5. The centers of Lemoine circles built as above belong to the line OK, where O is the center of the circle circumscribed to the triangle ABC.

Bibliography.

[1] *Exercices de Géométrie*, par F.G.M., Huitième édition, Paris VI^e, Librairie Générale, 77, Rue Le Vaugirard.

[2] Ion Paatrascu, Florentin Smarandache: *Variance on topics of Plane Geometry*, Educational Publishing, Columbus, Ohio, 2013.