

Geometric properties of Smarandache ruled surfaces generated by integral binormal curves in Euclidean 3-space

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ABSTRACT

This paper investigates the geometric properties of Smarandache ruled surfaces generated by integral binormal curves in the Euclidean 3-space \mathbb{E}^3 . Specifically, we study four types of Smarandache ruled surfaces: the tn , tb , nb , and tnb surfaces, each defined by different combinations of the tangent, normal, and binormal vectors of the integral curves. For each type of surface, we derive the parametric representations and compute the fundamental geometric properties, including the striction lines, distribution parameters, and the first and second fundamental forms. Additionally, we provide explicit expressions for the Gaussian and mean curvatures, which characterize the local shape of the surfaces. We also analyze the geodesic curvature, normal curvature, and geodesic torsion associated with the base curves on these surfaces. Furthermore, we establish necessary and sufficient conditions for these surfaces to be developable or minimal. The paper concludes with detailed conditions under which the base curves can be classified as geodesic or asymptotic lines on the surfaces. The results are supported by rigorous proofs and illustrative examples, offering a comprehensive understanding of the geometric behavior of these Smarandache ruled surfaces.

1. Introduction

The study of ruled surfaces has long captivated mathematicians and engineers due to their fundamental role in differential geometry and their extensive applications in computer-aided design, architectural structures, and mechanical engineering systems.¹ These surfaces, formed by the continuous motion of straight lines along predetermined curves, exhibit remarkable geometric characteristics that bridge theoretical mathematics with practical applications.

Among the various types of ruled surfaces, Smarandache ruled surfaces represent a particularly intriguing class. These surfaces are constructed using linear combinations of the Frenet frame vectors — tangent, normal, and binormal — associated with a given curve. The concept derives from Florentin Smarandache's pioneering contributions to extending classical geometric frameworks, leading to surfaces with distinctive properties that set them apart from conventional ruled surface constructions.^{2,3}

Recent mathematical investigations have increasingly focused on integral curves, which emerge as solutions to differential equations governing the evolution of systems along specified vector fields. The examination of integral binormal and integral normal curves has opened

new research directions in surface generation theory, as these curves serve as fundamental building blocks for constructing Smarandache ruled surfaces.^{4–7} When the binormal or normal vector fields of a base curve are integrated, the resulting integral curves become the foundation for sophisticated surface constructions.

The theoretical development of Smarandache curves has undergone significant advancement in the past decade. Yücesan et al.⁸ established comprehensive analytical frameworks for these curves in both Euclidean and non-Euclidean geometries, providing explicit formulations for their curvature and torsion characteristics. This foundational work was extended by Kahraman et al.,⁹ who explored the connections between integral curves and minimal surfaces, offering novel perspectives on their geometric behavior. Furthermore, Turgut and Yılmaz¹⁰ investigated Smarandache curves within Minkowski space-time, thereby expanding the theoretical foundations into relativistic geometric contexts. Additional contributions to the field include studies by Eren and Ersoy¹¹ on the characterization of Smarandache curves with modified orthogonal frames.

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The practical significance of these mathematical constructs extends across multiple disciplines. In robotics applications, Taş¹² demonstrated how Smarandache ruled surfaces can effectively model complex trajectory paths for automated systems. Architectural implementations have been examined by Senyurt et al.,¹³ who studied developable Smarandache ruled surfaces and their potential applications in structural engineering. The computer graphics industry has similarly benefited from these theoretical developments, as highlighted by Abdel-Aziz and Saad,¹⁴ who explored innovative surface design possibilities offered by Smarandache ruled surface geometries.

This research specifically examines four distinct categories of Smarandache ruled surfaces: tn , tb , nb , and tnb surfaces. Each category represents a unique combination of Frenet frame vectors, resulting in surfaces with markedly different geometric properties. The tn surface incorporates tangent and normal vectors, while the tb surface combines tangent and binormal vectors. The nb surface unifies normal and binormal vectors, and the tnb surface integrates all three fundamental vectors simultaneously. These surfaces possess significant practical relevance in applications where the complex interplay between curvature, torsion, and surface geometry plays a critical role.^{13,15-19}

The geometric analysis of these surfaces across various spatial contexts has been extensively documented. Önder and Kocayigit²⁰ investigated Smarandache curves in Galilean space, while other researchers have explored their properties in different geometric environments.²¹⁻²⁴ The fundamental question of surface developability — essential for applications requiring surfaces to be flattened without distortion, such as in sheet metal fabrication — remains a central concern in both theoretical investigations and practical implementations.^{1,25}

Minimal surfaces, which locally minimize area, present another compelling aspect of this investigation. These surfaces hold particular significance in physics, materials science, and architectural design due to their optimization properties.²⁵ Understanding the conditions under which Smarandache ruled surfaces become minimal provides valuable insights for applications ranging from soap film studies to architectural membrane design. Recent developments in this area include work by Elsharkawy^{26,27} on generalized curves and equiform geometry.

The classification of curves residing on surfaces represents a crucial component of differential geometry. When base curves can be characterized as geodesic lines, asymptotic lines, or principal lines, they reveal fundamental information about the surface's intrinsic geometric structure. Recent work by Senyurt and Ayvaci²⁸ has examined special Smarandache ruled surfaces according to alternative frame constructions, while previous investigations by the same research group have studied various aspects of these surfaces in Euclidean 3-space. Modern research includes studies by Li et al.²⁹⁻³² on modified sweeping surfaces and conformal interactions, as well as work by Zhu et al.^{30,31} on directional developable surfaces and polynomial curve sweeping.

This paper contributes several significant advances to the existing literature on ruled surfaces. We present a comprehensive analytical framework for examining Smarandache ruled surfaces generated by integral binormal curves in Euclidean 3-space \mathbb{E}^3 . Our systematic methodology includes the derivation of parametric representations, computation of striction lines and distribution parameters, and calculation of fundamental geometric properties, including first and second fundamental forms.

The practical implications of our findings extend beyond pure mathematical theory. The developability conditions we establish have direct applications in manufacturing processes where surfaces must be formed from planar materials. Similarly, our criteria for minimal surfaces provide valuable tools for optimization problems in engineering and physics. The classification of base curves as geodesic or asymptotic lines offers insights particularly relevant for applications in robotics and computer graphics, where path planning and surface parameterization are crucial considerations.

Our investigation reveals that the geometric behavior of Smarandache ruled surfaces generated by integral binormal curves exhibits rich

mathematical structure. The interplay between the original curve's curvature and torsion properties and the resulting surface characteristics demonstrates the profound connections between curve theory and surface geometry. These relationships become particularly evident when examining the conditions for special surface types, such as developable or minimal surfaces.

The organization of this paper reflects our systematic approach to understanding these complex geometric objects. Following the preliminary section, which establishes necessary definitions and fundamental concepts, we present detailed analyses of four types of Smarandache ruled surfaces. Each surface type receives comprehensive treatment, including derivation of its parametric form, computation of geometric invariants, and analysis of special properties. We conclude with illustrative examples that demonstrate the practical application of our theoretical results.

Our results contribute to the expanding field of Smarandache geometry while providing practical tools for applications in engineering and computer science. The explicit formulas we derive for curvatures, the conditions we establish for special surface types, and the criteria we develop for curve classification all represent advances in both theoretical understanding and practical applicability. These contributions position our work as a valuable resource for researchers and practitioners working with ruled surfaces across various disciplines.

2. Preliminaries

In this section, we introduce the fundamental concepts and notations used throughout the paper. We begin with the basic definitions of curves, Frenet frames, and ruled surfaces in Euclidean 3-space \mathbb{E}^3 . We then discuss integral curves, Smarandache ruled surfaces, and the geometric invariants associated with these surfaces.

2.1. Curves and Frenet frames in \mathbb{E}^3

Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be a regular curve parameterized by arc length s . The curve α is said to be of unit speed if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{E}^3 .²⁵ The Frenet frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ along $\alpha(s)$ consists of the tangent, normal, and binormal vectors, respectively, defined as follows:

$$\mathbf{t}(s) = \alpha'(s), \quad \mathbf{n}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}, \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$

The curvature $\kappa(s)$ and torsion $\tau(s)$ of $\alpha(s)$ are given by:

$$\kappa(s) = \|\alpha''(s)\|, \quad \tau(s) = \langle \mathbf{n}'(s), \mathbf{b}(s) \rangle.$$

The Frenet–Serret equations describe the evolution of the Frenet frame along the curve:

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}. \tag{2.1}$$

2.2. Integral curves

Integral curves are curves obtained by integrating a vector field along a base curve. In this paper, we focus on two types of integral curves:

- **Integral Binormal Curves:** Given a base curve $\alpha(s)$, the integral binormal curve $\beta(s)$ is defined as:

$$\beta(s) = \int \mathbf{b}_\alpha(s) ds,$$

where $\mathbf{b}_\alpha(s)$ is the binormal vector of $\alpha(s)$.

- **Integral Normal Curves:** Similarly, the integral normal curve $\gamma(s)$ is defined as:

$$\gamma(s) = \int \mathbf{n}_\alpha(s) ds,$$

where $\mathbf{n}_\alpha(s)$ is the normal vector of $\alpha(s)$.

2.3. Ruled surfaces

A ruled surface $X(s, v)$ in \mathbb{E}^3 is a surface generated by the motion of a straight line (called a ruling) along a base curve $\alpha(s)$. The parametric representation of a ruled surface is given by:

$$X(s, v) = \alpha(s) + v \mathbf{y}(s),$$

where $\mathbf{y}(s)$ is the direction vector of the ruling. The striction curve $\alpha^*(s)$ and the distribution parameter $\lambda(s)$ of the ruled surface are defined as:

$$\alpha^*(s) = \alpha(s) + \frac{\langle \mathbf{t}(s), \mathbf{y}'(s) \rangle}{\|\mathbf{y}'(s)\|^2} \mathbf{y}(s), \tag{2.2}$$

$$\lambda(s) = \frac{\det[\mathbf{t}(s), \mathbf{y}(s), \mathbf{y}'(s)]}{\|\mathbf{y}'(s)\|^2}. \tag{2.3}$$

The ruled surface is said to be developable if $\lambda(s) = 0$, and cylindrical if $\|\mathbf{y}'(s)\| = 0$.

2.4. Smarandache ruled surfaces

Smarandache ruled surfaces are a special class of ruled surfaces constructed by combining the tangent (\mathbf{t}), normal (\mathbf{n}), and binormal (\mathbf{b}) vectors of a base curve. In this paper, we study four types of Smarandache ruled surfaces:

- **tn Surface:** Generated by combining \mathbf{t} and \mathbf{n} of the integral curve.
- **tb Surface:** Generated by combining \mathbf{t} and \mathbf{b} of the integral curve.
- **nb Surface:** Generated by combining \mathbf{n} and \mathbf{b} of the integral curve.
- **tnb Surface:** Generated by combining \mathbf{t} , \mathbf{n} , and \mathbf{b} of the integral curve.

2.5. Fundamental forms and curvatures

The first fundamental form I and the second fundamental form II of a surface $X(s, v)$ are given by:

$$I = g_{11} ds^2 + 2g_{12} ds dv + g_{22} dv^2,$$

$$II = l_{11} ds^2 + 2l_{12} ds dv + l_{22} dv^2,$$

where the coefficients g_{ij} and l_{ij} are defined as:

$$g_{11} = \langle X_s, X_s \rangle, \quad g_{12} = \langle X_s, X_v \rangle, \quad g_{22} = \langle X_v, X_v \rangle, \tag{2.4}$$

$$l_{11} = \langle X_{ss}, \mathbf{N} \rangle, \quad l_{12} = \langle X_{sv}, \mathbf{N} \rangle, \quad l_{22} = \langle X_{vv}, \mathbf{N} \rangle. \tag{2.5}$$

Here, \mathbf{N} is the unit normal vector of the surface, given by:

$$\mathbf{N} = \frac{X_s \times X_v}{\|X_s \times X_v\|}.$$

The Gaussian curvature K and mean curvature H of the surface are defined as:

$$K = \frac{l_{11}l_{22} - l_{12}^2}{g_{11}g_{22} - g_{12}^2}, \quad H = \frac{g_{11}l_{22} - 2g_{12}l_{12} + g_{22}l_{11}}{2(g_{11}g_{22} - g_{12}^2)}. \tag{2.6}$$

2.6. Geometric invariants of curves on surfaces

For a curve $\alpha(s)$ lying on a surface $X(s, v)$, the geodesic curvature κ_g , normal curvature κ_n , and geodesic torsion τ_g are defined as:

$$\kappa_g = \langle \mathbf{N} \times \mathbf{t}, \mathbf{t}' \rangle, \quad \kappa_n = \langle \mathbf{N}, \alpha''(s) \rangle, \quad \tau_g = \langle \mathbf{N} \times \mathbf{N}', \mathbf{t} \rangle. \tag{2.7}$$

A curve $\alpha(s)$ on a surface X is called:

- A **geodesic** if $\kappa_g = 0$.
- An **asymptotic line** if $\kappa_n = 0$.
- A **principal line** if $\tau_g = 0$.

2.7. Developable and minimal surfaces

A ruled surface is said to be:

- **Developable** if its Gaussian curvature vanishes ($K = 0$).
- **Minimal** if its mean curvature vanishes ($H = 0$).

3. The Smarandache ruled surface of the integral binormal curves

In this section, we study the Smarandache ruled surface of the integral binormal curves in E^3 . Also, we obtain the striction line and the distribution parameter. Further, we obtain the first and the second fundamental forms of this surface. In addition, the Gaussian and mean curvatures are obtained.

Definition 3.1. The integral binormal curves $\beta(s)$ is given by

$$\beta(s) = \int b_\alpha ds \quad \& \quad b_\alpha = \beta'(s). \tag{3.1}$$

Proposition 3.1. If the curve $\beta(s)$ is a b_α -integral curve of a curve α , let $T_\beta, N_\beta, B_\beta$ be the Frenet frame of the curve β and let $T_\alpha, N_\alpha, B_\alpha$ be the Frenet frame of α , then $T_\beta, N_\beta, B_\beta, \kappa_\beta$ and τ_β can be written in terms of the curve α as:

$$\begin{aligned} t_\beta &= b_\alpha, & n_\beta &= -n_\alpha, & b_\beta &= t_\alpha, \\ \kappa_\beta &= \tau_\alpha, & \tau_\beta &= \kappa_\alpha. \end{aligned} \tag{3.2}$$

Proof. Since $\beta(s) = \int b_\alpha ds$, by differentiating Eq. (3.1), twice with respect to s , we obtain

$$t_\beta = b_\alpha, \tag{3.3}$$

$$n_\beta = -n_\alpha. \tag{3.4}$$

By taking the cross product of Eqs. (3.3) and (3.4), we have $b_\beta = t_\alpha$. Using the definitions of the curvature and the torsion, we can conclude that $\kappa_\beta = \tau_\alpha$ and $\tau_\beta = \kappa_\alpha$. \square

3.1. tn-Smarandache ruled surface of the integral binormal curves

We introduce the TN ruled surface based on the Frenet frame with the integral binormal curve. We calculate the first, second, and third fundamental forms of the surface. In addition, we examine the Gaussian and mean curvatures, the normal and geodesic curvatures, and geodesic torsion.

Definition 3.2. Let $\beta(s)$ be the integral normal curves, and let X^1 be a normal ruled surface generated by $\beta(s)$ according to the Frenet frame. The parametric description of X^1 can be defined as:

$$X^1(s, v) = \beta(s) + \frac{v}{\sqrt{2}} [t_\beta + n_\beta], \tag{3.5}$$

where $\beta(s)$ denotes the base curve.

Theorem 3.1. If $X^1(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the striction line and distribution parameter are given respectively by:

$$\begin{aligned} \beta^{*1} &= \int b_\alpha ds + \frac{1}{\sqrt{2}} \frac{\kappa_\alpha + \kappa'_\alpha + 2\tau_\alpha \tau'_\alpha}{\kappa_\alpha + 2\tau_\alpha^2} (b_\alpha - n_\alpha), \\ \lambda^1 &= \frac{\kappa_\alpha}{\kappa_\alpha^2 + 2\tau_\alpha^2}. \end{aligned}$$

Proof. Here

$$y(s) = \frac{1}{\sqrt{2}} (t_\beta + n_\beta)$$

$$= \frac{1}{\sqrt{2}}(b_\alpha - n_\alpha). \tag{3.6}$$

By taking the derivative twice (3.6) with respect to s , we have:

$$t_y = \frac{1}{\sqrt{2}}(\kappa_\alpha t_\alpha - \tau_\alpha n_\alpha - \tau_\alpha b_\alpha),$$

$$t'_y = \frac{1}{\sqrt{2}}[(\kappa'_\alpha + \tau_\alpha \kappa_\alpha)t_\alpha + (\kappa_\alpha^2 - \tau'_\alpha + \tau_\alpha^2)n_\alpha - (\tau_\alpha^2 + \tau'_\alpha)b_\alpha].$$

Hence, $\langle t_y, t'_y \rangle = \frac{1}{\sqrt{2}}[\kappa_\alpha \kappa'_\alpha + 2\tau_\alpha \tau'_\alpha]$. Also, $\|y'(s)\|^2 = \frac{1}{\sqrt{2}}(\kappa_\alpha^2 + 2\tau_\alpha^2)$.

Therefore, $\frac{\langle t_y, t'_y \rangle}{\|y'(s)\|^2} y(s) = \frac{1}{\sqrt{2}} \frac{\kappa_\alpha \kappa'_\alpha + 2\tau_\alpha \tau'_\alpha}{\kappa_\alpha^2 + 2\tau_\alpha^2} (b_\alpha - n_\alpha)$.

By using Eq. (2.2), we can write the striction line as:

$$\beta^{*1} = \int b_\alpha ds + \frac{1}{\sqrt{2}} \frac{\kappa_\alpha + \kappa'_\alpha + 2\tau_\alpha \tau'_\alpha}{\kappa_\alpha + 2\tau_\alpha^2} (b_\alpha - n_\alpha).$$

By using Eq. (2.3), we can easily conclude that:

$$\lambda^1 = \frac{\kappa_\alpha}{\kappa_\alpha + 2\tau_\alpha^2}. \quad \square$$

Theorem 3.2. If $X^1(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the first and second fundamental forms are provided of $X^1(s, v)$ are given, respectively, by:

$$I = [\frac{v^2 \kappa_\alpha^2}{2} + v^2 \tau_\alpha^2 + 1 - \sqrt{2}v\tau_\alpha] ds^2 + \sqrt{2} ds dv + dv^2, \tag{3.7}$$

$$II = \frac{1}{A} [[\frac{v\kappa'_\alpha}{2} + v\tau_\alpha \kappa_\alpha - \frac{v^2 \tau_\alpha \kappa'_\alpha}{\sqrt{2}} - \frac{v^2 \tau_\alpha^2 \text{phak}_\alpha}{\sqrt{2}} - \frac{v^2 \kappa_\alpha^3}{2\sqrt{2}} + \frac{v^2 \tau'_\alpha \kappa_\alpha}{\sqrt{2}}] ds^2 + \kappa_\alpha ds dv], \tag{3.8}$$

where $A = \sqrt{(\frac{1}{\sqrt{2}} - v\tau_\alpha)^2 + \frac{v^2 \kappa_\alpha^2}{2}}$.

Proof. Assuming that $X^1(s, v)$ is the tangent normal ruled surface, then the partial derivatives of $X^1(s, v)$ with respect to s and v are obtained by:

$$\begin{aligned} X^1_s(s, v) &= [\frac{v\kappa_\alpha}{\sqrt{2}}]t_\alpha - [\frac{v\tau_\alpha}{\sqrt{2}}]n_\alpha + [1 - \frac{v\tau_\alpha}{\sqrt{2}}]b_\alpha, \\ X^1_v(s, v) &= \frac{1}{\sqrt{2}}[b_\alpha - n_\alpha], \end{aligned} \tag{3.9}$$

$$X^1_{ss}(s, v) = [\frac{v\kappa'_\alpha}{\sqrt{2}} + \frac{v\tau_\alpha \kappa_\alpha}{\sqrt{2}}]t_\alpha + [\frac{v\kappa_\alpha^2}{\sqrt{2}} - \frac{v\tau'_\alpha}{\sqrt{2}} - \tau_\alpha + \frac{v\tau_\alpha^2}{\sqrt{2}}]n_\alpha - [\frac{v\tau_\alpha^2}{\sqrt{2}} + \frac{v\tau'_\alpha}{\sqrt{2}}]b_\alpha, \tag{3.10}$$

$$X^1_{sv}(s, v) = \frac{1}{\sqrt{2}}[\kappa_\alpha t_\alpha - \tau_\alpha n_\alpha - \tau_\alpha b_\alpha],$$

$$X^1_{vv}(s, v) = 0.$$

We can conclude that the unit normal of N^1 can be given by:

$$N^1 = \frac{1}{A} [(\frac{1}{\sqrt{2}} - v\tau_\alpha)t_\alpha - \frac{v\kappa_\alpha}{2}n_\alpha - \frac{v\kappa_\alpha}{2}b_\alpha]. \tag{3.11}$$

The coefficients of the first fundamental form, using Eq. (2.4), are:

$$I = [\frac{v^2 \kappa_\alpha^2}{2} + v^2 \tau_\alpha^2 + 1 - \sqrt{2}v\tau_\alpha] ds^2 + \sqrt{2} ds dv + dv^2.$$

Also, the second fundamental form using Eq. (2.5), we obtain:

$$II = \frac{1}{A} [[\frac{v\kappa'_\alpha}{2} + v\tau_\alpha \kappa_\alpha - \frac{v^2 \tau_\alpha \kappa'_\alpha}{\sqrt{2}} - \frac{v^2 \tau_\alpha^2 \text{phak}_\alpha}{\sqrt{2}} - \frac{v^2 \kappa_\alpha^3}{2\sqrt{2}} + \frac{v^2 \tau'_\alpha \kappa_\alpha}{\sqrt{2}}] ds^2 + \kappa_\alpha ds dv]. \quad \square$$

Theorem 3.3. If $X^1(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the mean curvature and the Gaussian curvature are given, respectively, by:

$$H_{X^1(s,v)} = \frac{-\frac{1}{\sqrt{2}}(\frac{\kappa_\alpha}{A}) + L_{11}}{2(\frac{v^2 \kappa_\alpha^2}{2}) + v^2 \tau_\alpha^2 - \sqrt{2}v\tau_\alpha + \frac{1}{2}},$$

$$K_{X^1(s,v)} = \frac{-\kappa_\alpha^2}{4A^2[(\frac{v^2 \kappa_\alpha^2}{2}) + v^2 \tau_\alpha^2 - \sqrt{2}v\tau_\alpha + \frac{1}{2}]}$$

Corollary 3.1. If $X^1(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the normal and geodesic curvatures for $\beta(s)$ on the surface are given, respectively, by:

$$\kappa_n^{X^1} = \frac{v\tau_\alpha \kappa_\alpha}{2A}, \quad \kappa_g^{X^1} = \frac{\tau_\alpha}{A} (\frac{1}{\sqrt{2}} - v\tau_\alpha).$$

Corollary 3.2. If $X^1(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the surface $X^1(s, v)$ is

- developable (minimum surface) if and only if $\kappa_\alpha = 0$. (α is straight line)
- minimal if and only if

$$-\kappa_\alpha + \frac{v\kappa'_\alpha}{\sqrt{2}} + \sqrt{2}v\tau_\alpha \kappa_\alpha - v^2 \tau_\alpha \kappa'_\alpha - v^2 \tau_\alpha^2 \kappa_\alpha - \frac{v^2 \kappa_\alpha^3}{2} + v^2 \tau'_\alpha \kappa_\alpha = 0.$$

Corollary 3.3. If $X^1(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, the curve $\beta(s)$ is

- geodesic if and only if $\tau_\alpha = \frac{1}{\sqrt{2}v}$.
- asymptotic curve if and only if $v\tau_\alpha \kappa_\alpha = 0$.

3.2. tb-Smarandache ruled surface of the integral binormal curves

Definition 3.3. Let $\beta(s)$ be the integral normal curves and X^2 be a normal ruled surface generated by $\beta(s)$ according to the Frenet frame. The parametric description of X^2 can be defined as:

$$X^2(s, v) = \beta(s) + \frac{v}{\sqrt{2}}[t_\beta + b_\beta], \tag{3.12}$$

where $\beta(s)$ denotes the base curve.

Theorem 3.4. If $X^2(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the striction line and distribution parameter are given, respectively, by:

$$\begin{aligned} \beta^{*2} &= \int b_\alpha ds + \frac{1}{\sqrt{2}} \frac{\kappa'_\alpha - \tau'_\alpha}{\kappa_\alpha - \tau_\alpha} (t_\alpha + b_\alpha), \\ \lambda^2 &= \frac{1}{\kappa_\alpha - \tau_\alpha}. \end{aligned}$$

Theorem 3.5. If $X^2(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the first and second fundamental forms are provided of $X^2(s, v)$ are given, respectively, by:

$$I = [1 + \frac{v^2}{2}(\kappa_\alpha - \tau_\alpha)^2] ds^2 + \sqrt{2} ds dv + dv^2, \tag{3.13}$$

$$II = \frac{1}{B} [-[\frac{v^2}{2\sqrt{2}}(\kappa_\alpha - \tau_\alpha)^2(\kappa_\alpha + \tau_\alpha) + \frac{v\tau_\alpha}{2}(\kappa'_\alpha - \tau'_\alpha) - \frac{\tau_\alpha}{\sqrt{2}}] ds^2 + \frac{\kappa_\alpha - \tau_\alpha}{2} ds dv], \tag{3.14}$$

where $B = \frac{1}{2}(1 + v^2(\kappa_\alpha - \tau_\alpha)^2)$.

Proof. Assuming that $X^2(s, v)$ is the tangent normal ruled surface, then the partial derivatives of $X^2(s, v)$ with respect to s and v are obtained by:

$$X^2_s(s, v) = \frac{v}{\sqrt{2}}(\kappa_\alpha - \tau_\alpha)n_\alpha + b_\alpha, \tag{3.15}$$

$$X^2_v(s, v) = \frac{1}{\sqrt{2}}[t_\alpha + b_\alpha], \tag{3.16}$$

$$X^2_{ss}(s, v) = -\frac{v\kappa_\alpha}{\sqrt{2}}(\kappa_\alpha - \tau_\alpha)t_\alpha + [-\tau + \frac{v}{\sqrt{2}}(\kappa'_\alpha - \tau'_\alpha)]n_\alpha + \frac{v\tau_\alpha}{\sqrt{2}}(\kappa_\alpha - \tau_\alpha)b_\alpha, \tag{3.17}$$

$$X_{sv}^2(s, v) = \frac{1}{\sqrt{2}}[\kappa_\alpha - \tau_\alpha]n_\alpha,$$

$$X_{vv}^2(s, v) = 0.$$

We can conclude that the unit normal of N_2 can be given by:

$$N_2 = \frac{1}{B}[\frac{v}{2}(\kappa_\alpha - \tau_\alpha)t_\alpha - \frac{1}{\sqrt{2}}n_\alpha - \frac{v}{2}(\kappa_\alpha - \tau_\alpha)b_\alpha]. \tag{3.18}$$

The coefficients of the first fundamental form, using Eq. (2.4), are

$$I = [1 + \frac{v^2}{2}(\kappa_\alpha - \tau_\alpha)^2]ds^2 + \sqrt{2}dsdv + dv^2.$$

Also, the second fundamental form using Eq. (2.5), we obtain:

$$II = \frac{1}{B}[-\frac{v^2}{2\sqrt{2}}(\kappa_\alpha - \tau_\alpha)^2(\kappa_\alpha + \tau_\alpha) + \frac{v\tau_\alpha}{2}(\kappa'_\alpha - \tau'_\alpha) + \frac{\tau_\alpha}{\sqrt{2}}]ds^2 + \frac{\kappa_\alpha - \tau_\alpha}{2} dsdv]. \quad \square$$

Theorem 3.6. If $X^2(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the mean curvature and the Gaussian curvature are given, respectively, by:

$$H_{X^2(s,v)} = -\frac{(\kappa_\alpha - \tau_\alpha) + \frac{v^2}{2}(\kappa_\alpha - \tau_\alpha)^2(\kappa_\alpha + \tau_\alpha) + \frac{v\tau_\alpha}{\sqrt{2}}(\kappa'_\alpha - \tau'_\alpha) - \tau_\alpha}{\sqrt{2B}[1 + \frac{v^2}{2}(\kappa_\alpha - \tau_\alpha)^2]},$$

$$K_{X^2(s,v)} = \frac{-(\kappa_\alpha - \tau_\alpha)^2}{2B^2[1 + v^2(\kappa_\alpha - \tau_\alpha)^2]}.$$

Corollary 3.4. If $X^2(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the normal and geodesic curvatures for $\beta(s)$ on the surface are given, respectively, by:

$$\kappa_n^{X^2} = \frac{-\tau_\alpha}{\sqrt{2}B}, \quad \kappa_g^{X^2} = \frac{-v\kappa_\alpha}{2B}(\kappa_\alpha - \tau_\alpha).$$

Corollary 3.5. If $X^2(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the surface $X^2(s, v)$ is

- developable (minimum surface) if and only if $\kappa_\alpha = \tau_\alpha$. (α is a straight line)
- minimal if and only if

$$-\frac{v^2}{2\sqrt{2}}(\kappa_\alpha - \tau_\alpha)^2(\kappa_\alpha + \tau_\alpha) - \frac{v\tau_\alpha}{2}(\kappa'_\alpha - \tau'_\alpha) - \frac{\tau_\alpha}{\sqrt{2}} = \frac{\kappa_\alpha - \tau_\alpha}{\sqrt{2}}.$$

Corollary 3.6. If $X^2(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, the curve $\beta(s)$ is

- geodesic if and only if $\tau_\alpha = \kappa_\alpha$ or $\kappa_\alpha = 0$.
- asymptotic curve if and only if $\tau_\alpha = 0$.

3.3. nb-Smarandache ruled surface of the integral binormal curves

Definition 3.4. Let $\beta(s)$ be the integral normal curves and let X^3 be a normal ruled surface generated by $\beta(s)$ according to the Frenet frame. The parametric description of X^1 can be defined as:

$$X^3(s, v) = \beta(s) + \frac{v}{\sqrt{2}}[n_\beta + b_\beta], \tag{3.19}$$

where $\beta(s)$ denotes the base curve.

Theorem 3.7. If $X^3(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the striction line and distribution parameter are given, respectively, by:

$$\beta^{*3} = \int b_\alpha ds + \frac{1}{\sqrt{2}} \frac{2\kappa_\alpha \kappa'_\alpha + \tau_\alpha \tau'_\alpha}{2\kappa_\alpha^2 + \tau_\alpha^2} (t_\alpha - n_\alpha),$$

$$\lambda^3 = \frac{2\kappa_\alpha}{2\kappa_\alpha^2 + \tau_\alpha^2}.$$

Theorem 3.8. If $X^3(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the first and second fundamental forms are provided of $X^3(s, v)$ are given, respectively, by:

$$I = [\frac{v^2\tau_\alpha^2}{2} + v^2\kappa_\alpha^2 + 1 - \frac{2}{\sqrt{2}}v\tau_\alpha]ds^2 + dv^2, \tag{3.20}$$

$$II = \frac{1}{C\sqrt{2}}[[v\kappa'_\alpha(\sqrt{2}-v\tau_\alpha)+(1-\frac{v}{\sqrt{2}}\tau_\alpha)(-\tau_\alpha+\frac{v}{\sqrt{2}}\tau_\alpha^2)-v^2\kappa_\alpha(\kappa_\alpha\tau_\alpha-\tau'_\alpha)]ds^2+2\kappa_\alpha dsdv], \tag{3.21}$$

where $C = \sqrt{1 - \frac{v}{\sqrt{2}}\tau_\alpha^2 + v^2\kappa_\alpha^2}$.

Proof. Assuming that $X^3(s, v)$ is the tangent normal ruled surface, then the partial derivatives of $X^3(s, v)$ with respect to s and v are obtained by:

$$X_s^3(s, v) = [\frac{v\kappa'_\alpha}{\sqrt{2}}]t_\alpha + [\frac{v\kappa_\alpha}{\sqrt{2}}]n_\alpha + [1 - \frac{v\tau_\alpha}{\sqrt{2}}]b_\alpha,$$

$$X_v^3(s, v) = \frac{1}{\sqrt{2}}[t_\alpha - n_\alpha],$$

$$X_{ss}^3(s, v) = [\frac{v\kappa''_\alpha}{\sqrt{2}} - \frac{v\kappa_\alpha\kappa'_\alpha}{\sqrt{2}}]t_\alpha + [\frac{v\kappa'_\alpha}{\sqrt{2}} + \frac{v\kappa_\alpha}{\sqrt{2}} - \tau_\alpha + \frac{v\tau_\alpha^2}{\sqrt{2}}]n_\alpha + [\frac{v\kappa_\alpha\tau_\alpha}{\sqrt{2}} - \frac{v\tau'_\alpha}{\sqrt{2}}]b_\alpha, \tag{3.22}$$

$$X_{sv}^3(s, v) = \frac{1}{\sqrt{2}}[\kappa_\alpha t_\alpha + \kappa_\alpha n_\alpha - \tau_\alpha b_\alpha],$$

$$X_{vv}^3(s, v) = 0.$$

We can conclude that the unit normal of N^3 can be given by:

$$N^3 = \frac{1}{C\sqrt{2}}[(1 - \frac{v}{\sqrt{2}}\tau_\alpha)(t_\alpha + n_\alpha) - \sqrt{2}v\kappa_\alpha b_\alpha]. \tag{3.23}$$

The coefficients of the first fundamental form, using Eq. (2.4), are

$$I = [\frac{v^2\tau_\alpha^2}{2} + v^2\kappa_\alpha^2 + 1 - \frac{2}{\sqrt{2}}v\tau_\alpha]ds^2 + dv^2.$$

Also, the coefficients of the second fundamental form using Eq. (2.5) are

$$II = \frac{1}{C\sqrt{2}}[[v\kappa'_\alpha(\sqrt{2}-v\tau_\alpha)+(1-\frac{v}{\sqrt{2}}\tau_\alpha)(-\tau_\alpha+\frac{v}{\sqrt{2}}\tau_\alpha^2)-v^2\kappa_\alpha(\kappa_\alpha\tau_\alpha-\tau'_\alpha)]ds^2 + 2\kappa_\alpha dsdv], \quad \square$$

Theorem 3.9. If $X^3(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the mean curvature and the Gaussian curvature are given, respectively, by:

$$H_{X^3(s,v)} = \frac{L_{11}}{2[(\frac{v^2\tau_\alpha^2}{2}) + v^2\kappa_\alpha^2 - \sqrt{2}v\tau_\alpha + 1]},$$

$$K_{X^3(s,v)} = \frac{-\kappa_\alpha^2}{C^2[v^2\kappa_\alpha^2 + 1 + \frac{v^2}{2}\tau_\alpha^2 - \sqrt{2}v\tau_\alpha]}.$$

Corollary 3.7. If $X^3(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the normal and geodesic curvatures for $\beta(s)$ on the surface are given, respectively, by:

$$\kappa_n^{X^3} = \frac{-\tau_\alpha}{\sqrt{2}C}(1 - \frac{v}{\sqrt{2}}\tau_\alpha), \quad \kappa_g^{X^3} = \frac{-\kappa_\alpha}{\sqrt{2}C}(1 - \frac{v}{\sqrt{2}}\tau_\alpha).$$

Corollary 3.8. If $X^3(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the surface $X^3(s, v)$ is

- developable (minimum surface) if and only if $\kappa_\alpha = 0$. (α is straight line.)
 - minimal if and only if
- $$v\kappa'_\alpha(\sqrt{2}-v\tau_\alpha) + (1 - \frac{v}{\sqrt{2}}\tau_\alpha)(-\tau_\alpha + \frac{v}{\sqrt{2}}\tau_\alpha^2) - v^2\kappa_\alpha(\kappa_\alpha\tau_\alpha - \tau'_\alpha) = 0.$$

Corollary 3.9. If $X^3(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, the curve $\beta(s)$ is

- geodesic if and only if $\tau_\alpha = \frac{\sqrt{2}}{v}$ or $\kappa_\alpha = 0$.
- asymptotic curve if and only if $\tau_\alpha = \frac{\sqrt{2}}{v}$ or $\tau_\alpha = 0$.

3.4. *tnb-Smarandache ruled surface of the integral binormal curves*

Definition 3.5. Let $\beta(s)$ be the integral normal curves and let X^4 be a normal ruled surface generated by $\beta(s)$ according to the Frenet frame. The parametric description of X^4 can be defined as:

$$X^4(s, v) = \beta(s) + \frac{v}{\sqrt{3}}[t_\beta + n_\beta + b_\beta], \tag{3.24}$$

where $\beta(s)$ denotes the base curve.

Theorem 3.10. If $X^4(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the striction line and distribution parameter are given, respectively, by:

$$\beta^{*4} = \int b_\alpha ds + \frac{1}{2\sqrt{3}} \frac{2\kappa_\alpha \kappa'_\alpha + 2\tau_\alpha \tau'_\alpha - \kappa_\alpha \tau'_\alpha - \kappa'_\alpha \tau_\alpha}{\kappa_\alpha^2 + \tau_\alpha^2 - \kappa_\alpha \tau_\alpha} (t_\alpha - n_\alpha + b_\alpha),$$

$$\lambda^4 = \frac{2\kappa_\alpha - \tau_\alpha}{2(\kappa_\alpha^2 + \tau_\alpha^2 - \kappa_\alpha \tau_\alpha)}.$$

Theorem 3.11. If $X^4(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the first and second fundamental forms are provided of $X^4(s, v)$ are given, respectively, by:

$$I = [\frac{2}{3}v^2(\kappa_\alpha^2 + \tau_\alpha^2 - \kappa_\alpha \tau_\alpha) + 1 - \frac{2}{\sqrt{3}}\tau_\alpha v]ds^2 + \frac{2}{\sqrt{3}}dsdv + dv^2, \tag{3.25}$$

$$II = \frac{1}{D\sqrt{3}} [(\frac{v}{3}\kappa_\alpha - \frac{2v}{3}\tau_\alpha + \frac{1}{\sqrt{3}})(v\kappa'_\alpha - v\kappa_\alpha^2 + v\tau_\alpha \kappa_\alpha) + (\frac{-v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{v\tau_\alpha}{3})(v\kappa_\alpha^2 + v\kappa'_\alpha - v\tau'_\alpha - \sqrt{3}\tau_\alpha + \tau_\alpha^2 v) - (\frac{2}{3}v\kappa_\alpha - \frac{v}{3}\tau_\alpha)(v\tau_\alpha \kappa_\alpha - v\tau_\alpha^2 - \tau'_\alpha v)]ds^2 + \frac{2}{3}(2\kappa_\alpha - \tau_\alpha)dsdv,$$

where $D = [\frac{v}{3}\kappa_\alpha - \frac{2v}{3}\tau_\alpha + \frac{1}{\sqrt{3}}]^2 + [-\frac{v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{v\tau_\alpha}{3}]^2 + [\frac{2v}{3}\kappa_\alpha - \frac{v}{3}\tau_\alpha]^2$.

Proof. Assuming that $X^4(s, v)$ is the tangent normal ruled surface, then the partial derivatives of $X^4(s, v)$ with respect to s and v are obtained by:

$$X^4_s(s, v) = [\frac{v\kappa'_\alpha}{\sqrt{3}}]t_\alpha - [\frac{v}{\sqrt{3}}(\kappa_\alpha - \tau_\alpha)]n_\alpha + [1 - \frac{v\tau'_\alpha}{\sqrt{3}}]b_\alpha,$$

$$X^4_v(s, v) = \frac{1}{\sqrt{3}}[t_\alpha - n_\alpha + b_\alpha],$$

$$X^4_{ss}(s, v) = [\frac{v\kappa''_\alpha}{\sqrt{3}} - \frac{v\kappa_\alpha^2}{\sqrt{3}} + \frac{v\tau_\alpha \kappa_\alpha}{\sqrt{3}}]t_\alpha + [\frac{v\kappa_\alpha^2}{\sqrt{3}} + \frac{v\kappa'_\alpha}{\sqrt{3}} - \frac{v\tau'_\alpha}{\sqrt{3}} - \tau_\alpha + \frac{v\tau_\alpha^2}{\sqrt{3}}]n_\alpha + [\frac{v\tau_\alpha \kappa_\alpha}{\sqrt{3}} - \frac{v\tau_\alpha^2}{\sqrt{3}} - \frac{v\tau'_\alpha}{\sqrt{3}}]b_\alpha,$$

$$X^4_{sv}(s, v) = \frac{1}{\sqrt{3}}[\kappa_\alpha t_\alpha + (\kappa_\alpha - \tau_\alpha)n_\alpha - \tau_\alpha b_\alpha],$$

$$X^4_{vv}(s, v) = 0.$$

We can conclude that the unit normal of N^4 can be given by:

$$N^4 = \frac{1}{D} [(\frac{1}{\sqrt{3}} - \frac{2}{3}v\tau_\alpha + \frac{v}{3}\kappa_\alpha)t_\alpha + (-\frac{v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{v}{3}\tau_\alpha)n_\alpha + (\frac{v}{3}\tau_\alpha - \frac{2v}{3}\kappa_\alpha)b_\alpha]. \tag{3.26}$$

The first fundamental form is denoted by:

$$I = [\frac{2}{3}v^2(\kappa_\alpha^2 + \tau_\alpha^2 - \kappa_\alpha \tau_\alpha) + 1 - \frac{2}{\sqrt{3}}\tau_\alpha v]ds^2 + \frac{2}{\sqrt{3}}dsdv + dv^2.$$

Also, the second fundamental form is given by:

$$II = \frac{1}{D\sqrt{3}} [(\frac{v}{3}\kappa_\alpha - \frac{2v}{3}\tau_\alpha + \frac{1}{\sqrt{3}})(v\kappa'_\alpha - v\kappa_\alpha^2 + v\tau_\alpha \kappa_\alpha) + (\frac{-v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{v\tau_\alpha}{3})(v\kappa_\alpha^2 + v\kappa'_\alpha - v\tau'_\alpha - \sqrt{3}\tau_\alpha + \tau_\alpha^2 v) - (\frac{2}{3}v\kappa_\alpha - \frac{v}{3}\tau_\alpha)(v\tau_\alpha \kappa_\alpha - v\tau_\alpha^2 - \tau'_\alpha v)]ds^2 + \frac{2}{3}(2\kappa_\alpha - \tau_\alpha)dsdv, \quad \square$$

Theorem 3.12. If $X^4(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the mean curvature and the Gaussian curvature are given, respectively, by:

$$H_{X^4(s,v)} = \frac{\frac{-2}{\sqrt{3}}(\frac{2\kappa_\alpha - \tau_\alpha}{3D}) + L_{11}}{2(\frac{2}{3}v^2(\kappa_\alpha^2 + \tau_\alpha^2 - \kappa_\alpha \tau_\alpha) + 1 - \frac{2}{\sqrt{3}}v\tau_\alpha - \frac{1}{3})},$$

$$K_{X^4(s,v)} = \frac{-(2\kappa_\alpha - \tau_\alpha)^2}{9D^2(\frac{2}{3}v^2(\kappa_\alpha^2 + \tau_\alpha^2 - \kappa_\alpha \tau_\alpha) + 1 - \frac{2}{\sqrt{3}}v\tau_\alpha - \frac{1}{3})}.$$

Corollary 3.10. If $X^4(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the normal and geodesic curvatures for $\beta(s)$ on the surface are given, respectively, by:

$$\kappa_n^{X^4} = \frac{-\tau_\alpha}{D}(-\frac{v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{v}{3}\tau_\alpha), \quad \kappa_g^{X^4} = \frac{-\kappa_\alpha}{D}(\frac{v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{2v}{3}\tau_\alpha).$$

Corollary 3.11. If $X^4(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then the surface $X^4(s, v)$ is

- developable (minimum surface) if and only if $2\kappa_\alpha - \tau_\alpha = 0$. (α is straight line.)
- minimal if and only if $\frac{-2}{3}(2\kappa_\alpha - \tau_\alpha) + (\frac{v}{3}\kappa_\alpha - \frac{2v}{3}\tau_\alpha + \frac{1}{\sqrt{3}})(v\kappa'_\alpha - v\kappa_\alpha^2 + v\tau_\alpha \kappa_\alpha)$

$$+ (\frac{-v}{3}\kappa_\alpha + \frac{1}{\sqrt{3}} - \frac{v\tau_\alpha}{3})(v\kappa_\alpha^2 + v\kappa'_\alpha - v\tau'_\alpha - \sqrt{3}\tau_\alpha + \tau_\alpha^2 v) - (\frac{2}{3}v\kappa_\alpha - \frac{v}{3}\tau_\alpha)(v\tau_\alpha \kappa_\alpha - v\tau_\alpha^2 - \tau'_\alpha v) = 0.$$

Corollary 3.12. If $X^4(s, v)$ is a Smarandache ruled surface generated by the binormal-integral curve $\beta(s)$, then curve $\beta(s)$ is

- geodesic if and only if $\kappa_\alpha = 0$ or $\frac{v}{3}\kappa_\alpha - \frac{2}{3}v\tau_\alpha + \frac{1}{\sqrt{3}} = 0$.
- asymptotic curve if and only if $\tau_\alpha = 0$ or $-\frac{v}{3}\kappa_\alpha - \frac{1}{3}v\tau_\alpha + \frac{1}{\sqrt{3}} = 0$.

4. Examples

Let $\alpha(s)$ be a circular helix in \mathbb{E}^3 parameterized by arc length s . The parametric equations of $\alpha(s)$ are:

$$\alpha(s) = (A \cos(\frac{s}{c}), A \sin(\frac{s}{c}), \frac{Bs}{c}),$$

where $c = \sqrt{A^2 + B^2}$, A is the radius of the helix, and B determines the pitch. The curvature κ_α and torsion τ_α of $\alpha(s)$ are:

$$\kappa_\alpha = \frac{A}{A^2 + B^2}, \quad \tau_\alpha = \frac{B}{A^2 + B^2}.$$

The binormal vector $\mathbf{b}_\alpha(s)$ of $\alpha(s)$ is:

$$\mathbf{b}_\alpha(s) = (\frac{B}{c} \sin(\frac{s}{c}), -\frac{B}{c} \cos(\frac{s}{c}), \frac{A}{c}).$$

The integral binormal curve $\beta(s)$ is obtained by integrating $\mathbf{b}_\alpha(s)$:

$$\beta(s) = \int \mathbf{b}_\alpha(s) ds = (-\frac{Bc}{A} \cos(\frac{s}{c}), -\frac{Bc}{A} \sin(\frac{s}{c}), \frac{As}{c}).$$

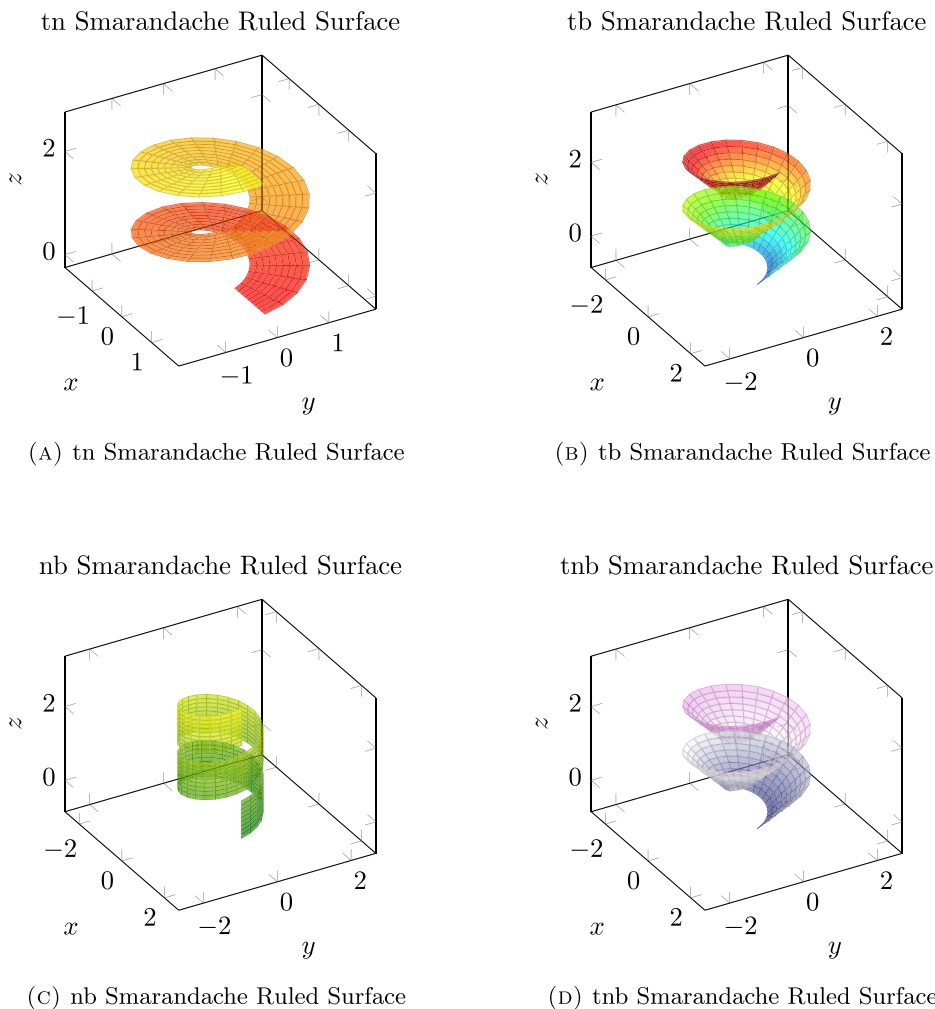


Fig. 1. Four types of Smarandache ruled surfaces.

Tangent vector $\mathbf{t}_\beta(s)$:

$$\mathbf{t}_\beta(s) = \beta'(s) = \left(\frac{B}{A} \sin\left(\frac{s}{c}\right), -\frac{B}{A} \cos\left(\frac{s}{c}\right), \frac{A}{c} \right).$$

The curvature $\kappa_\beta(s)$ is denoted by:

$$\kappa_\beta(s) = \|\mathbf{t}'_\beta(s)\| = \frac{B}{Ac}.$$

The normal vector $\mathbf{n}_\beta(s)$ is denoted by:

$$\mathbf{n}_\beta(s) = \frac{\mathbf{t}'_\beta(s)}{\|\mathbf{t}'_\beta(s)\|} = \left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0 \right).$$

The binormal vector $\mathbf{b}_\beta(s)$ is denoted by:

$$\mathbf{b}_\beta(s) = \mathbf{t}_\beta(s) \times \mathbf{n}_\beta(s) = \left(-\frac{A}{c} \sin\left(\frac{s}{c}\right), \frac{A}{c} \cos\left(\frac{s}{c}\right), \frac{B}{c} \right).$$

4.1. Smarandache ruled surfaces

Using the Frenet frame $\{\mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta\}$ of $\beta(s)$, we construct the four types of Smarandache ruled surfaces given in Fig. 1:

1. *tn* Smarandache ruled surface:

$$X^1(s, v) = \beta(s) + \frac{v}{\sqrt{2}} (\mathbf{t}_\beta(s) + \mathbf{n}_\beta(s)).$$

2. *tb* Smarandache ruled surface:

$$X^2(s, v) = \beta(s) + \frac{v}{\sqrt{2}} (\mathbf{t}_\beta(s) + \mathbf{b}_\beta(s)).$$

3. *nb* Smarandache ruled surface:

$$X^3(s, v) = \beta(s) + \frac{v}{\sqrt{2}} (\mathbf{n}_\beta(s) + \mathbf{b}_\beta(s)).$$

4. *tnb* Smarandache ruled surface:

$$X^4(s, v) = \beta(s) + \frac{v}{\sqrt{3}} (\mathbf{t}_\beta(s) + \mathbf{n}_\beta(s) + \mathbf{b}_\beta(s)).$$

These surfaces can now be analyzed for their geometric properties, such as Gaussian curvature, mean curvature, and developability, as described in the paper.

5. Conclusion

In this paper, we have explored the geometric properties of Smarandache ruled surfaces generated by integral binormal curves in Euclidean 3-space \mathbb{E}^3 . By focusing on four types of Smarandache ruled surfaces *tn*, *tb*, *nb*, and *tnb*, we derived their parametric representations and computed fundamental geometric properties, including striction lines, distribution parameters, and the first and second fundamental forms. Additionally, we provided explicit expressions for the Gaussian and mean curvatures, which characterize the local shape of these surfaces. Our analysis also included the geodesic curvature, normal curvature, and geodesic torsion associated with the base curves on these surfaces.

One of the key contributions of this work is the establishment of necessary and sufficient conditions for these surfaces to be developable or minimal. We also classified the base curves as geodesic or asymptotic lines under specific conditions.

The results presented in this paper not only advance the theoretical understanding of Smarandache ruled surfaces but also provide practical tools for their application in various scientific and engineering disciplines. The conditions for developability, minimality, and the classification of base curves as geodesic or asymptotic lines offer valuable insights for applications in geometry and engineering.

While this paper has laid a solid foundation for the study of Smarandache ruled surfaces generated by integral binormal curves, several avenues for future research remain open. One promising direction is the extension of this work to higher-dimensional Euclidean spaces or non-Euclidean spaces, such as Minkowski space. Investigating the geometric properties of Smarandache ruled surfaces in these spaces could yield new insights and broaden the scope of their applications.

Another area of future research is the exploration of the applications of these surfaces in physics and engineering. For example, further studies could focus on the role of Smarandache ruled surfaces in the design of minimal surfaces and developable surfaces, which have important applications in material science, architecture, and robotics.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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