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Higher-Degree Asymptotes of a Rational-Polynomial Function

Florentin Smarandache
Mathematics, Physics, and Natural Science Division
University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA
E-mail: smarand@unm.edu

Abstract: By a straight-forward method we extend the horizontal and slant asymptotes to the higher -degree asymptotes of a function, and we give several examples and prove a theorem.

Keywords: Horizontal Asymptote, Slant Asymptote, Higher-Degree Asymptotes

1. Introduction

Let $f : R \to R$ be a rational function, where R is the set of real numbers, with the numerator $P_m(x)$ and the denominator $P_n(x)$ being polynomials:

$$f(x) = \frac{P_m(x)}{P_n(x)} = \frac{a_m x^m + a_{m-1}x^{m-1} + ... + a_1 x^1 + a_0}{b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x^1 + b_0}$$, where $a_i \in R, 0 \le i \le m,$ and $b_j \in R, 0 \le j \le n$, and $m \ge 0, n \ge 1$ are integers, with $a_m \ne 0, b_n \ne 0$.

(i) Horizontal Asymptote (Degree Zero)

If $m < n$ then the function f(x) has the horizontal asymptote
A(x) = 0 (the x-axis line).

If $m = n$, then the function f(x) has the horizontal asymptote

$$A(x) = \frac{a_n}{b_n},$$

which is also a line.

Yet, $\frac{a_n}{b_n}$ is the quotient of the division of function's numerator by its denominator:

$$(a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0) \div (b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x^1 + b_0).$$

{The division's remainder does not interest us.}

(ii) Slant Asymptote (Degree One)

If $m = n + 1$, we also divide the numerator by the denominator,

$$(a_{n+1} x^{n+1} + a_n x^n + ... + a_1 x^1 + a_0) \div (b_n x^n + b_{n-1}x^{n-1} + ... + b_1 x^1 + b_0)$$

and we get the quotient as degree one slant line:

$$A(x) = \frac{a_{n+1}}{b_n} x + \frac{a_n b_n - a_{n+1} b_{n-1}}{b_n^2}$$

(iii) **Parabolic Asymptote (Degree Two)**

If $m = n + 2$, dividing the numerator by the denominator, we get a quotient of degree two (a parabola):

$A(x) = c_2 x^2 + c_1 x + c_0$, where $c_2, c_1, c_0 \in R$, and $c_2 \neq 0$.

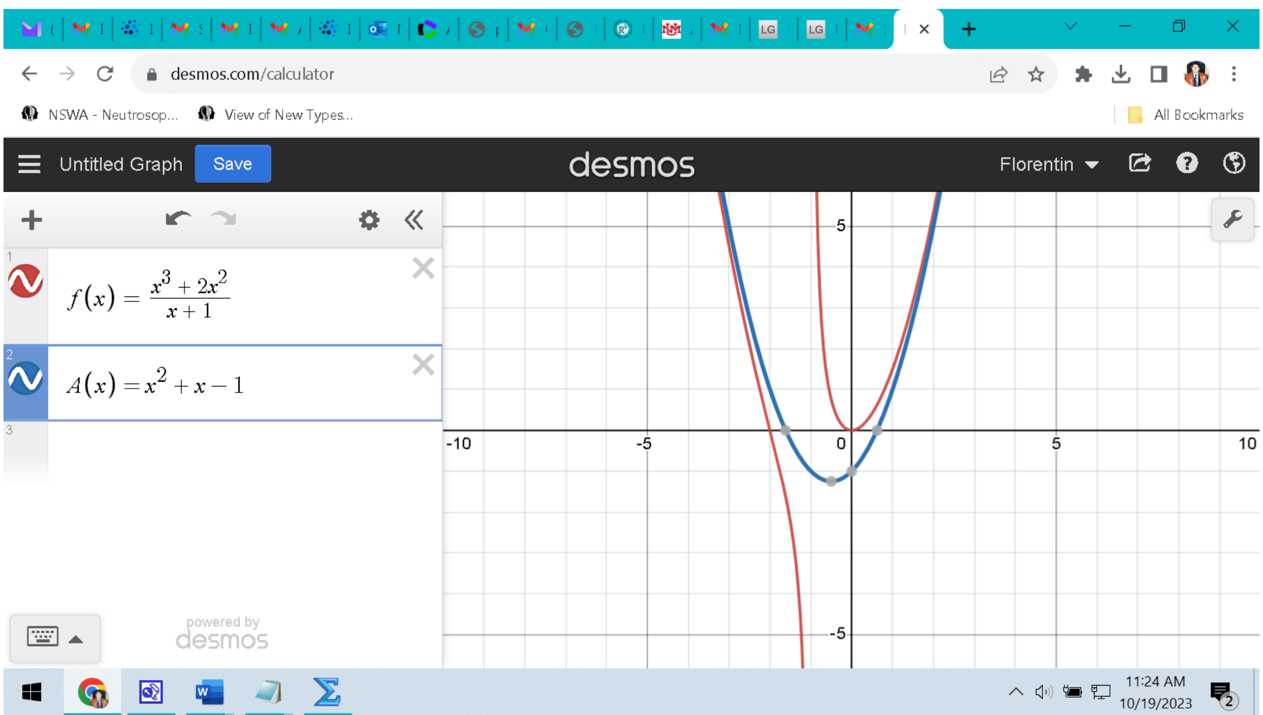
Example:

$$f(x) = \frac{x^3 + 2x^2}{x+1}$$

has a parabolic asymptote:

$$A(x) = x^2 + x - 1$$

See the below graphs:



(iv) **Cubic Asymptote (Degree Three)**

If $m = n + 3$, dividing the numerator by the denominator, we get a quotient of degree three (a cubic function).

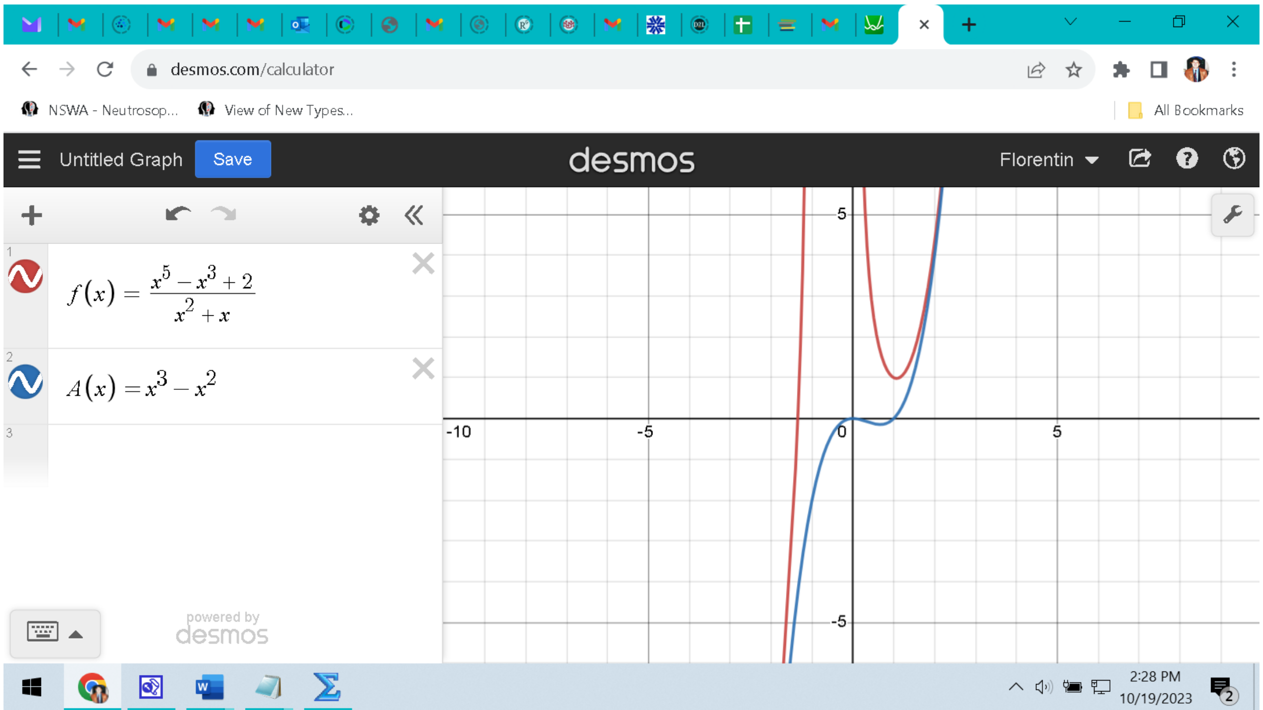
Example:

$$f(x) = \frac{x^5 - x^3 + 2}{x^2 + x}$$

has a cubic asymptote:

$$A(x) = x^3 - x^2$$

See the below graphs:



(v) In general, the **Higher-Degree Asymptote (Degree k ≥ 0)**.
If $m = n + k$, dividing the numerator by the denominator, we get a quotient of degree k. Thus, the k-Degree Asymptote has the form:

$$A(x) = c_k x^k + c_{k-1} x^{k-1} + ... + c_1 x + c_o$$

2. Theorem

Let *f(x)* be a rational function whose numerator and denominator are polynomials, and *A(x)* be its Higher-Degree Asymptote of degree $k \geq 0$, where k is an integer:

Then:

$$\lim_{x \to \pm\infty} [f(x) - A(x)] = 0.$$

Proof

It is obvious that the function f(x) is gradually approaching its asymptote when x approaches positive and negative infinity, which is just the definition of the asymptote in general.

Let's show it using calculus:

Assume $f(x) = \frac{P_m(x)}{P_n(x)} = \frac{a_m x^m + a_{m-1} x^{m-1} + ... + a_1 x^1 + a_0}{b_n x^n + b_{n-1} x^{n-1} + ... + b_1 x^1 + b_0}$.

By division one gets: $\frac{P_m(x)}{P_n(x)} = A(x) + \frac{B_r(x)}{P_n(x)}$, where *A(x)* is the quotient polynomial (which coincides with the asymptote), and $B_r(x)$ is the remainder polynomial of degree $r < n$.

Whence one has:

$$\lim_{x\to\pm\infty}[f(x)-A(x)] = \lim_{x\to\pm\infty}[\frac{P_m(x)}{P_n(x)}-A(x)] = \lim_{x\to\pm\infty}[(A(x)+\frac{B_r(x)}{P_n(x)})-A(x)] = \lim_{x\to\pm\infty}[\frac{B_r(x)}{P_n(x)}] = 0$$

Because the degree of the top polynomial is strictly smaller than the degree of the bottom polynomial, $r < n$.

Reference

[1] William L. Briggs, Lyle Cochran, Bernard Gillett, Eric P. Schulz, Calculus. Early Transcendentals, Pearson, New York, NY, USA, 2019.