

## THE HYPERBOLIC MENELAUS THEOREM IN THE POINCARÉ DISC MODEL OF HYPERBOLIC GEOMETRY

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**Abstract.** In this note, we present the hyperbolic Menelaus theorem in the Poincaré disc of hyperbolic geometry.

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### 1. Introduction

Hyperbolic Geometry appeared in the first half of the 19<sup>th</sup> century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-euclidian geometry. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. Here, in this study, we present a proof of Menelaus's theorem in the Poincaré disc model of hyperbolic geometry.

The well-known Menelaus theorem states that if  $l$  is a line not through any vertex of a triangle  $ABC$  such that  $l$  meets  $BC$  in  $D$ ,  $CA$  in  $E$ , and  $AB$  in  $F$ , then  $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$  [1]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by A. Johnson [2], N.A. Court [3], C. Coșniță [4], A. Ungar [5]. F. Smarandache (1983) has generalized the Theorem of Menelaus for any polygon with  $n \geq 4$  sides as follows: If a line  $l$  intersects the  $n$ -gon  $A_1A_2\dots A_n$  sides  $A_1A_2, A_2A_3, \dots$ , and  $A_nA_1$  respectively in the points  $M_1, M_2, \dots$ , and  $M_n$ , then  $\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \dots \cdot \frac{M_nA_n}{M_nA_1} = 1$  [6].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let  $D$  denote the unit disc in the complex  $z$ -plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of  $D$  is

$$z \rightarrow e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z},$$

which induces the Möbius addition  $\oplus$  in  $D$ , allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \rightarrow z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}$$

followed by a rotation. Here  $\theta \in \mathbb{R}$  is a real number,  $z, z_0 \in D$ , and  $\bar{z}_0$  is the complex conjugate of  $z_0$ . Let  $Aut(D, \oplus)$  be the automorphism group of the grupoid  $(D, \oplus)$ . If we define

$$gyr : D \times D \rightarrow Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space  $(G, \oplus, \otimes)$  is a gyro-commutative gyro-group  $(G, \oplus)$  that obeys the following axioms:

- (1)  $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$  for all points  $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$ .
- (2)  $G$  admits a scalar multiplication,  $\otimes$ , possessing the following properties.  
For all real numbers  $r, r_1, r_2 \in \mathbb{R}$  and all points  $\mathbf{a} \in G$ :

$$(G1) \quad 1 \otimes \mathbf{a} = \mathbf{a}$$

$$(G2) \quad (r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$$

$$(G3) \quad (r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$$

$$(G4) \quad \frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

$$(G5) \quad \text{gyr}[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}$$

$$(G6) \quad \text{gyr}[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$$

- (3) Real vector space structure  $(\|G\|, \oplus, \otimes)$  for the set  $\|G\|$  of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition  $\oplus$  and scalar multiplication  $\otimes$ , such that for all  $r \in \mathbb{R}$  and  $\mathbf{a}, \mathbf{b} \in G$ ,

$$(G7) \quad \|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$$

$$(G8) \quad \|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|.$$

**Theorem 1 (The law of gyrosines in Möbius gyrovector spaces).** *Let  $ABC$  be a gyrotriangle in a Möbius gyrovector space  $(V_s, \oplus, \otimes)$  with vertices  $A, B, C \in V_s$ , sides  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$ , and side gyrolengths  $a, b, c \in (-s, s)$ ,  $\mathbf{a} = \ominus B \oplus C$ ,  $\mathbf{b} = \ominus C \oplus A$ ,  $\mathbf{c} = \ominus A \oplus B$ ,  $a = \|\mathbf{a}\|$ ,  $b = \|\mathbf{b}\|$ ,  $c = \|\mathbf{c}\|$ , and with gyroangles  $\alpha, \beta$ , and  $\gamma$  at the vertices  $A, B$ , and  $C$ . Then*

$$\frac{a_\gamma}{\sin \alpha} = \frac{b_\gamma}{\sin \beta} = \frac{c_\gamma}{\sin \gamma},$$

where  $v_\gamma = \frac{v}{1 - \frac{v^2}{s^2}}$  [7, p. 267].

**Definition 2** The hyperbolic distance function in  $D$  is defined by the equation

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \bar{a}b} \right|.$$

Here,  $a \ominus b = a \oplus (-b)$ , for  $a, b \in D$ .

For further details we refer to the recent book of A.Ungar [5].

## 2. Main results

In this section, we prove the Menelaus's theorem in the Poincaré disc model of hyperbolic geometry.

**Theorem 3 (The Menelaus's Theorem for Hyperbolic Gyrotriangle).** *If  $l$  is a gyroline not through any vertex of an gyrotriangle  $ABC$  such that  $l$  meets  $BC$  in  $D$ ,  $CA$  in  $E$ , and  $AB$  in  $F$ , then*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

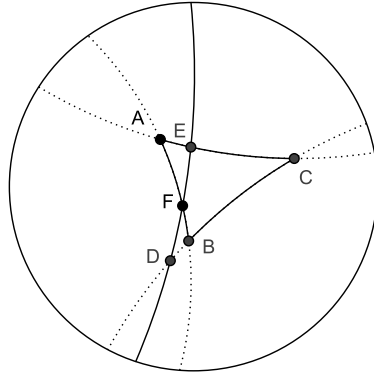


Figure 1

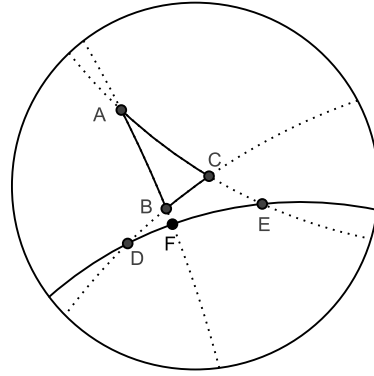


Figure 2

**Proof.** In function of the position of the gyroline  $l$  intersect internally a side of  $ABC$  triangle and the other two externally (See Figure 1), or the line  $l$  intersect all three sides externally (See Figure 2).

If we consider the first case, the law of gyrosines (See Theorem 1), gives for the gyrotriangles  $AEF$ ,  $BFD$ , and  $CDE$ , respectively

$$(1) \quad \frac{(AE)_\gamma}{(AF)_\gamma} = \frac{\sin \widehat{AFE}}{\sin \widehat{AEF}},$$

$$(2) \quad \frac{(BF)_\gamma}{(BD)_\gamma} = \frac{\sin \widehat{FDB}}{\sin \widehat{DFB}},$$

and

$$(3) \quad \frac{(CD)_\gamma}{(CE)_\gamma} = \frac{\sin \widehat{DEC}}{\sin \widehat{EDC}},$$

where  $\sin \widehat{AFE} = \sin \widehat{DFB}$ ,  $\sin \widehat{EDC} = \sin \widehat{FDB}$ , and  $\sin \widehat{AEF} = \sin \widehat{DEC}$ , since gyroangles  $\widehat{AEF}$  and  $\widehat{DEC}$  are supplementary. Hence, by (1), (2) and (3), we have

$$(4) \quad \frac{(AE)_\gamma}{(AF)_\gamma} \cdot \frac{(BF)_\gamma}{(BD)_\gamma} \cdot \frac{(CD)_\gamma}{(CE)_\gamma} = \frac{\sin \widehat{AFE}}{\sin \widehat{AEF}} \cdot \frac{\sin \widehat{FDB}}{\sin \widehat{DFB}} \cdot \frac{\sin \widehat{DEC}}{\sin \widehat{EDC}} = 1,$$

the conclusion follows. The second case is treated similar to the first. ■

Naturally, one may wonder whether the converse of the Menelaus theorem exists.

**Theorem 4 (Converse of Menelaus's Theorem for Hyperbolic Gyro-triangle).** *If  $D$  lies on the gyroline  $BC$ ,  $E$  on  $CA$ , and  $F$  on  $AB$  such that*

$$(5) \quad \frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

*then  $D, E$ , and  $F$  are collinear.*

**Proof.** Relabelling if necessary, we may assume that the gyropoint  $D$  lies beyond  $B$  on  $BC$ . If  $E$  lies between  $C$  and  $A$ , then the gyroline  $ED$  cuts the gyroside  $AB$ , at  $F'$  say. Applying Menelaus's theorem to the gyrotriangle  $ABC$  and the gyroline  $E - F' - D$ , we get

$$(6) \quad \frac{(AF')_\gamma}{(BF')_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

From (5) and (6), we get  $\frac{(AF)_\gamma}{(BF)_\gamma} = \frac{(AF')_\gamma}{(BF')_\gamma}$ . This equation holds for  $F = F'$ . Indeed, if we take  $x := |\ominus A \oplus F'|$  and  $c := |\ominus A \oplus B|$ , then we get  $c \ominus x = |\ominus F' \oplus B|$ . For  $x \in (-1, 1)$  define

$$(7) \quad f(x) = \frac{x}{1-x^2} : \frac{c \ominus x}{1-(c \ominus x)^2}.$$

Because  $c \ominus x = \frac{c-x}{1-cx}$ , then  $f(x) = \frac{x(1-c^2)}{(c-x)(1-cx)}$ . Since the following equality holds

$$(8) \quad f(x) - f(y) = \frac{c(1-c^2)(1-xy)}{(c-x)(1-cx)(c-y)(1-cy)}(x-y),$$

we get  $f(x)$  is an injective function and this implies  $F = F'$ , so  $D, E, F$  are collinear.

There are still two possible cases. The first is if we suppose that the gyropoint  $F$  lies on the gyroside  $AB$ , then the gyroline  $DF$  cuts the gyrosegment  $AC$  in the gyropoint  $E'$ . The second possibility is that  $E$  is not on the gyroside  $AC$ ,  $E$  lies beyond  $C$ . Then  $DE$  cuts the gyroline  $AB$  in the gyropoint  $F'$ . In each case a similar application of Menelaus gives the result. ■

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