INTEGER NUMBER SOLUTIONS OF LINEAR SYSTEMS

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Definitions and Properties of the Integer Solution of a Linear System

Let's consider

(1) $$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = \overline{1,m}$$

a linear system with all coefficients being integer numbers (the case with rational coefficients is reduced to the same).

Definition 1. $x_j = x_j^0,\ j = \overline{1,n}$ is a particular integer solution of (1) if $x_j^0 \in \mathbf{Z},\ j = \overline{1,n}$ and $\sum_{j=1}^{n} a_{ij} x_j^0 = b_i,\ i = \overline{1,m}$.

Let's consider the functions $f_j : \mathbf{Z}^h \to \mathbf{Z},\ j = \overline{1,n}$, where $h \in \mathrm{N}^*$.

Definition 2. $x_j = f_j(k_1,...,k_h),\ j = \overline{1,n}$ is the general integer solution for (1) if:

(a) $\sum_{j=1}^{n} a_{ij} f_j(k_1,...,k_h) = b_i,\ i = \overline{1,m}$, irrespective of $k_1,...,k_h \in \mathbf{Z}$;

(b) Irrespective of $x_j = x_j^0,\ j = \overline{1,n}$ a particular integer solution of (1) there is $(k_1^0,...,k_h^0) \in \mathbf{Z}$ such that $f_j(k_1^0,...,k_h^0) = x_j,\ j = \overline{1,n}$. (In other words the general solution that comprises all the other solutions.)

Property 1.
A general solution of a linear system of m equations with n unknowns, $r(A) = m < n$, is undetermined $n - m$ -times.

Proof:
We assume by reduction ad absurdum that it is of order r, $1 \le r \le n - m$ (the case $r = 0$, i.e., when the solution is particular, is trivial). It follows that the general solution is of the form:

(S$_1$)

$$\begin{cases} x_1 = u_{11}p_1 + \ldots + u_{1r}p_r + v_1 \\ \vdots \\ x_n = u_{n1}p_1 + \ldots + u_{nr}p_r + v_n, \quad u_{ih}, \ \forall i \in \mathbf{Z} \\ p_h = \text{parameters} \in \mathbf{Z} \end{cases}$$

We prove that the solution is undetermined $n-m$ -times.

The homogeneous linear system (1), resolved in r has the solution:

$$\begin{cases} x_1 = \dfrac{D^1_{m+1}}{D}x_{m+1} + \ldots + \dfrac{D^1_n}{D}x_n \\ \vdots \\ x_m = \dfrac{D^m_{m+1}}{D}x_{m+1} + \ldots + \dfrac{D^m_n}{D}x_n \end{cases}$$

Let $x_i = x_i^0, \ i = \overline{1,n}$, be a particular solution of the linear system (1).

Considering

$$\begin{cases} x_{m+1} = D \cdot k_{m+1} \\ \vdots \\ x_n = D \cdot k_n \end{cases}$$

we obtain the solution

$$\begin{cases} x_1 = D^1_{m+1} \cdot k_{m+1} + \ldots + D^1_n \cdot k_n + x_1^0 \\ \vdots \\ x_m = D^m_{m+1} \cdot k_{m+1} + \ldots + D^m_n \cdot k_n + x_m^0 \\ x_{m+1} = D \cdot k_{m+1} + x_{m+1}^0 \\ \vdots \\ x_n = D \cdot k_n + x_n^0, \qquad\qquad k_j = \text{parameters} \in \mathbf{Z} \end{cases}$$

which depends on the $n-m$ independent parameters, for the system (1). Let the solution be undetermined $n-m$ -times:

(S$_2$)

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1 \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + u_{nn-m}k_{n-m} + d_n \\ c_{ij}, \ d_i \in \mathbf{Z}, \ k_j = \text{parameters} \in \mathbf{Z} \end{cases}$$

(There are such solutions, we have proved it before.) Let the system be:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

x_i = unknowns $\in \mathbf{Z}$, $a_{ij}, \ b_i \in \mathbf{Z}$.

I. The case $b_i = 0, \ i = \overline{1,m}$ results in a homogenous linear system:

$$\begin{aligned}
&a_{i1}x_i + \ldots + a_{in}x_n = 0;\ i = \overline{1,m}.\\
(S_2)\quad &\Rightarrow a_{i1}\left(c_{i1}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1\right) + \ldots + a_{in}\left(c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n\right) = 0\\
&0 = \left(a_{i1}c_{11} + \ldots + a_{in}c_{n1}\right)k_1 + \ldots + \left(a_{i1}c_{1n-m} + \ldots + a_{in}c_{nn-m}\right)k_{n-m} + \left(a_{i1}d_1 + \ldots + a_{in}d_n\right)\\
&\forall k_j \in \mathbf{Z}
\end{aligned}$$

For $k_1 = \ldots = k_{n-m} = 0 \Rightarrow a_{i1}d_1 + \ldots + a_{in}d_n = 0$.

For $k_1 = \ldots = k_{h-1} = k_{h+1} = \ldots = k_{n-m} = 0$ and $k_h = 1 \ \Rightarrow$

$\Rightarrow \left(a_{i1}c_{ih} + \ldots + a_{in}c_{nh}\right) + \left(a_{i1}d_1 + \ldots + a_{in}d_d^{(n)}\right) = 0 \Rightarrow$

$a_{i1}c_{ih} + \ldots + a_{in}c_{nh} = 0,\ \forall\ i = \overline{1,m},\ \forall\ h = \overline{1,n-m}$.

The vectors

$$V_h = \begin{pmatrix} c_{1h} \\ : \\ : \\ c_{nh} \end{pmatrix},\ \ h = \overline{1,n-m}$$

are the particular solutions of the system.

$V_h,\ h = \overline{1,n-m}$ also linearly independent because the solution is undetermined $n-m$ -times $V_1,\ldots,V_{n-m}\ +d$ is a linear variety that includes the solutions of the system obtained from (S_2).

Similarly for (S_1) we deduce that

$$U_s = \begin{pmatrix} U_{1s} \\ : \\ : \\ U_{ns} \end{pmatrix},\ s = \overline{1,r}$$

are particular solutions of the given system and are linearly independent, because (S_1) is undetermined $n-m$ -times, and $V = \begin{pmatrix} V_1 \\ : \\ : \\ V_n \end{pmatrix}$ is a solution of the given system.

Case (a) $U_1,\ldots,U_r,\ V$ = linearly dependent, it follows that $U_1,\ldots,U_r$ is a free sub-module of order $r < n - m$ of solutions of the given system, then, it follows that there are solutions that belong to $V_1,\ldots,V_{n-m}\ +d$ and which do not belong to $U_1,\ldots,U_r$, a fact which contradicts the assumption that (S_1) is the general solution.

Case (b) $U_1,\ldots,U_r,\ V$ = linearly independent.

$U_1,\ldots,U_r$ +V is a linear variety that comprises the solutions of the given system, which were obtained from (S_1). It follows that the solution belongs to $V_1,\ldots,V_{n-m}\ +d$ and does not belong to $\langle U_1,...,U_r\rangle$ +V , a fact which is a contradiction to the assumption that (S₁) is the general solution.

II. When there is an $i\in\overline{1,m}$ with $b_i \neq 0$ then non-homogeneous linear system

$$a_{i1}x_i + ... + a_{in}x_n = b_1,\ \ i=\overline{1,m}$$

(S₂) $\Rightarrow a_{i1}\left(c_{11}k_1 + ... + c_{1n-m}k_{n-m} + d_1\right) + ... + a_{in}\left(c_{n1}k_1 + ... + c_{nn-m}k_{n-m} + d_n\right) = b_i$

it follows that

$$\Rightarrow \left(a_{i1}c_{11} + ... + a_{in}c_{n1}\right)k_1 + ... + \left(a_{i1}c_{1n-m} + ... + a_{in}c_{nn-m}\right)k_{n-m} + \left(a_{i1}d_1 + ... + a_{in}d_n\right) = b_i$$

For $k_1 = ... = k_{n-m} = 0 \Rightarrow a_{i1}d_1 + ... + a_{in}d_n = b_1$;

For $k_1 = ... = k_{j-1} = k_{j+1} = ... = k_{n-m} = 0$ and $k_j = 1 \Rightarrow$

$\Rightarrow$ $a_{i1}c_{1j} + ... + a_{in}c_{nj}$ + $a_{in}d_1 + ... + a_{in}d_n$ $= b_i$ it follows that

$$\begin{cases} a_{i1}c_{1j} + ... + a_{in}c_{nj} = 0 \\ a_{i1}d_1 + ... + a_{in}d_n = b_i \end{cases};\ \ \forall i=\overline{1,m},\ \forall j=\overline{1,n-m}.$$

$V_j = \begin{pmatrix} c_{1j} \\ : \\ c_{nj} \end{pmatrix}$, $j=\overline{1,n-m}$, are linearly independent because the solution (S₂) is undetermined $n-m$ -times.

(?!) $V_j,\ j=\overline{1,n-m}$, and $d = \begin{pmatrix} d_1 \\ : \\ d_n \end{pmatrix}$

are linearly independent.

We assume that they are not linearly independent. It follows that

$$d = s_1V_1 + ... + s_{n-m}V_{n-m} = \begin{pmatrix} s_1c_{11} + ... + s_{n-m}c_{1n-m} \\ : \\ s_1c_{n1} + ... + s_{n-m}c_{nn-m} \end{pmatrix}.$$

Irrespective of $i=\overline{1,m}$:

$$b_1 = a_{i1}d_1 + ... + a_{in}d_n = a_{i1}\left(s_1c_{11} + ... + s_{n-m}c_{1n-m}\right) + ... + a_{in}\left(s_1c_{n1} + ... + s_{n-m}c_{nn-m}\right) =$$
$$= \left(a_{i1}c_{11} + ... + a_{in}c_{n1}\right)s_1 + ... + \left(a_{i1}c_{1n-m} + ... + a_{in}c_{nn-m}\right)s_{n-m} = 0.$$

Then, $b_i = 0$, irrespective of $i=\overline{1,m}$, contradicts the hypothesis (that there is an $i\in\overline{1,m}$, $b_i \neq 0$). It follows that $V_1,...,V_{n-m}, d$ are linearly independent.

$\langle V_1,...,V_{n-m}\rangle + d$ is a linear variety that contains the solutions of the non-homogeneous system, solutions obtained from (S₂). Similarly it follows that $\langle G_1,...,G_r\rangle + V$ is a linear variety containing the solutions of the non-homogeneous system, obtained from (S₁).

n - $m > r$ it follows that there are solutions of the system that belong to

"?!" means "to prove that"

$V_1,\ldots,V_{n-m} \; +d$ and which do not belong to $G_1,\ldots,G_r \; +V$, this contradicts the fact that (S_1) is the general solution. Then, it shows that the general solution depends on the $n-m$ independent parameters.

Theorem 1. The general solution of a non-homogeneous linear system is equal to the general solution of an associated linear system plus a particular solution of the non-homogeneous system.

Proof:

Let's consider the homogeneous linear solution:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ : \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases}, \quad (AX = 0)$$

with the general solution:

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1 \\ : \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n \end{cases}$$

and

$$\begin{cases} x_1 = x_1^0 \\ : \\ x_n = x_n^0 \end{cases}$$

with the general solution a particular solution of the non-homogeneous linear system $AX = b$;

$$(?!) \qquad \begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d + x_1^0 \\ : \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n + x_n^0 \end{cases}$$

is a solution of the non-homogeneous linear system.

We note:

$$A = \begin{pmatrix} a_{11}\ldots a_{1n} \\ : \\ a_{m1}\ldots a_{mn} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ : \\ x_n \end{pmatrix}, \; b = \begin{pmatrix} b_1 \\ : \\ b_m \end{pmatrix}, \; 0 = \begin{pmatrix} 0 \\ : \\ 0 \end{pmatrix}$$

(vector of dimension m),

$$K = \begin{pmatrix} k_1 \\ : \\ k_{n-m} \end{pmatrix}, \; C = \begin{pmatrix} c_{11}\ldots c_{1n-m} \\ : \\ c_{n1}\ldots c_{nn-m} \end{pmatrix}, \; d = \begin{pmatrix} d_1 \\ : \\ d_n \end{pmatrix}, \; x^0 = \begin{pmatrix} x_1^0 \\ : \\ x_n^0 \end{pmatrix};$$

$$AX = A\left(Ck + d + x^0\right) = A\left(Ck + d\right) + AX^0 = b + 0 = b$$

We will prove that irrespective of

$$x_1 = y_1^0$$
$$\vdots$$
$$x_n = y_n^0$$

there is a particular solution of the non-homogeneous system

$$\begin{cases} k_1 = k_1^0 \in \mathbf{Z} \\ \vdots \\ k_{n-m} = k_{n-m}^0 \in \mathbf{Z} \end{cases},$$

with the property:

$$\begin{cases} x_1 = c_{11}k_1^0 + \ldots + c_{1n}k_{n-m}^0 + d_1 + x_1^0 = y_1^0 \\ \vdots \\ x_n = c_{n1}k_1^0 + \ldots + c_{nn-m}k_{n-m}^0 + d_1 + x_n^0 = y_n^0 \end{cases}$$

We note $Y^0 = \begin{pmatrix} y_1^0 \\ \vdots \\ y_n^0 \end{pmatrix}$.

We'll prove that those $k_j^0 \in \mathbf{Z},\ j = \overline{1, n-m}$ are those for which $A\left(CX^0 + d\right) = 0$ (there are such $X_j^0 \in \mathbf{Z}$ because

$$\begin{cases} x_1 = 0 \\ \vdots \\ x_n = 0 \end{cases}$$

is a particular solution of the homogeneous linear system and $X = CK + d$ is a general solution of the non-homogeneous linear system)

$$A\ CK^0 + d + X^0 - Y^0\ = A\ CK^0 + d\ + AX^0 - AY^0 = 0 + b - b = 0\ .$$

Property 2 The general solution of the homogeneous linear system can be written under the form:

(SG)

(2) $$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} \end{cases}$$

k_j is a parameter that belongs to $\mathbf{Z}$ (with $d_1 = d_2 = \ldots = d_n = 0$).

Poof:

(SG) = general solution. It results that (SG) is undetermined $(n-m)$-times.

Let's consider that (SG) is of the form

$$(3)\qquad \begin{cases} x_1 = c_{11}p_1 + \ldots + c_{1n-m}p_{n-m} + d_1 \\ \vdots \\ x_n = c_{n1}p_1 + \ldots + c_{nn-m}p_{n-m} + d_n \end{cases}$$

with not all $d_i = 0$; we'll prove that it can be written under the form (2); the system has the trivial solution

$$\begin{cases} x_1 = 0 \in \mathbf{Z} \\ \vdots \\ x_n = 0 \in \mathbf{Z} \end{cases};$$

it results that there are $p_j \in \mathbf{Z},\ j = \overline{1, n-m}$,

$$(4)\qquad \begin{cases} x_1 = c_{11}p_1^0 + \ldots + c_{1n-m}p_{n-m}^0 + d_1 = 0 \\ \vdots \\ x_n = c_{n1}p_1^0 + \ldots + c_{nn-m}p_{n-m}^0 + d_n = 0 \end{cases}$$

Substituting $p_j = k_j + p_j^0,\ j = \overline{1, n-m}$ in (3)

$$\left.\begin{array}{l} k_j \in \mathbf{Z} \\ p_j^0 \in \mathbf{Z} \end{array}\right\} \Rightarrow p_j \in \mathbf{Z},$$

$$\left.\begin{array}{l} p_j \in \mathbf{Z} \\ p_j^0 \in \mathbf{Z} \end{array}\right\} \Rightarrow k_j = p_j - p_j^0 \in \mathbf{Z}$$

which means that that they do not make any restrictions.

It results that

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + \left(c_{11}p_1^0 + \ldots + c_{1n-m}p_{n-m}^0 + d_1\right) \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + \left(c_{n1}p_1^0 + \ldots + c_{nn-m}p_{n-m}^0 + d_n\right) \end{cases}$$

But

$$c_{h1}p_1^0 + \ldots + c_{hn-m}p_{n-m}^0 + d_h = 0,\ h = \overline{1, n} \text{ (from (4))}.$$

Then the general solution is of the form:

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} \end{cases}$$

k_j = parameters $\in \mathbf{Z}$, $j = \overline{1, n-m}$; it results that $d_1 = d_2 = \ldots = d_n = 0$.

Theorem 2. Let's consider the homogeneous linear system:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases},$$

$r(A) = m$, $(a_{h1}, \ldots, a_{hn}) = 1$, $h = \overline{1, m}$ and the general solution

$$\begin{cases} x_1 = c_{11}k_1 + ... + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + ... + c_{nn-m}k_{n-m} \end{cases}$$

then

$$(a_{h1}, ..., a_{hi-1}, a_{hi+1}, ..., a_{hn}) \mid (c_{i1}, ..., c_{in-m})$$

irrespective of $h = \overline{1, m}$ and $i = \overline{1, n}$.

Proof:

Let's consider some arbitrary $h \in \overline{1, m}$ and some arbitrary $i \in \overline{1, n}$;

$$a_{h1}x_1 + ... + a_{hi-1}x_{i-1} + a_{hi+1}x_{i+1} + ... + a_{hn}x_n = a_{hi}x_i .$$

Because

$$(a_{h1}, ..., a_{hi-1}, a_{hi+1}, ..., a_{hn}) \mid a_{hi}$$

it results that

$$d = (a_{h1}, ..., a_{hi-1}, a_{hi+1}, ..., a_{hn}) \mid x_i$$

irrespective of the value of x_i in the vector of particular solutions.

For $k_2 = k_3 = ... = k_{n-m} = 0$ and $k_1 = 1$ we obtain the particular solution:

$$\begin{cases} x_1 = c_{11} \\ \vdots \\ x_i = c_{i1} \Rightarrow d \mid c_{i1} \\ \vdots \\ x_n = c_{n1} \end{cases}$$

For $k_1 = k_2 = ... = k_{n-m-1} = 0$ and $k_{n-m} = 1$ it results the following particular solution:

$$\begin{cases} x_1 = c_{1n-m} \\ \vdots \\ x_i = c_{in-m} \Rightarrow d \mid c_{in-m}; \\ \vdots \\ x_n = c_{nn-m} \end{cases}$$

hence

$$d \mid c_{ij},\ j = \overline{1, n-m} \Rightarrow d \mid (c_{i1}, ..., c_{in-m}).$$

Theorem 3.

If

$$\begin{cases} x_1 = c_{11}k_1 + ... + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + ... + c_{nn-m}k_{n-m} \end{cases}$$

k_j = parameters $\in \mathbf{Z}$, $c_{ij} \in \mathbf{Z}$ being given, is the general solution of the homogeneous linear system

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n = 0 \\ : \\ a_{m1}x_1 + ... + a_{mn}x_n = 0 \end{cases}, \quad r(A) = m < n$$

then $c_{1j},...,c_{nj} \; = 1, \; \forall j = \overline{1, n-m}$.

Proof:

We assume, by reduction ad absurdum, that there is $j_0 \in \overline{1, n-m}$: $c_{1j_0},...,c_{nj_0} \; = d$ we consider the maximal co-divisor > 0; we reduce to the case when the maximal co-divisor is $-d$ to the case when it is equal to d (non restrictive hypothesis); then the general solution can be written under the form:

$$(5) \qquad \begin{cases} x_1 = c_{11}k_1 + ... + c'_{1j_0} dk_{j_0} + ... + c_{1n-m}k_{n-m} \\ : \\ x_n = c_{n1}k_1 + ... + c'_{nj_0} dk_{j_0} + ... + c_{nn-m}k_{n-m} \end{cases}$$

where $d = \; c_{ij_0},..,c_{nj_0}$, $c_{ij_0} = d \cdot c'_{ij_0}$ and $c'_{ij_0},...,c'_{nj_0} \; = 1$.

We prove that

$$\begin{cases} x_1 = c'_{1j_0} \\ : \\ x_n = c'_{nj_0} \end{cases}$$

is a particular solution of the homogeneous linear system.

We'll note:

$$C = \begin{pmatrix} c_{11} & ... & c'_{ij_0} & d & ... & c_{1n-m} \\ : & & : & & & : \\ c_{n1} & ... & c'_{nj_0} & d & ... & c_{nn-m} \end{pmatrix}, \; k = \begin{pmatrix} k_1 \\ : \\ k_{j_0} \\ : \\ k_{n-m} \end{pmatrix}$$

$x = C \cdot k$ the general solution.

We know that $AX = 0 \Rightarrow A(CK) = 0$, $A = \begin{pmatrix} a_{11} ... a_{1n} \\ : \\ a_{n1} ... a_{mn} \end{pmatrix}$.

We assume that the principal variables are $x_1,...,x_m$ (if not, we have to renumber). It follows that $x_{m+1},...,x_n$ are the secondary variables.

For $k_1 = ... = k_{j_0-1} = k_{j_0+1} = ... = k_{n-m} = 0$ and $k_{j_0} = 1$ we obtain a particular solution of the system

$$\begin{cases} x_1 = c'_{1j_0} d \\ \vdots \\ x_n = c'_{nj_0} d \end{cases} \Rightarrow 0 = A \begin{pmatrix} c'_{1j_0} d \\ \vdots \\ c'_{nj_0} d \end{pmatrix} = d \cdot A \begin{pmatrix} c'_{1j_0} \\ \vdots \\ c'_{nj_0} \end{pmatrix} \Rightarrow A \begin{pmatrix} c'_{1j_0} \\ \vdots \\ c'_{nj_0} \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = c'_{1j_0} \\ \vdots \\ x_n = c'_{nj_0} \end{cases}$$

is the particular solution of the system.

We'll prove that this particular solution cannot be obtained by

(6) $$\begin{cases} x_1 = c_{11}k_1 + \dots + c'_{1j_0} dk_{j_0} + \dots + c_{1n-m}k_{n-m} = c'_{1j_0} \\ \vdots \\ x_n = c_{n1}k_1 + \dots + c'_{nj_0} dk_{j_0} + \dots + c_{nn-m}k_{n-m} = c'_{nj_0} \end{cases}$$

(7) $$\begin{cases} x_{m+1} = c_{m+1}k_1 + \dots + c'_{m+1} dk_{j_0} + \dots + c_{m+1,n-m}k_{n-m} = c'_{m+1j_0} \\ \vdots \\ x_n = c_{n1}k_1 + \dots + c'_{nj_0} dk_{j_0} + \dots + c_{nn-m}k_{n-m} = c'_{nj_0} \end{cases}$$

$$\Rightarrow \quad k_{j_0} = \frac{\begin{vmatrix} c_{m+1,1} & \dots & c_{m+1j} & \dots & c_{m+1,n-m} \\ \vdots & & \vdots & 0. & \vdots \\ c_{h,1} & \dots & c_{nj} & \dots & c_{n,n-m} \end{vmatrix}}{\begin{vmatrix} c_{m+1,1} & \dots & c'_{m+1j_0} d & \dots & c_{m+1,n-m} \\ \vdots & & \vdots & 0. & \vdots \\ c_{h,1} & \dots & c'_{nj} d & \dots & c_{n,n-m} \end{vmatrix}} = \frac{1}{d} \notin \mathbf{Z}$$

(because $d \neq 1$).

It is important to point out the fact that those $k_j = k_j^0$, $j = \overline{1, n-m}$, that satisfy the system (7) also satisfy the system (6), because, otherwise (6) would not satisfy the definition of the solution of a linear system of equations (i.e., considering the system (7) the hypothesis was not restrictive). From $X_{j_0} \in \mathbf{Z}$ follows that (6) is not the general solution of the homogeneous linear system contrary to the hypothesis); then $c_{1j}, \dots, c_{nj} = 1$, irrespective of $j = \overline{1, n-m}$.

Property 3. Let's consider the linear system

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in \mathbf{Z}$, $r(A) = m < n$, x_j = unknowns $\in \mathbf{Z}$

Resolved in $\mathbf{P}$, we obtain

$$\begin{cases} x_1 = f_1(x_{m+1},...,x_n) \\ : \\ x_m = f_m(x_{m+1},...,x_n) \end{cases}, \ x_1,...,x_m \text{ are the main variables,}$$

where f_i are linear functions of the form:

$$f_i = \frac{c_{m+1}^i x_{m+1} + ... + c_n^i x_n + e_i}{d_i},$$

where $c_{m+j}^i,\ d_i,\ e_i \in \mathbf{Z};\ i = \overline{1,m},\ j = \overline{1,n-m}$.

If $\frac{e_i}{d_i} \in \mathbf{Z}$ irrespective of $i = \overline{1,m}$ then the linear system has integer solution.

Proof:

For $1 \le i \le m,\ x_i \in \mathbf{Z}$, then $f_j \in \mathrm{Z}$. Let's consider

$$\begin{cases} x_{m+1} = u_{m+1}k_{m+1} \\ : \\ x_n = u_n k_n \\ : \\ x_1 = v_{m+1}^1 k_{m+1} + ... + v_n^1 k_n + \frac{e_1}{d_1} \\ : \\ x_m = v_{m+1}^m k_{m+1} + ... + v_n^m k_n + \frac{e_m}{d_m} \end{cases}$$

a solution, where u_{m+1} is the maximal co-divisor of the denominators of the fractions $\frac{c_{m+j}^i}{d_i},\ i = \overline{1,m},\ j = \overline{1,n-m}$ calculated after their complete simplification.

$v_{m+j}^i = \frac{c_{m+j}^i u_{m+j}}{d_i} \in \mathbf{Z}$ is a solution undetermined $(n-m)$-times which depends on $n-m$ independent parameters $(k_{m+1},...,k_n)$ but is not a general solution.

Property 4. Under the conditions of property 3, if there is an $i_0 \in \overline{1,m}: f_{i_0} = u_{m+1}^{i_0} x_{m+1} + ... + u_n^{i_0} x_n + \frac{e_{i_0}}{d_{i_0}}$ with $u_{m+j}^{i_0} \in \mathbf{Z}$, $j = \overline{1,n-m}$, and $\frac{e_{i_0}}{d_{i_0}} \notin \mathbf{Z}$ then the system does not have integer solution.

Proof:

$\forall x_{m+1},...,x_n$ in $\mathbf{Z}$, it results that $x_{i_0} \notin \mathbf{Z}$.

Theorem 4. Let's consider the linear system

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in \mathbf{Z}$, $x_j =$ unknowns $\in \mathbf{Z}$, $r(A) = m < n$. If there are indices $1 \le i_1 < \ldots < i_m \le n$, $i_h \in \{1, 2, .., n\}$, $h = \overline{1, m}$, with the property:

$$\Delta = \begin{vmatrix} a_{1i_1} & \ldots & a_{1i_m} \\ \vdots & & \vdots \\ a_{mi_1} & \ldots & a_{mi_m} \end{vmatrix} \ne 0 \text{ and}$$

$$\Delta_{x_{i_1}} = \begin{vmatrix} b_1 & a_{1i_2} & \ldots & a_{1i_m} \\ \vdots & \vdots & & \vdots \\ b_m & a_{mi_2} & \ldots & a_{mi_m} \end{vmatrix} \text{ is divided by } \Delta$$

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$$\Delta_{x_{i_m}} = \begin{vmatrix} a_{1i_1} & \ldots & a_{1i_{m-1}} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{mi_1} & \ldots & a_{mi_{m-1}} & b_m \end{vmatrix} \text{ is divided by } \Delta$$

then the system has integer number solutions.

Proof:

We use property 3

$$d_i = \Delta,\ i = \overline{1, m};\ e_{i_h} = \Delta_{x_{i_h}},\ h = \overline{1, m}$$

Note 1. It is not true in the reverse case.

Consequence 1. Any homogeneous linear system has integer number solutions (besides the trivial one); $r(A) = m < n$.

Proof:

$$\Delta_{x_{i_h}} = 0 \vdots \Delta, \text{ irrespective of } h = \overline{1, m}.$$

Consequence 2. If $\Delta = \pm 1$, it follows that the linear system has integer number solutions.

Proof:

$\Delta_{x_{i_h}} \vdots (\pm 1)$, irrespective of $h = \overline{1, m}$;

$\Delta_{x_{i_h}} \in \mathbf{Z}$.