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Integer Number Solutions of Linear Systems

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INTEGER NUMBER SOLUTIONS OF LINEAR SYSTEMS

Definitions and Properties of the Integer Solution of a Linear System

Let's consider

$$(1) \qquad \sum_{j=1}^{n} a_{ij} x_j = b_i\,, \quad i = \overline{1,m}$$

a linear system with all coefficients being integer numbers (the case with rational coefficients is reduced to the same).

Definition 1. $x_j = x_j^0,\ j = \overline{1,n}$, is a particular integer solution of (1) if $x_j^0 \in \mathbb{Z},\ j = \overline{1,n}$ and $\sum_{j=1}^{n} a_{ij} x_j^0 = b_i,\ \ i = \overline{1,m}$.

Let's consider the functions $f_j : \mathbb{Z}^h \to \mathbb{Z},\ \ j = \overline{1,n}$, where $h \in \mathbb{N}^*$.

Definition 2. $x_j = f_j(k_1,...,k_h),\ j = \overline{1,n}$, is the general integer solution for (1) if:

(a) $\sum_{j=1}^{n} a_{ij} f_j(k_1,...,k_h) = b_i,\ \ i = \overline{1,m}$, irrespective of $\left(k_1,...,k_h\right) \in \mathbb{Z}$;

(b) Irrespective of $x_j = x_j^0,\ \ j = \overline{1,n}$ a particular integer solution of (1) there is $(k_1^0,...,k_h^0) \in \mathbb{Z}$ such that $f_j(k_1^0,...,k_h^0) = x_j,\ j = \overline{1,n}$. (In other words the general solution that comprises all the other solutions.)

Property 1.

A general solution of a linear system of m equations with n unknowns, $r(A) = m < n$, is undetermined $\left(n-m\right)$-times.

Proof:

We assume by reduction ad absurdum that it is of order r, $1 \le r \le n-m$ (the case $r = 0$, i.e., when the solution is particular, is trivial). It follows that the general solution is of the form:

$$(\mathrm{S}_1) \qquad \begin{cases} x_1 = u_{11} p_1 + ... + u_{1r} p_r + v_1 \\ \vdots \\ x_n = u_{n1} p_1 + ... + u_{nr} p_r + v_n, \quad u_{ih},\ \forall i \in \mathbb{Z} \\ p_h = \text{parameters} \in \mathbb{Z} \end{cases}$$

We prove that the solution is undetermined $\left(n-m\right)$-times.

The homogeneous linear system (1), resolved in r has the solution:

$$\begin{cases} x_1 = \frac{D_{m+1}^1}{D} x_{m+1} + \ldots + \frac{D_n^1}{D} x_n \\ \vdots \\ x_m = \frac{D_{m+1}^m}{D} x_{m+1} + \ldots + \frac{D_n^m}{D} x_n \end{cases}$$

Let $x_i = x_i^0,\ i = \overline{1,n}$, be a particular solution of the linear system (1).

Considering

$$\begin{cases} x_{m+1} = D \cdot k_{m+1} \\ \vdots \\ x_n = D \cdot k_n \end{cases}$$

we obtain the solution

$$\begin{cases} x_1 = D_{m+1}^1 \cdot k_{m+1} + \ldots + D_n^1 \cdot k_n + x_1^0 \\ \vdots \\ x_m = D_{m+1}^m \cdot k_{m+1} + \ldots + D_n^m \cdot k_n + x_m^0 \\ x_{m+1} = D \cdot k_{m+1} + x_{m+1}^0 \\ \vdots \\ x_n = D \cdot k_n + x_n^0, \qquad\qquad k_j = \text{parameters} \in \mathbb{Z} \end{cases}$$

which depends on the $n-m$ independent parameters, for the system (1). Let the solution be undetermined $(n-m)$-times:

(S2)
$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1 \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + u_{nn-m}k_{n-m} + d_n \\ c_{ij},\ d_i \in \mathbb{Z},\ k_j = \text{parameters} \in \mathbb{Z} \end{cases}$$

(There are such solutions, we have proved it before.) Let the system be:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

x_i = unknowns $\in \mathbb{Z}$, $a_{ij},\ b_i \in \mathbb{Z}$.

I. The case $b_i = 0,\ \ i = \overline{1,m}$ results in a homogenous linear system:

$a_{i1}x_i + \ldots + a_{in}x_n = 0;\ i = \overline{1,m}$.

(S2) $\Rightarrow a_{i1}\left(c_{i1}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1\right) + \ldots + a_{in}\left(c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n\right) = 0$

$0 = \left(a_{i1}c_{11} + \ldots + a_{in}c_{n1}\right)k_1 + \ldots + \left(a_{i1}c_{1n-m} + \ldots + a_{in}c_{nn-m}\right)k_{n-m} + \left(a_{i1}d_1 + \ldots + a_{in}d_n\right)$

$\forall k_j \in \mathbb{Z}$

For $k_1 = \ldots = k_{n-m} = 0 \Rightarrow a_{i1}d_1 + \ldots + a_{in}d_n = 0$.

For $k_1 = \ldots = k_{h-1} = k_{h+1} = \ldots = k_{n-m} = 0$ and $k_h = 1 \ \Rightarrow$

$$\Rightarrow \left(a_{i1}c_{ih} + ... + a_{in}c_{nh}\right) + \left(a_{i1}d_1 + ... + a_{in}d_d^{(n)}\right) = 0 \Rightarrow$$

$$a_{i1}c_{ih} + ... + a_{in}c_{nh} = 0,\ \forall\ i = \overline{1,m},\ \forall\ h = \overline{1,n-m}.$$

The vectors

$$V_h = \begin{pmatrix} c_{1h} \\ : \\ : \\ c_{nh} \end{pmatrix},\ \ h = \overline{1,n-m}$$

are the particular solutions of the system.

$V_h,\ h = \overline{1,n-m}$ also linearly independent because the solution is undetermined $(n-m)$-times $\{V_1,...,V_{n-m}\} + d$ is a linear variety that includes the solutions of the system obtained from (S_2).

Similarly for (S_1) we deduce that

$$U_s = \begin{pmatrix} U_{1s} \\ : \\ : \\ U_{ns} \end{pmatrix},\ s = \overline{1,r}$$

are particular solutions of the given system and are linearly independent, because (S_1) is undetermined $(n-m)$-times, and $V = \begin{pmatrix} V_1 \\ : \\ : \\ V_n \end{pmatrix}$ is a solution of the given system.

Case (a) $U_1,...,U_r,\ V =$ linearly dependent, it follows that $\{U_1,...,U_r\}$ is a free sub-module of order $r < n - m$ of solutions of the given system, then, it follows that there are solutions that belong to $\{V_1,...,V_{n-m}\} + d$ and which do not belong to $\{U_1,...,U_r\}$, a fact which contradicts the assumption that (S_1) is the general solution.

Case (b) $U_1,...,U_r,\ V =$ linearly independent.

$\{U_1,...,U_r\}$+V is a linear variety that comprises the solutions of the given system, which were obtained from (S_1). It follows that the solution belongs to $\{V_1,...,V_{n-m}\} + d$ and does not belong to $\{U_1,...,U_r\}$+V, a fact which is a contradiction to the assumption that (S_1) is the general solution.

II. When there is an $i \in \overline{1,m}$ with $b_i \neq 0$ then non-homogeneous linear system

$$a_{i1}x_i + ... + a_{in}x_n = b_1,\ i = \overline{1,m}$$

(S_2) $\Rightarrow a_{i1}\left(c_{11}k_1 + ... + c_{1n-m}k_{n-m} + d_1\right) + ... + a_{in}\left(c_{n1}k_1 + ... + c_{nn-m}k_{n-m} + d_n\right) = b_i$

it follows that

$$\Rightarrow \left(a_{i1}c_{11} + ... + a_{in}c_{n1}\right)k_1 + ... + \left(a_{i1}c_{1n-m} + ... + a_{in}c_{nn-m}\right)k_{n-m} + \left(a_{i1}d_1 + ... + a_{in}d_n\right) = b_i$$

For $k_1 = ... = k_{n-m} = 0 \Rightarrow a_{i1}d_1 + ... + a_{in}d_n = b_1$;

For $k_1 = ... = k_{j-1} = k_{j+1} = ... = k_{n-m} = 0$ and $k_j = 1 \Rightarrow$

$\Rightarrow \left(a_{i1}c_{1j} + ... + a_{in}c_{nj}\right) + \left(a_{in}d_1 + ... + a_{in}d_n\right) = b_i$ it follows that

$$\begin{cases} a_{i1}c_{1j} + ... + a_{in}c_{nj} = 0 \\ a_{i1}d_1 + ... + a_{in}d_n = b_i \end{cases}; \quad \forall i = \overline{1,m},\ \forall j = \overline{1,n-m}.$$

$V_j = \begin{pmatrix} c_{1j} \\ : \\ c_{nj} \end{pmatrix}$, $j = \overline{1,n-m}$, are linearly independent because the solution (S_2) is undetermined $(n-m)$-times.

(?!) V_j, $j = \overline{1,n-m}$, and $d = \begin{pmatrix} d_1 \\ : \\ d_n \end{pmatrix}$

are linearly independent.

We assume that they are not linearly independent. It follows that

$$d = s_1V_1 + ... + s_{n-m}V_{n-m} = \begin{pmatrix} s_1c_{11} + ... + s_{n-m}c_{1n-m} \\ : \\ s_1c_{n1} + ... + s_{n-m}c_{nn-m} \end{pmatrix}.$$

Irrespective of $i = \overline{1,m}$:

$$b_1 = a_{i1}d_1 + ... + a_{in}d_n = a_{i1}\left(s_1c_{11} + ... + s_{n-m}c_{1n-m}\right) + ... + a_{in}\left(s_1c_{n1} + ... + s_{n-m}c_{nn-m}\right) =$$
$$= \left(a_{i1}c_{11} + ... + a_{in}c_{n1}\right)s_1 + ... + \left(a_{i1}c_{1n-m} + ... + a_{in}c_{nn-m}\right)s_{n-m} = 0.$$

Then, $b_i = 0$, irrespective of $i = \overline{1,m}$, contradicts the hypothesis (that there is an $i \in \overline{1,m}$, $b_i \neq 0$). It follows that $V_1,...,V_{n-m}, d$ are linearly independent.

$\{V_1,...,V_{n-m}\} + d$ is a linear variety that contains the solutions of the non-homogeneous system, solutions obtained from (S_2). Similarly it follows that $\{G_1,...,G_r\} + V$ is a linear variety containing the solutions of the non-homogeneous system, obtained from (S_1).

$n - m > r$ it follows that there are solutions of the system that belong to

"?!" means "to prove that"

$\{V_1,...,V_{n-m}\} + d$ and which do not belong to $\{G_1,...,G_r\} + V$, this contradicts the fact that (S_1) is the general solution. Then, it shows that the general solution depends on the $n - m$ independent parameters.

Theorem 1. The general solution of a non-homogeneous linear system is equal to the general solution of an associated linear system plus a particular solution of the non-homogeneous system.

Proof:

Let's consider the homogeneous linear solution:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases}, \quad (AX = 0)$$

with the general solution:

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d_1 \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n \end{cases}$$

and

$$\begin{cases} x_1 = x_1^0 \\ \vdots \\ x_n = x_n^0 \end{cases}$$

with the general solution a particular solution of the non-homogeneous linear system $AX = b$;

$$(?!) \qquad \begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + d + x_1^0 \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + d_n + x_n^0 \end{cases}$$

is a solution of the non-homogeneous linear system.

We note:

$$A = \begin{pmatrix} a_{11} \ldots a_{1n} \\ \vdots \\ a_{m1} \ldots a_{mn} \end{pmatrix}, \; X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \; b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \; 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

(vector of dimension m),

$$K = \begin{pmatrix} k_1 \\ \vdots \\ k_{n-m} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} \ldots c_{1n-m} \\ \vdots \\ c_{n1} \ldots c_{nn-m} \end{pmatrix}, \; d = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \; x^0 = \begin{pmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{pmatrix};$$

$$AX = A\left(Ck + d + x^0\right) = A\left(Ck + d\right) + AX^0 = b + 0 = b$$

We will prove that irrespective of

$$\begin{matrix} x_1 = y_1^0 \\ \vdots \\ x_n = y_n^0 \end{matrix}$$

there is a particular solution of the non-homogeneous system

$$\begin{cases} k_1 = k_1^0 \in \mathbb{Z} \\ \vdots \\ k_{n-m} = k_{n-m}^0 \in \mathbb{Z} \end{cases},$$

with the property:

$$\begin{cases} x_1 = c_{11}k_1^0 + \ldots + c_{1n}k_{n-m}^0 + d_1 + x_1^0 = y_1^0 \\ \vdots \\ x_n = c_{n1}k_1^0 + \ldots + c_{nn-m}k_{n-m}^0 + d_1 + x_n^0 = y_n^0 \end{cases}$$

We note $Y^0 = \begin{pmatrix} y_1^0 \\ \vdots \\ y_n^0 \end{pmatrix}$.

We'll prove that those $k_j^0 \in \mathbb{Z},\ j = \overline{1, n-m}$ are those for which $A\left(CX^0 + d\right) = 0$ (there are such $X_j^0 \in \mathbb{Z}$ because

$$\begin{cases} x_1 = 0 \\ \vdots \\ x_n = 0 \end{cases}$$

is a particular solution of the homogeneous linear system and $X = CK + d$ is a general solution of the non-homogeneous linear system)

$$A\left(CK^0 + d + X^0 - Y^0\right) = A\left(CK^0 + d\right) + AX^0 - AY^0 = 0 + b - b = 0 \quad .$$

Property 2 The general solution of the homogeneous linear system can be written under the form:

(SG)

(2) $$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} \end{cases}$$

k_j is a parameter that belongs to $\mathbb{Z}$ (with $d_1 = d_2 = \ldots = d_n = 0$).

Poof:

(SG) = general solution. It results that (SG) is undetermined $(n-m)$ -times.

Let's consider that (SG) is of the form

(3) $$\begin{cases} x_1 = c_{11}p_1 + \ldots + c_{1n-m}p_{n-m} + d_1 \\ \vdots \\ x_n = c_{n1}p_1 + \ldots + c_{nn-m}p_{n-m} + d_n \end{cases}$$

with not all $d_i = 0$; we'll prove that it can be written under the form (2); the system has the trivial solution

$$\begin{cases} x_1 = 0 \in \mathbb{Z} \\ \vdots \\ x_n = 0 \in \mathbb{Z} \end{cases};$$

it results that there are $p_j \in \mathbb{Z},\ j = \overline{1, n-m}$,

$$(4)\qquad \begin{cases} x_1 = c_{11}p_1^0 + \ldots + c_{1n-m}p_{n-m}^0 + d_1 = 0 \\ \vdots \\ x_n = c_{n1}p_1^0 + \ldots + c_{nn-m}p_{n-m}^0 + d_n = 0 \end{cases}$$

Substituting $p_j = k_j + p_j^0$, $j = \overline{1, n-m}$ in (3)

$$\left.\begin{matrix} k_j \in \mathbb{Z} \\ p_j^0 \in \mathbb{Z} \end{matrix}\right\} \Rightarrow p_j \in \mathbb{Z},$$

$$\left.\begin{matrix} p_j \in \mathbb{Z} \\ p_j^0 \in \mathbb{Z} \end{matrix}\right\} \Rightarrow k_j = p_j - p_j^0 \in \mathbb{Z}$$

which means that that they do not make any restrictions.

It results that

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} + \left(c_{11}p_1^0 + \ldots + c_{1n-m}p_{n-m}^0 + d_1\right) \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} + \left(c_{n1}p_1^0 + \ldots + c_{nn-m}p_{n-m}^0 + d_n\right) \end{cases}$$

But

$$c_{h1}p_1^0 + \ldots + c_{hn-m}p_{n-m}^0 + d_h = 0,\ h = \overline{1,n} \text{ (from (4))}.$$

Then the general solution is of the form:

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} \end{cases}$$

k_j = parameters $\in \mathbb{Z}$, $j = \overline{1, n-m}$; it results that $d_1 = d_2 = \ldots = d_n = 0$.

Theorem 2. Let's consider the homogeneous linear system:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases},$$

$r(A) = m$, $\left(a_{h1}, \ldots, a_{hn}\right) = 1$, $h = \overline{1,m}$ and the general solution

$$\begin{cases} x_1 = c_{11}k_1 + \ldots + c_{1n-m}k_{n-m} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c_{nn-m}k_{n-m} \end{cases}$$

then

$$\left(a_{h1}, \ldots, a_{hi-1}, a_{hi+1}, \ldots, a_{hn}\right) \Big| \left(c_{i1}, \ldots, c_{in-m}\right)$$

irrespective of $h = \overline{1,m}$ and $i = \overline{1,n}$.

Proof:

Let's consider some arbitrary $h \in \overline{1,m}$ and some arbitrary $i \in \overline{1,n}$;

$$a_{h1}x_1 + \ldots + a_{hi-1}x_{i-1} + a_{hi+1}x_{i+1} + \ldots + a_{hn}x_n = a_{hi}x_i.$$

Because

$$\left(a_{h1},...,a_{hi-1},a_{hi+1},...,a_{hn}\right)\big| a_{hi}$$

it results that

$$d=\left(a_{h1},...,a_{hi-1},a_{hi+1},...,a_{hn}\right)\big| x_i$$

irrespective of the value of x_i in the vector of particular solutions.

For $k_2 = k_3 = ... = k_{n-m} = 0$ and $k_1 = 1$ we obtain the particular solution:

$$\begin{cases} x_1 = c_{11} \\ : \\ x_i = c_{i1} \;\; \Rightarrow d \mid c_{i1} \\ : \\ x_n = c_{n1} \end{cases}$$

For $k_1 = k_2 = ... = k_{n-m-1} = 0$ and $k_{n-m} = 1$ it results the following particular solution:

$$\begin{cases} x_1 = c_{1n-m} \\ : \\ x_i = c_{in-m} \;\; \Rightarrow d \mid c_{in-m}\,; \\ : \\ x_n = c_{nn-m} \end{cases}$$

hence

$$d \mid c_{ij},\ j=\overline{1,n-m} \Rightarrow d\big|\left(c_{i1},...,c_{in-m}\right).$$

Theorem 3.

If

$$\begin{cases} x_1 = c_{11}k_1 + ... + c_{1n-m}k_{n-m} \\ : \\ x_n = c_{n1}k_1 + ... + c_{nn-m}k_{n-m} \end{cases}$$

k_j = parameters $\in \mathbb{Z}$, $c_{ij} \in \mathbb{Z}$ being given, is the general solution of the homogeneous linear system

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n = 0 \\ : \\ a_{m1}x_1 + ... + a_{mn}x_n = 0 \end{cases}, \quad r(A) = m < n$$

then $\left(c_{1j},...,c_{nj}\right)=1,\ \ \forall j=\overline{1,n-m}$.

Proof:

We assume, by reduction ad absurdum, that there is $j_0 \in \overline{1,n-m} : \left(c_{1j_0},...,c_{nj_0}\right) = d$ we consider the maximal co-divisor >0; we reduce to the case when the maximal co-divisor is $-d$ to the case when it is equal to d (non restrictive hypothesis); then the general solution can be written under the form:

$$(5)\qquad \begin{cases} x_1 = c_{11}k_1 + ... + c'_{1j_0} dk_{j_0} + ... + c_{1n-m}k_{n-m} \\ : \\ x_n = c_{n1}k_1 + ... + c'_{nj_0} dk_{j_0} + ... + c_{nn-m}k_{n-m} \end{cases}$$

where $d = \left(c_{ij_0}, .., c_{nj_0}\right)$, $c_{ij_0} = d \cdot c'_{ij_0}$ and $\left(c'_{ij_0}, ..., c'_{nj_0}\right) = 1$.

We prove that

$$\begin{cases} x_1 = c'_{1j_0} \\ : \\ x_n = c'_{nj_0} \end{cases}$$

is a particular solution of the homogeneous linear system.

We'll note:

$$C = \begin{pmatrix} c_{11} & ... & c'_{ij_0} & d & ... & c_{1n-m} \\ : & & : & & & : \\ c_{n1} & ... & c'_{nj_0} & d & ... & c_{nn-m} \end{pmatrix},\ k = \begin{pmatrix} k_1 \\ : \\ k_{j_0} \\ : \\ k_{n-m} \end{pmatrix}$$

$x = C \cdot k$ the general solution.

We know that $AX = 0 \Rightarrow A(CK) = 0$, $A = \begin{pmatrix} a_{11} ... a_{1n} \\ : \\ a_{n1} ... a_{mn} \end{pmatrix}$.

We assume that the principal variables are $x_1, ..., x_m$ (if not, we have to renumber). It follows that $x_{m+1}, ..., x_n$ are the secondary variables.

For $k_1 = ... = k_{j_0 -1} = k_{j_0 +1} = ... = k_{n-m} = 0$ and $k_{j_0} = 1$ we obtain a particular solution of the system

$$\begin{cases} x_1 = c'_{1j_0} d \\ : \\ x_n = c'_{nj_0} d \end{cases} \Rightarrow 0 = A \begin{pmatrix} c'_{1j_0} d \\ : \\ c'_{nj_0} d \end{pmatrix} = d \cdot A \begin{pmatrix} c'_{1j_0} \\ : \\ c'_{nj_0} \end{pmatrix} \Rightarrow A \begin{pmatrix} c'_{1j_0} \\ : \\ c'_{nj_0} \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = c'_{1j_0} \\ : \\ x_n = c'_{nj_0} \end{cases}$$

is the particular solution of the system.

We'll prove that this particular solution cannot be obtained by

$$(6)\qquad \begin{cases} x_1 = c_{11}k_1 + ... + c'_{1j_0} dk_{j_0} + ... + c_{1n-m}k_{n-m} = c'_{1j_0} \\ : \\ x_n = c_{n1}k_1 + ... + c'_{nj_0} dk_{j_0} + ... + c_{nn-m}k_{n-m} = c'_{nj_0} \end{cases}$$

(7) $$\begin{cases} x_{m+1} = c_{m+1}k_1 + \ldots + c'_{m+1}dk_{j_0} + \ldots + c_{m+1,n-m}k_{n-m} = c'_{m+1j_0} \\ \vdots \\ x_n = c_{n1}k_1 + \ldots + c'_{nj_0}dk_{j_0} + \ldots + c_{nn-m}k_{n-m} = c'_{nj_0} \end{cases}$$

$$\Rightarrow \quad k_{j_0} = \frac{\begin{vmatrix} c_{m+1,1} & \ldots & c_{m+1j} & \ldots & c_{m+1,n-m} \\ \vdots & & \vdots & 0. & \vdots \\ c_{h,1} & \ldots & c_{nj} & \ldots & c_{n,n-m} \end{vmatrix}}{\begin{vmatrix} c_{m+1,1} & \ldots & c'_{m+1j_0}d & \ldots & c_{m+1,n-m} \\ \vdots & & \vdots & 0. & \vdots \\ c_{h,1} & \ldots & c'_{nj}d & \ldots & c_{n,n-m} \end{vmatrix}} = \frac{1}{d} \notin \mathbb{Z}$$

(because $d \neq 1$).

It is important to point out the fact that those $k_j = k_j^0$, $j = \overline{1, n-m}$, that satisfy the system (7) also satisfy the system (6), because, otherwise (6) would not satisfy the definition of the solution of a linear system of equations (i.e., considering the system (7) the hypothesis was not restrictive). From $X_{j_0} \in \mathbb{Z}$ follows that (6) is not the general solution of the homogeneous linear system contrary to the hypothesis); then $\left(c_{1j}, \ldots, c_{nj}\right) = 1$, irrespective of $j = \overline{1, n-m}$.

Property 3. Let's consider the linear system

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in \mathbb{Z}$, $r(A) = m < n$, $x_j =$ unknowns $\in \mathbb{Z}$

Resolved in $\mathbb{R}$, we obtain

$$\begin{cases} x_1 = f_1\left(x_{m+1}, \ldots, x_n\right) \\ \vdots \\ x_m = f_m\left(x_{m+1}, \ldots, x_n\right) \end{cases}, \quad x_1, \ldots, x_m \text{ are the main variables,}$$

where f_i are linear functions of the form:

$$f_i = \frac{c^i_{m+1}x_{m+1} + \ldots + c^i_n x_n + e_i}{d_i},$$

where c^i_{m+j}, d_i, $e_i \in \mathbb{Z}$; $i = \overline{1, m}$, $j = \overline{1, n-m}$.

If $\frac{e_i}{d_i} \in \mathbb{Z}$ irrespective of $i = \overline{1, m}$ then the linear system has integer solution.

Proof:

For $1 \leq i \leq m$, $x_i \in \mathbb{Z}$, then $f_j \in \mathbb{Z}$. Let's consider

$$\begin{cases} x_{m+1} = u_{m+1}k_{m+1} \\ \vdots \\ x_n = u_n k_n \\ \vdots \\ x_1 = v_{m+1}^1 k_{m+1} + ... + v_n^1 k_n + \dfrac{e_1}{d_1} \\ \vdots \\ x_m = v_{m+1}^m k_{m+1} + ... + v_n^m k_n + \dfrac{e_m}{d_m} \end{cases}$$

a solution, where u_{m+1} is the maximal co-divisor of the denominators of the fractions $\dfrac{c_{m+j}^i}{d_i}$, $i = \overline{1,m}$, $j = \overline{1,n-m}$ calculated after their complete simplification.

$v_{m+j}^i = \dfrac{c_{m+j}^i u_{m+j}}{d_i} \in \mathbb{Z}$ is a $(n-m)$-times undetermined solution which depends on $n-m$ independent parameters $(k_{m+1},...,k_n)$ but is not a general solution.

Property 4. Under the conditions of property 3, if there is an $i_0 \in \overline{1,m} : f_{i_0} = u_{m+1}^{i_0} x_{m+1} + ... + u_n^{i_0} x_n + \dfrac{e_{i_0}}{d_{i_0}}$ with $u_{m+j}^{i_0} \in \mathbb{Z}$, $j = \overline{1,n-m}$, and $\dfrac{e_{i_0}}{d_{i_0}} \notin \mathbb{Z}$ then the system does not have integer solution.

Proof:

$\forall x_{m+1},...,x_n$ in $\mathbb{Z}$, it results that $x_{i_0} \notin \mathbb{Z}$.

Theorem 4. Let's consider the linear system

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{cases}$$

$a_{ij}, b_i \in \mathbb{Z}$, $x_j =$ unknowns $\in \mathbb{Z}$, $r(A) = m < n$. If there are indices $1 \le i_1 < ... < i_m \le n$, $i_h \in \{1,2,..,n\}$, $h = \overline{1,m}$, with the property:

$$\Delta = \begin{vmatrix} a_{1i_1} & ... & a_{1i_m} \\ \vdots & & \vdots \\ a_{mi_1} & ... & a_{mi_m} \end{vmatrix} \neq 0 \text{ and}$$

$$\Delta_{x_{i_1}} = \begin{vmatrix} b_1 & a_{1i_2} & \dots & a_{1i_m} \\ \vdots & \vdots & & \vdots \\ b_m & a_{mi_2} & \dots & a_{mi_m} \end{vmatrix} \text{ is divided by } \Delta$$

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$$\Delta_{x_{i_m}} = \begin{vmatrix} a_{1i_1} & \dots & a_{1i_{m-1}} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{mi_1} & \dots & a_{mi_{m-1}} & b_m \end{vmatrix} \text{ is divided by } \Delta$$

then the system has integer number solutions.

Proof:

We use property 3

$$d_i = \Delta,\ i = \overline{1,m};\ e_{i_h} = \Delta_{x_{i_h}},\ h = \overline{1,m}$$

Note 1. It is not true in the reverse case.

Consequence 1. Any homogeneous linear system has integer number solutions (besides the trivial one); $r(A) = m < n$.

Proof:

$$\Delta_{x_{i_h}} = 0 \vdots \Delta \text{, irrespective of } h = \overline{1,m}.$$

Consequence 2. If $\Delta = \pm 1$, it follows that the linear system has integer number solutions.

Proof:

$\Delta_{x_{i_h}} \vdots (\pm 1)$, irrespective of $h = \overline{1,m}$;

$\Delta_{x_{i_h}} \in \mathbb{Z}$.