Chapter 8
Introduction to Plithogenic Subgroup

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ABSTRACT

This chapter gives some essential scopes to study some plithogenic algebraic structures. Here the notion of plithogenic subgroup has been introduced and explored. It has been shown that subgroups defined earlier in the crisp, fuzzy, intuitionistic fuzzy, as well as neutrosophic environments, can also be represented as plithogenic fuzzy subgroups. Furthermore, introducing function in plithogenic setting, some homomorphic characteristics of plithogenic fuzzy subgroup have been studied. Also, the notions of plithogenic intuitionistic fuzzy subgroup, plithogenic neutrosophic subgroup have been introduced and their homomorphic characteristics have been analyzed.

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PREFACE

This chapter is written for the advancement of Neutrosophic theory and Plithogenic theory, which can be helpful for scientists and researchers in mathematics, computer science and also other disciplines. It provides a detailed introduction and fundamental ideas of plithogenic subgroup. The prerequisites are some knowledge of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic set theories. Also, some understandings in fuzzy, intuitionistic fuzzy and neutrosophic algebraic structures are required. In this chapter, the notion of plithogenic subgroup is introduced and explored with proper examples. It has been shown that subgroups defined earlier in the crisp, fuzzy, intuitionistic fuzzy, as well as neutrosophic environments, can also be represented as plithogenic fuzzy subgroups. Furthermore, some homomorphic characteristics of plithogenic fuzzy subgroup are studied. Also, the notions of plithogenic intuitionistic fuzzy subgroup, plithogenic neutrosophic subgroup are introduced and their homomorphic characteristics are studied.

1. INTRODUCTION

Crisp set (CS) theory has certain drawbacks. It is quite insufficient in case of handling real-life problems. Fuzzy set (FS) theory (Zadeh, 1965) is more reliable in tackling such scenarios. Since the very beginning of FS theory, many researchers have carried out that perception in various realistic problems. But eventually some other set theories have emerged like intuitionistic fuzzy set (IFS) (Atanassov, 1986), neutrosophic set (NS) (Smarandache, 2005), Pythagorean FS (Yager, 2013), Plithogenic set (PS) (Smarandache, 2017), etc., which are capable of handling uncertainty better than FSs. As a result, these set theories are preferred by most of the researchers to solve different real-life problems in which uncertainty plays a crucial role. Actually, NS is a generalization of IFS, which is further a generalization of FS. Smarandache's contributions towards the development of NS theory are remarkable. For instance, he has contributed in developing neutrosophic measure and probability (Smarandache, 2013), calculus (Smarandache & Khalid, 2015), psychology (Smarandache, 2018), etc. Also, NS theory has a vast area of applications. Furthermore, Smarandache has introduced the notion of PS (Smarandache, 2018) theory which is a generalization of CS, FS, IFS and NS theories. He has further generalized PS and developed the notions of refined PS, plithogenic multiset, plithogenic bipolar set, plithogenic tripolar set, plithogenic multipolar set, plithogenic complex set, etc. Furthermore, he has developed plithogenic logic, probability and statistics (Smarandache, 2000; Smarandache, 2017) and shown that all those notions are generalizations of crisp logic, probability and statistics. Presently, PS theory is extensively applied in various

Table 1. Significance and influences of NS and PS in various fields

Author and Year	Contributions in Various Fields
(Vlachos & Sergiadis, 2007)	Implemented NS in pattern recognition problem.
(Guo & Cheng, 2009)	Implemented neutrosophic approach to image segmentation.
(Smarandache, 2013)	Mentioned some applications of neutrosophic logic in physics.
(Majumdar, 2015)	Implemented NS in decision-making problem.
(Kumar et. al., 2015)	Implemented neutrosophic cognitive maps in medical diagnosis.
(Deli et. al., 2015)	Applied neutrosophic refined sets in medical diagnosis.
(Broumi et. al., 2016)	Solved neutrosophic shortest path problem.
(Smarandache, 2017)	Introduced plithogenic set, logic, probability and statistics.
(Kumar et. al., 2018)	Solved neutrosophic shortest path problem.
(Smarandache, 2018)	Introduced aggregation plithogenic operator in physical fields.
(Smarandache, 2018)	Introduced physical plithogenic set.
(Smarandache, 2018)	Extended soft set to hypersoft set and introduced plithogenic hypersoft set.
(Kumar et. al., 2019)	Solved neutrosophic shortest path problem.
(Kumar et. al., 2019)	Solved neutrosophic Transportation problem.

decision-making problems as well as in other applied fields. The following Table 1 consists of some important contributions in NS and PS theory.

Again, plithogenic number (PN) has been introduced and some essential operations on PN, like, the summation of PNs, multiplication of PNs, power of a PN, etc., have been developed by Samandache. Furthermore, he has introduced some important measure functions like, dice similarity plithogenic number measure (based on (Ye, 2014)), cosine similarity plithogenic number measure, Hamming plithogenic number distance, Euclidean plithogenic number distance, Jaccard similarity plithogenic number measure (Smarandache, 2017), etc. Also, in plithogenic probability theory the concepts of plithogenic fuzzy probability, plithogenic intuitionistic fuzzy probability, and plithogenic neutrosophic probability have been proposed which are essential tools to handle various probabilistic problems.

This Chapter has been arranged as following: In Segment 2, literature surveys of the fuzzy subgroup (FSG), intuitionistic fuzzy subgroup (IFSG), neutrosophic subgroup (NSG) are given. In Segment 3, some preliminary notions like PS, the preeminence of PS over other set theories, homomorphic characteristics of FSG, IFSG, and NSG are discussed. In Segment 4, different aspects of plithogenic subgroup (PSG), like, plithogenic fuzzy subgroup (PFSG), plithogenic intuitionistic fuzzy subgroup (PIFSG), plithogenic neutrosophic subgroup (PNSG) are introduced and also, the effects of homomorphism on those notions are mentioned. Finally, in segment 5 the conclusion is given and some scopes of future researches are mentioned.

2. LITERATURE SURVEY

To tackle drawbacks and insufficiency of CS theory FS was introduced. But that too has certain limitations and hence IFS and further NS were introduced. Presently, Pythagorean FS (Yager, 2013) and PS (Smarandache, 2018) are more popular for their uncertainty handling capability. As a result, these set theories are preferred by most of the researchers to solve different realistic problems in which uncertainty is involved.

In this segment, we have discussed FS, IFS, NS and also some other essential notions like FSG, IFSG, NSG, level set, level subgroup, T-norm, T-conorm, etc. All these notions play vital roles in developing plithogenic subgroup (PSG).

Definition 2.1 (Zadeh, 1965) A FS σ of a CS U is a function from U to [0,1] i.e., $\sigma : U \to [0,1]$.

Definition 2.2 (Atanassov, 1986) A IFS $\boldsymbol{\gamma}$ of a CS U is denoted as

$$\gamma = \{(m, t_\gamma(m), f_\gamma(m)) : m \in U\},$$

where both t_γ and f_γ are FSs of *U,* known as the respective degree of membership and non-membership of any element $m \in U$. Here for every $m \in U$, t_γ and f_γ satisfy the condition

$$0 \leq t_\gamma(m) + f_\gamma(m) \leq 1.$$

Definition 2.3 (Smarandache, 1999) A NS η of a CS U is denoted as

$$\eta = \{(m, t_\eta(m), i_\eta(m), f_\eta(m)) : m \in U\},$$

where

$$t_\eta, i_\eta, f_\eta : U \to]^{-}0, 1^{+}[$$

are the respective degree of truth, indeterminacy and falsity of any element $m \in U$. Here for every $m \in U$ t_η, i_η and f_η satisfy the condition

$$^{-}0 \leq t_\eta(m) + i_\eta(m) + f_\eta(m) \leq 3^{+}.$$

Definition 2.4 (Zadeh, 1965) Let α be a FS of *U*. Then $\forall\, t \in [0,1]$ the set

$$\alpha_t = \{x \in U : \alpha(x) \geq t\}$$

is called a level subset (*t*-level subset) of α.

Definition 2.5 (Gupta & Qi, 1991) A function $T : [0,1] \rightarrow [0,1]$ is termed as T-norm iff $\forall\, m, u, t \in [0,1]$ subsequent conditions are fulfilled:

(i) $T(m,1) = m$

(ii) $T(m,u) = T(u,m)$

(iii) $T(m,u) \leq T(t,u)$ *if* $m \leq t$

(iv) $T(m, T(u,t)) = T(T(m,u), t)$

Definition 2.6 (Gupta & Qi, 1991) A function $T^{*} : [0,1] \rightarrow [0,1]$ is termed as T-conorm iff $\forall\, m, u, t \in [0,1]$ subsequent conditions are fulfilled:

(i) $T^{*}(m,0) = m$

(ii) $T^{*}(m,u) = T^{*}(u,m)$

(iii) $T^{*}(m,u) \leq T^{*}(t,u)$ *if* $m \leq t$

(iv) $T^{*}(m, T^{*}(u,t)) = T^{*}(T^{*}(m,u), t)$

In the next subsection, the notions of FSG, IFSG, and NSG are discussed and also, some of their basic fundamental properties are given.

2.1. Fuzzy Subgroup, Intuitionistic Fuzzy Subgroup and Neutrosophic Subgroup

Definition 2.7 (Rosenfeld, 1971) A FS α of a group *P* is termed as a FSG of *P* iff $\forall\, m, u \in P,$ the subsequent conditions are fulfilled:

(i) $\alpha(mu) \geq min\{\alpha(m), \alpha(u)\}$

(ii) $\alpha(m^{-1}) \geq \alpha(m)$.

Here $\alpha(m^{-1}) = \alpha(m)$ and $\alpha(m) \leq \alpha(e),$ where *e* represents the neutral element of *P*. Also, in the above definition if only condition (i) is satisfied by α then we call it a fuzzy subgroupoid.

Theorem 2.1 (Rosenfeld, 1971) α is a FSG of *U* iff

$\forall\, m, u \in U \;\; \alpha(mu^{-1}) \geq min\{\alpha(m), \alpha(u)\}$.

Definition 2.8 (Das, 1981) Let α be a FSG of a group *P*. Then $\forall\, t \in [0,1]$ and $\alpha(e) \geq t$ the subgroups α_t are called level subgroups of α.

Definition 2.9 (Biswas, 1989) An IFS

$$\gamma = \{(m, t_\gamma(m), f_\gamma(m)) : m \in U\}$$

of a crisp set *U* is called an IFSG of *U* iff $\forall m, u \in U$

(i) $t_\gamma(mu^{-1}) \geq \min\{t_\gamma(m), t_\gamma(u)\}$

(ii) $f_\gamma(mu^{-1}) \leq \max\{f_\gamma(m), f_\gamma(u)\}$

The collection of all IFSG will be denoted as IFSG(*U*).

Example 2.1 Let $U = \{1, -1, i, -i\}$ and δ be a NS of *U* such that

$\gamma = \{(1, 0.6, 0.4), (-1, 0.7, 0.3), (i, 0.8, 0.2), (-i, 0.8, 0.2)\}$.

Notice that $\gamma \in$ IFSG (U).

Definition 2.10 (Çetkin & Aygün, 2015) Let U be a group and δ be a NS of U. δ is called a NSG of U iff the subsequent conditions are fulfilled:

(i) $\delta(m \cdot u) \geq \min\{\delta(m), \delta(u)\}$, i.e.

$$t_\delta(m \cdot u) \geq \min\{t_\delta(m), t_\delta(u)\},$$

$$i_\delta(m \cdot u) \geq \min\{i_\delta(m), i_\delta(u)\}$$

and

$$f_\delta(m \cdot u) \leq \max\{f_\delta(m), f_\delta(u)\}$$

(ii) $\delta(m^{-1}) \geq \delta(m)$ i.e.

$$t_\delta(m^{-1}) \geq t_\delta(u),$$

$$i_\delta(m^{-1}) \geq i_\delta(u)$$

and

$$f_\delta(m^{-1}) \leq f_\delta(u)$$

The collection of all NSG will be denoted as NSG(U). Here notice that t_δ and i_δ are following Definition 2.7 i.e. both of them are actually FSGs of U.

Example 2.2 (Çetkin & Aygün, 2015) Let $U = \{1, -1, i, -i\}$ and δ be a NS of U such that

$$\delta = \{(1, 0.6, 0.5, 0.4), (-1, 0.7, 0.4, 0.3), (i, 0.8, 0.4, 0.2), (-i, 0.8, 0.4, 0.2)\}.$$

Notice that δ∈NSG(*U*).

Theorem 2.2 (Çetkin & Aygün, 2015) Let *U* be a group and δ be a NS of *U*. Then δ∈NSG(*U*) iff

$$\forall m,u \in U \;\; \delta(m \cdot u^{-1}) \geq \min\{\delta(m),\delta(u)\}\ .$$

Theorem 2.3 (Çetkin & Aygün, 2015) δ∈NSG (*U*) iff $\forall p \in [0,1]$ the *p*-level sets $(t_\delta)_p$, $(i_\delta)_p$ and *p*-lower-level set $(\overline{f}_\delta)_p$ are CSGs of *U*.

Definition 2.11 (Çetkin & Aygün, 2015) Let *U* be a group and δ be a NS of *U*. δ is called a neutrosophic normal subgroup (NNSG) of *U* iff

$$\forall m,u \in U \;\; \delta(m \cdot u \cdot m^{-1}) \leq \delta(u)$$

i.e.

$$t_\delta(m \cdot u \cdot m^{-1}) \leq t_\delta(u)\,,$$

$$i_\delta(m \cdot u \cdot m^{-1}) \leq i_\delta(u)$$

and

$$f_\delta(m \cdot u \cdot m^{-1}) \geq f_\delta(u)\,.$$

The collection of all NNSG of *U* will be denoted as NNSG (*U*). Some more references that can be helpful to various authors are (Kandasamy & Smarandache, 2004; Gayen et. al., 2019; Kumar et. al., 2019; Gayen et. al, 2019; Broumi et. al., 2014; Kumar et. al., 2020 Kumar et. al., 2019b; Kumar et. al., 2017.. In the Table 2, some sources have been mentioned which have some major contributions in the fields of FSG, IFSG, and NSG.

2.2. Motivation of the Work

From the above discussions, it is clear that the studies of FSG, IFSG, as well as NSG, have generated many fruitful research fields. Some researchers have studied their normal versions, homomorphic characteristics and different other algebraic structures. Also, some authors have implemented the soft set theory in these notions

Table 2. Significance and influences of some authors in FSG, IFSG, and NSG

Author and Year	Different Contributions in FSG, IFSG, and NSG
(Rosenfeld, 1971)	Introduced FSG.
(Das, 1981)	Introduced level subgroup.
(Anthony & Sherwood, 1979)	Introduced FSG using general T-norm.
(Foster, 1979)	Introduced product of FSGs.
(Anthony & Sherwood, 1982)	Introduced subgroup generated and function generated FSG.
(Sherwood, 1983)	Studied product of FSGs.
(Mukherjee & Bhattacharya, 1984)	Introduced fuzzy normal subgroups and cosets.
(Biswas, 1989)	Introduced IFSG.
(Eroğlu, 1989)	Studied homomorphic image of FSG.
(Kim & Kim, 1996)	Studied fuzzy symmetric groups.
(Ray, 1999)	Studied some properties on the product of FSGs.
(Hur et. al., 2004)	Introduced Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets.
(Yuan et. al., 2010)	Introduced (α,β)-IFSG.
(Sharma, 2011)	Studied homomorphism of IFSG.
(Çetkin & Aygün, 2015)	Introduced NSG and neutrosophic normal subgroup and studied some fundamental properties by introducing homomorphism in them.

and studied their fundamental properties. Presently, the PS theory has grabbed a lot of attention due to its uncertainty handling nature. Again, this set theory is more general than CS, FS, IFS, as well as NS theories. So, the notions of PSGs i.e. PFSG, PIFSG and PNSG can become effective research fields. Furthermore, whether PFSG, PIFSG, and PNSG will act as generalizations of the crisp subgroup (CSG), FSG, IFSG, and NSG or not that is needed to be discussed. Again, their normal forms, homomorphic properties and also some other essential algebraic structures are needed to be studied. In this chapter, the subsequent research gaps are discussed:

- Still, the notion of PFSG is undefined.
- Whether CSG, FSG, IFSG, NSG can be represented by PFSG or not, that is needed to be analyzed.
- Also, some other essential PSGs like PIFSG, PNSG are needed to be introduced and studied.
- Furthermore, homomorphic characteristics PFSG, PIFSG, PNSG are still unexplored.

Therefore, this motivates us to introduce these notions of PFSG, PIFSG, and PNSG and analyze their algebraic properties.

2.3. Contribution of the Work

On the basis of the above gaps, the purpose of this article is to provide some essential definitions, examples, theories, propositions, etc. in the fields of PSG i.e. PFSG, PIFSG, and PNSG. Also, effectiveness and excellence of PSG in comparison with CSG, FSG, IFSG, and NSG will be mentioned. Furthermore, some important analysis like homomorphic characteristics of these notions will be discussed. The following are some purposes that are planned and executed during this research work.

- To define PFSG and study its algebraic properties.
- To check whether PFSG is a generalization of CSG, FSG, IFSG, and NSG or not.
- To define PIFSG and study its algebraic properties.
- To define PNSG and study its algebraic properties.
- To study some homomorphic characteristics of PFSG, PIFSG, PNSG.

3. DESCRIPTION OF THE WORK

3.1. Research Problem

So far, many researchers have studied different algebraic structures and fundamental properties of FSG, IFSG, and NSG. We know that homomorphic functions preserve algebraic structures. Therefore, to study those essential algebraic properties one need to study the effects of homomorphism on them. Several researchers have already introduced and studied homomorphism in the environments of FSG, IFSG and NSG. Also, some researchers have introduced the normal forms of FSG, IFSG, NSG and studied their homomorphic properties. Till now the concept of PFSG is undefined and unexplored. Again, depending upon the different degree of appurtenance and degree of dissimilarity functions some other PSG can be introduced, like PIFSG, PNSG, etc. In addition, PS is a generalized version of CS, FS, IFS as well as NS and hence PSGs i.e. PFSG, PIFSG, and PNSG have the potentials to become a generalized version of FSG, IFSG, and NSG. Smarandache has showed that with only one set theory i.e. PS theory all the other set theories can be developed. Similarly, with only PSG the notions of FSG, IFSG, and NSG can be developed.

Again, not only these notions are needed to be defined but also some essential analysis of homomorphic images, pre-images, etc. are needed to be analyzed. However, before introducing homomorphism in PSG, PIFSG and PNSG one first need to understand the behavior of any mapping in plithogenic environment. In this chapter, these essential notions of PSGs have been introduced and analyzed with proper examples. In the following preliminary subsection, some essential notions have been discussed, which were introduced earlier.

3.1.1. Preliminaries

The term plithogenic means pertaining to genesis or evolution or creation. A PS is a set in which its elements are characterized by one or more attributes and each attribute consists of some values. In PS a relation between an element and any attribute's value is denoted as $d(m,u)$, which is known as the degree of appurtenance function. Also, the relation between any two attribute's values is denoted as $c(u,u)$, which is known as the degree of contradiction or dissimilarity function. The following is a formal definition of a PS:

Definition 3.1 (Smarandache, 2018) Let *U* be a universal set and $P \subseteq U$. A PS is denoted as

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F}),$$

where ψ be an attribute or appurtenance, V_ψ is the corresponding range of attribute's value,

$$p_{d_F} : P \times V_\psi \to [0,1]^s$$

is the degree of appurtenance function (DAF) and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]^t$$

is the corresponding degree of contradiction function (DCF). Here $s,t \in \{1,2,3\}$.

Note that, in the above definition for $s=1$ and $t=1$ p_{d_F} will become a fuzzy DAF (FDAF) and p_{c_F} will become a fuzzy DCF (FDCF). In general, for simplicity, we consider only FDAF and FDCF. In the case of FDCF p_{c_F} satisfies the following axioms:

$$\forall (u_i, u_j) \in V_\psi \times V_\psi \ p_{c_F}(u_i, u_i) = 0$$

and

$$p_{c_F}(u_i, u_j) = p_{c_F}(u_j, u_i).$$

Again, to increase more accuracy one may wish to take $s = 2$ or $s = 3$ with $t = 1$. In that case, we will have

$$p_{d_{IF}} : P \times V_\psi \rightarrow [0,1]^2$$

(intuitionistic fuzzy DAF (IFDAF)) and

$$p_{d_N} : P \times V_\psi \rightarrow [0,1]^3$$

(neutrosophic DAF (NDAF)) along with p_{c_F} as FDCF. Again, to generalize further and increase the level of accuracy and complexity one may wish to take $t = 2$ or $t = 3$ i.e.

$$p_{c_F} : V_\psi \times V_\psi \rightarrow [0,1]^2$$

(intuitionistic fuzzy DCF (IFDCF)) or

$$p_{c_F} : V_\psi \times V_\psi \rightarrow [0,1]^3$$

(neutrosophic DCF (NDCF)).

CS, FS, IFS and NS are characterized by a single attribute which has one value for CS (membership (M)), two values for FS (M, nonmembership (NM)) and three values for NS (M, indeterminacy (I), NM) and hence using PS one can easily denote any CS, FS, IFS, and NS. So, PS is a generalization of these sets.

The following are some preliminary homomorphic characteristics of FSG, IFSG, and NSG. These fundamental characteristics will help one to understand the effects of homomorphism in PFSG, PIFSG, and PNSG.

Definition 3.2 (Anthony & Sherwood, 1979) A FS α of U is said to have supremum property if for any $\alpha' \subseteq \alpha\ \exists m_0 \in \alpha'$ such that $\alpha(m_0) = \sup_{m\in\alpha'} \alpha(m)$.

Theorem 3.1 (Anthony & Sherwood, 1979) Let α be a fuzzy subgroupoid of P on the basis of a continuous t-norm T and l be a homomorphism on P, then the image (supremum image) of α is a fuzzy subgroupoid on $l(P)$ with respect to T.

Theorem 3.2 (Rosenfeld, 1971) Homomorphic image or pre-image of any FSG having supremum property is a FSG.

Theorem 3.3 (Sharma, 2011) Let g be a homomorphism of a group U_1 into another group U_2 then preimage of an IFSG γ of U_2 i.e. $g^{-1}(\gamma)$ is an IFSG of U_1.

Theorem 3.4 (Sharma, 2011) Let g be a surjective homomorphism of a group U_1 to another group U_2, then the image of an IFSG γ of U_1 i.e. $g(\gamma)$ is an IFSG of U_2.

Theorem 3.5 (Çetkin & Aygün, 2015) Homomorphic image or pre-image of any NSG is a NSG.

Theorem 3.6 (Çetkin & Aygün, 2015) Let δ∈NNSG(U) and l be a homomorphism on U. Then the homomorphic pre-image of δ i.e. $l^{-1}(\delta) \in$ NNSG(U).

Theorem 3.7 (Çetkin & Aygün, 2015) Let δ∈NNSG(U) and l be a surjective homomorphism on U. Then the homomorphic image of δ i.e. $l(\delta) \in$ NNSG(U).

Since PS is a generalization of CS, FS, IFS and NS one may guess superiority of PSG over CSG, FSG, IFSG, and NSG. In the next section, we have introduced different types of PSGs and studied their homomorphic characteristics.

4. PROPOSED NOTIONS OF PLITHOGENIC SUBGROUPS

4.1. Plithogenic Fuzzy Subgroup

Definition 4.1 Let

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F})$$

be a PS of a group U. Where ψ is an attribute, V_ψ is a range of all attribute's values,

$$p_{d_F} : P \times V_\psi \to [0,1]$$

is the corresponding FDAF and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

is the corresponding FDCF. Then P_s is called a PFSG of *U* iff p_{d_F} is a fuzzy subgroup i.e. in other words iff

$$\forall (m_1, u_1), (m_2, u_2) \in P \times V_\psi$$

the subsequent conditions are fulfilled:

(i) $p_{d_F}((m_1, u_1) \cdot (m_2, u_2)) \geq \min\{p_{d_F}(m_1, u_1), p_{d_F}(m_2, u_2)\}$ and

(ii) $p_{d_F}((m_1, u_1)^{-1}) \geq p_{d_F}(m_1, u_1)$

A set of all PFSG of a group *U* is denoted as PFSG(*U*).

Example 4.1 Let

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F})$$

be a PS of a group *U,* where $P = \{m, u, mu, e\}$ be the Klein's four group, ψ be an attribute, $V_\psi = \{a, e'\}$ be a group consisting of two attribute values (here $a^2 = e'$ and e' is the neutral element). Also, let

$$p_{d_F} : P \times V_\psi \to [0,1]$$

and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

are respectively corresponding FDAF and FDCF defined in Table 3 and Table 4.

Then $P_s \in$ PFSG(*U*).

Table 3. FDAF

p_{d_F}	*m*	*u*	*mu*	*e*
a	0.2	0.5	0.3	0.2
e'	0.3	0.2	0.2	0.2

Note that in Definition 4.1 both p_{d_F} and p_{c_F} are FSs but only on p_{d_F} the conditions for FSG has been assigned because p_{c_F} will become a FSG only if

$\forall (u_i, u_j) \in V_\psi \; p_{c_F}(u_i, u_j) = 0$.

Example 4.2 Let

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F})$$

bet a PS of a group *U*, where $P = \{1, -1, i, -i\}$ be a cyclic group, ψ be an attribute, $V_\psi = \{m, u, mu, e\}$ be the Klein four-group. Also, let

$$p_{d_F} : P \times V_\psi \to [0,1]$$

and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

are respectively corresponding FDAF and FDCF given in Table 5 and Table 6.

Theorem 4.1 $P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F}) \in$ PFSG(*U*) iff

Table 4. FDCF

p_{c_F}	*a*	e'
a	0	0.5
e'	0.5	0

Table 5. FDAF

$\alpha(p)_{d_{IF}}$	*m*	*u*	*mu*	*e*
1	0.4	0.4	0.6	0.2
-1	0.4	0.4	0.7	0.2
i	0.4	0.4	0.8	0.2
-*i*	0.4	0.4	0.8	0.2

$$p_{d_F}((m_1,u_1)\cdot(m_2,u_2)^{-1}) \geq \min\{p_{d_F}(m_1,u_1), p_{d_F}(m_2,u_2)\}$$

Proof: Using Theorem 2.1 this can be easily proved.

Using PS one can easily handle indeterminate, uncertain and incongruous data. As a result, it has become more general than CS, FS, IFS as well as NS. Hence, it is quite evident that PFSG can have the potentials to become more general than CSG, FSG, IFSG, and NSG. In the next section, the preeminence aspects of PFSG have been discussed with proper justifications.

4.1.1. PFSG as a Generalization of Other Subgroups

Proposition 4.1 Any CSG is a PFSG (given that the corresponding attribute value set is a union of two singleton crisp groups).

Proof: Let *C* be a CSG of a group *U*. So, $C \subseteq U$ and hence *C* is a PS of *U*. Here, one may consider corresponding $\psi =$ "appurtenance", $V_\psi = \{\mathrm{M}, \mathrm{NM}\}$ with cardinality 2,

$$p_{d_F} : C \times V_\psi \to [0,1]$$

Table 6. FDCF

c_{d_F}	*m*	*u*	*mu*	*e*
m	0	0.2	0.3	0.5
u	0.2	0	0.9	0.8
mu	0.3	0.9	0	0.6
e	0.5	0.8	0.6	0

and

$p_{c_F} : V_\psi \times V_\psi \to [0,1]$.

Here

$p_{c_F}(\mathrm{M}, \mathrm{M}) = 0$,

$p_{c_F}(\mathrm{NM}, \mathrm{NM}) = 0$

and

$p_{c_F}(\mathrm{M}, \mathrm{NM}) = 1.$

Also, V_ψ can be considered as

$V_\psi = \{\mathrm{M}\} \cup \{\mathrm{NM}\}$,

where $\{\mathrm{M}\}$ and $\{\mathrm{NM}\}$ are singleton CSGs. Also, $p_{d_F}(m, \mathrm{M}) = 1$ and $p_{d_F}(m, \mathrm{NM}) = 0.$

Now,

$\forall\, m_1, m_2 \in C m_1 m_2^{-1} \in C$

i.e. $p_{d_F}(m_1, \mathrm{M}) = 1$ and $p_{d_F}(m_2, \mathrm{M}) = 1$ imply that

$p_{d_F}(m_1 m_2^{-1}, \mathrm{M}) = 1.$

Where from it can be easily proved that

$p_{d_F}(m_1, \mathrm{M}) \cdot (m_2, \mathrm{M})^{-1} = 1 \geq \min\{1,1\} = \min\{p_{d_F}(m_1, \mathrm{M}), p_{d_F}(m_2, \mathrm{M})\}.$

Similarly, $\forall\, m_1, m_2 \notin C$ it can be proved that

$$p_{d_F}(m_1, \mathrm{NM}) \cdot (m_2, \mathrm{NM})^{-1} = 0 \geq \min\{0,0\} = \min\{p_{d_F}(m_1, \mathrm{NM}), p_{d_F}(m_2, \mathrm{NM})\}.$$

Hence, $C \in$PFSG(U).

Proposition 4.2 Any FSG is a PFSG (given that the corresponding attribute value set is a singleton CSG).

Proof: Let α∈FSG(U). So, α∈FS(U) and hence α∈PS(U). Here one may consider corresponding ψ = "appurtenance", V_ψ = {M} with cardinality 1,

$$p_{d_F} : C_\alpha \times V_\psi \to [0,1] \, (C_\alpha = \{m \in U : (m, \alpha(m)) \in \alpha\})$$

and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

with p_{c_F} (M, M) = 0. Note that here

$$p_{d_F}(m, \mathrm{M}) = \alpha(m) \in [0,1].$$

Then by Theorem 2.1, $\forall\, m_1, m_2 \in C_\alpha$,

$$\alpha(m_1 m_2^{-1}) \geq min\{\alpha(m_1), \alpha(m_2)\}$$

$$\Rightarrow p_{d_F}(m_1 m_2^{-1}, \mathrm{M}) \geq min\{p_{d_F}(m_1, \mathrm{M}), p_{d_F}(m_2, \mathrm{M})\}$$

$$\Rightarrow p_{d_F}((m_1, \mathrm{M}) \cdot (m_2^{-1}, \mathrm{M})) \geq min\{p_{d_F}(m_1, \mathrm{M}), p_{d_F}(m_2, \mathrm{M})\}$$

$$\Rightarrow p_{d_F}((m_1, \mathrm{M}) \cdot (m_2, \mathrm{M})^{-1}) \geq min\{p_{d_F}(m_1, \mathrm{M}), p_{d_F}(m_2, \mathrm{M})\}$$

Hence, from it can be concluded that, α∈PFSG(U).

Proposition 4.3 Any IFSG is a PFSG (given that the corresponding attribute value set is a union of two singleton CSG).

Proof: Let

$\gamma = (m, t_\gamma(m), f_\gamma(m)) \in \text{IFSG}(U)$.

So, γ∈IFS(U)and hence γ∈PS(U). Here one may consider $\psi =$ "appurtenance", $V_\psi = \{\text{M}, \text{NM}\}$ with cardinality 2,

$$p_{d_F} : C_\gamma \times V_\psi \to [0,1]$$

$$(C_\gamma = \{m \in U : (m, t_\gamma(m), f_\gamma(m)) \in \gamma\})$$

and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

(here $p_{c_F}(\text{M},\text{M}) = 0$ $p_{c_F}(\text{NM},\text{NM}) = 0$ and $p_{c_F}(\text{M},\text{NM}) = 1.$)

Here, V_ψ can be considered as $V_\psi = \{\text{M}\} \cup \{\text{NM}\}$, where $\{\text{M}\}$ and $\{\text{NM}\}$ are singleton CSGs and

$$p_{d_F}(m, \text{M}) + p_{d_F}(m, \text{NM}) \le 1.$$

Note that,

$$p_{d_F}(m, \text{M}) = t_\gamma(m) \in [0,1]$$

and

$$p_{d_F}(m, \text{NM}) = f_\gamma(m) \in [0,1].$$

Then by Theorem 2.1, $\forall\, m_1, m_2 \in C_\gamma$,

$$t_\gamma(m_1 m_2^{-1}) \ge min\{t_\gamma(m_1), t_\gamma(m_2)\}$$

$$\Rightarrow p_{d_F}(m_1 m_2^{-1}, \text{M}) \ge min\{p_{d_F}(m_1, \text{M}), p_{d_F}(m_2, \text{M})\}$$

$$\Rightarrow p_{d_F}((m_1, \mathrm{M}) \cdot (m_2^{-1}, \mathrm{M})) \geq min\{p_{d_F}(m_1, \mathrm{M}), p_{d_F}(m_2, \mathrm{M})\}$$

$$\Rightarrow p_{d_F}((m_1, \mathrm{M}) \cdot (m_2, \mathrm{M})^{-1}) \geq min\{p_{d_F}(m_1, \mathrm{M}), p_{d_F}(m_2, \mathrm{M})\}$$

Again,

$$f_\gamma(m_1 m_2^{-1}) \leq \max\{f_\gamma(m_1), f_\gamma(m_2)\}$$

$$\Rightarrow p_{d_F}(m_1 m_2^{-1}, \mathrm{NM}) \leq \max\{p_{d_F}(m_1, \mathrm{NM}), p_{d_F}(m_2, \mathrm{NM})\}$$

$$\Rightarrow p_{d_F}((m_1, \mathrm{NM}) \cdot (m_2^{-1}, \mathrm{NM})) \leq \max\{p_{d_F}(m_1, \mathrm{NM}), p_{d_F}(m_2, \mathrm{NM})\}$$

$$\Rightarrow p_{d_F}((m_1, \mathrm{NM}) \cdot (m_2, \mathrm{NM})^{-1}) \leq \max\{p_{d_F}(m_1, \mathrm{NM}), p_{d_F}(m_2, \mathrm{NM})\}$$

Hence, from and γ∈PFSG(*U*).

Proposition 4.4 Any NSG is a PFSG (given that the corresponding attribute value set is a union of three singleton CSGs).

Proof: Using Proposition 4.2 and Proposition 4.3 this can be easily proved.

To understand structures of different algebraic objects, one needs to study functions that preserve those algebraic structures i.e. one must study the effects of homomorphism on those algebraic entities. In the next section image and preimage of any PS under a function has been defined. Also, the homomorphic image, preimage of any PSG have been introduced and efficiently discussed.

4.1.2. Homomorphism on PFSG

Let ψ be an attribute and V_ψ be the corresponding range of attribute's values and

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F})$$

be a PS of a group *U*. Also, let *f* be a function defined on $U \cup V_\psi$. Then the image of P_s is denoted as

$$P_s' = (P', \psi, V_\psi', p_{d_F}', p_{c_F}'),$$

where $p'_{d_F} : P' \times V'_\psi \to [0,1]$ is defined as

$$p'_{d_F}(m_2, u_2) = \sup_{\substack{m_1 \in f^{-1}(m_2) \\ u_1 \in f^{-1}(u_2)}} p_{d_F}(m_1, u_1)$$

and $p'_{c_F} : V'_\psi \times V'_\psi \to [0,1]$ is defined as

$$p'_{c_F}(u_3, u_4) = \sup_{\substack{u_1 \in f^{-1}(u_3) \\ u_2 \in f^{-1}(u_4)}} p_{c_F}(u_1, u_2).$$

Also, if

$$P'_s = (P', \psi, V'_\psi, p'_{d_F}, p'_{c_F})$$

is a PS of $f(U)$ then the preimage of P'_s will be denoted as

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F}),$$

where $p_{d_F} : P \times V_\psi \to [0,1]$ is defined as

$$p_{d_F}(m_1, u_1) = p'_{d_F}(f(m_1), f(u_1)),\ \forall (m_1, u_1) \in P \times V_\psi$$

and $p_{c_F} : V_\psi \times V_\psi \to [0,1]$ is defined as

$$p_{c_F}(u_1, u_2) = p'_{c_F}(f(u_1), f(u_2)),\ \forall (u_1, u_2) \in V_\psi \times V_\psi.$$

Theorem 4.2 Homomorphic preimage of a PFSG is a PFSG.

Proof: Let ψ be an attribute and V'_ψ be the corresponding range of attribute's values and

$$P'_s = (P', \psi, V'_\psi,\ p'_{d_F}, p'_{c_F})$$

be a PFSG of a group $f(U)$, where f is a homomorphism on $U \cup V_\psi$. Hence,

$$p'_{d_F} : P' \times V'_\psi \to [0,1]$$

is a FSG and

$$p'_{c_F} : V'_\psi \times V'_\psi \to [0,1]$$

is a FS.

Then preimage of P'_s is denoted as

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F}),$$

where

$p_{d_F} : P \times V_\psi \to [0,1]$ is defined as

$$p_{d_F}(m_1, u_1) = p'_{d_F}(f(m_1), f(u_1)) \ \forall (m_1, u_1) \in P \times V_\psi$$

and $p_{c_F} : V_\psi \times V_\psi \to [0,1]$ is defined as

$$p_{c_F}(u_1, u_2) = p'_{c_F}(f(u_1), f(u_2)) \ \forall (u_1, u_2) \in V_\psi \times V_\psi.$$

Let

$$(m_1, u_1), (m'_1, u'_1) \in P \times V_\psi,$$

then

$$p_{d_F}((m_1, u_1) \cdot (m'_1, u'_1)) = p_{d_F}(m_1 m'_1, u_1 u'_1)$$

$$= p'_{d_F}(f(m_1 m'_1), f(u_1 u'_1))$$

$= p'_{d_F}(f(m_1)f(m'_1), f(u_1)f(u'_1))$

$= p'_{d_F}(((f(m_1), f(u_1)) \cdot (f(m'_1), f(u'_1)))$

$\geq \min\{p'_{d_F}(f(m_1), f(u_1)),\ p'_{d_F}(f(m'_1), f(u'_1))\}$ (As p'_{d_F} is a FSG)

$= \min\{p_{d_F}(m_1, u_1),\ p_{d_F}(m'_1, u'_1)\}$

Also,

$p_{d_F}(m_1, u_1)^{-1} = p_{d_F}(m_1^{-1}, u_1^{-1})$

$= p'_{d_F}(f(m_1^{-1}), f(u_1^{-1}))$

$= p'_{d_F}(f(m_1)^{-1}, f(u_1)^{-1})$

$= p'_{d_F}(f(m_1), f(u_1))^{-1}$

$\geq p'_{d_F}(f(m_1), f(u_1))$

$= p_{d_F}(m_1, u_1)$

So, by and p_{d_F} is a FSG. Again, as p'_{c_F} is a FS its preimage under *f* i.e. p_{c_F} is a FS and hence, P_s is a PFSG of *U*.

Theorem 4.3 Homomorphic image of a PFSG is a PFSG (provided for FDAF and FDCF supremum property holds).

Proof: Let ψ be an attribute and V_ψ be the corresponding range of attribute's values and

$P_s = (P, \psi, V_\psi,\ p_{d_F}, p_{c_F})$

be a PFSG of a group U. Also, let f be a homomorphism defined on $U \cup V_{\psi}$.

Then the image of P_s is denoted as

$$P'_s = (P', \psi, V'_{\psi}, p'_{d_F}, p'_{c_F}),$$

where $p'_{d_F} : P' \times V'_{\psi} \to [0,1]$ is defined as

$$p'_{d_F}(m_2, u_2) = \sup_{\substack{m_1 \in f^{-1}(m_2) \\ u_1 \in f^{-1}(u_2)}} p_{d_F}(m_1, u_1)$$

and $p'_{c_F} : V'_{\psi} \times V'_{\psi} \to [0,1]$ is defined as

$$p'_{c_F}(u_3, u_4) = \sup_{\substack{u_1 \in f^{-1}(u_3) \\ u_2 \in f^{-1}(u_4)}} p_{c_F}(u_1, u_2).$$

Then

$$\forall (f(m_1), f(u_1)), (f(m'_1), f(u'_1)) \in P' \times V'_{\psi},$$

$\exists\, m_0 \in f^{-1}(f(m_1))$ and $\exists\, u_0 \in f^{-1}(f(u_1))$ such that

$$p_{d_F}(m_0, u_0) = \sup_{\substack{m_t \in f^{-1}(f(m_1)) \\ u_t \in f^{-1}(f(u_1))}} p_{d_F}(m_t, u_t).$$

Also, $\exists\, m'_0 \in f^{-1}(f(m'_1))$ such that

$$p_{d_F}(m'_0, u'_0) = \sup_{\substack{m_t \in f^{-1}(f(m'_1)) \\ u_t \in f^{-1}(f(u'_1))}} p_{d_F}(m_t, u_t).$$

So,

$$p'_{d_F}(f(m_1), f(u_1)) \cdot (f(m'_1), f(u'_1)) = p'_{d_F}(f(m_1) \cdot f(m'_1), f(u_1) \cdot f(u'_1))$$

$$= \sup_{\substack{m_z \in f^{-1}(f(m_1)\cdot f(m_1')) \\ u_z \in f^{-1}(f(u_1)\cdot f(u_1'))}} p_{d_F}(m_z, u_z)$$

$$\geq \min\{p_{d_F}(m_0, u_0), p_{d_F}(m_0', u_0')\}$$

$$= \min\{p_{d_F}'(f(m_1), f(u_1)), p_{d_F}'(f(m_1'), f(u_1'))\}$$

Again,

$$p_{d_F}'(f(m_1), f(u_1))^{-1} = p_{d_F}'(f(m_1)^{-1}, f(u_1)^{-1})$$

$$= \sup_{\substack{m_z \in f^{-1}(f(m_1)^{-1}) \\ u_z \in f^{-1}(f(u_1)^{-1})}} p_{d_F}(m_z, u_z)\ p_{c_F}$$

$$= p_{d_F}(m_0{}^{-1}, u_0{}^{-1})$$

$$= p_{d_F}(m_0, u_0)^{-1}$$

$$\geq p_{d_F}(m_0, u_0)$$

$$= p_{d_F}'(f(m_1), f(u_1))$$

So, by and p_{d_F}' is a FSG. Also, as p_{c_F} is a FS its image under f i.e. p_{c_F}' is a FS and hence, P_s' is a PFSG of $f(U)$.

In Definition 4.1 only FDAF (p_{d_F}) and FDCF (p_{c_F}) has been used. Instead of using a fuzzy environment, intuitionistic fuzzy or neutrosophic environments can also be used. In that case, the degree of appurtenance functions will be IFDAF ($p_{d_{IF}}$) or NDAF (p_{d_N}). Also, different degrees of contradiction functions like IFDCF ($p_{c_{IF}}$) or NDCF (p_{c_N}) can be used. In this chapter, for simplicity, only FDCF (p_{c_F}) has been considered. However, one may always use some complicated degree of contradiction functions, like, $p_{c_{IF}}$ or p_{c_N} to increase the level of accuracy as well as complexity as per their requirements. In the next section, using $p_{d_{IF}}$ and the notion PIFSG has been introduced and some of their homomorphic characteristics have been mentioned.

4.2. Plithogenic Intuitionistic Fuzzy Subgroup

Definition 4.2 Let

$$P_s = (P, \psi, V_\psi, p_{d_{IF}}, p_{c_F})$$

be a PS of a group *U*. Where ψ is an attribute, V_ψ is a range of all attribute's values,

$$p_{d_{IF}} : P \times V_\psi \to [0,1]^2$$

is the corresponding IFDAF and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

is the corresponding FDCF. Then P_s is called a PIFSG of *U* iff

$$p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\}$$

is an IFSG i.e. in other words iff

$$\forall (m_1, u_1), (m_2, u_2) \in P \times V_\psi$$

the subsequent conditions are fulfilled:

(i) $\alpha(p)_{d_{IF}}(m_1, u_1) \cdot (m_2, u_2) \geq \min\{\alpha(p)_{d_{IF}}(m_1, u_1), \alpha(p)_{d_{IF}}(m_2, u_2)\}$

(ii) $\alpha(p)_{d_{IF}}(m_1, u_1)^{-1} \geq \alpha(p)_{d_{IF}}(m_1, u_1)$

(iii) $\beta(p)_{d_{IF}}(m_1, u_1) \cdot (m_2, u_2) \leq \max\{\beta(p)_{d_{IF}}(m_1, u_1), \beta(p)_{d_{IF}}(m_2, u_2)\}$

Table 7. IFDAF with membership

$\boldsymbol{\alpha}(\boldsymbol{p})_{d_{IF}}$	*m*	*u*	*mu*	*e*
1	0.4	0.4	0.8	0.2
-1	0.4	0.4	0.5	0.2
i	0	0	0	0
-*i*	0	0	0	0

(iv) $\beta(p)_{d_{IF}}(m_1, u_1)^{-1} \leq \beta(p)_{d_{IF}}(m_1, u_1)$

A set of all PIFSG of a group *U* is denoted as PIFSG(*U*). Note that in Definition 4.2 for simplicity, the attribute value contradiction function p_{c_F} has been chosen. But to generalize Definition 4.2 one may use $p_{c_{IF}}$.

Example 4.3 Let

$$P_s = (P, \psi, V_\psi, p_{d_{IF}}, p_{c_F})$$

bet a PS of a group *U,* where $P = \{1, -1, i, -i\}$ be a cyclic group, ψ be an attribute, $V_\psi = \{m, u, mu, e\}$ be the Klein four-group. Also, let

$$p_{d_{IF}} : P \times V_\psi \to [0,1]^2 \, (\, p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\}\,)$$

and $p_{c_F} : V_\psi \times V_\psi \to [0,1]$ are respectively the corresponding IFDAF and FDCF, which are given in Table 7, and Table 9.

Table 8. IFDAF with nonmembership

$\boldsymbol{\beta}(\boldsymbol{p})_{d_{IF}}$	*m*	*u*	*mu*	*e*
1	0.6	0.6	0.2	0.8
-1	0.6	0.6	0.5	0.8
i	1	1	1	1
-*i*	1	1	1	1

Table 9. FDCF

c_{d_F}	m	u	mu	e
m	0	0.2	0.9	0.5
u	0.2	0	0.4	0.3
mu	0.9	0.4	0	0.2
e	0.5	0.3	0.2	0

Then $P_s \in \text{PIFSG}(U)$.

Example 4.4 Let

$$P_s = (P, \psi, V_\psi, p_{d_F}, p_{c_F})$$

bet a PS of a group *U,* where $P = \{m, u, mu, e\}$ be the Klein four-group, ψ be an attribute and

$$V_\psi = \{1, -1, i, -i\}$$

be a cyclic group. Also, let

$$p_{d_{IF}} : P \times V_\psi \rightarrow [0,1]^2 \, (p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\})$$

and

$$p_{c_F} : V_\psi \times V_\psi \rightarrow [0,1]$$

are the respective IFDAF and FDCF, which are given in Table 10, Table 11 and Table 12.

4.2.1. Homomorphism on PIFSG

Let ψ be an attribute and V_ψ be the corresponding range of attribute's values and

$$P_s = (P, \psi, V_\psi, p_{d_{IF}}, p_{c_F})$$

Table 10. IFDAF with membership

$\alpha(p)_{d_{IF}}$	**1**	**-1**	*i*	**-***i*
m	0.3	0.4	0.4	0.4
u	0.3	0.4	0.4	0.4
mu	0.3	0.4	0.4	0.4

Table 11. IFDAF with nonmembership

$\beta(p)_{d_{IF}}$	**1**	**-1**	*i*	**-***i*
m	0.7	0.6	0.6	0.6
u	0.7	0.6	0.6	0.6
mu	0.7	0.6	0.6	0.6
e	0.8	0.8	0.8	0.8

$\alpha(p)_{d_{IF}}$	**1**	**-1**	*i*	**-***i*
e	0.2	0.2	0.2	0.2

Table 12. FDCF

c_{d_F}	**1**	**-1**	*i*	**-***i*
1	0	0.5	0.3	0.8
-1	0.5	0	0.7	0.2
i	0.3	0.7	0	1
-*i*	0.8	0.2	1	0

be a PS of a group *U,* where

$$p_{d_{IF}} : P \times V_{\psi} \to [0,1]^2 \,(p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_{\psi}\})$$

is the corresponding IFDAF and

$$p_{c_F} : V_{\psi} \times V_{\psi} \to [0,1]$$

is the corresponding FDCF. Also, let *f* be a function defined on $U \cup V_{\psi}$. Then the image of P_s is denoted as

$$P'_s = (P', \psi, V'_{\psi}, p'_{d_{IF}}, p'_{c_F}),$$

where

$$p'_{d_{IF}} : P' \times V'_{\psi} \to [0,1]^2 \,(p'_{d_{IF}} = \{((m,u), \alpha(p')_{d_{IF}}, \beta(p')_{d_{IF}}) : (m,u) \in P' \times V'_{\psi}\})$$

is defined as

$$\alpha(p')_{d_{IF}}(m_2, u_2) = \sup_{\substack{m_1 \in f^{-1}(m_2) \\ u_1 \in f^{-1}(u_2)}} \alpha(p)_{d_{IF}}(m_1, u_1),$$

$$\beta(p')_{d_{IF}}(m_2, u_2) = \inf_{\substack{m_1 \in f^{-1}(m_2) \\ u_1 \in f^{-1}(u_2)}} \beta(p)_{d_{IF}}(m_1, u_1).$$

Again, $p'_{c_F} : V'_{\psi} \times V'_{\psi} \to [0,1]$ is defined as

$$p'_{c_F}(u_3, u_4) = \sup_{\substack{u_1 \in f^{-1}(u_3) \\ u_2 \in f^{-1}(u_4)}} p_{c_F}(u_1, u_2).$$

Also, if

$$P'_s = (P', \psi, V'_{\psi}, p'_{d_{IF}}, p'_{c_F})$$

is a PS of $f(U)$ then the preimage of P'_s will be denoted as

$$P_s = (P, \psi, V_\psi, p_{d_{IF}}, p_{c_F}).$$

Here

$$p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\},$$

where

$$\alpha(p)_{d_{IF}}(m_1, u_1) = \alpha(p')_{d_{IF}}(f(m_1), f(u_1)),$$

$$\beta(p)_{d_{IF}}(m_1, u_1) = \beta(p')_{d_{IF}}(f(m_1), f(u_1)) \forall (m_1, u_1) \in P \times V_\psi$$

and p_{c_F} is defined as

$$\forall (u_1, u_2) \in V_\psi \times V_\psi,\ p_{c_F}(u_1, u_2) = p'_{c_F}(f(u_1), f(u_2)).$$

Theorem 4.4 Homomorphic preimage of a PIFSG is a PIFSG.

Proof: Let ψ be an attribute and V'_ψ be the corresponding range of attribute's values and

$$P'_s = (P', \psi, V'_\psi, p'_{d_{IF}}, p'_{c_F})$$

be a PFSG of a group $f(U)$, where *f* is a homomorphism on $U \cup V_\psi$. Here,

$$p'_{d_{IF}} : P' \times V'_\psi \to [0,1]^2$$

is defined as

$$p'_{d_{IF}} = \{((m,u), \alpha(p')_{d_{IF}}, \beta(p')_{d_{IF}}) : (m,u) \in P' \times V'_\psi\}.$$

Then preimage of P'_s is denoted as

$P_s = (P, \psi, V_\psi, p_{d_{IF}}, p_{c_F})$.

Here

$p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\}$,

where

$\alpha(p)_{d_{IF}}(m_1, u_1) = \alpha(p')_{d_{IF}}(f(m_1), f(u_1))$,

$\beta(p)_{d_{IF}}(m_1, u_1) = \beta(p')_{d_{IF}}(f(m_1), f(u_1)) \forall (m_1, u_1) \in P \times V_\psi$

and p_{c_F} is defined as

$\forall (u_1, u_2) \in V_\psi \times V_\psi,\ p_{c_F}(u_1, u_2) = p'_{c_F}(f(u_1), f(u_2))$.

Then

$\forall (m_1, u_1), (m'_1, u'_1) \in P \times V_\psi$,

$\alpha(p)_{d_{IF}}((m_1, u_1) \cdot (m'_1, u'_1)) = \alpha(p)_{d_{IF}}(m_1 m'_1, u_1 u'_1)$

$= \alpha(p')_{d_{IF}}(f(m_1 m'_1), f(u_1 u'_1))$

$= \alpha(p')_{d_{IF}}(f(m_1) f(m'_1), f(u_1) f(u'_1))$

$= \alpha(p')_{d_{IF}}((f(m_1), f(u_1)) \cdot (f(m'_1), f(u'_1)))$

$\geq \min\{\alpha(p')_{d_{IF}}(f(m_1), f(u_1)),\ \alpha(p')_{d_{IF}}(f(m'_1), f(u'_1))\}$

$$= \min\{\alpha(p)_{d_{IF}}(m_1, u_1),\ \alpha(p)_{d_{IF}}(m_1', u_1')\}$$

Also,

$$\alpha(p)_{d_{IF}}(m_1, u_1)^{-1} = \alpha(p)_{d_{IF}}(m_1^{-1}, u_1^{-1})$$

$$= \alpha(p')_{d_{IF}}(f(m_1^{-1}), f(u_1^{-1}))$$

$$= \alpha(p')_{d_{IF}}(f(m_1)^{-1}, f(u_1)^{-1})$$

$$= \alpha(p')_{d_{IF}}(f(m_1), f(u_1))^{-1}$$

$$\geq \alpha(p')_{d_{IF}}(f(m_1), f(u_1))$$

$$= \alpha(p)_{d_{IF}}(m_1, u_1)$$

So, by and $\alpha(p)_{d_F}$ satisfies conditions (i) and (ii) of Definition 4.2.

Again,

$$\forall (m_1, u_1), (m_1', u_1') \in P \times V_\psi,$$

$$\beta(p)_{d_{IF}}((m_1, u_1) \cdot (m_1', u_1')) = \beta(p)_{d_{IF}}(m_1 m_1', u_1 u_1')$$

$$= \beta(p')_{d_{IF}}(f(m_1 m_1'), f(u_1 u_1'))$$

$$= \beta(p')_{d_{IF}}(f(m_1) f(m_1'), f(u_1) f(u_1'))$$

$$= \beta(p')_{d_{IF}}((f(m_1), f(u_1)) \cdot (f(m_1'), f(u_1')))$$

$$\leq \max\{\beta(p')_{d_{IF}}(f(m_1), f(u_1)),\ \beta(p')_{d_{IF}}(f(m_1'), f(u_1'))\}$$

$$= \max\{\beta(p)_{d_{IF}}(m_1, u_1),\ \beta(p)_{d_{IF}}(m_1', u_1')\}$$

Also,

$$\beta(p)_{d_{IF}}(m_1, u_1)^{-1} = \beta(p)_{d_{IF}}(m_1^{-1}, u_1^{-1})$$

$$= \beta(p')_{d_{IF}}(f(m_1^{-1}), f(u_1^{-1}))$$

$$= \beta(p')_{d_{IF}}(f(m_1)^{-1}, f(u_1)^{-1})$$

$$= \beta(p')_{d_{IF}}(f(m_1), f(u_1))^{-1}$$

$$\leq \beta(p')_{d_{IF}}(f(m_1), f(u_1))$$

$$= \beta(p)_{d_{IF}}(m_1, u_1)$$

So, by and $\beta(p)_{d_{IF}}$ satisfies conditions (iii) and (iv) of Definition 4.2.

Hence,

$$p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\}$$

forms an IFSG. Again, as p'_{c_F} is a FS its preimage under i.e. p_{c_F} is a FS and hence, P_s is a PIFSG of *U*.

Theorem 4.5 Homomorphic image of a PIFSG is a PIFSG (provided for $\alpha(p)_{d_{IF}}$ and p_{c_F} supremum property hold and for $\beta(p)_{d_{IF}}$ infimum property holds).

Proof: Let ψ be an attribute and V_ψ be the corresponding range of attribute's values and

$$P_s = (P, \psi, V_\psi, p_{d_{IF}}, p_{c_F})$$

be a PFSG of a group *U* and *f* be a homomorphism defined on $U \cup V_\psi$.

Here,

$$p_{d_{IF}} : P \times V_\psi \to [0,1]^2$$

is defined as

$$p_{d_{IF}} = \{((m,u), \alpha(p)_{d_{IF}}, \beta(p)_{d_{IF}}) : (m,u) \in P \times V_\psi\}.$$

Then the image of P_s is denoted as

$$P_s' = (P', \psi, V_\psi', p_{d_{IF}}', p_{c_F}').$$

Here

$$p_{d_{IF}}' = \{((m,u), \alpha(p')_{d_{IF}}, \beta(p')_{d_{IF}}) : (m,u) \in P' \times V_\psi'\},$$

where

$$\alpha(p')_{d_{IF}}(m_2, u_2) = \sup_{\substack{m_1 \in f^{-1}(m_2) \\ u_1 \in f^{-1}(u_2)}} \alpha(p)_{d_{IF}}(m_1, u_1)$$

and

$$\beta(p')_{d_{IF}}(m_2, u_2) = \inf_{\substack{m_1 \in f^{-1}(m_2) \\ u_1 \in f^{-1}(u_2)}} \beta(p)_{d_{IF}}(m_1, u_1).$$

Also, p_{c_F}' is defined as

$$p_{c_F}'(u_3, u_4) = \sup_{\substack{u_1 \in f^{-1}(u_3) \\ u_2 \in f^{-1}(u_4)}} p_{c_F}(u_1, u_2).$$

Then

$$\forall (f(m_1), f(u_1)), (f(m_1'), f(u_1')) \in P' \times V_\psi',$$

$$\exists m_0 \in f^{-1}(f(m_1)) \text{ and } \exists u_0 \in f^{-1}(f(u_1))$$

such that

$$\alpha(p)_{d_{IF}}(m_0, u_0) = \sup_{\substack{m_t \in f^{-1}(f(m_1)) \\ u_t \in f^{-1}(f(u_1))}} \alpha(p)_{d_{IF}}(m_t, u_t).$$

Also,

$$\exists m_0' \in f^{-1}(f(m_1')) \text{ and } \exists u_0' \in f^{-1}(f(u_1'))$$

such that

$$\alpha(p)_{d_{IF}}(m_0', u_0') = \sup_{\substack{m_t \in f^{-1}(f(m_1')) \\ u_t \in f^{-1}(f(u_1'))}} \alpha(p)_{d_{IF}}(m_t, u_t).$$

So,

$$\alpha(p')_{d_{IF}}(f(m_1), f(u_1)) \cdot (f(m_1'), f(u_1')) = \alpha(p')_{d_{IF}}(f(m_1) \cdot f(m_1'), f(u_1) \cdot f(u_1'))$$

$$= \sup_{\substack{m_z \in f^{-1}(f(m_1) \cdot f(m_1')) \\ u_z \in f^{-1}(f(u_1) \cdot f(u_1'))}} \alpha(p)_{d_{IF}}(m_z, u_z)$$

$$\geq \min\{\alpha(p)_{d_{IF}}(m_0, u_0), \alpha(p)_{d_{IF}}(m_0', u_0')\}$$

$$= \min\{\alpha(p')_{d_{IF}}(f(m_1), f(u_1)), \alpha(p')_{d_{IF}}(f(m_1'), f(u_1'))\}$$

Again,

$$\alpha(p')_{d_{IF}}(f(m_1), f(u_1))^{-1} = \alpha(p')_{d_{IF}}(f(m_1)^{-1}, f(u_1)^{-1})$$

$$= \sup_{\substack{m_z \in f^{-1}(f(m_1)^{-1}) \\ u_z \in f^{-1}(f(u_1)^{-1})}} \alpha(p)_{d_{IF}}(m_z, u_z)$$

$$= \alpha(p)_{d_{IF}}(m_0^{-1}, u_0^{-1})$$

$$= \alpha(p)_{d_{IF}}(m_0, u_0)^{-1}$$

$$\geq \alpha(p)_{d_{IF}}(m_0, u_0)$$

$$= \alpha(p')_{d_{IF}}(f(m_1), f(u_1))$$

So, by and $\alpha(p')_{d_{IF}}$ satisfies conditions (i) and (ii) of Definition 4.2.

Again, let

$$\forall (f(m_1), f(u_1)), (f(m_1'), f(u_1')) \in P' \times V_\psi',$$

$\exists\, m_0 \in f^{-1}(f(m_1))$ and $\exists\, u_0 \in f^{-1}(f(u_1))$

such that

$$\beta(p)_{d_{IF}}(m_0, u_0) = \inf_{\substack{m_t \in f^{-1}(f(m_1)) \\ u_t \in f^{-1}(f(u_1))}} \beta(p)_{d_{IF}}(m_t, u_t).$$

Also,

$\exists\, m_0' \in f^{-1}(f(m_1'))$ and $\exists\, u_0' \in f^{-1}(f(u_1'))$

such that

$$\beta(p)_{d_{IF}}(m_0', u_0') = \inf_{\substack{m_t \in f^{-1}(f(m_1')) \\ u_t \in f^{-1}(f(u_1'))}} \beta(p)_{d_{IF}}(m_t, u_t).$$

$$\beta(p')_{d_{IF}}(f(m_1), f(u_1)) \cdot (f(m_1'), f(u_1')) = \beta(p')_{d_{IF}}(f(m_1) \cdot f(m_1'), f(u_1) \cdot f(u_1'))$$

$$= \inf_{\substack{m_z \in f^{-1}(f(m_1) \cdot f(m_1')) \\ u_z \in f^{-1}(f(u_1) \cdot f(u_1'))}} \beta(p)_{d_{IF}}(m_z, u_z)$$

$$\leq \max\{\beta(p)_{d_{IF}}(m_0, u_0), \beta(p)_{d_{IF}}(m_0', u_0')\}$$

$$= \max\{\beta(p')_{d_{IF}}(f(m_1), f(u_1)), \beta(p')_{d_{IF}}(f(m_1'), f(u_1'))\}$$

Again,

$$\beta(p')_{d_{IF}}(f(m_1), f(u_1))^{-1} = \beta(p')_{d_{IF}}(f(m_1)^{-1}, f(u_1)^{-1})$$

$$= \inf_{\substack{m_z \in f^{-1}(f(m_1)^{-1}) \\ u_z \in f^{-1}(f(u_1)^{-1})}} \alpha(p)_{d_{IF}}(m_z, u_z)$$

$$= \beta(p)_{d_{IF}}(m_0^{-1}, u_0^{-1})$$

$$= \beta(p)_{d_{IF}}(m_0, u_0)^{-1}$$

$$\leq \beta(p)_{d_{IF}}(m_0, u_0)$$

$$= \beta(p')_{d_{IF}}(f(m_1), f(u_1))$$

So, by and $\beta(p')_{d_F}$ satisfies conditions (iii) and (iv) of Definition 4.2.

Hence,

$$p'_{d_{IF}} = \{((m,u), \alpha(p')_{d_{IF}}, \beta(p')_{d_{IF}}) : (m,u) \in P' \times V'_\psi\}$$

forms an IFSG. Again, as p_{c_F} is a FS its preimage under *f* i.e. p_{c_F} is a FS and hence, P_s is a PIFSG of *U*.

Notice that in Definition 4.2 IFDAF ($p_{d_{IF}}$) and FDCF (p_{c_F}) has been used. But, here instead of using an intuitionistic fuzzy setting one may wish to use a neutrosophic environment. In that case, the degree of appurtenance functions will be NDAF (p_{d_N}). Also, one may wish to use other degrees of contradiction functions like IFDCF ($p_{c_{IF}}$) or NDCF (p_{c_N}). In the next section using p_{d_N} and p_{c_F} the notion PNSG has been introduced and some of their homomorphic characteristics have been discussed.

4.3. Plithogenic Neutrosophic Subgroup

Definition 4.3 Let

$$P_s = (P, \psi, V_\psi, p_{d_N}, p_{c_F})$$

be a PS of a group *U*. Where ψ is an attribute, V_ψ is a range of all attribute's values,

$$p_{d_N} : P \times V_\psi \to [0,1]^3$$

is the corresponding NDAF and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

is the corresponding FDCF. Then P_s is called a PNSG of *U* iff

$$p_{d_N} = \{((m,u), t(p)_{d_N}, i(p)_{d_N}, f(p)_{d_N}) : (m,u) \in P \times V_\psi\}$$

is a neutrosophic subgroup i.e. in other words iff

$$\forall (m_1,u_1),(m_2,u_2) \in P \times V_\psi,$$

the subsequent conditions are fulfilled:

(i) $t(p)_{d_{IF}}((m_1,u_1)\cdot(m_2,u_2)^{-1}) \geq \min\{t(p)_{d_{IF}}(m_1,u_1), t(p)_{d_{IF}}(m_2,u_2)\}$

Table 13. NDAF with truth

$t(p)_{d_N}$	*m*	*u*	*mu*	e'
a	0.4	0.4	0.5	0.2

Table 14. NDAF with indeterminacy

$i(p)_{d_N}$	*m*	*u*	*mu*	e'
a	0.3	0.3	0.3	0.2
e	0.4	0.4	0.5	0.2

$t(p)_{d_N}$	*m*	*u*	*mu*	e'
e	0.4	0.4	0.6	0.2

(ii) $i(p)_{d_{IF}}((m_1,u_1)\cdot(m_2,u_2)^{-1}) \geq \min\{i(p)_{d_{IF}}(m_1,u_1), i(p)_{d_{IF}}(m_2,u_2)\}$

(iii) $f(p)_{d_{IF}}((m_1,u_1)\cdot(m_2,u_2)^{-1}) \leq \max\{f(p)_{d_{IF}}(m_1,u_1), f(p)_{d_{IF}}(m_2,u_2)\}$

A set of all PNSG of a group *U* is denoted as PNSG(*U*). Note that in Definition 4.3 for simplicity the attribute value contradiction function p_{c_F} has been chosen. But to generalize Definition 4.3 one may use p_{c_N}.

Example 4.5 Let

Table 15. NDAF with falsity

$f(p)_{d_N}$	*m*	*u*	*mu*	e'
a	0.6	0.6	0.5	0.8
e	0.6	0.6	0.4	0.8

Table 16. FDCF

p_{c_F}	m	u	mu	e'
m	0	0.1	0.9	1
u	0.1	0	0.5	0.7
mu	0.9	0.5	0	0.2
e'	1	0.7	0.2	0

$$P_s = (P, \psi, V_\psi, p_{d_N}, p_{c_F})$$

bet a PS of a group *U,* where $P = \{a, e\}$ ($a^2 = e$) be a group, ψ be an attribute, $V_\psi = \{m, u, mu, e'\}$ be the Klein's four group. Also, let

$$p_{d_N} : P \times V_\psi \to [0,1]^3 \,(\, p_{d_N} = \{((m,u),\ t(p)_{d_N}, i(p)_{d_N}, f(p)_{d_N}) : (m,u) \in P \times V_\psi\}\,)$$

and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

are respectively the corresponding NDAF and FDCF mentioned in Table 13, Table 14, Table 15 and Table 16.

Then $P_s \in$ PNSG(*U*).

4.3.1. Homomorphism on PNSG

Let ψ be an attribute and V_ψ be the corresponding range of attribute's values and

$$P_s = (P, \psi, V_\psi, p_{d_N}, p_{c_F})$$

be a PS of a group *U,* where

$$p_{d_N} : P \times V_\psi \to [0,1]^3 \,(\, p_{d_N} = \{((m,u), t(p)_{d_N}, i(p)_{d_N}, f(p)_{d_N}) : (m,u) \in P \times V_\psi\}\,)$$

is the corresponding NDAF and

$$p_{c_F} : V_\psi \times V_\psi \to [0,1]$$

is the corresponding FDCF. Also, let *g* be a function defined on $U \cup V_\psi$. Then the image of P_s is denoted as

$$P'_s = (P', \psi, V'_\psi, p'_{d_N}, p'_{c_F}),$$

where

$$p'_{d_N} : P' \times V'_\psi \\ \to [0,1]^3 \, (p'_{d_N} = \{((m,u), t(p')_{d_N}, i(p')_{d_N}, f(p')_{d_N}) : (m,u) \in P' \times V'_\psi\})$$

is defined as

$$t(p')_{d_N}(m_2,u_2) = \sup_{\substack{m_1 \in g^{-1}(m_2) \\ u_1 \in g^{-1}(u_2)}} t(p)_{d_N}(m_1,u_1),$$

$$i(p')_{d_N}(m_2,u_2) = \sup_{\substack{m_1 \in g^{-1}(m_2) \\ u_1 \in g^{-1}(u_2)}} i(p)_{d_N}(m_1,u_1),$$

$$f(p')_{d_N}(m_2,u_2) = \inf_{\substack{m_1 \in g^{-1}(m_2) \\ u_1 \in g^{-1}(u_2)}} f(p)_{d_N}(m_1,u_1)$$

and $p'_{c_F} : V'_\psi \times V'_\psi \to [0,1]$ is defined as

$$p'_{c_F}(u_3,u_4) = \sup_{\substack{u_1 \in g^{-1}(u_3) \\ u_2 \in g^{-1}(u_4)}} p_{c_F}(u_1,u_2).$$

Also, if

$$P'_s = (P', \psi, V'_\psi, p'_{d_N}, p'_{c_F})$$

is a PS of $\mathrm{g}(U)$ then the preimage of P_s' will be denoted as

$P_s = (P, \psi, V_\psi, p_{d_N}, p_{c_F})$.

Here

$$p_{d_N} = \{((m,u), t(p)_{d_N}, i(p)_{d_N}, f(p)_{d_N}) : (m,u) \in P \times V_\psi\},$$

where

$$\forall (m_1, u_1) \in P \times V_\psi, t(p)_{d_N}(m_1, u_1) = t(p')_{d_N}(\mathrm{g}(m_1), \mathrm{g}(u_1)),$$

$$i(p)_{d_N}(m_1, u_1) = i(p')_{d_N}(\mathrm{g}(m_1), \mathrm{g}(u_1)),$$

$$f(p)_{d_N}(m_1, u_1) = f(p')_{d_N}(g(m_1), g(u_1))$$

and p_{c_F} is defined as

$$\forall (u_1, u_2) \in V_\psi \times V_\psi, p_{c_F}(u_1, u_2) = p'_{c_F}(\mathrm{g}(u_1), \mathrm{g}(u_2)).$$

Theorem 4.6 Homomorphic preimage of a PNSG is a PNSG.

Proof: Using Theorem 4.2 and Theorem 4.4 this can be easily proved.

Theorem 4.7 Homomorphic image of a PNSG is a PNSG (provided for $f(p)_{d_N}$, $i(p)_{d_N}$ and p_{c_F} supremum property hold and for $f(p)_{d_N}$ infimum property holds).

Proof: Using Theorem 4.3 and Theorem 4.5 this can be easily proved.

CONCLUSION

Group theory is a fundamental part of abstract algebra. To study algebraic characteristics of any object we need to understand functions which preserve its algebraic characteristics i.e. we need to study homomorphism. The notion of plithogenic subgroup is nothing but generalization of crisp subgroup, fuzzy subgroup, intuitionistic fuzzy subgroup, and neutrosophic subgroup. Hence, plithogenic subgroups have been introduced and the effects of homomorphism on them have been studied. Most of the definitions mentioned in this chapter can further be generalized by using general T-norm and T-conorm. Again, most of the theorems, propositions mentioned here can also be proved by using those triangular norms. In future, one may extend research by introducing normal forms of different plithogenic subgroups i.e. normal plithogenic fuzzy subgroup, normal plithogenic intuitionistic fuzzy subgroup, and normal plithogenic neutrosophic subgroup. Again, one can study their fundamental properties and homomorphic characteristics. In addition, one may introduce the notion of soft set theory in plithogenic subgroup and further generalize them.

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