Lemoine Circles

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In this article, we get to **Lemoine's circles** in a different manner than the known one.

Theorem 1.

Let ABC a triangle and K its simedian center. We take through K the parallel A_1A_2 to BC, $A_1 \in (AB)$, $A_2 \in (AC)$; through A_2 we take the antiparallels A_2B_1 to AB in relation to CA and CB, $B_1 \in (BC)$; through B_1 we take the parallel B_1B_2 to AC, $B_2 \in AB$; through B_2 we take the antiparallels B_1C_1 to BC, $C_1 \in (AC)$, and through C_1 we take the parallel C_1C_2 to AB, $C_1 \in (BC)$. Then:

i. C_2A_1 is an antiparallel of AC;
ii. $B_1B_2 \cap C_1C_2 = \{K\}$;
iii. The points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclical (*the first Lemoine circle*).

Proof.

i. The quadrilateral BC_2KA is a parallelogram, and its center, i.e. the middle of the segment (C_2A_1), belongs to the simedian BK; it follows that C_2A_2 is an antiparallel to AC (see *Figure 1*).
ii. Let $\{K'\} = A_1A_2 \cap B_1B_2$, because the quadrilateral $K'B_1CA_2$ is a parallelogram; it follows that CK' is a simedian; on the other hand, CK is a simedian, and since $K, K' \in A_1A_2$, it follows that we have $K' = K$.

iii. B_2C_1 being an antiparallel to BC and $A_1A_2 \parallel BC$, it means that B_2C_1 is an antiparallel to A_1A_2, so the points B_2, C_1, A_2, A_1 are concyclical.

Fom $B_1B_2 \parallel AC$, $\sphericalangle B_2C_1A \equiv \sphericalangle ABC$, $\sphericalangle B_1A_2C \equiv \sphericalangle ABC$ we get that the quadrilateral $B_2C_1A_2B_1$ is an isosceles trapezoid, so the points B_2, C_1, A_2, B_1 are concyclical.

Analogously, it can be shown that the quadrilateral $C_2B_1A_2A_1$ is an isosceles trapezoid, therefore the points C_2, B_1, A_2, A_1 are concyclical.

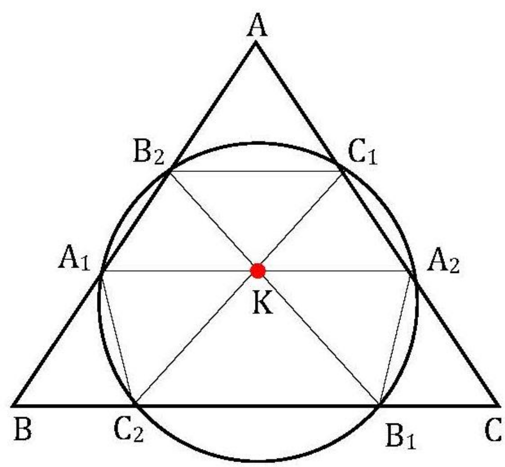

Figure 1

From the previous three quartets of concyclical points, it results the concyclicity of the points belonging to *the first Lemoine circle*.

Theorem 2.

In the scalene triagle ABC, let K be the simedian center. We take from K the antiparallel A_1A_2 to BC; $A_1 \in AB, A_2 \in AC$; through A_2 we build $A_2B_1 \parallel AB$; $B_1 \in (BC)$, then through B_1 we build B_1B_2 the antiparallel to AC, $B_2 \in (AB)$, and through B_2 we build $B_2C_1 \parallel BC$, $C_1 \in AC$, and, finally, through C_1 we take the antiparallel C_1C_2 to AB, $C_2 \in (BC)$. Then:

i. $C_2A_1 \parallel AC$;

ii. $B_1B_2 \cap C_1C_2 = \{K\}$;

iii. The points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclical (*the second Lemoine circle*).

Proof.

i. Let $\{K'\} = A_1A_2 \cap B_1B_2$, having $\sphericalangle AA_1A_2 = \sphericalangle ACB$ and $\sphericalangle BB_1B_2 \equiv \sphericalangle BAC$ because A_1A_2 și B_1B_2 are antiparallels to BC, AC, respectively, it follows that $\sphericalangle K'A_1B_2 \equiv \sphericalangle K'B_2A_1$, so $K'A_1 = K'B_2$; having $A_1B_2 \parallel B_1A_2$ as well, it follows that also $K'A_2 = K'B_1$, so $A_1A_2 = B_1B_2$. Because C_1C_2 and B_1B_2 are antiparallels to AB and AC, we have $K"C_2 = K"B_1$; we noted $\{K"\} = B_1B_2 \cap C_1C_2$; since $C_1B_2 \parallel B_1C_2$, we have that the triangle $K"C_1B_2$ is also isosceles, therefore $K"C_1 = C_1B_2$, and we get that $B_1B_2 = C_1C_2$. Let $\{K'''\} = A_1A_2 \cap C_1C_2$; since A_1A_2 and C_1C_2 are antiparallels to BC and AB, we get that the triangle $K'''A_2C_1$ is isosceles, so $K'''A_2 = K'''C_1$, but $A_1A_2 = C_1C_2$ implies that $K'''C_2 = K'''A_1$, then $\sphericalangle K'''A_1C_2 \equiv \sphericalangle K'''A_2C_1$ and, accordingly, $C_2A_1 \parallel AC$.

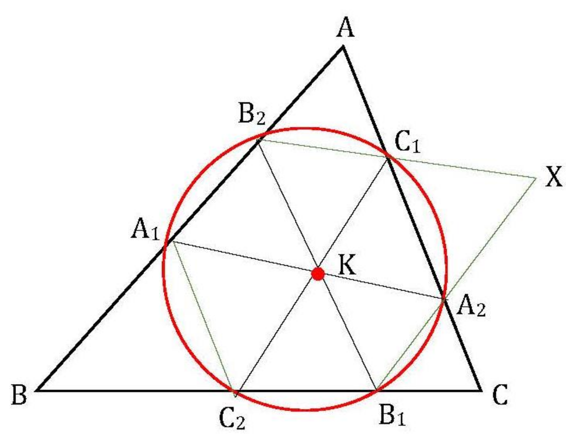

Figure 2

ii. We noted $\{K'\} = A_1A_2 \cap B_1B_2$; let $\{X\} = B_2C_1 \cap B_1A_2$; obviously, BB_1XB_2 is a parallelogram; if K_0 is the middle of (B_1B_2), then BK_0 is a simedian, since B_1B_2 is an antiparallel to AC, and the middle of the antiparallels of AC are situated on the simedian BK. If $K_0 \neq$K, then $K_0K \parallel A_1B_2$ (because $A_1A_2 = B_1B_2$ and $B_1A_2 \parallel A_1B_2$), on the other hand, $B, K_0,$K are collinear (they belong to the simedian BK), therefore K_0K intersects AB in B, which is absurd, so $K_0 =$K, and, accordingly, $B_1B_2 \cap A_1A_2 = \{K\}$. Analogously, we prove that $C_1C_2 \cap A_1A_2 = \{K\}$, so $B_1B_2 \cap C_1C_2 = \{K\}$.

iii. K is the middle of the congruent antiparalells A_1A_2, B_1B_2, C_1C_2, so $KA_1 = KA_2 = KB_1 = KB_2 = KC_1 = KC_2$. The simedian center K is the center of *the second Lemoine circle*.

Remark.

The center of *the first Lemoine circle* is the middle of the segment $[OK]$, where O is the center of the circle circumscribed to the triangle ABC. Indeed, the perpendiculars taken from A, B, C on the antiparallels B_2C_1, A_1C_2, B_1A_2 respectively pass through O, the center of the circumscribed circle (the antiparallels have the directions of the tangents taken to the circumscribed circle in A, B, C). The mediatrix of the segment B_2C_1 pass though the middle of B_2C_1, which coincides with the middle of AK, so is the middle line in the triangle AKO passing through the middle of (OK). Analogously, it follows that the mediatrix of A_1C_2 pass through the middle L_1 of $[OK]$.

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