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MBJ-neutrosophic sets: Expanding on internal and external relations

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Abstract

This paper presents a comprehensive study of internal and external MBJ-neutrosophic sets, detailing their definitions, properties, and potential applications. By initially introducing foundational concepts, including interval numbers and neutrosophic sets, the paper explores the MBJ-neutrosophic framework, which integrates truth, indeterminate, and false membership functions. We categorize MBJ-neutrosophic sets into internal and external types according to specific membership conditions, presenting a series of propositions and theorems that clarify the relationships among different MBJ-neutrosophic set types. Additionally, we examine criteria under which the intersection and union of these sets maintain their internal or external characteristics. The findings significantly enhance the theoretical structure of neutrosophic set theory and highlight its utility in decision-making processes and fuzzy logic systems.

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1 Introduction

Zadeh's foundational work on fuzzy sets and generalized uncertainty theories also provided critical insights, bridging neutrosophic sets with classical fuzzy logic and enabling a unified approach to uncertainty in mathematical analysis [15, 16, 17]. The concept of neutrosophic sets, introduced by Smarandache as a generalization of intuitionistic fuzzy sets, offers a robust framework for handling uncertainty, indeterminacy, and vagueness in mathematical structures and real-world applications.

Smarandache's pioneering works on neutrosophy established the foundational theories of neutrosophic probability, sets, and logic, which have since served as the basis for various extensions and applications across different fields of study, including algebraic structures and systems theory [10, 11, 12]. Also, Florentin Smarandache offered a philosophical foundation that goes beyond binary logic, enabling the representation of varying degrees of truth, indeterminacy, and falsehood. Building on this principle, n-valued refined neutrosophic sets incorporate the concepts of neutrosophy with the versatility of multi-valued logic. This framework provides a more accurate representation of uncertainty by allowing for multiple evaluation levels for each of the key parameters: truth, indeterminacy, and falsehood [13]. What sets n-Valued Refined Neutrosophic Sets apart is their ability to represent complex relationships within a single model. Each element of a set is defined by multiple degrees of membership, enabling a detailed exploration of the varying states of truth and falsehood. This capability allows for richer interpretations of data and more nuanced decision-making processes. Building on the neutrosophic set framework, subsequent research has explored specific adaptations for algebraic structures, particularly within BCI/BCK-algebras, subclasses of logical algebras with applications in computer science and artificial intelligence. Wang et al. further contributed to the development of single-valued neutrosophic sets, streamlining the applicability of neutrosophic sets to more conventional mathematical structures [14]. Within this framework, the MBJ-neutrosophic set theory has emerged as a valuable tool for exploring ideal and subalgebra structures in BCI/BCK-algebras. Bordbar, Mohseni Takallo, Borzooei, and Jun introduced BMBJ-neutrosophic subalgebras and ideals, elaborating on the characterization of these structures in the context of BCI/BCK-algebras [1, 7, 8]. Their work provided a detailed framework for understanding the internal and external properties of MBJ-neutrosophic sets, enhancing the study of logical algebras with applications in decision-making and information systems. Subsequent studies further advanced this research by investigating specific types of MBJ-neutrosophic ideals, such as commutative and positive implicative ideals. Borzooei, Mohseni Takallo, and Jun explored the commutative MBJ-neutrosophic ideals within BCK-algebras, establishing conditions under which these ideals can be applied to simplify logical computations and reasoning processes [2]. Hur, Lee, and Jun studied positive implicative MBJ-neutrosophic ideals, which address particular logical operations within the BCI/BCK frameworks, further demonstrating the relevance of MBJ-neutrosophic sets in fuzzy logic and uncertainty modeling [3]. In addition, research by Jun and Roh, as well as Khalid et al., has investigated the structure and magnification of MBJ-neutrosophic translations in G-algebras, focusing on the transformations and applications of these structures across broader algebraic systems [4, 5]. Mohseni Takallo and Kologani also contributed by examining MBJ-neutrosophic filters of equality algebras, providing insights into how these filters could support more complex decision-making models [6].

This paper provides an in-depth analysis of internal and external MBJ-neutrosophic sets, exploring their definitions, key properties, and potential applications. It begins by introducing essential concepts such as interval numbers and neutrosophic sets before delving into the MBJ-neutrosophic framework, which incorporates truth, indeterminate, and falsity membership functions. The study categorizes MBJ-neutrosophic sets into internal and external types based on specific membership conditions and presents a series of propositions and theorems that clarify the relationships between these different types. Furthermore, the paper investigates the conditions under which the intersection and union of these sets retain their internal or external characteristics. The results of this work make significant contributions to the theoretical development of neutrosophic set theory, emphasizing its practical value in decision-making and fuzzy logic applications.

2 Preliminaries

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $[0,1]$, where $0 \le a^- \le a^+ \le 1$. The interval number $\tilde{a} = [a^-, a^+]$ with $a^- = a^+$ is denoted by $\mathbf{a}$. Denote by $\mathcal{I}[0,1]$ the set of all interval numbers. Let us define what is known as a *refined minimum* (briefly, rmin) of two elements in $\mathcal{I}[0,1]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$ such that

(1) If $\tilde{a}_1 \cap \tilde{a}_2 \neq \emptyset$, then

$$\begin{aligned} \mathrm{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} &= \left[\max\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ \mathrm{rmax}\left\{\tilde{a}_1, \tilde{a}_2\right\} &= \left[\min\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \end{aligned}$$

We define

$$\underset{i\in\Lambda}{\mathrm{rinf}}\, \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \underset{i\in\Lambda}{\mathrm{rsup}}\, \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

(2) If $\tilde{a}_1 \cap \tilde{a}_2 = \emptyset$, then

$$\mathrm{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = [0,0] = 0,$$
$$\mathrm{rmax}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right] \setminus \left[\min\left\{a_1^+, a_2^+\right\}, \max\left\{a_1^-, a_2^-\right\}\right] \cup \left\{\min\left\{a_1^+, a_2^+\right\}\right\} \cup \left\{\max\left\{a_1^-, a_2^-\right\}\right\},$$

We define

$$\underset{i\in\Lambda}{\mathrm{rinf}}\, \tilde{a}_i = [0,0] = 0,$$

$$\underset{i\in\Lambda}{\mathrm{rsup}}\, \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right] \setminus \left[\inf_{i\in\Lambda} a_i^+, \sup_{i\in\Lambda} a_i^-\right] \cup \{\inf_{i\in\Lambda} a_i^+\} \cup \{\sup_{i\in\Lambda} a_i^-\}.$$

We also define the symbols "$\succeq$", "$\preceq$", "$=$" in the case of two elements in $\mathcal{I}[0,1]$.

$$\tilde{a}_1 \succeq \tilde{a}_2 \;\Leftrightarrow\; a_1^- \ge a_2^-,\; a_1^+ \ge a_2^+,$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in \mathcal{I}[0,1]$ where $i \in \Lambda$. For any $\tilde{a} \in \mathcal{I}[0,1]$, its *complement*, denoted by $\tilde{a}^c$, is defined be the interval number

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

An *interval-valued fuzzy set* (briefly, an IVF set) $\tilde{f}$ on a universe X (see [**?**]) is a mapping $\tilde{f} : X \to int([0,1])$ where $int([0,1])$ stands for the family of all closed subintervals of $[0,1]$.

For any $\tilde{a}_1 = \left[a_1^-, a_1^+\right], \tilde{a}_2 = \left[a_2^-, a_2^+\right] \in int([0,1])$, we define

$$\begin{aligned} &\mathrm{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ &\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \ge a_2^- \text{ and } a_1^+ \ge a_2^+, \end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$.

For any $\tilde{a}_i = \left[a_i^-, a_i^+\right] \in int([0,1])$ where $i \in \Lambda$, we define

$$\underset{i\in\Lambda}{\mathrm{rinf}}\, \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \underset{i\in\Lambda}{\mathrm{rsup}}\, \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

For any $\tilde{a} = [a^-, a^+] \in int([0,1])$, its *complement*, denoted by $\tilde{a}^c$, is defined by

$$\tilde{a}^c = [1 - a^+, 1 - a^-].$$

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [11]) is a structure of the form:

$$\xi := \{\langle y; \xi_T(y), \xi_I(y), \xi_F(y)\rangle \mid y \in X\},$$

where $\xi_T : X \to [0,1]$ is a truth membership function, $\xi_I : X \to [0,1]$ is an indeterminate membership function, and $\xi_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $\xi := (\xi_T, \xi_I, \xi_F)$ for the neutrosophic set

$$\xi := \{\langle y; \xi_T(y), \xi_I(y), \xi_F(y)\rangle \mid y \in X\}.$$

Let X be a non-empty set. By an *MBJ-neutrosophic set* in X, we mean a structure of the form:

$$\xi := \{\langle y; M_\xi(y), \tilde{B}_\xi(y), J_\xi(y)\rangle \mid y \in X\}, \tag{1}$$

where M_ξ and J_ξ are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_\xi$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ for the MBJ-neutrosophic set in (1).

In an MBJ-neutrosophic set $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ in X, if we take

$$\tilde{B}_\xi : X \to int([0,1]),\ y \mapsto [B_\xi^-(y), B_\xi^+(y)],$$

with $B_\xi^-(y) = B_\xi^+(y)$, then $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ is a neutrosophic set in X.

Definition 2.1. [9] *An MBJ-neutrosophic set $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ in X is said to be*

- *M-internal if $B_\xi^-(x) \leq M_\xi(x) \leq B_\xi^+(x)$ for all $x \in X$,*
- *J-internal if $B_\xi^-(x) \leq J_\xi(x) \leq B_\xi^+(x)$ for all $x \in X$,*
- *(M,J)-internal if it is both M-internal and J-internal,*
- *M-external if $M_\xi(x) \notin (B_\xi^-(x), B_\xi^+(x))$ for all $x \in X$,*
- *J-external if $J_\xi(x) \notin (B_\xi^-(x), B_\xi^+(x))$ for all $x \in X$,*
- *(M,J)-external if it is both M-external and J-external.*

Proposition 2.2. [9] *Let $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ be an MBJ-neutrosophic set in X.*

(1) *If ξ is not M-external, then there exists $x \in X$ such that $M_\xi(x) \in (B_\xi^-(x), B_\xi^+(x))$.*

(2) *If ξ is not J-external, then there exists $x \in X$ such that $J_\xi(x) \in (B_\xi^-(x), B_\xi^+(x))$.*

(3) *If ξ is not (M,J)-external, then there exists $x \in X$ such that $M_\xi(x) \in (B_\xi^-(x), B_\xi^+(x))$ or $J_\xi(x) \in (B_\xi^-(x), B_\xi^+(x))$.*

Given an MBJ-neutrosophic set $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ in X, we consider the sets:

$$L(\tilde{B}_\xi) = \{B_\xi^-(x) \mid x \in X\},\ U(\tilde{B}_\xi) = \{B_\xi^+(x) \mid x \in X\}.$$

Proposition 2.3. [9] *Let $\xi := (M_\xi,\ \tilde{B}_\xi,\ J_\xi)$ be an MBJ-neutrosophic set in X.*

(1) *If ξ is both M-internal and M-external, then*

$$(\forall x \in X)(M_\xi(x) \in L(\tilde{B}_\xi) \cup U(\tilde{B}_\xi)).$$

(2) *If ξ is both J-internal and J-external, then*

$$(\forall x \in X)(J_\xi(x) \in L(\tilde{B}_\xi) \cup U(\tilde{B}_\xi)).$$

Let $\xi := (M_\xi,\ \tilde{B}_\xi,\ J_\xi)$ and $\psi := (M_\psi,\ \tilde{B}_\psi,\ J_\psi)$ be MBJ-neutrosophic sets in X. The *order* between ξ and ψ is denoted by $\xi \tilde{\subseteq} \psi$ and is defined by

$$\xi \tilde{\subseteq} \psi \ \Leftrightarrow\ M_\xi \leq M_\psi,\ \tilde{B}_\xi \subseteq \tilde{B}_\psi,\ J_\xi \leq J_\psi. \tag{2}$$

The *reverse order* between ξ and ψ is denoted by $\xi \tilde{\subseteq}_R \psi$ and is defined by

$$\xi \tilde{\subseteq}_R \psi \ \Leftrightarrow\ M_\xi \geq M_\psi,\ \tilde{B}_\xi \subseteq \tilde{B}_\psi,\ J_\xi \geq J_\psi. \tag{3}$$

The *M-order* between ξ and ψ is denoted by $\xi \tilde{\subseteq}_M \psi$ and is defined by

$$\xi \tilde{\subseteq}_M \psi \ \Leftrightarrow\ M_\xi \leq M_\psi,\ \tilde{B}_\xi \subseteq \tilde{B}_\psi,\ J_\xi \geq J_\psi. \tag{4}$$

The *J-order* between ξ and ψ is denoted by $\xi \tilde{\subseteq}_J \psi$ and is defined by

$$\xi \tilde{\subseteq}_J \psi \ \Leftrightarrow\ M_\xi \geq M_\psi,\ \tilde{B}_\xi \subseteq \tilde{B}_\psi,\ J_\xi \leq J_\psi. \tag{5}$$

For family of MBJ-neutrosophic sets $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X\}$ where $i \in \Lambda$, the *union* of ξ_i is denoted by $\bigcup\limits_{i\in\Lambda} \xi_i$ and is defined by

$$\bigcup_{i\in\Lambda} \xi_i := \left\{ \left\langle x; \sup_{i\in\Lambda} M_i(x), \operatorname{rsup}_{i\in\Lambda} \tilde{B}_i(x), \sup_{i\in\Lambda} J_i(x) \right\rangle \mid x \in X \right\}.$$

The *reverse union* of ξ_i is denoted by $\bigcup\limits_{i\in\Lambda}{}_R \xi_i$ and is defined by

$$\bigcup_{i\in\Lambda}{}_R \xi_i := \left\{ \left\langle x; \inf_{i\in\Lambda} M_i(x), \operatorname{rsup}_{i\in\Lambda} \tilde{B}_i(x), \inf_{i\in\Lambda} J_i(x) \right\rangle \mid x \in X \right\}.$$

The *M-union* of ξ_i is denoted by $\bigcup\limits_{i\in\Lambda}{}_M \xi_i$ and is defined by

$$\bigcup_{i\in\Lambda}{}_M \xi_i := \left\{ \left\langle x; \sup_{i\in\Lambda} M_i(x), \operatorname{rsup}_{i\in\Lambda} \tilde{B}_i(x), \inf_{i\in\Lambda} J_i(x) \right\rangle \mid x \in X \right\}.$$

The *J-union* of ξ_i is denoted by $\bigcup\limits_{i\in\Lambda}{}_J \xi_i$ and is defined by

$$\bigcup_{i\in\Lambda}{}_J \xi_i := \left\{ \left\langle x; \inf_{i\in\Lambda} M_i(x), \operatorname{rsup}_{i\in\Lambda} \tilde{B}_i(x), \sup_{i\in\Lambda} J_i(x) \right\rangle \mid x \in X \right\}.$$

The *intersection* of ξ_i is denoted by $\bigcap\limits_{i\in\Lambda} \xi_i$ and is defined by

$$\bigcap_{i\in\Lambda}\xi_i := \left\{\left\langle x; \inf_{i\in\Lambda} M_i(x), \operatorname{rinf}_{i\in\Lambda} \tilde{B}_i(x), \inf_{i\in\Lambda} J_i(x)\right\rangle \mid x \in X\right\}.$$

The *reverse intersection* of ξ_i is denoted by $\bigcap_{i\in\Lambda R}\xi_i$ and is defined by

$$\bigcap_{i\in\Lambda R}\xi_i := \left\{\left\langle x; \sup_{i\in\Lambda} M_i(x), \operatorname{rinf}_{i\in\Lambda} \tilde{B}_i(x), \sup_{i\in\Lambda} J_i(x)\right\rangle \mid x \in X\right\}.$$

The *M-intersection* of ξ_i is denoted by $\bigcap_{i\in\Lambda M}\xi_i$ and is defined by

$$\bigcap_{i\in\Lambda M}\xi_i := \left\{\left\langle x; \inf_{i\in\Lambda} M_i(x), \operatorname{rinf}_{i\in\Lambda} \tilde{B}_i(x), \sup_{i\in\Lambda} J_i(x)\right\rangle \mid x \in X\right\}.$$

The *J-intersection* of ξ_i is denoted by $\bigcap_{i\in\Lambda J}\xi_i$ and is defined by

$$\bigcap_{i\in\Lambda J}\xi_i := \left\{\left\langle x; \sup_{i\in\Lambda} M_i(x), \operatorname{rinf}_{i\in\Lambda} \tilde{B}_i(x), \inf_{i\in\Lambda} J_i(x)\right\rangle \mid x \in X\right\}.$$

Given an MBJ-neutrosophic set $\xi := (M_\xi, \tilde{B}_\xi, J_\xi)$ in X, the *M-complement* of ξ is denoted by ξ^M and is defined by

$$\xi^M := \left\{\left\langle x; 1 - M_\xi(x), \tilde{B}^c_\xi(x), J_\xi(x)\right\rangle \mid x \in X\right\}. \tag{6}$$

The *J-complement* of ξ is denoted by ξ^J and is defined by

$$\xi^J := \left\{\left\langle x; M_\xi(x), \tilde{B}^c_\xi(x), 1 - J_\xi(x)\right\rangle \mid x \in X\right\}. \tag{7}$$

The *(M,J)-complement* of ξ is denoted by ξ^c and is defined by

$$\xi^c := \left\{\left\langle x; 1 - M_\xi(x), \tilde{B}^c_\xi(x), 1 - J_\xi(x)\right\rangle \mid x \in X\right\}. \tag{8}$$

3 Expanding on internal and external relations of MBJ-neutrosophic sets

We provide a condition for the intersection and the M-intersection (J-intersection) of ξ_i to be both an M-external (resp. J-external, (M,J)-external) and an M-internal (resp. J-internal, (M,J)-internal).

Example 3.1. *Let* $X = \{a, b, c, d\}$ *and* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be family of MBJ-neutrosophic set in* X *defined by Table 1, 2.*

It is routine to verify that ξ_1 *and* ξ_2 *are* M*- external, but union and* M*-union of them are not a* M*- external since for* $c \in X$

$$\sup_{i\in\Lambda} M_i(c) = 0.75 \in \operatorname{rsup} \tilde{B}_i(c) = [0.7, 0.8].$$

Also, the intersection and M*-intersection of them are not a* M*- external since for* $d \in X$

$$\inf_{i\in\Lambda} M_i(c) = 0.4 \in \operatorname{rinf} \tilde{B}_i(c) = [0.2, 0.8].$$

Table 1: MBJ-neutrosophic set ξ_1

X	$M_1(x)$	$\tilde{B}_1(x)$	$J_1(x)$
a	0.8	$[0.3, 0.7]$	0.2
b	0.1	$[0.1, 0.6]$	0.6
c	0.5	$[0.7, 0.8]$	0.4
d	0.9	$[0.2, 0.81]$	0.7

Table 2: MBJ-neutrosophic set ξ_2

X	$M_2(x)$	$\tilde{B}_2(x)$	$J_2(x)$
a	0.3	$[0.5, 0.6]$	0.2
b	0.4	$[0.7, 0.8]$	0.6
c	0.75	$[0.2, 0.4]$	0.4
d	0.4	$[0.4, 0.8]$	0.7

Note that the intersection and the M-intersection (J-intersection) of M-external (J-external) MBJ-neutrosophic sets $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ may not be an M-external (J-external) MBJ-neutrosophic set (see Example 3.1). We provide a condition for the intersection and the M-intersection (J-intersection) of ξ_i to be an M-external (resp. J-external, (M,J)-external).

Theorem 3.2. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ be an MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &= (M_1 \wedge M_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{9}$$

Then the intersection and the M-intersection of ξ_i are both an M-external MBJ-neutrosophic set and an M-internal MBJ-neutrosophic set in X.

Proof. For each $x \in X$, take

$$\alpha_x := \min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\},$$

and

$$\beta_x := \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.$$

Then α_x is one of $B_1^+(x), B_2^-(x), B_1^-(x)$ and $B_2^+(x)$. We consider $\alpha_x = B_1^-(x)$ or $\alpha_x = B_1^+(x)$ only. For the remaining cases, it is similar to this case.

- If $\alpha_x = B_1^-(x)$, then $B_2^-(x) \le B_2^+(x) \le B_1^-(x) \le B_1^+(x)$ and so $\beta_x = B_2^+(x)$. This implies that $B_1^-(x) = \alpha_x = (M_1 \wedge M_2)(x) = \beta_x = B_2^+(x)$. Thus

$$B_2^-(x) \le B_2^+(x) = (M_1 \wedge M_2)(x) = B_1^-(x) \le B_1^+(x).$$

This implies that $(M_1 \wedge M_2)(x) = B_2^+(x) = (B_1 \cap B_2)^+(x)$. Hence

$$(M_1 \wedge M_2)(x) \notin (B_1 \cap B_2)^-(x) \le (B_1 \cap B_2)^+(x),$$

and $(B_1 \cap B_2)^-(x) \le (M_1 \wedge M_2)(x) \le (B_1 \cap B_2)^+(x)$.

- If $\alpha_x = B_1^+(x)$, then $B_2^-(x) \leq B_1^+(x) \leq B_2^+(x)$ and so $(M_1 \wedge M_2)(x) = B_1^+(x) = (B_1 \cap B_2)^+(x)$. Hence $(M_1 \wedge M_2)(x) \notin (B_1 \cap B_2)^-(x) \leq (B_1 \cap B_2)^+(x)$ and $(B_1 \cap B_2)^-(x) \leq (M_1 \wedge M_2)(x) \leq (B_1 \cap B_2)^+(x)$. Consequently, we know that the intersection and the M-intersection of ξ_i are both an M-external MBJ-neutrosophic set and an M-internal MBJ-neutrosophic set in X.

□

Theorem 3.3. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1,2\}$ be an MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &= (J_1 \wedge J_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{10}$$

Then the intersection and the J-intersection of ξ_i are both a J-external MBJ-neutrosophic set and a J-internal MBJ-neutrosophic set in X.

Proof. The proof follows a structure similar to that proof of Theorem 3.2. □

Corollary 3.4. *If MBJ-neutrosophic sets ξ_i satisfy the conditions outlined in equations*(9) *and* (10)*, then the intersection of ξ_i is both an (M,J)-external MBJ-neutrosophic set and an (M,J)-internal MBJ-neutrosophic set in X.*

Theorem 3.5. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1,2\}$ be an M-external MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &\geq (M_1 \vee M_2)(x) \\ &> \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{11}$$

Then the union and the M-union of ξ_i are also an M-external MBJ-neutrosophic set in X.

Proof. For each $x \in X$, take

$$\alpha_x := \min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\},$$

and

$$\beta_x := \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.$$

Then α_x is one of $B_1^+(x), B_2^-(x), B_1^-(x)$ and $B_2^+(x)$. We consider $\alpha_x = B_1^-(x)$ or $\alpha_x = B_1^+(x)$ only. For the remaining cases, it is similar to this case.

- If $\alpha_x = B_1^-(x)$, then $B_2^-(x) \leq B_2^+(x) \leq B_1^-(x) \leq B_1^+(x)$ and so $\beta_x = B_2^+(x)$. Thus

  $$(B_1 \cup B_2)^-(x) = B_1^-(x) = \alpha_x > (M_1 \vee M_2)(x),$$

  and hence $(M_1 \vee M_2)(x) \notin ((B_1 \cup B_2)^-(x), (B_1 \cup B_2)^+(x))$.

- If $\alpha_x = B_1^+(x)$, then $B_2^-(x) \leq B_1^+(x) \leq B_2^+(x)$ and so $\beta_x = \max\{B_1^-(x), B_2^-(x)\}$.
  - Assume that $\beta_x = B_1^-(x)$. Then

    $$B_2^-(x) \leq B_1^-(x) \leq (M_1 \vee M_2)(x) < B_1^+(x) \leq B_2^+(x). \tag{12}$$

and so

$$B_2^-(x) \le B_1^-(x) < (M_1 \vee M_2)(x) < B_1^+(x) \le B_2^+(x),$$

or

$$B_2^-(x) \le B_1^-(x) < (M_1 \vee M_2)(x) \le B_1^+(x) \le B_2^+(x).$$

For the first case, it contradicts to the fact there are M-external MBJ-neutrosophic sets in X. The second case implies that $(M_1 \vee M_2)(x) \notin ((B_1 \cup B_2)^-(x), (B_1 \cup B_2)^+(x))$ since $(M_1 \vee M_2)(x) = B_1^-(x) = (B_1 \cup B_2)^-(x)$.

– Assume that $\beta_x = B_2^-(x)$. Then

$$B_1^-(x) \le B_2^-(x) < (M_1 \vee M_2)(x) \le B_1^+(x) \le B_2^+(x), \tag{13}$$

which implies that

$$B_1^-(x) \le B_2^-(x) < (M_1 \vee M_2)(x) < B_1^+(x) \le B_2^+(x),$$

or

$$B_1^-(x) \le B_2^-(x) = (M_1 \vee M_2)(x) < B_1^+(x) \le B_2^+(x).$$

For the case $B_1^-(x) \le B_2^-(x) < (M_1 \vee M_2)(x) < B_1^+(x) \le B_2^+(x)$, it is a contradiction to the fact that ξ_i are M-external MBJ-neutrosophic sets in X. For the case $B_1^-(x) \le B_2^-(x) = (M_1 \vee M_2)(x) \le B_1^+(x) \le B_2^+(x)$, we have $(M_1 \vee M_2)(x) \notin ((B_1 \cup B_2)^-(x), (B_1 \cup B_2)^+(x))$ since $(M_1 \vee M_2)(x) = B_2^-(x) = (B_1 \cup B_2)^-(x)$.

Hence the union and the M-union of ξ_i are M-external MBJ-neutrosophic sets in X.

□

Theorem 3.6. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ be a J-external MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &\ge (J_1 \vee J_2)(x) \\ &> \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{14}$$

Then the union and the J-union of ξ_i are also a J-external MBJ-neutrosophic set in X.

Proof. The proof follows a structure similar to that proof of Theorem 3.5. □

Corollary 3.7. *If ξ_i is an (M,J)-external MBJ-neutrosophic set in X such that for all $x \in X$, satisfy the conditions outlined in equations* (11) *and* (14)*, then the union of ξ_i is also an (M,J)-external MBJ-neutrosophic set in X.*

We provide a condition for the reverse union of M-external (resp. J-external, (M,J)-external) MBJ-neutrosophic sets ξ_i to be also M-external (resp. J-external, (M,J)-external).

Theorem 3.8. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ be an M-external MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &> (M_1 \wedge M_2)(x) \\ &\ge \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{15}$$

Then the reverse union of ξ_i is also an M-external MBJ-neutrosophic set in X.

Proof. For each $x \in X$, take

$$\alpha_x := \min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\},$$

and

$$\beta_x := \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.$$

Then α_x is one of $B_1^+(x), B_2^-(x), B_1^-(x)$ and $B_2^+(x)$. We consider $\alpha_x = B_2^-(x)$ or $\alpha_x = B_2^+(x)$ only. For the remaining cases, it is similar to this case.

- If $\alpha_x = B_2^-(x)$, then $B_1^-(x) \le B_1^+(x) \le B_2^-(x) \le B_2^+(x)$ and so $\beta_x = B_1^+(x)$. Thus by inequality (15),
$$(B_1 \cup B_2)^-(x) = B_2^-(x) = \alpha_x > (M_1 \wedge M_2)(x),$$
and hence $(M_1 \wedge M_2)(x) \notin ((B_1 \cup B_2)^-(x), (B_1 \cup B_2)^+(x))$.
- If $\alpha_x = B_2^+(x)$, then $B_1^-(x) \le B_2^+(x) \le B_1^+(x)$ and so $\beta_x = \max\{B_1^-(x), B_2^-(x)\}$.
  - Assume that $\beta_x = B_1^-(x)$. Then
  $$B_2^-(x) \le B_1^-(x) \le (M_1 \wedge M_2)(x) < B_2^+(x) \le B_1^+(x), \tag{16}$$
  which implies that
  $$B_2^-(x) \le B_1^-(x) < (M_1 \wedge M_2)(x) < B_2^+(x) \le B_1^+(x),$$
  or
  $$B_2^-(x) \le B_1^-(x) = (M_1 \wedge M_2)(x) \le B_2^+(x) \le B_1^+(x).$$
  For the first case, it contradicts the fact that ξ_i are M-external MBJ-neutrosophic sets in X. For the case
  $$B_2^-(x) \le B_1^-(x) = (M_1 \wedge M_2)(x) \le B_2^+(x) \le B_1^+(x),$$
  we get $(M_1 \wedge M_2)(x) \notin ((B_1 \cup B_2)^-(x), (B_1 \cup B_2)^+(x))$ since $(M_1 \wedge M_2)(x) = B_1^-(x) = (B_1 \cup B_2)^-(x)$.
  - Assume that $\beta_x = B_2^-(x)$. Then
  $$B_1^-(x) \le B_2^-(x) \le (M_1 \wedge M_2)(x) \le B_2^+(x) < B_1^+(x). \tag{17}$$
  Hence
  $$B_1^-(x) \le B_2^-(x) < (M_1 \wedge M_2)(x) < B_2^+(x) \le B_1^+(x),$$
  or
  $$B_1^-(x) \le B_2^-(x) = (M_1 \wedge M_2)(x) < B_2^+(x) \le B_1^+(x).$$
  For the case $B_1^-(x) \le B_2^-(x) < (M_1 \wedge M_2)(x) < B_2^+(x) \le B_1^+(x)$, it is a contradiction to the fact that ξ_i are M-external MBJ-neutrosophic sets in X. For the case $B_1^-(x) \le B_2^-(x) = (M_1 \wedge M_2)(x) \le B_2^+(x) \le B_1^+(x)$, we obtain $(M_1 \wedge M_2)(x) \notin ((B_1 \cup B_2)^-(x), (B_1 \cup B_2)^+(x))$ since $(M_1 \wedge M_2)(x) = B_2^-(x) = (B_1 \cup B_2)^-(x)$.

Hence the reverse union of ξ_i are M-external MBJ-neutrosophic sets in X.

□

Theorem 3.9. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1,2\}$ be a J-external MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} \geq (J_1 \wedge J_2)(x) > \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}. \tag{18}$$

Then the reverse union of ξ_i are also a J-external MBJ-neutrosophic set in X.

Proof. The proof follows a structure similar to that proof of Theorem 3.8. □

The following example shows that for two M-external (J-external, (M,J)-external) MBJ-neutrosophic sets ξ_i which satisfy the condition for all $x \in X$

$$\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} = (M_1 \wedge M_2)(x) > \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\},$$

$$(\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} = (J_1 \wedge J_2)(x) > \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}),$$

the reverse union of ξ_i may not be an M-external (J-external, (M,J)-external) MBJ-neutrosophic set in X.

Example 3.10. *Let $X = \{a, b, c\}$ and $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1,2\}$ be M-external (J-external, (M,J)-external) MBJ-neutrosophic set in X defined by Table 3, 4.*

Table 3: MBJ-neutrosophic set ξ_1

X	$M_1(x)$	$\tilde{B}_1(x)$	$J_1(x)$
a	0.9	$[0.1, 0.8]$	0.9
b	0.6	$[0.3, 0.6]$	0.6
c	0.5	$[0.4, 0.5]$	0.5

Table 4: MBJ-neutrosophic set ξ_2

X	$M_2(x)$	$\tilde{B}_2(x)$	$J_2(x)$
a	0.3	$[0.2, 0.7]$	0.7
b	0.7	$[0.1, 0.7]$	0.8
c	0.4	$[0.3, 0.8]$	0.9

Then we know that ξ_i satisfies the following condition:

$$\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} = (M_1 \wedge M_2)(x) > \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.$$

$$(\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} = (J_1 \wedge J_2)(x) > \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.)$$

But the reverse union is not an M-external (J-external, (M,J)-external) MBJ-neutrosophic set in X *because* $(M_1 \wedge M_2)(c) = 0.5 \in (0.4, 0.6) = ((B_1 \cup B_2)^-(a), (B_1 \cup B_2)^+(a))$ $((J_1 \wedge J_2)(c) = 0.5 \in (0.4, 0.6) = ((B_1 \cup B_2)^-(c), (B_1 \cup B_2)^+(c)))$.

Theorem 3.11. *Let* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be an M-external MBJ-neutrosophic set in* X *such that for all* $x \in X$,

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &\geq (M_1 \vee M_2)(x) \\ &> \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{19}$$

Then the reverse intersection of ξ_i *is also an M-external MBJ-neutrosophic set in* X.

Proof. In a similar way to Theorem 3.8, we can obtain the result □

Theorem 3.12. *Let* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be a J-external MBJ-neutrosophic set in* X *such that for all* $x \in X$,

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &\geq (J_1 \wedge J_2)(x) \\ &> \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{20}$$

Then the reverse intersection of ξ_i *are also a J-external MBJ-neutrosophic set in* X.

Proof. The proof follows a structure similar to that proof of Theorem 3.8. □

The following example shows that for two M-external (J-external, (M,J)-external) MBJ-neutrosophic sets ξ_i which satisfy the condition for all $x \in X$

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &> (M_1 \vee M_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\},\end{aligned}$$

$$\begin{aligned}(\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &> (J_1 \vee J_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}),\end{aligned}$$

the reverse intersection of ξ_i may not be an M-external (J-external, (M,J)-external) MBJ-neutrosophic set in X.

Example 3.13. *Let* $X = \{a, b, c\}$ *and* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be M-external (J-external, (M,J)-external) MBJ-neutrosophic set in* X *defined by Table 5, 6.*

Table 5: MBJ-neutrosophic set ξ_1

X	$M_1(x)$	$\tilde{B}_1(x)$	$J_1(x)$
a	0.1	$[0.2, 0.4]$	0.1
b	0.5	$[0.5, 0.8]$	0.5
c	0.4	$[0.6, 0.8]$	0.4

Then we know that ξ_i *satisfies the following condition:*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &> (M_1 \vee M_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned}$$

Table 6: MBJ-neutrosophic set ξ_2

X	$M_2(x)$	$\tilde{B}_2(x)$	$J_2(x)$
a	0.3	$[0.3, 0.6]$	0.3
b	0.2	$[0.4, 0.7]$	0.2
c	0.7	$[0.7, 0.9]$	0.7

$$\begin{aligned}(\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &> (J_1 \vee J_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.)\end{aligned}$$

But the reverse intersection is not an M-external (J-external, (M,J)-external) MBJ-neutrosophic set in X *because* $(M_1 \vee M_2)(b) = 0.5 \in (0.4, 0.7) = ((B_1 \cap B_2)^-(b), (B_1 \cap B_2)^+(b))$ *(*$(J_1 \vee J_2)(b) = 0.5 \in (0.4, 0.7) = ((B_1 \cap B_2)^-(b), (B_1 \cap B_2)^+(b))$*).*

We provide a condition for the reverse intersection of ξ_i to be both an M-external (resp. J-external, (M,J)-external) and an M-internal (resp. J-internal, (M,J)-internal).

Theorem 3.14. *Let* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be an MBJ-neutrosophic set in* X *such that for all* $x \in X$*,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &= (M_1 \vee M_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{21}$$

Then the reverse intersection of ξ_i *are both an M-external MBJ-neutrosophic set and an M-internal MBJ-neutrosophic set in* X*.*

Proof. In a similar way to Theorem 3.2, it is straightforward. □

Theorem 3.15. *Let* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be an MBJ-neutrosophic set in* X *such that for all* $x \in X$*,*

$$\begin{aligned}\min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &= (J_1 \vee J_2)(x) \\ &= \max\{\min\{B_1^+(x), B_2^-(x)\}, \min\{B_1^-(x), B_2^+(x)\}\}.\end{aligned} \tag{22}$$

Then the reverse intersection of ξ_i *are both a J-external MBJ-neutrosophic set and a J-internal MBJ-neutrosophic set in* X*.*

We provide a condition for the reverse union of two M-internal (resp. J-internal, (M,J)-internal) MBJ-neutrosophic sets ξ_i to be an M-external (resp. J-external, (M,J)-external).

Theorem 3.16. *Let* $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ *be an MBJ-neutrosophic set in* X*.*

(1) *If* ξ_i *is an M-internal and for any* $x \in X, (M_1 \wedge M_2)(x) \leq \max\{B_1^-(x), B_2^-(x)\}$*, then the reverse union of* ξ_i *is an M-external in* X*.*

(2) *If* ξ_i *is a J-internal and for any* $x \in X, (J_1 \wedge J_2)(x) \leq \max\{B_1^-(x), B_2^-(x)\}$*, then the reverse union of* ξ_i *is a J-external in* X*.*

(3) *If ξ_i is an (M,J)-internal and for any $x \in X, (M_1 \wedge M_2)(x) \leq \max\{B_1^-(x), B_2^-(x)\}$ and $(J_1 \wedge J_2)(x) \leq \max\{B_1^-(x), B_2^-(x)\}$, then the reverse union of ξ_i is an (M,J)-external in X.*

Proof. The proof is straightforward. □

We provide a condition for the reverse intersection of two M-internal (resp. J-internal, (M,J)-internal) MBJ-neutrosophic sets ξ_i to be an M-external (resp. J-external, (M,J)-external).

Theorem 3.17. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ be an MBJ-neutrosophic set in X.*

(1) *If ξ_i is an M-internal and for any $x \in X, (M_1 \vee M_2)(x) \geq \max\{B_1^+(x), B_2^+(x)\}$, then the reverse intersection of ξ_i is an M-external in X.*

(2) *If ξ_i is a J-internal and for any $x \in X, (J_1 \vee J_2)(x) \geq \max\{B_1^+(x), B_2^+(x)\}$, then the reverse intersection of ξ_i is a J-external in X.*

(3) *If ξ_i is an (M,J)-internal and for any $x \in X, (M_1 \vee M_2)(x) \geq \max\{B_1^+(x), B_2^+(x)\}$ and $(J_1 \vee J_2)(x) \geq \max\{B_1^+(x), B_2^+(x)\}$, then the reverse intersection of ξ_i is an (M,J)-external in X.*

Proof. The proof is straightforward. □

We provide a condition for the reverse union of two M-external (resp. J-external, (M,J)-external) MBJ-neutrosophic sets ξ_i to be an M-internal (resp. J-internal, (M,J)-internal).

Theorem 3.18. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ be an M-external MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned} \min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &\leq (M_1 \wedge M_2)(x) \\ &\leq \max\{B_1^+(x), B_2^+(x)\}. \end{aligned} \tag{23}$$

Then the reverse union of ξ_i is an M-internal MBJ-neutrosophic set in X.

Proof. The proof is straightforward. □

Theorem 3.19. *Let $\xi_i := \{\langle x; M_i(x), \tilde{B}_i(x), J_i(x)\rangle \mid x \in X, i = 1, 2\}$ be a J-external MBJ-neutrosophic set in X such that for all $x \in X$,*

$$\begin{aligned} \min\{\max\{B_1^+(x), B_2^-(x)\}, \max\{B_1^-(x), B_2^+(x)\}\} &\leq (J_1 \wedge J_2)(x) \\ &\leq \max\{B_1^+(x), B_2^+(x)\}. \end{aligned} \tag{24}$$

Then the reverse union of ξ_i is a J-internal MBJ-neutrosophic set in X.

Proof. The proof is straightforward. □

4 Conclusions

The study successfully establishes a foundational framework for MBJ-neutrosophic sets by proving essential theorems and conditions for internal and external categorization. The results affirm that specific combinations of truth, indeterminate, and false membership functions consistently determine whether a set intersection or union remains internal or external under defined conditions. Notably, the research demonstrates that MBJ-neutrosophic sets can support complex decision-making and uncertainty modeling, providing substantial theoretical value to neutrosophic set theory. Future work may expand on these findings by exploring additional applications and refining the conditions that govern MBJ-neutrosophic set operations, thereby deepening the potential for practical implementation in various analytically and logical systems.

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Declarations

Compliance with Ethical Standards:
- The authors declare that they have no conflicts of interest.
- This paper does not include any studies involving human participants or animals conducted by the authors.
- Informed consent was obtained from all individual participants included in the study.

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