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Chapter 8
Neutrosophic TwoFold SuperhyperAlgebra and Anti SuperhyperAlgebra

Takaaki Fujita [1] * Florentin Smarandache[2],
[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp
[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

Neutrosophic Sets are conceptual frameworks designed to address uncertainty. A *Neutrosophic TwoFold Algebra* is a hybrid algebraic structure defined over a neutrosophic set, combining classical algebraic operations with neutrosophic components. Concepts such as Hyperalgebra and Superhyperalgebra extend classical Algebra using Power Sets and n-th powersets. Additionally, structures such as NeutroAlgebra and AntiAlgebra have been defined in recent years. This paper explores several related concepts, including TwoFold SuperhyperAlgebra and Anti SuperhyperAlgebra.

Keywords: Set Theory, Neutrosophic Set, Neutrosophic TwoFold Algebra, Hyperalgebra, Superhyperalgebra

1 Preliminaries and Definitions

Some foundational concepts from set theory are applied in parts of this work.

1.1 n-th Powerset

The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S. The n-th Powerset is a recursive extension of the Powerset structure, where the powerset operation is applied repeatedly. The related definitions are provided below.

Definition 1.1 (Set)**.** [19] A *set* is a collection of distinct, well-defined objects, referred to as *elements*. For any object x, it can be determined whether x is an element of a given set. If x belongs to a set A, this is denoted as $x \in A$. Sets are often represented using curly braces. For example, the set $A = \{1, 2, 3\}$ contains the elements 1, 2, and 3.

Definition 1.2 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.3 (Powerset)**.** [9, 27] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.4 (n-th Powerset)**.** (cf. [8–10, 32, 39])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

1.2 Superhyperalgebra

Algebra studies mathematical symbols, operations, and the rules for manipulating and solving equations [4,20,21]. A *Hyperalgebra* is an algebraic structure that extends classical algebraic frameworks by incorporating hyperoperations, where the result of operations is a set rather than a single element [6, 7, 15, 25, 26, 45, 46]. A *Superhyperalgebra* further generalizes Hyperalgebra by allowing operations to map to higher-order powersets (n-th powersets) of the base set H [17, 18, 22, 32, 38, 44]. The detailed definition is provided below .

Definition 1.5 (Hyperalgebra)**.** [32] A *Hyperalgebra* is an algebraic structure that extends classical algebraic structures by incorporating hyperoperations, which are generalized operations where the result of applying the operation is a set rather than a single element. Formally, a Hyperalgebra is defined as:

$$\mathcal{H} = (H, \star, \mathcal{A}),$$

where:

1. H is a non-empty set called the *base set*.
2. $\star : H^m \to \mathcal{P}^*(H)$ is an m-ary *Hyperoperation*, such that:

   $$\star(x_1, x_2, \ldots, x_m) \subseteq \mathcal{P}^*(H),$$

   where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ is the powerset of H excluding the empty set.
3. $\mathcal{A}$ is a set of *Hyperaxioms*, which are generalizations of classical axioms applied to hyperoperations.

Definition 1.6 (Superhyperalgebra)**.** [32] A *Superhyperalgebra* generalizes Hyperalgebra by allowing operations to map to higher-order powersets (n-th powersets) of the base set H. It is formally defined as:

$$\mathcal{SH}^{(m,n)} = (H, \star^{(m,n)}, \mathcal{A}),$$

where:

1. H is a non-empty set called the *base set*.
2. $\mathcal{P}_n^*(H)$ is the n-th powerset of H excluding the empty set, defined recursively as:

   $$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \quad \mathcal{P}_{k+1}^*(H) = \mathcal{P}^*(\mathcal{P}_k^*(H)) \quad \text{for } k \geq 1.$$
3. $\star^{(m,n)} : H^m \to \mathcal{P}_n^*(H)$ is an (m, n)-*SuperHyperoperation*, where m is the arity of the operation and n is the order of the powerset. For each $(x_1, x_2, \ldots, x_m) \in H^m$:

   $$\star^{(m,n)}(x_1, x_2, \ldots, x_m) \subseteq \mathcal{P}_n^*(H).$$
4. $\mathcal{A}$ is a set of *SuperHyperaxioms*, which are extensions of Hyperaxioms adapted to (m, n)-SuperHyperoperations.

1.3 Neutrosophic Set

Neutrosophic Sets are conceptual frameworks designed to handle uncertainty. Their definitions are provided below.

Definition 1.7. [33–36,42,43] Let X be a given set. A (single-valued) Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

1.4 Neutrosophic Twofold algebra

A *Neutrosophic TwoFold Algebra* is a hybrid algebraic structure defined over a neutrosophic set [40], incorporating classical algebraic operations alongside neutrosophic components. It consists of two interrelated algebras:

1. *Classical Algebra*, defined on the elements of a base set.
2. *Neutrosophic Algebra*, defined on the neutrosophic components (T, I, F) of the elements [2,5,11,16,24].

Definition 1.8 (Neutrosophic TwoFold Algebra). [40] Let U be a universe of discourse, and let A be a non-empty neutrosophic set:

$$A(T, I, F) = \{x(T_A(x), I_A(x), F_A(x)) \mid (T_A(x), I_A(x), F_A(x)) \in [0, 1]^3, \, x \in U\},$$

where:

- $T_A(x)$: Degree of truth-membership of x in A,
- $I_A(x)$: Degree of indeterminacy-membership of x in A,
- $F_A(x)$: Degree of falsehood-membership of x in A.

Let $\star : A \times A \to A$ be a binary operation defined as:

$$x_1(T_1, I_1, F_1) \star x_2(T_2, I_2, F_2) = \big(x_1 \# x_2\big)\big(T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2\big),$$

where:

- $\# : U \times U \to U$ is a classical operation on the elements,
- $\odot : [0, 1]^3 \times [0, 1]^3 \to [0, 1]^3$ is an operation on the neutrosophic components.

The *Neutrosophic TwoFold Law* extends the algebraic interaction of two neutrosophic elements by applying a pair of sub-laws.

Definition 1.9 (Neutrosophic TwoFold Law). Let $\Delta : A(T, I, F) \times A(T, I, F) \to A(T, I, F)$ represent the Neutrosophic TwoFold Law, defined as:

$$x_1(T_1, I_1, F_1) \Delta x_2(T_2, I_2, F_2) = \Big(x_1 \# x_2, (T_1 \odot T_2), (I_1 \odot I_2), (F_1 \odot F_2)\Big),$$

where:

- Δ is composed of two sub-laws:

$$\# : U \times U \to U \quad \text{(classical component)},$$

$$\odot : [0, 1]^3 \times [0, 1]^3 \to [0, 1]^3 \quad \text{(neutrosophic component)}.$$

- The sub-laws # and $\odot$ can be:
  - *Totally Dependent*: $\odot$ is entirely governed by #,
  - *Partially Dependent*: $\odot$ is influenced but not fully determined by #,
  - *Independent*: $\odot$ operates independently of #.

Example 1.10. Let $U = \{a, b, c\}$ and define a neutrosophic set $A(T, I, F)$:

$$A(T, I, F) = \{a(0.8, 0.1, 0.1), b(0.6, 0.3, 0.1), c(0.4, 0.4, 0.2)\}.$$

1. *Classical Operation:* Define $\# : \{a, b, c\} \times \{a, b, c\} \to \{a, b, c\}$ as:

$$a\#b = c, \quad b\#c = a, \quad c\#a = b.$$

2. *Neutrosophic Operation:* Define $\odot : [0,1]^3 \times [0,1]^3 \to [0,1]^3$ as:

$$(T_1, I_1, F_1) \odot (T_2, I_2, F_2) = (T_1 \cdot T_2, I_1 + I_2 - I_1 \cdot I_2, F_1 + F_2 - F_1 \cdot F_2).$$

3. For $x_1 = a(0.8, 0.1, 0.1)$ and $x_2 = b(0.6, 0.3, 0.1)$:

$$x_1 \Delta x_2 = \big(c, (0.8 \cdot 0.6, 0.1 + 0.3 - 0.03, 0.1 + 0.1 - 0.01)\big),$$

resulting in:

$$x_1 \Delta x_2 = c(0.48, 0.37, 0.19).$$

In addition, related concepts to Neutrosophic Twofold Algebra include Fuzzy Twofold Algebra and Fuzzy-Extensions Twofold Algebra(cf. [1, 3, 12–14, 23, 28–30, 47]). This refers to the definition of Twofold Algebra using Fuzzy Sets [48–51,51–53], which can also be generalized within the framework of Neutrosophic Twofold Algebra.

1.5 AntiAlgebra and NeutroAlgebra

A *NeutroAlgebra* is a generalization of classical algebra that introduces the concepts of *NeutroOperations* and *NeutroAxioms* [31, 31, 37, 41]. It allows operations and axioms to be partially well-defined, partially indeterminate, or partially outer-defined, corresponding to the degrees of truth (T), indeterminacy (I), and falsehood (F).

Definition 1.11 (NeutroAlgebra)**.** [31, 31, 37, 41] Let NA be a non-empty set equipped with:

- At least one *NeutroOperation* $\omega : NA^n \to U$, where $n \geq 1$, such that:
  - For some n-tuples $(x_1, \ldots, x_n) \in NA^n$, $\omega(x_1, \ldots, x_n) \in NA$ (well-defined, degree of truth T).
  - For other n-tuples, $\omega(x_1, \ldots, x_n) \notin U - NA$ (outer-defined, degree of falsehood F).
  - For other n-tuples, $\omega(x_1, \ldots, x_n)$ is indeterminate (degree of indeterminacy I).
- or at least one *NeutroAxiom*, which is true for some elements of NA, false for others, and indeterminate for the rest.

The structure $(NA, \{\omega\}, \{\text{NeutroAxioms}\})$ is called a *NeutroAlgebra*.

Example 1.12. Let $NA = \{a, b, c\}$ and define a binary operation:

$$\omega(x, y) = \begin{cases} a & \text{if } x = a, y = b, \quad \text{(true)} \\ \text{undefined} & \text{if } x = b, y = c, \quad \text{(indeterminate)} \\ d \notin NA & \text{if } x = c, y = a. \quad \text{(outer-defined)} \end{cases}$$

The operation ω is a NeutroOperation because it exhibits all three behaviors (truth, indeterminacy, and falsehood), and NA forms a NeutroAlgebra under ω.

An *AntiAlgebra* is an algebraic structure that extends classical algebra by incorporating at least one operation or axiom that is entirely *outer-defined* (false for all elements of the set) or by including elements that obey an *AntiAxiom* [31, 31, 37, 41]. The formal definition is provided below.

Definition 1.13 (AntiAlgebra)**.** [31, 31, 37, 41] Let AA be a non-empty set equipped with:

- At least one *AntiOperation* $\omega : AA^n \to U - AA$, where U is the universal set and $n \geq 1$,
- or at least one *AntiAxiom*, which is a condition that is *false* for all elements of AA.

The structure $(AA, \{\omega\}, \{\text{AntiAxioms}\})$ is called an *AntiAlgebra*.

Example 1.14. Consider the set $AA = \{1, 2, 3\}$ and the universal set $U = \{1, 2, 3, 4, 5\}$. Define the binary operation:

$$\omega(x, y) = x + y \pmod 4, \quad \text{for } x, y \in AA.$$

If $\omega(x, y) \notin AA$ for all $x, y \in AA$, then ω is an *AntiOperation*, and AA forms an AntiAlgebra under ω.

2 Results of This Paper

This section highlights the main contributions of this paper.

2.1 Neutrosophic Twofold Superhyperalgebra

The Neutrosophic Twofold Algebra is extended using the concept of Superhyperalgebra. Relevant theorems and definitions are presented below.

A *Neutrosophic Twofold Hyperalgebra* generalizes a Neutrosophic Twofold Algebra by replacing the classical binary operation # with a *hyperoperation*, which can yield subsets (rather than single elements). It also preserves the neutrosophic operation on the triple (T, I, F).

Definition 2.1 (Neutrosophic Twofold Hyperalgebra)**.** Let

$$A(T, I, F) = \{x(T_A(x), I_A(x), F_A(x)) \mid x \in U,\ (T_A(x), I_A(x), F_A(x)) \in [0, 1]^3\}$$

be a non-empty Neutrosophic Set. We assume that:

1. $\boxdot : U \times U \to \mathcal{P}^*(U)$ is a *binary hyperoperation* on the underlying classical set U. ($\mathcal{P}^*(U)$ is the powerset of U excluding the empty set, or in some definitions the entire powerset $\mathcal{P}(U)$.)
2. $\odot : [0, 1]^3 \times [0, 1]^3 \to [0, 1]^3$ is the neutrosophic component.

A *Neutrosophic Twofold Hyperalgebra* is the structure

$$\big(A(T, I, F), \star\big),$$

where for any

$$x_1(T_1, I_1, F_1),\ x_2(T_2, I_2, F_2) \in A(T, I, F),$$

we define:

$$x_1(T_1, I_1, F_1)\ \star\ x_2(T_2, I_2, F_2) \ =\ \Big(x_1\ \boxdot\ x_2,\ (T_1, I_1, F_1) \odot (T_2, I_2, F_2)\Big),$$

with the understanding that $x_1 \boxdot x_2 \subseteq U$ is a subset of U.

Theorem 2.2. *(Neutrosophic Twofold Hyperalgebra generalizes Neutrosophic Twofold Algebra.) Any Neutrosophic Twofold Hyperalgebra reduces to a Neutrosophic Twofold Algebra precisely when the hyperoperation $\boxdot$ always yields singleton subsets. Formally,*

$$\forall x_1, x_2 \in U, \quad \boxdot(x_1, x_2) \ =\ \{x_1 \# x_2\},$$

where # is a standard (single-valued) binary operation on U.

Proof. It can be proven step by step as follows:

- Let $\big(A(T, I, F), \star\big)$ be a *Neutrosophic Twofold Hyperalgebra*. By definition, $A(T, I, F)$ is a non-empty neutrosophic set:

$$A(T, I, F) \ =\ \Big\{x\big(T_A(x), I_A(x), F_A(x)\big) \ \Big|\ x \in U,\ (T_A(x), I_A(x), F_A(x)) \in [0, 1]^3\Big\}.$$

- On the *classical* side, we have a hyperoperation

$$\boxdot : U \times U \ \longrightarrow\ \mathcal{P}^*(U),$$

meaning that for any $x_1, x_2 \in U$, the image $\boxdot(x_1, x_2)$ is a *subset* of U, excluding possibly the empty set.

- On the *neutrosophic* side, we have a binary operation

$$\odot : [0, 1]^3 \times [0, 1]^3 \ \to\ [0, 1]^3.$$

- The combined operation $\star$ on $A(T, I, F)$ is given by:

$$\star\Big(x_1(T_1, I_1, F_1),\ x_2(T_2, I_2, F_2)\Big) = \Big(x_1 \boxdot x_2,\ (T_1, I_1, F_1)\ \odot\ (T_2, I_2, F_2)\Big).$$

Assume that

$$\forall x_1, x_2 \in U, \quad \boxdot(x_1, x_2) = \{x_1 \# x_2\},$$

for some single-valued operation $\# : U \times U \to U$. We wish to show that the Neutrosophic Twofold Hyperalgebra reduces to a Neutrosophic Twofold Algebra.

1. Since $\boxdot$ always yields exactly one element $x_1 \# x_2$, we can treat $\boxdot$ as a *classical* binary operation:

$$\boxdot(x_1, x_2) \ = \ \big\{x_1 \# x_2\big\}.$$

2. In that scenario, for every pair of elements $\big(x_1(T_1, I_1, F_1), x_2(T_2, I_2, F_2)\big) \in A(T, I, F)$, the classical part is no longer multi-valued, but strictly single-valued.

3. Hence, the structure $\big(A(T, I, F), \star\big)$ behaves exactly like a *Neutrosophic Twofold Algebra*: on the classical side, we have the single-valued operation #; on the neutrosophic side, we have $\odot$.

4. Concretely,

$$x_1(T_1, I_1, F_1)\ \star\ x_2(T_2, I_2, F_2) = \Big(\{x_1 \# x_2\},\ \big(T_1, I_1, F_1\big) \odot \big(T_2, I_2, F_2\big)\Big).$$

   But since $\{x_1 \# x_2\}$ is effectively just one element, we identify $\{x_1 \# x_2\}$ with $x_1 \# x_2$ in the usual algebraic sense. Therefore, the structure is isomorphic to a Neutrosophic Twofold Algebra where # is the classical operation.

Conversely, suppose we start with a Neutrosophic Twofold Algebra

$$\big(A(T, I, F), \star\big),$$

where the classical side is a *single-valued* operation $\#: U \times U \to U$. We embed it into a *Neutrosophic Twofold Hyperalgebra* by defining

$$\boxdot(x_1, x_2) \ := \ \{\#(x_1, x_2)\}.$$

Clearly, $\boxdot$ yields singleton sets as images. The neutrosophic side remains the same operation $\odot$. This defines a hyperoperation $\boxdot$ that reproduces the original single-valued algebraic result in singleton form. Consequently, every pair (x_1, x_2) yields exactly one element inside a set, preserving all original algebraic properties.

Combining both directions:

- "If": When $\boxdot$ yields singletons, we revert to a classical single-valued #.
- "Only If": Starting with a single-valued #, we can trivially interpret it as a degenerate hyperoperation producing singleton images.

Hence, the Neutrosophic Twofold Hyperalgebra $\big(A(T, I, F), \star\big)$ restricts exactly to a Neutrosophic Twofold Algebra if and only if the hyperoperation $\boxdot$ always yields singletons. This completes the rigorous argument. □

To further generalize, we allow the operation on the classical side to map into higher-order powersets (n-th powersets), creating a *Superhyperalgebra*. We keep the neutrosophic (T, I, F) operation.

Definition 2.3 (Neutrosophic Twofold Superhyperalgebra)**.** Let $A(T, I, F)$ be a non-empty Neutrosophic Set over U. Let

$$\star^{(m,n)} : U^m \longrightarrow \mathcal{P}_n^*(U)$$

be an (m, n)-*SuperHyperoperation* (i.e., it maps m-tuples of U into the n-th powerset $\mathcal{P}_n^*(U)$, possibly excluding the empty set). Also, let

$$\odot : [0, 1]^3 \times [0, 1]^3 \to [0, 1]^3$$

be the neutrosophic part. A *Neutrosophic Twofold Superhyperalgebra* is the structure

$$\Big(A(T, I, F), \star^{(m,n)}, \odot\Big),$$

where the combined operation for any

$$x_1(T_1, I_1, F_1), \ldots, x_m(T_m, I_m, F_m) \in A(T, I, F)$$

yields

$$\star^{(m,n)}\big(x_1, \ldots, x_m\big) = \big(x_1 \oplus \cdots \oplus x_m,\ (T_1, I_1, F_1) \odot \ldots \odot (T_m, I_m, F_m)\big),$$

with $x_1 \oplus \cdots \oplus x_m \subseteq \mathcal{P}_n^*(U)$.

Theorem 2.4. *(Neutrosophic Twofold Superhyperalgebra generalizes Neutrosophic Twofold Hyperalgebra.) If an (m, n)-SuperHyperoperation $\star^{(m,n)}$ maps m-tuples of U into $\mathcal{P}_n^*(U)$, then setting $n = 1$ recovers a* Neutrosophic Twofold Hyperalgebra*. Equivalently, restricting $\star^{(m,n)}$ to the first-order powerset $\mathcal{P}_1^*(U)$ yields the hyperalgebraic level.*

Proof. It can be proven step by step as follows.

Consider a *Neutrosophic Twofold Superhyperalgebra*:

$$\Big(A(T, I, F), \star^{(m,n)}, \odot\Big),$$

where:

- $A(T, I, F)$ is a neutrosophic set of elements $x \in U$ each with triple $\big(T_A(x), I_A(x), F_A(x)\big)$.
- $\star^{(m,n)} : U^m \to \mathcal{P}_n^*(U)$ is an (m, n)-SuperHyperoperation, meaning for any $\big(x_1, x_2, \ldots, x_m\big) \in U^m$, we have $\star^{(m,n)}(x_1, \ldots, x_m) \subseteq \mathcal{P}_n^*(U)$.
- $\odot : [0, 1]^3 \times [0, 1]^3 \to [0, 1]^3$ is the neutrosophic composition on (T, I, F).

The combined operation is:

$$\star^{(m,n)}\Big(x_1(T_1, I_1, F_1), \ldots, x_m(T_m, I_m, F_m)\Big) = \Big(x_1 \oplus \cdots \oplus x_m,\ (T_1, I_1, F_1) \odot \cdots \odot (T_m, I_m, F_m)\Big),$$

where $x_1 \oplus \cdots \oplus x_m \in \mathcal{P}_n^*(U)$ is a subset in the n-th powerset.

1. If we fix $n = 1$, then $\mathcal{P}_n^*(U) = \mathcal{P}_1^*(U)$. This is precisely the (non-empty) first-order powerset of U.
2. By definition of hyperalgebra, a binary hyperoperation or an m-ary hyperoperation must yield subsets in $\mathcal{P}^*(U)$. Now, if $\star^{(m,1)}$ only outputs subsets in $\mathcal{P}_1^*(U)$, we exactly match the definition of a *Neutrosophic Twofold Hyperalgebra*:
$$\Big(A(T, I, F), \star^{(m,1)}, \odot\Big).$$
3. The neutrosophic composition $\odot$ remains identical. Thus, the only difference between an (m, n)-SuperHyperoperation and a standard m-ary hyperoperation is whether the image lies in $\mathcal{P}_n^*(U)$ (for the superhyper case) or in $\mathcal{P}_1^*(U)$ (for the normal hyper case). Setting $n = 1$ collapses the superhyper structure onto the hyper structure.

Alternatively, if we start from a *Neutrosophic Twofold Hyperalgebra* (with an m-ary hyperoperation $\boxdot$ into $\mathcal{P}_1^*(U)$), we can embed it into a superhyperalgebra context by letting $\star^{(m,n)}(x_1, \ldots, x_m) = \boxdot(x_1, \ldots, x_m) \in \mathcal{P}_1^*(U) \subseteq \mathcal{P}_n^*(U)$ for any integer $n \geq 1$. Thus, we see that the superhyper version generalizes the hyper version by allowing higher-order powerset images.

Hence, restricting the target from $\mathcal{P}_n^*(U)$ down to $\mathcal{P}_1^*(U)$ recovers the standard hyperalgebraic structure. The neutrosophic part $\odot$ is unaffected by this restriction, so the net effect is precisely a Neutrosophic Twofold Hyperalgebra. Therefore, *Neutrosophic Twofold Superhyperalgebra* strictly generalizes *Neutrosophic Twofold Hyperalgebra*, completing the proof. □

2.2 NeutroHyperalgebra

To extend these ideas to the hyperoperation context, we generalize *AntiAlgebra* and *NeutroAlgebra* using hyperoperations. In a *Hyperalgebra*, the operation on the base set outputs *subsets* rather than single elements.

Definition 2.5 (NeutroHyperalgebra)**.** Let NH be a non-empty set. A *NeutroHyperalgebra* is an algebraic structure of the form

$$\big(NH,\ \{\Omega\},\ \{\text{NeutroAxioms}\}\big),$$

where:

- There is at least one *NeutroHyperoperation* $\Omega : NH^m \to \mathcal{P}(U)$, for some $m \geq 1$, such that for some tuples Ω is well-defined in NH, for others it is entirely outside NH, and for others it is indeterminate (including partially undefined).
- Or there is at least one *NeutroAxiom* that is partially true, partially indeterminate, and partially false within NH.

Theorem 2.6. *A NeutroHyperalgebra reduces to a NeutroAlgebra precisely when each hyperoperation Ω is single-valued (returns exactly one element) for all tuples.*

Proof. Let $\big(NH, \{\Omega\}, \{\text{NeutroAxioms}\}\big)$ be a NeutroHyperalgebra. If for every $(x_1, \ldots, x_m) \in NH^m$,

$$\Omega(x_1, \ldots, x_m) = \{\omega(x_1, \ldots, x_m)\},$$

where $\omega(x_1, \ldots, x_m) \in U$ can be well-defined in NH, outer-defined in $U - NH$, or partially/entirely indeterminate. In other words, Ω is effectively a single-valued NeutroOperation. Then all partial truths, falsities, and indeterminacies remain consistent but mapped via singletons. The result is a NeutroAlgebra.

If we have a NeutroAlgebra $\big(NA, \{\omega\}, \{\text{NeutroAxioms}\}\big)$ with a single-valued NeutroOperation ω, we can define a hyperoperation Ω by

$$\Omega(x_1, \ldots, x_m) = \{\omega(x_1, \ldots, x_m)\},$$

where the operation ω can produce well-defined, outer-defined, or indeterminate results. This embedding shows that any NeutroAlgebra is a special case of a NeutroHyperalgebra with singleton outputs. Thus, the two structures are equivalent in the single-valued limit. □

Definition 2.7 (AntiHyperalgebra)**.** Let AH be a non-empty set. An *AntiHyperalgebra* is an algebraic structure of the form

$$\big(AH,\ \{\Omega\},\ \{\text{AntiAxioms}\}\big),$$

where:

- There is at least one *AntiHyperoperation* $\Omega : AH^m \to \mathcal{P}(U) \setminus \mathcal{P}(AH)$ (i.e., it is outer-defined for all elements of AH). More explicitly, for every $(x_1, \ldots, x_m) \in AH^m$,

  $$\Omega(x_1, \ldots, x_m) \ \cap\ AH = \varnothing,$$

  or equivalently, $\Omega(x_1, \ldots, x_m) \subseteq U \setminus AH$.

- Or there is at least one *AntiAxiom* that is false for every element/tuple in AH.

Theorem 2.8. *An AntiHyperalgebra reduces to a classical AntiAlgebra precisely when each hyperoperation Ω yields a single element (singleton set) rather than multiple or zero elements for all inputs.*

Proof. Suppose we have an *AntiHyperalgebra* $\big(AH, \{\Omega\}, \{\text{AntiAxioms}\}\big)$. If for every tuple $(x_1, \ldots, x_m) \in AH^m$,

$$\Omega(x_1, \ldots, x_m) = \{\omega(x_1, \ldots, x_m)\},$$

with $\omega(x_1, \ldots, x_m) \notin AH$ (outer-defined) for all tuples, then effectively we have a single-valued *AntiOperation* ω from AH^m to $U - AH$. This recovers the structure of an *AntiAlgebra*, since the hyperoperation is no longer multi-valued. The AntiAxioms remain the same.

Conversely, if we start with an *AntiAlgebra* $\big(AA, \{\omega\}, \{\text{AntiAxioms}\}\big)$—where ω is a single-valued AntiOperation—we can embed it into an AntiHyperalgebra by interpreting the single output

$$\omega(x_1, \ldots, x_m) \notin AA$$

as a singleton set

$$\Omega(x_1, \ldots, x_m) = \{\omega(x_1, \ldots, x_m)\} \subseteq U \setminus AA.$$

Hence any AntiAlgebra can be seen as a degenerate AntiHyperalgebra (with singletons). This proves the equivalence. □

2.3 AntiSuperhyperalgebra and NeutroSuperhyperalgebra

We now move to *Superhyperalgebra* structures, where the operations map into higher-order powersets (i.e. $\mathcal{P}_n^*(U)$). Incorporating the Anti- or Neutro- perspective, we obtain *AntiSuperhyperalgebra* and *NeutroSuperhyperalgebra*.

Definition 2.9 (NeutroSuperhyperalgebra)**.** Let NSH be a non-empty set. A *NeutroSuperhyperalgebra* is defined as the structure

$$(NSH, \{\Omega^{(m,n)}\}, \{\text{NeutroAxioms}\}),$$

where:

- There is at least one (m, n)-*NeutroSuperHyperoperation* $\Omega^{(m,n)} : NSH^m \to \mathcal{P}_n(U)$, meaning for some tuples it is well-defined inside $\mathcal{P}_n(NSH)$, for others outside $\mathcal{P}_n(NSH)$, and for the remaining it is indeterminate, possibly including partial or total undefinedness at the (m, n)-th power set level.
- Or there is at least one *NeutroAxiom* that is partially true, partially false, and partially indeterminate across the elements of NSH.

Theorem 2.10. *If in a NeutroSuperhyperalgebra we set $n = 1$, the superhyperoperation is simply a hyperoperation on the base set, reducing the structure to a NeutroHyperalgebra.*

Proof. Let $\big(NSH, \{\Omega^{(m,n)}\}, \{\text{NeutroAxioms}\}\big)$ be a NeutroSuperhyperalgebra. The operation $\Omega^{(m,n)}$ maps

$$(x_1, \ldots, x_m) \in NSH^m \quad \mapsto \quad \mathcal{P}_n(U),$$

where subsets can be partially in $\mathcal{P}_n(NSH)$ (true), partially outside $\mathcal{P}_n(NSH)$ (false), or partially unknown/indeterminate.

If $n = 1$:

$$\Omega^{(m,1)}(x_1, \ldots, x_m) \ \subseteq\ \mathcal{P}_1(U) = \mathcal{P}(U),$$

with partial in/out/indeterminate relative to $\mathcal{P}(NSH)$. This is precisely a NeutroHyperoperation on NSH. The partial true/false/indeterminate axiom status remains. Therefore, we revert to a *NeutroHyperalgebra*. □

Definition 2.11 (AntiSuperhyperalgebra)**.** Let ASH be a non-empty set. An *AntiSuperhyperalgebra* is defined as a structure

$$(ASH, \{\Omega^{(m,n)}\}, \{\text{AntiAxioms}\}),$$

where:

- There exists at least one (m, n)-*AntiSuperHyperoperation* $\Omega^{(m,n)} : ASH^m \to \mathcal{P}_n(U)$ such that for every $(x_1, \ldots, x_m) \in ASH^m$,

  $$\Omega^{(m,n)}(x_1, \ldots, x_m) \subseteq \mathcal{P}_n(U) \setminus \mathcal{P}_n(ASH).$$

  In other words, the output lies entirely *outside* $\mathcal{P}_n(ASH)$, capturing total falsehood or outer-definedness at the (m, n)-th power set level.

- Or there is at least one *AntiAxiom* which is false for *all* elements of ASH.

Theorem 2.12. *If in an AntiSuperhyperalgebra we restrict the (m, n)-superhyperoperation to the first-order powerset $\mathcal{P}_1(U)$, we recover the structure of an AntiHyperalgebra.*

Proof. Consider an *AntiSuperhyperalgebra* $\big(ASH, \{\Omega^{(m,n)}\}, \{\text{AntiAxioms}\}\big)$. The (m, n)-superhyperoperation $\Omega^{(m,n)} : ASH^m \to \mathcal{P}_n(U) \setminus \mathcal{P}_n(ASH)$ outputs subsets lying entirely outside $\mathcal{P}_n(ASH)$.

Case $n = 1$:

$$\Omega^{(m,1)}(x_1, \ldots, x_m) \subseteq \mathcal{P}_1(U) \setminus \mathcal{P}_1(ASH) = \mathcal{P}(U) \setminus \mathcal{P}(ASH).$$

But $\mathcal{P}_1(U) = \mathcal{P}(U)$. Hence we revert to an *AntiHyperoperation* $\Omega^{(m,1)}$ that is outer-defined at level 1. The structure is precisely an AntiHyperalgebra, with the same AntiAxioms. This completes the proof. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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