On The Topological Space of Some n- Refined Neutrosophic Real Intervals and Its Open Sets For $4 \leq n \leq 5$

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Abstract

This paper is dedicated to studying for the first time the building of a topological space based on the intervals defined over 4-refined neutrosophic real numbers and 5-refined neutrosophic real numbers, where we define a special partial order relation on these rings, and we use it to study the structure of the corresponding intervals generated from this relation. Also, we characterize the formula of open sets through these two topological spaces with some illustrated examples.

Keywords: 4-refined neutrosophic numbers; 5-refined neutrosophic numbers; Partial order relation; Neutrosophic interval topology; Open set

1. Introduction

Many neutrosophic structures were built over the idea of splitting the indeterminacy element to many different sub-indeterminacies. Many authors used this idea to study the generalized structures related to this splitting operation such that algebraic rings, matrices, spaces, and modules [4-8]. In the literature, we find many different refined neutrosophic structures. For example refined neutrosophic structures [11, 19] n-refined neutrosophic structures, and n-cyclic refined neutrosophic structures [20-22]. Neutrosophic topological spaces were studied by many authors [1-3], especially open sets, closed sets, and compact sets [9-10, 15, 17]. The concept of n-refined neutrosophic rings was presented in [13]. These rings are considered as an extension of refined neutrosophic rings. Recently AL-Husban et al. Discuss the new structure such neutrosophic set and its application [23-38].

In this work, we focus on the partial order relations over 4-refined neutrosophic real numbers and 5-refined neutrosophic real numbers, where we define a special partial order relation on these rings, and we use it to study the structure of the corresponding intervals generated from this relation. Also, we characterize the formula of open sets through these two topological spaces with some illustrated examples.

2. Main discussion

Definition 2.1

Let $\mu = \{x + yI_1 + zI_2 + tI_3 + qI_4 \ ; x, y, z, t, q \ \in \mathbb{R}\}$ and

$I_1.I_2 = I_2.I_1 = I_1.I_3 = I_3.I_1 = I_1.I_4 = I_4.I_1 = I_1,\ I_2.I_3 = I_3.I_2 = \ I_2,$

$I_4 = I_4.I_2 = I_2, I_3.I_4 = I_4.I_3 = I_3, {I_1}^2 = I_1, {I_2}^2 = I_2, {I_3}^2 = I_3, {I_4}^2 = I_4.$

μ is called the real 4-refined neutrosophic ring.

Remark 1

For $x = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in \mu$, we denote it by $x = x_0 + \sum_{i=1}^{4} x_iI_i$.

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Theorem 2.1

μ is partially ordered set.

Proof:

We define the relation $(\leq)$ as follows:

$u=u_0+\sum_{i=1}^{4}u_iI_i\leq v=v_0+\sum_{i=1}^{4}v_iI_i$, if and only if:

$$\begin{cases} u_0\leq v_0 \\ \sum_{i=0}^{4}u_i\leq\sum_{i=0}^{4}v_i \\ \sum_{i\neq1}u_i\leq\sum_{i\neq1}v_i \\ \sum_{i\neq1,2}u_i\leq\sum_{i\neq1,2}v_i \\ \sum_{i\neq1,2,3}u_i\leq\sum_{i\neq1,2,3}v_i \end{cases}.$$

We will show that $(\leq)$ is a partial order relation:

Let $u=u_0+\sum_{i=1}^{4}u_iI_i\,, v=v_0+\sum_{i=1}^{4}v_iI_i\,, w=w_0+\sum_{i=1}^{4}w_iI_i\,, u\leq u$ that is because:

$$\begin{cases} u_0\leq u_0 \\ \sum_{i=0}^{4}u_i\leq\sum_{i=0}^{4}u_i \\ u_0+u_2+u_3+u_4\leq u_0+u_2+u_3+u_4 \\ u_0+u_3+u_4\leq u_0+u_3+u_4 \\ u_0+u_4\leq u_0+u_4 \end{cases}$$

If $u\leq v$ and $v\leq u$, then:

$$\begin{cases} u_0\leq v_0\,, v_0\leq u_0 \\ \sum_{i=0}^{4}u_i\leq\sum_{i=0}^{4}v_i\,, \sum_{i=0}^{4}v_i\leq\sum_{i=0}^{4}u_i \\ u_0+u_2+u_3+u_4\leq v_0+v_2+v_3+v_4 \\ v_0+v_2+v_3+v_4\leq u_0+u_2+u_3+u_4 \\ u_0+u_3+u_4\leq v_0+v_3+v_4,\quad v_0+v_3+v_4\leq u_0+u_3+u_4 \\ u_0+u_4\leq v_0+v_4\ \ , v_0+v_4\leq u_0+u_4 \end{cases}$$

Hence:

$$\begin{cases} u_0=v_0 \\ u_0+u_4=v_0+v_4 \\ u_0+u_3+u_4=v_0+v_3+v_4 \\ u_0+u_2+u_3+u_4=v_0+v_2+v_3+v_4 \\ u_0+u_1+u_2+u_3+u_4=v_0+v_1+v_2+v_3+v_4 \end{cases}\Longrightarrow\begin{cases} u_0=v_0, u_1=v_1 \\ u_2=v_2, u_3=v_3 \\ u_4=v_4 \end{cases}\Longrightarrow u=v.$$

If $u\leq v$, and $v\leq w$, then:

$$\begin{cases} u_0\leq v_0, \sum_{i=0}^{4}u_i\leq\sum_{i=0}^{4}v_i \\ u_0+u_2+u_3+u_4\leq v_0+v_2+v_3+v_4 \\ u_0+u_3+u_4\leq v_0+v_3+v_4 \\ u_0+u_4\leq v_0+v_4 \end{cases}$$

And $$\begin{cases} v_0\leq w_0 \\ \sum_{i=0}^{4}v_i\leq\sum_{i=0}^{4}w_i \\ v_0+v_2+v_3+v_4\leq w_0+w_2+w_3+w_4 \\ v_0+v_3+v_4\leq w_0+w_3+w_4 \\ v_0+v_4\leq w_0+w_4 \end{cases}$$

Thus: $$\begin{cases} u_0\leq w_0 \\ \sum_{i=0}^{4}u_i\leq\sum_{i=0}^{4}w_i \\ u_0+u_4\leq w_0+w_4 \\ u_0+u_3+u_4\leq w_0+w_3+w_4 \\ u_0+u_2+u_3+u_4\leq w_0+w_2+w_3+w_4 \end{cases}$$

So that $u\leq w$.

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Definition 2.2

Let u, v be two 4-refined neutrosophic real numbers with $u \leq v$, then we define:

$]u, v[\, = \{x \in \mu \ ; u < x < v\}$,

$[u, v[\, = \{x \in \mu \ ; u \leq x < v\}$,

$]u, v] = \{x \in \mu \ ; u < x \leq v\}$,

$[u, v] = \{x \in \mu \ ; u \leq x \leq v\}$.

We say that $]u, v[$ is an open 4-refined neutrosophic interval, and $[u, v]$ is a closed one.

Theorem 2.2

Let $]u, v[,]w, t[$ be two 4-refined neutrosophic open intervals, then:

$]u, v[\, \cap \,]w, t[$ is an open interval of $\emptyset$.

Proof:

If $u = u_0 + \sum_{i=1}^{4} u_i I_i \, , v = v_0 + \sum_{i=1}^{4} v_i I_i \, , w = w_0 + \sum_{i=1}^{4} w_i I_i \, , t = t_0 + \sum_{i=1}^{4} t_i I_i$, and $u < v \, , w < t$ with$]u, v[\, \cap \,]w, t[\, \neq \emptyset$,

then there exists $m = m_0 + \sum_{i=1}^{4} m_i I_i$ such that:

$m \in \,]u, v[, m \in \,]w, t[$.

Define:

$q = q_0 + \sum_{i=1}^{4} q_i I_i \, , p = p_0 + \sum_{i=1}^{4} p_i I_i$, with:

$q_0 = \max(u_0, w_0), q_4 = \max(u_0 + u_4, w_0 + w_4) - \max(u_0, w_0) \, , q_3 = \max(u_0 + u_3 + u_4, w_0 + w_3 + w_4) - \max(u_0 + u_4, w_0 + w_4), q_2 = \max(u_0 + u_2 + u_3 + u_4, w_0 + w_2 + w_3 + w_4) - \max(u_0 + u_3 + u_4, w_0 + w_3 + w_4)$,$q_1 = \max(\sum_{i=0}^{4} u_i, \sum_{i=0}^{4} w_i) - \max(u_0 + u_2 + u_3 + u_4, w_0 + w_2 + w_3 + w_4)$

$p_0 = \min(v_0, t_0), p_4 = \min(v_0 + v_4, t_0 + t_4) - \min(v_0, t_0) \, , p_3 = \min(v_0 + v_3 + v_4, t_0 + t_3 + t_4) - \min(v_0 + v_4, t_0 + t_4), p_2 = \min\,(v_0 + v_2 + v_3 + v_4, t_0 + t_2 + t_3 + t_4) - \min(v_0 + v_3 + v_4, t_0 + t_3 + t_4)$,$p_1 = \min(\sum_{i=0}^{4} v_i, \sum_{i=0}^{4} w_i) - \min\,(v_0 + v_2 + v_3 + v_4, t_0 + t_2 + t_3 + t_4)$, we have:

$m > q$ that is because:

$$\begin{cases} m_0 > u_0 \, , m_0 > w_0 \Longrightarrow m_0 > q_0 \\ \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} u_i \, , \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} w_i \Longrightarrow \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} q_i \\ m_0 + m_4 > u_0 + u_4 \ \ , m_0 + m_4 > w_0 + w_4 \Longrightarrow m_0 + m_4 > q_0 + q_4 \\ m_0 + m_3 + m_4 > u_0 + u_3 + u_4, \ \ m_0 + m_3 + m_4 > w_0 + w_3 + w_4, \\ m_0 + m_3 + m_4 > q_0 + q_3 + q_4, m_0 + m_2 + m_3 + m_4 > u_0 + u_2 + u_3 + u_4 \\ m_0 + m_2 + m_3 + m_4 > w_0 + w_2 + w_3 + w_4, m_0 + m_2 + m_3 + m_4 > q_0 + q_2 + q_3 + q_4 \end{cases}$$

$m < p$ that is because:

$$\begin{cases} m_0 < v_0 \, , m_0 < t_0 \Longrightarrow m_0 < p_0 \\ \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} v_i \, , \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} t_i \Longrightarrow \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} p_i \\ m_0 + m_4 < v_0 + v_4 \ \ , m_0 + m_4 < t_0 + t_4, m_0 + m_4 < p_0 + p_4 \\ m_0 + m_3 + m_4 < v_0 + v_3 + v_4, \ \ m_0 + m_3 + m_4 < t_0 + t_3 + t_4, \\ m_0 + m_3 + m_4 < p_0 + p_3 + p_4, m_0 + m_2 + m_3 + m_4 < v_0 + v_2 + v_3 + v_4 \\ m_0 + m_2 + m_3 + m_4 < t_0 + t_2 + t_3 + t_4, m_0 + m_2 + m_3 + m_4 < p_0 + p_2 + p_3 + p_4 \end{cases}$$

Thus: $]u, v[\, \cap \,]w, t[\, \subseteq \,]q, p[$

Now, let $m = m_0 + \sum_{i=1}^{4} m_i I_i \in \,]q, p[$, then:

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$$m > q \Longrightarrow \begin{cases} m_0 > u_0 \,, m_0 > w_0 \\ \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} u_i \,, \sum_{i=0}^{4} m_i > \sum_{i=0}^{4} w_i \\ m_0 + m_4 > u_0 + u_4 \ \,, m_0 + m_4 > w_0 + w_4 \\ m_0 + m_3 + m_4 > u_0 + u_3 + u_4, \ \ m_0 + m_3 + m_4 > w_0 + w_3 + w_4 \\ m_0 + m_2 + m_3 + m_4 > u_0 + u_2 + u_3 + u_4 \\ m_0 + m_2 + m_3 + m_4 > w_0 + w_2 + w_3 + w_4 \end{cases} \Longrightarrow m > u \text{ and } m > w.$$

$$m < p \ \Longrightarrow \begin{cases} m_0 < v_0 \,, m_0 < t_0, \\ \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} v_i \,, \sum_{i=0}^{4} m_i < \sum_{i=0}^{4} t_i, \\ m_0 + m_4 < v_0 + v_4 \ \,, m_0 + m_4 < t_0 + t_4, \\ m_0 + m_3 + m_4 < v_0 + v_3 + v_4, \ \ m_0 + m_3 + m_4 < t_0 + t_3 + t_4, \\ m_0 + m_2 + m_3 + m_4 < v_0 + v_2 + v_3 + v_4, \\ m_0 + m_2 + m_3 + m_4 < t_0 + t_2 + t_3 + t_4, \end{cases} \Longrightarrow m < t \text{ and } m < v.$$

Hence $]q,p[\subseteq]u,v[\cap \ ]w,t[$ and $]u,v[\cap]w,t[=]q,p[$.

Remark 2

$[u,v] \cap [w,t] = [q,p]$, i.e. the intersection of two 4-refined neutrosophic real closed intervals is a closed interval or $\emptyset$.

Definition 2.3

We define $\tau = \{\mu, \emptyset, s_i\}$; $s_i = \bigcup_{i \in I}]u_i, v_i[$, with $u_i, v_i \in \mu$.

Theorem 2.3

(μ, τ) is a topological space.

Proof:

$\mu \in \tau, \emptyset \in \tau$.

Let $s_i, s_j \in \tau$, then:

$s_i = \bigcup_{i \in I}]u_i, v_i[\,, s_j = \bigcup_{j \in I}]u_j, v_j[$, then:

$$s_i \cap s_j = \bigcup_{k \in K}]p_k, q_k[\in \tau.$$

Let $\{(s_i)_j\}$ be a family of τ, then:

$\bigcup_{j \in J} (s_i)_j = \bigcup_{j \in J} (\bigcup_{i \in I}]u_i, v_i[) \in \tau$, thus τ is a topology and (μ, τ) is a topological space.

Example 2.1

Consider: $u_1 = 2 + I_1 + I_2 + I_3 + I_4 \,, v_1 = 3 + 2I_1 + I_2 + 3I_3 + 2I_4, u_2 = 1 + 2I_1 + I_2 + I_3 + 2I_4, v_1 = 2 + 3I_1 + 3I_2 + I_3 + 3I_4$, we have: $u_1 < v_1, u_2 < v_2,]u_1, v_1[\in \tau,]u_2, v_2[\in \tau,]u_1, v_1[\cup]u_2, v_2[\in \tau,]u_1, v_1[\cap]u_2, v_2[=]q, p[$;

$q = 2 + 2I_1 + I_2 + I_3 + I_4 \,, p = 2 + 2I_1 + 3I_2 + I_3 + 3I_4$.

Remark 3

Open sets of μ are the union of open intervals.

The complements of open sets are closed sets.

Definition 2.4

Let $\mu = \{x + yI_1 + zI_2 + tI_3 + qI_4 + sI_5 \ ; x, y, z, t, q, s \ \in \mathbb{R}\}$ and

$I_1.I_2 = I_2.I_1 = I_1.I_3 = I_3.I_1 = I_1.I_4 = I_4.I_1 = I_5.I_1 = I_1, \ I_2.I_3 = I_3.I_2 = \ I_2,$

$I_4 = I_4.I_2 = I_5.I_2 = I_2, I_3.I_4 = I_4.I_3 = I_5.I_3 = I_3, {I_1}^2 = I_1, {I_2}^2 = I_2, {I_3}^2 = I_3, {I_4}^2 = I_4, {I_5}^2 = I_5.$

$I_5 I_4 = I_4 I_5 = I_4.$

μ is called the real 5-refined neutrosophic ring.

Remark 4

For $x = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4 + x_5 I_5 \in \mu$, we denote it by $x = x_0 + \sum_{i=1}^{5} x_i I_i$.

Theorem 2.4

μ is partially ordered set.

Proof:

We define the relation $(\leq)$ as follows:

$u = u_0 + \sum_{i=1}^{5} u_i I_i \leq v = v_0 + \sum_{i=1}^{5} v_i I_i$, if and only if:

$$\begin{cases} u_0 \leq v_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} v_i \\ \sum_{i\neq 1} u_i \leq \sum_{i\neq 1} v_i \\ \sum_{i\neq 1,2} u_i \leq \sum_{i\neq 1,2} v_i \\ \sum_{i\neq 1,2,3} u_i \leq \sum_{i\neq 1,2,3} v_i \\ \sum_{i\neq 1,2,3,4} u_i \leq \sum_{i\neq 1,2,3,4} v_i \end{cases}.$$

We will show that $(\leq)$ is a partial order relation:

Let $u = u_0 + \sum_{i=1}^{5} u_i I_i$, $v = v_0 + \sum_{i=1}^{5} v_i I_i$, $w = w_0 + \sum_{i=1}^{5} w_i I_i$, $u \leq u$ that is because:

$$\begin{cases} u_0 \leq u_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} u_i \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq u_0 + u_2 + u_3 + u_4 + u_5 \\ u_0 + u_3 + u_4 + u_5 \leq u_0 + u_3 + u_4 + u_5 \\ u_0 + u_4 + u_5 \leq u_0 + u_4 + u_5 \\ u_0 + u_5 \leq u_0 + u_5 \end{cases}$$

If $u \leq v$ and $v \leq u$, then:

$$\begin{cases} u_0 \leq v_0 , v_0 \leq u_0 \\ \displaystyle\sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} v_i , \sum_{i=0}^{5} v_i \leq \sum_{i=0}^{5} u_i \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq v_0 + v_2 + v_3 + v_4 + v_5 \\ v_0 + v_2 + v_3 + v_4 + v_5 \leq u_0 + u_2 + u_3 + u_4 + u_5 \\ u_0 + u_3 + u_4 + u_5 \leq v_0 + v_3 + v_4 + v_5, \quad v_0 + v_3 + v_4 + v_5 \leq u_0 + u_3 + u_4 + u_5 \\ u_0 + u_4 + u_5 \leq v_0 + v_4 + v_5 \ , v_0 + v_4 + v_5 \leq u_0 + u_4 + u_5 \\ u_0 + u_5 \leq v_0 + v_5, v_0 + v_5 \leq u_0 + u_5 \end{cases}$$

Hence:

$$\begin{cases} u_0 = v_0 \\ u_0 + u_4 + u_5 = v_0 + v_4 + v_5 \\ u_0 + u_3 + u_4 + u_5 = v_0 + v_3 + v_4 + v_5 \\ u_0 + u_2 + u_3 + u_4 + u_5 = v_0 + v_2 + v_3 + v_4 + v_5 \\ u_0 + u_1 + u_2 + u_3 + u_4 + u_5 = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 \\ u_0 + u_5 = v_0 + v_5 \end{cases} \Rightarrow \begin{cases} u_0 = v_0, u_1 = v_1 \\ u_2 = v_2, u_3 = v_3 \\ u_4 = v_4 \\ u_5 = v_5 \end{cases} \Rightarrow u = v.$$

If $u \leq v$, and $v \leq w$, then:

$$\begin{cases} u_0 \leq v_0, \displaystyle\sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} v_i \\ u_0 + u_2 + u_3 + u_4 + u_5 \leq v_0 + v_2 + v_3 + v_4 + v_5 \\ u_0 + u_3 + u_4 + u_5 \leq v_0 + v_3 + v_4 + v_5 \\ u_0 + u_4 + u_5 \leq v_0 + v_4 + v_5 \\ u_0 + u_5 \leq v_0 + v_5 \end{cases}$$

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$$\text{And}\begin{cases} v_0 \leq w_0 \\ \sum_{i=0}^{5} v_i \leq \sum_{i=0}^{5} w_i \\ v_0+v_2+v_3+v_4+v_5 \leq w_0+w_2+w_3+w_4+w_5 \\ v_0+v_3+v_4+v_5 \leq w_0+w_3+w_4+w_5 \\ v_0+v_4+v_5 \leq w_0+w_4+w_5 \\ v_0+v_5 \leq w_0+w_5 \end{cases}$$

$$\text{Thus:}\begin{cases} u_0 \leq w_0 \\ \sum_{i=0}^{5} u_i \leq \sum_{i=0}^{5} w_i \\ u_0+u_4+u_5 \leq w_0+w_4+w_5 \\ u_0+u_3+u_4+u_5 \leq w_0+w_3+w_4+w_5 \\ u_0+u_2+u_3+u_4+u_5 \leq w_0+w_2+w_3+w_4+w_5 \\ u_0+u_5 \leq w_0+w_5 \end{cases}$$

So that $u \leq w$.

Definition 2.5

Let u, v be two 5-refined neutrosophic real numbers with $u \leq v$, then we define:

$]u, v[= \{x \in \mu \ ; u < x < v\}$,

$[u, v[= \{x \in \mu \ ; u \leq x < v\}$,

$]u, v] = \{x \in \mu \ ; u < x \leq v\}$,

$[u, v] = \{x \in \mu \ ; u \leq x \leq v\}$.

We say that $]u, v[$ is an open 5-refined neutrosophic interval, and $[u, v]$ is a closed one.

Theorem 2.5

Let $]u, v[,]w, t[$ be two 5-refined neutrosophic open intervals, then:

$]u, v[\cap \ ]w, t[$ is an open interval of $\emptyset$.

Proof:

If $u = u_0 + \sum_{i=1}^{5} u_i I_i\,, v = v_0 + \sum_{i=1}^{5} v_i I_i\,, w = w_0 + \sum_{i=1}^{5} w_i I_i\,, t = t_0 + \sum_{i=1}^{5} t_i I_i$, and $u < v\,, w < t$ with$]u, v[\cap \ ]w, t[\neq \emptyset$,

then there exists $m = m_0 + \sum_{i=1}^{5} m_i I_i$ such that: $m \in \ ]u, v[, m \in \ ]w, t[$.

Define:

$q = q_0 + \sum_{i=1}^{5} q_i I_i\,, p = p_0 + \sum_{i=1}^{5} p_i I_i$, with:

$q_0 = \max(u_0, w_0), q_4 = \max(u_0+u_4+u_5, w_0+w_4+w_5) - \max(u_0+u_5, w_0+w_5)\,, q_3 = \max(u_0+u_3+u_4+u_5, w_0+w_3+w_4+w_5) - \max(u_0+u_4+u_5, w_0+w_4+w_5), q_2 = \max(u_0+u_2+u_3+u_4+u_5, w_0+w_2+w_3+w_4+w_5) - \max(u_0+u_3+u_4+u_5, w_0+w_3+w_4+w_5),q_1 = \max\left(\sum_{i=0}^{5} u_i, \sum_{i=0}^{5} w_i\right) - \max(u_0+u_2+u_3+u_4+u_5, w_0+w_2+w_3+w_4+w_5), q_5 = \max(u_0+u_5, w_0+w_5) - \max(u_0, w_0)$,

$p_0 = \min(v_0, t_0), p_4 = \min(v_0+v_4+v_5, t_0+t_4+t_5) - \min(v_0, t_0)\,, p_3 = \min(v_0+v_3+v_4+v_5, t_0+t_3+t_4+t_5) - \min(v_0+v_4+v_5, t_0+t_4+t_5), p_2 = \min\,(v_0+v_2+v_3+v_4+v_5, t_0+t_2+t_3+t_4+t_5) - \min(v_0+v_3+v_4+v_5, t_0+t_3+t_4+t_5),p_1 = \min\left(\sum_{i=0}^{5} v_i, \sum_{i=0}^{5} w_i\right) - \min\,(v_0+v_2+v_3+v_4+v_5, t_0+t_2+t_3+t_4+t_5), p_5 = \min(v_0+v_5, t_0+t_5) - \min(v_0, t_0)$,, we have:

$m > q$ that is because:

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$$\left\{\begin{array}{c}
m_0 > u_0\,, m_0 > w_0 \Longrightarrow m_0 > q_0 \\
\sum_{i=0}^{5} m_i > \sum_{i=0}^{5} u_i\,, \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} w_i \Longrightarrow \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} q_i \\
m_0 + m_4 + m_5 > u_0 + u_4 + u_5 \;\; , m_0 + m_4 + m_5 > w_0 + w_4 + w_5 \Longrightarrow m_0 + m_4 + m_5 > q_0 + q_4 + q_5 \\
m_0 + m_3 + m_4 + m_5 > u_0 + u_3 + u_4 + u_5, \;\; m_0 + m_3 + m_4 + m_5 > w_0 + w_3 + w_4 + w_5, \\
m_0 + m_3 + m_4 + m_5 > q_0 + q_3 + q_4, m_0 + m_2 + m_3 + m_4 + m_5 > u_0 + u_2 + u_3 + u_4 + u_5 \\
m_0 + m_2 + m_3 + m_4 + m_5 > w_0 + w_2 + w_3 + w_4 + w_5, m_0 + m_2 + m_3 + m_4 + m_5 > q_0 + q_2 + q_3 + q_4 + q_5 \\
m_0 + m_5 > u_0 + u_5 \;\; , m_0 + m_5 > w_0 + w_5 \Longrightarrow m_0 + m_5 > q_0 + q_5
\end{array}\right.$$

$m < p$ that is because:

$$\left\{\begin{array}{c}
m_0 < v_0\,, m_0 < t_0 \Longrightarrow m_0 < p_0 \\
\sum_{i=0}^{5} m_i < \sum_{i=0}^{5} v_i\,, \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} t_i \Longrightarrow \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} p_i \\
m_0 + m_4 + m_5 < v_0 + v_4 + v_5 \;\; , m_0 + m_4 + m_5 < t_0 + t_4 + t_5, m_0 + m_4 + m_5 < p_0 + p_4 + p_5 \\
m_0 + m_3 + m_4 + m_5 < v_0 + v_3 + v_4 + v_5, \;\; m_0 + m_3 + m_4 + m_5 < t_0 + t_3 + t_4 + t_5, \\
m_0 + m_3 + m_4 + m_5 < p_0 + p_3 + p_4 + p_5, m_0 + m_2 + m_3 + m_4 + m_5 < v_0 + v_2 + v_3 + v_4 + v_5 \\
m_0 + m_2 + m_3 + m_4 + m_5 < t_0 + t_2 + t_3 + t_4 + t_5, m_0 + m_2 + m_3 + m_4 + m_5 < p_0 + p_2 + p_3 + p_4 + p_5 \\
m_0 + m_5 < v_0 + v_5 \;\; , m_0 + m_5 < t_0 + t_5, m_0 + m_5 < p_0 + p_5
\end{array}\right.$$

Thus: $]u, v[\,\cap\,]w, t[\,\subseteq\,]q, p[$

Now, let $m = m_0 + \sum_{i=1}^{5} m_i I_i \in\,]q, p[$, then:

$$m > q \Longrightarrow \left\{\begin{array}{c}
m_0 > u_0\,, m_0 > w_0 \\
\sum_{i=0}^{5} m_i > \sum_{i=0}^{5} u_i\,, \sum_{i=0}^{5} m_i > \sum_{i=0}^{5} w_i \\
m_0 + m_4 + m_5 > u_0 + u_4 \; + u_5\,, m_0 + m_4 + m_5 > w_0 + w_4 + w_5 \\
m_0 + m_3 + m_4 > u_0 + u_3 + u_4 + u_5, \;\; m_0 + m_3 + m_4 > w_0 + w_3 + w_4 + w_5 \\
m_0 + m_2 + m_3 + m_4 + m_5 > u_0 + u_2 + u_3 + u_4 \\
m_0 + m_2 + m_3 + m_4 + m_5 > w_0 + w_2 + w_3 + w_4 + w_5
\end{array}\right. \Longrightarrow m > u \text{ and } m >$$

w.

$$m < p \;\Longrightarrow\; \left\{\begin{array}{c}
m_0 < v_0\,, m_0 < t_0, \\
\sum_{i=0}^{5} m_i < \sum_{i=0}^{5} v_i\,, \sum_{i=0}^{5} m_i < \sum_{i=0}^{5} t_i, \\
m_0 + m_4 + m_5 < v_0 + v_4 + v_5 \;\; , m_0 + m_4 + m_5 < t_0 + t_4 + t_5, \\
m_0 + m_3 + m_4 + m_5 < v_0 + v_3 + v_4 + v_5, \;\; m_0 + m_3 + m_4 + m_5 < t_0 + t_3 + t_4 + t_5, \\
m_0 + m_2 + m_3 + m_4 + m_5 < v_0 + v_2 + v_3 + v_4 + v_5, \\
m_0 + m_2 + m_3 + m_4 + m_5 < t_0 + t_2 + t_3 + t_4 + t_5,
\end{array}\right. \Longrightarrow m <$$

t and $m < v$.

Hence $]q, p[\,\subseteq\,]u, v[\,\cap\,]w, t[$ and $]u, v[\,\cap\,]w, t[\,=\,]q, p[$.

Remark 5

$[u, v] \cap [w, t] = [q, p]$, i.e. the intersection of two 5-refined neutrosophic real closed intervals is a closed interval or $\emptyset$.

Definition 2.6

We define $\tau = \{\mu, \emptyset, s_i\}$; $s_i = \cup_{i \in I}]u_i, v_i[$, with $u_i, v_i \in \mu$.

Theorem 2.6

(μ, τ) is a topological space.

Proof:

$\mu \in \tau, \emptyset \in \tau$.

Let $s_i, s_j \in \tau$, then:

$s_i = \cup_{i \in I}]u_i, v_i[\,, s_j = \cup_{j \in I}]u_j, v_j[$, then:

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$s_i \cap s_j = \bigcup_{k\in K}]p_k, q_k[\in \tau.$

Let $\{(s_i)_j\}$ be a family of τ, then:

$\cup_{j\in J}(s_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[) \in \tau$, thus τ is a topology and (μ, τ) is a topological space.

Remark 6

Open sets of μ are the union of open intervals with respect to the partial order relation.

The complements of open sets are closed sets.

Example 2.2

Consider $u = 2 + I_1 + I_2 + I_3 + I_4 + I_5, v = 2 + 2I_1 + I_2 + 2I_3 + I_4 + 2I_5$,

Let $x = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 + x_5I_5 \in [u, v]$, then:

$$\begin{cases} x_0 = 2 \\ 3 \le x_0 + x_5 \le 4 \\ 4 \le x_0 + x_4 + x_5 \le 5 \\ 5 \le x_0 + x_3 + x_4 + x_5 \le 7 \\ 6 \le x_0 + x_2 + x_3 + x_4 + x_5 \le 8 \\ 7 \le x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \le 10 \end{cases}$$

Thus

$$\begin{cases} x_0 = 2 \\ 1 \le x_5 \le 2 \\ 2 \le x_4 + x_5 \le 3 \\ 3 \le x_3 + x_4 + x_5 \le 5 \\ 4 \le x_2 + x_3 + x_4 + x_5 \le 6 \\ 5 \le x_1 + x_2 + x_3 + x_4 + x_5 \le 8 \end{cases}$$

Theorem 2.7

Let (μ, τ) be the topological space of 4-refined neutrosophic numbers intervals, and (Δ, β) be the topological space of 5-refined neutrosophic numbers intervals, then:

$\big(\mu \times \Delta, (\tau, \beta)\big) = \{(\emptyset, \emptyset), (\mu, \Delta), (s_i, t_i); s_i \in \tau, t_i \in \beta\}$, is a topological space.

Proof:

$(\mu, \Delta) \in (\tau, \beta), (\emptyset, \emptyset) \in (\tau, \beta)$.

Let $(s_i, t_i), (s_j, t_j) \in (\tau, \beta)$, then:

$s_i = \cup_{i\in I}]u_i, v_i[, s_j = \cup_{j\in I}]u_j, v_j[, t_i = \cup_{i\in I}]m_i, n_i[, t_j = \cup_{j\in I}]m_j, n_j[$ then:

$(s_i, t_i) \cap (s_j, t_j) = (\bigcup_{k\in K}]p_k, q_k[, \bigcup_{k\in K}]h_k, f_k[) \in (\tau, \beta).$

Let $\{(s_i, t_i)_j\}$ be a family of (τ, β), then:

$\cup_{j\in J}(s_i, t_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[, \cup_{i\in I}]m_i, n_i[) \in (\tau, \beta)$, thus (τ, β) is a topology and $\big(\mu \times \Delta, (\tau, \beta)\big)$ is a topological space.

Theorem 2.8

Let (μ, τ) be the topological space of 4-refined neutrosophic numbers intervals, then:

$\big(\mu \times \mu, (\tau, \tau)\big) = \{(\emptyset, \emptyset), (\mu, \mu), (s_i, t_i); s_i \in \tau, t_i \in \tau\}$, is a topological space.

Proof:

$(\mu, \mu) \in (\tau, \tau), (\emptyset, \emptyset) \in (\tau, \tau)$.

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Let $(s_i, t_i), (s_j, t_j) \in (\tau, \tau)$, then:

$s_i = \cup_{i\in I}]u_i, v_i[\,, s_j = \cup_{j\in I}]u_j, v_j[\,, t_i = \cup_{i\in I}]m_i, n_i[\,, t_j = \cup_{j\in I}]m_j, n_j[$ then:

$(s_i, t_i) \cap (s_j, t_j) = (\bigcup_{k\in K}]p_k, q_k[\,, \bigcup_{k\in K}]h_k, f_k[\,) \in (\tau, \tau)$.

Let $\{(s_i, t_i)_j\}$ be a family of (τ, τ), then:

$\cup_{j\in J}(s_i, t_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[\,, \cup_{i\in I}]m_i, n_i[\,) \in (\tau, \tau)$, thus (τ, τ) is a topology and $(\mu \times \mu, (\tau, \tau))$ is a topological space.

Theorem 2.9

Let (μ, τ) be the topological space of 5-refined neutrosophic numbers intervals, then:

$(\mu \times \mu, (\tau, \tau)) = \{(\emptyset, \emptyset), (\mu, \mu), (s_i, t_i); s_i \in \tau, t_i \in \tau\}$, is a topological space.

Proof:

$(\mu, \mu) \in (\tau, \tau), (\emptyset, \emptyset) \in (\tau, \tau)$.

Let $(s_i, t_i), (s_j, t_j) \in (\tau, \tau)$, then:

$s_i = \cup_{i\in I}]u_i, v_i[\,, s_j = \cup_{j\in I}]u_j, v_j[\,, t_i = \cup_{i\in I}]m_i, n_i[\,, t_j = \cup_{j\in I}]m_j, n_j[$ then:

$(s_i, t_i) \cap (s_j, t_j) = (\bigcup_{k\in K}]p_k, q_k[\,, \bigcup_{k\in K}]h_k, f_k[\,) \in (\tau, \tau)$.

Let $\{(s_i, t_i)_j\}$ be a family of (τ, τ), then:

$\cup_{j\in J}(s_i, t_i)_j = \cup_{j\in J}(\cup_{i\in I}]u_i, v_i[\,, \cup_{i\in I}]m_i, n_i[\,) \in (\tau, \tau)$, thus (τ, τ) is a topology and $(\mu \times \mu, (\tau, \tau))$ is a topological space.

3. Conclusion

In this paper we studied for the first time the building of a topological space based on the intervals defined over 4-refined neutrosophic real numbers and 5-refined neutrosophic real numbers, where we defined a special partial order relation on these rings, and we use it to study the structure of the corresponding intervals generated from this relation. Also, we characterized the formula of open sets through these two topological spaces with some illustrated examples.

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