Positive integer solutions of an equation involving Euler, generalized Euler, and Smarandache functions

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Abstract: This study investigates the positive integer solutions of an equation involving the Smarandache function. The equation is given by $t\varphi(n)=\varphi_2(n)+S(n^{16})$, where $\varphi(n)$ represents the Euler function, $\varphi_e(n)$ represents the generalized Euler function with e as a positive integer, and $S(n)$ represents the Smarandache function. The solutions of this equation are discussed, and it is proven that the equation only has positive integer solutions when $t=1,2,3,6,7,9,12,17,18,19$. Furthermore, all positive integer solutions of the equation are provided.

1. Introduction

We make n be a positive integer. The Euler function $\varphi(n)$ is defined as the number of positive integers that are coprime with n among the numbers $1,2,\cdots,n$. The generalized Euler function $\varphi_e(n)$ is defined as the number of positive integers that are coprime with n among the numbers $1,2,\cdots,\left[\frac{n}{e}\right]$, where e is a positive integer and $\left[\frac{*}{*}\right]$ denotes the floor function. The Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n\,|\,m!$. The study of number-theoretic functions and their relation to the solvability of related indeterminate equations has long been a topic of interest in number theory [1]. Previous works have discussed the solvability of equations involving the Euler function $\varphi(n)$ [2-4], the generalized Euler function $\varphi_2(n)$ [5], and the Smarandache function $S(n)$ [6-10]. There is also research on the solvability of equations that combine the Euler function $\varphi(n)$, the generalized Euler function $\varphi_e(n)$, and the Smarandache function $S(n)$. For example, Zhang et al. [11] discuss the solvability of the equation $K\varphi(Y)=\varphi_2(Y)+S(Y^8)$, Jiang et al. [12] discuss the solvability of two equations in the form of $K\varphi(n)=\varphi_2(n)+S(n^m)$, and Zheng [13] discusses the solvability of the equation $m\varphi(n)=\varphi_2(n)+S(n^{10})$. In this paper, we will discuss the solvability of the equation $t\varphi(n)=M\varphi_2(n)+NS(n^L)$ when $L=16$, $M=N=1$,

$$t\varphi(n)=\varphi_2(n)+S(n^{16}) \qquad (1)$$

and provide all positive integer solutions to this equation.

2. Related lemmas

Lemma 2.1. If $n=\prod_{i=1}^{k} q_i^{\alpha_i}$, then $S(n)=\max\left(S\left(q_1^{\alpha_1}\right),S\left(q_2^{\alpha_2}\right),\cdots,S\left(q_k^{\alpha_k}\right)\right)$.

Lemma 2.2. [9] For a prime number p and a positive integer k , we have $S\left(p^k\right)\le kp$; in particular, when $k<p$, we have $S\left(p^k\right)=kp$.

Lemma 2.3. [14] For $n\ge 3$, we have $\varphi_2(n)=\frac{\varphi(n)}{2}$.

Lemma 2.4. [15] For $n\ge 3$, $\varphi(n)$ is an even number.

3. Main theorem and proof

Theorem 3.1. The indeterminate Equation (1) only has positive integer solutions when $t=1,2,3,6,7,9,12,17,18,19$, and

(1) When $t=1$, Equation (1) has positive integer solutions $n=1$, 847, 972, 1000, 1029, 1089, 1372, 1500, 1694,
2058, 2178;

(2) When $t=2$, Equation (1) has a positive integer solution $n=363,484,726$;

(3) When $t=3$, Equation (1) has a positive integer solution $n=81,162,169,338$;

(4) When $t=6$, Equation (1) has a positive integer solution $n=27,54$;

(5) When $t=7$, Equation (1) has a positive integer solution $n=24,25,50$;

(6) When $t=9$, Equation (1) has a positive integer solution $n=51,68,102$;

(7) When $t=12$, Equation (1) has a positive integer solution $n=9,18$;

(8) When $t=17$, Equation (1) has a positive integer solution $n=11,22$;

(9) When $t=18$, Equation (1) has a positive integer solution $n=5,10$;

(10) When $t=19$, Equation (1) has a positive integer solution $n=2$.

Proof. According to the definition of the generalized Euler function $\varphi_e(n)$, we have $\varphi_2(1)=0$ and $\varphi_2(2)=1$. For Equation (1) , when $n=1$, we have $t\varphi(1)=\varphi_2(1)+S\left(1^{16}\right)$, which gives $t=1$. Therefore, when $t=1$, $n=1$ is a solution to Equation (1) (Here, “solution” refers to positive integer solutions, the same applies to the following). When $n=2$, we have $t\varphi(2)=\varphi_2(2)+S\left(2^{16}\right)$, which gives $t=1+18=19$. Therefore, when $t=19$, $n=2$ is a solution to Equation (1) . For $n\ge 3$, we can assume that

$$n=\prod_{i=1}^{k} q_i^{\alpha_i}\ge 3$$

where $q_1<q_2<\cdots<q_k$ are prime numbers. According to Lemma 1, we have:

$$S(n)=\max\left(S\left(q_1^{\alpha_1}\right),S\left(q_2^{\alpha_2}\right),\cdots,S\left(q_k^{\alpha_k}\right)\right)=S\left(q^{\alpha}\right) \qquad (2)$$

where $n=q^{\alpha}n_1$, $(q,n_1)=1$. Therefore, we have:

$$\varphi(n)=\varphi\left(q^{\alpha}n_1\right)=\varphi(q^{\alpha})\varphi(n_1)=q^{\alpha-1}(q-1)\varphi(n_1) \qquad (3)$$

By Lemma 2.4, combined with Equations (2) and (3) , Equation (1) can be transformed into:

$$(2t-1)q^{\alpha-1}(q-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{16\alpha}\right) \qquad (4)$$

Next, we will discuss Equation (4) by classifying and analyzing the cases based on different values of α .

Case 1. When $\alpha=1$, we have $(2t-1)(q-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{16}\right)$.

When $q=2$, for Equation (4), we have $(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{16}\right)=36$. Therefore, we have $2t-1=1$, $\varphi\left(n_1\right)=36$, which gives $t=1$, $n_1=37,57,63,74,76,108,114,126$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=74,114,126$. However, after checking, none of $n=74,114,126$ is a solution to Equation (1). When $2t-1=3$, $\varphi\left(n_1\right)=12$, we have $t=2$, $n_1=13,21,26,28,36,42$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=26,42$. However, after checking, none of $n=26,42$ is a solution to Equation (1). When $2t-1=9$, $\varphi\left(n_1\right)=4$, we have $t=5$, $n_1=5,8,10,12$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=10$. However, after checking, $n=10$ is not a solution to Equation (1).

When $q=3$, for Equation (4), we have $(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(3^{16}\right)=36$. Therefore, we have $2t-1=1$, $\varphi\left(n_1\right)=36$, which gives $t=1$, $n_1=37,57,63,74,76,108,114,126$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=111,222,228$. However, after checking, none of $n=111,222,228$ is a solution to Equation (1). When $2t-1=3$, $\varphi\left(n_1\right)=12$, we have $t=2$, $n_1=13,21,26,28,36,42$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=39,78,84$. However, after checking, none of $n=39,78,84$ is a solution to Equation (1). When $2t-1=9$, $\varphi\left(n_1\right)=4$, we have $t=5$, $n_1=5,8,10,12$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=15,24,30$. However, after checking, none of $n=15,24,30$ is a solution to Equation (1).

When $q=5$, for Equation (4), we have $2(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(5^{16}\right)=70$, which means $(2t-1)\varphi(n_1)=35$. Therefore, we have $2t-1=35$, $\varphi\left(n_1\right)=1$, which gives $t=18$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=5,10$. After checking, we find that both $n=5$ and $n=10$ are solutions to Equation (1).

When $q=7$, for Equation (4), we have $3(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(7^{16}\right)=98$, which has no solution.

When $q=11$, for Equation (4), we have $5(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(11^{16}\right)=165$, which means $(2t-1)\varphi(n_1)=33$. Therefore, we have $2t-1=33$, $\varphi\left(n_1\right)=1$, which gives $t=17$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=11,22$. After checking, we find that both $n=11$ and $n=22$ are solutions to Equation (1).

When $q=13$, for Equation (4), we have $6(2t-1)\varphi(n_1)=S\left(n^{16}\right)=S\left(13^{16}\right)=195$, which has no solution.

When $q=17$, for Equation (4), we have $16(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(17^{16}\right)=2\times17\times16$, which means $(2t-1)\varphi(n_1)=34$. Therefore, we have $2t-1=1$, $\varphi\left(n_1\right)=34$, and $2t-1=17$, $\varphi\left(n_1\right)=2$. When $2t-1=1$, $\varphi\left(n_1\right)=34$, we have $t=1$, but since 34 is not an Euler quotient, there is no solution for $\varphi\left(n_1\right)=34$. When $2t-1=17$, $\varphi\left(n_1\right)=2$, we have $t=9$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $\left(q,n_1\right)=1$, we have $n=51,68,102$. After checking, we find that both $n=51$, $n=68$, and $n=102$ are solutions to Equation (1).

When $q=19$, for Equation (4), we have $18(2t-1)\varphi(n_1)=2S(n^{16})=2S(19^{16})=2\times19\times16$, which means $9(2t-1)\varphi(n_1)=304$. This equation has no solution.

When $q=23$, for Equation (4), we have $22(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(23^{16}\right)=2\times23\times16$, which means $11(2t-1)\varphi(n_1)=368$. This equation has no solution.

When $q=29$, for Equation (4), we have $28(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(29^{16}\right)=2\times29\times16$, which means $7(2t-1)\varphi(n_1)=232$. This equation has no solution.

When $q=31$, for Equation (4), we have $30(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(31^{16}\right)=2\times31\times16$, which means $15(2t-1)\varphi(n_1)=496$. This equation has no solution.

When $q\geq37$, for Equation (4), we have $(2t-1)(q-1)\varphi(n_1)=2S\left(q^{16}\right)=32q$, which means $\left[(2t-1)\varphi(n_1)-32\right](q-1)=32$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 2. When $\alpha=2$, we have $q(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{32}\right)$.

When $q=2$, for Equation (4), we have $2(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{32}\right)=2\times 34$, which means $(2t-1)\varphi(n_1)=34$. Therefore, we have $2t-1=1$, $\varphi(n_1)=34$, and $2t-1=17$, $\varphi(n_1)=2$. When $2t-1=1$, $\varphi(n_1)=34$, we have $t=1$. However, since 34 is not an Euler quotient, there is no solution for $\varphi(n_1)=34$. When $2t-1=17$, $\varphi(n_1)=2$, we have $t=9$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=12$. After checking, we find that $n=12$ is not a solution to Equation (1).

When $q=3$, for Equation (4), we have $6(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(3^{32}\right)=2\times 69$, which means $(2t-1)\varphi(n_1)=23$. Therefore, we have $2t-1=23$, $\varphi(n_1)=1$, and we have $t=12$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=9,18$. After checking, we find that both $n=9$ and $n=18$ are solutions to Equation (1).

When $q=5$, for Equation (4), we have $20(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(5^{32}\right)=2\times 130$, which means $(2t-1)\varphi(n_1)=13$. Therefore, we have $2t-1=13$, $\varphi(n_1)=1$, and we have $t=7$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=25,50$. After checking, we find that both $n=25$ and $n=50$ are solutions to Equation (1).

When $q=7$, for Equation (4), we have $42(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(7^{32}\right)=2\times 196$. This gives $3(2t-1)\varphi(n_1)=14$, which has no solution for Equation (1).

When $q=11$, for Equation (4), we have $110(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(11^{32}\right)=2\times 330$, which means $(2t-1)\varphi(n_1)=6$. Therefore, we have $2t-1=1$, $\varphi(n_1)=6$, and $2t-1=3$, $\varphi(n_1)=2$. When $2t-1=1$, $\varphi(n_1)=6$, we have $t=1$, $n_1=7,9,14,18$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=847,1089,1694,2178$. After checking, we find that $n=847,1089,1694,2178$ is a solution to Equation (1). When $2t-1=3$, $\varphi(n_1)=2$, we have $t=2$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=363,484,726$. After checking, we find that $n=363,484,726$ is a solution to Equation (1).

When $q=13$, for Equation (4), we have $156(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(13^{32}\right)=2\times 390$, which means $(2t-1)\varphi(n_1)=5$. Therefore, we have $2t-1=5$, $\varphi(n_1)=1$, and we have $t=3$, $n_1=1,2$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=169,338$. After checking, we find that both $n=169$ and $n=338$ are solutions to Equation (1).

When $q=17$, for Equation (4), we have $272(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(17^{32}\right)=2\times 527$. This gives $8(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=19$, for Equation (4), we have $342(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(19^{32}\right)=2\times 589$. This gives $9(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=23$, for Equation (4), we have $506(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(23^{32}\right)=2\times 713$. This gives $11(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=29$, for Equation (4), we have $812(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(29^{32}\right)=2\times 899$. This gives $14(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q=31$, for Equation (4), we have $930(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(31^{32}\right)=2\times 961$. This gives $15(2t-1)\varphi(n_1)=31$, which has no solution for Equation (1).

When $q \geq 37$, for Equation (4), we have $q(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{32})=2\times 32q$, which means $(2t-1)\varphi(n_1)(q-1)=64$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 3. When $\alpha=3$, we have $q^2(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{48})$.

When $q=2$, for Equation (4), we have $4(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{48})=2\times 52$, which means $(2t-1)\varphi(n_1)=26$. Therefore, we have $2t-1=1$, $\varphi(n_1)=26$, and $2t-1=13$, $\varphi(n_1)=2$. When $2t-1=1$, $\varphi(n_1)=26$, we have $t=1$. However, since 26 is not an Euler quotient, there is no solution for $\varphi(n_1)=26$. When $2t-1=13$, $\varphi(n_1)=2$, we have $t=7$, $n_1=3,4,6$. According to $n=q^\alpha n_1$, $(q,n_1)=1$, we have $n=24$. After checking, we find that $n=24$ is a solution to Equation (1).

When $q=3$, for Equation (4), we have $18(2t-1)\varphi(n_1)=2S(n^{16})=2S(3^{48})=2\times 99$, which means $(2t-1)\varphi(n_1)=11$. Therefore, we have $2t-1=11$, $\varphi(n_1)=1$, and we have $t=6$, $n_1=1,2$. According to $n=q^\alpha n_1$, $(q,n_1)=1$, we have $n=27,54$. After checking, we find that both $n=27$ and $n=54$ are solutions to Equation (1).

When $q=5$, for Equation (4), we have $100(2t-1)\varphi(n_1)=2S(n^{16})=2S(5^{48})=2\times 200$, which means $(2t-1)\varphi(n_1)=4$. Therefore, we have $2t-1=1$, $\varphi(n_1)=4$, and we have $t=1$, $n_1=5,8,10,12$. According to $n=q^\alpha n_1$, $(q,n_1)=1$, we have $n=1000,1500$. After checking, we find that both $n=1000$ and $n=1500$ are solutions to Equation (1).

When $q=7$, for Equation (4), we have $294(2t-1)\varphi(n_1)=2S(n^{16})=2S(7^{48})=2\times 294$, which means $(2t-1)\varphi(n_1)=2$. Therefore, we have $2t-1=1$, $\varphi(n_1)=2$, and we have $t=1$, $n_1=3,4,6$. According to $n=q^\alpha n_1$, $(q,n_1)=1$, we have $n=1029,1372,2058$. After checking, we find that $n=1029,1372,2058$ is a solution to to Equation (1).

When $q \geq 11$, for Equation (4), we have $q^2(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{48})\leq 96q$, which means $q(q-1)(2t-1)\varphi(n_1)\leq 96$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 4. When $\alpha=4$, we have $q^3(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{64})$.

When $q=2$, for Equation (4), we have $8(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{64})=2\times 66$, which means $2(2t-1)\varphi(n_1)=33$. This equation has no solution for Equation (1).

When $q=3$, for Equation (4), we have $54(2t-1)\varphi(n_1)=2S(n^{16})=2S(3^{64})=2\times 135$, which means $(2t-1)\varphi(n_1)=5$. Therefore, we have $2t-1=5$, $\varphi(n_1)=1$, and we have $t=3$, $n_1=1,2$. According to $n=q^\alpha n_1$, $(q,n_1)=1$, we have $n=81,162$. After checking, we find that both $n=81$ and $n=162$ are solutions to Equation (1).

When $q=5$, for Equation (4), we have $500(2t-1)\varphi(n_1)=2S(n^{16})=2S(5^{64})=2\times 260$, which means $25(2t-1)\varphi(n_1)=26$. This equation has no solution for Equation (1).

When $q \geq 7$, for Equation (4), we have $q^3(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{64})\leq 128q$, which means $q^2(q-1)(2t-1)\varphi(n_1)\leq 128$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 5. When $\alpha=5$, we have $q^4(q-1)(2t-1)\varphi(n_1)=2S(n^{16})=2S(q^{80})$.

When $q=2$, for Equation (4), we have $16(2t-1)\varphi(n_1)=2S(n^{16})=2S(2^{80})=2\times 84$, which means $2(2t-1)\varphi(n_1)=21$. This equation has no solution for Equation (1).

When $q=3$, for Equation (4), we have $162(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(3^{80}\right)=2\times 162$, which means $(2t-1)\varphi(n_1)=2$. Therefore, we have $2t-1=1$, $\varphi(n_1)=2$, and we have $t=1$, $n_1=3,4,6$. According to $n=q^{\alpha}n_1$, $(q,n_1)=1$, we have $n=972$. After checking, we find that $n=972$ is a solution to Equation (1).

When $q\geq 5$, for Equation (4), we have $q^4(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{80}\right)\leq 160q$, which means $q^3(q-1)(2t-1)\varphi(n_1)\leq 160$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 6. When $\alpha=6$, we have $q^5(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{96}\right)$.

When $q=2$, for Equation (4), we have $32(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{96}\right)=2\times 100$, which means $4(2t-1)\varphi(n_1)=25$. This equation has no solution for Equation (1).

When $q=3$, for Equation (4), we have $486(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(3^{96}\right)=2\times 198$, which means $27(2t-1)\varphi(n_1)=22$. This equation has no solution for Equation (1).

When $q\geq 5$, for Equation (4), we have $q^5(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{96}\right)\leq 192q$, which means $q^4(q-1)(2t-1)\varphi(n_1)\leq 192$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 7. When $\alpha=7$, we have $q^6(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{112}\right)$.

When $q=2$, for Equation (4), we have $64(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{112}\right)=2\times 116$, which means $8(2t-1)\varphi(n_1)=29$. This equation has no solution for Equation (1).

When $q\geq 3$, for Equation (4), we have $q^6(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{112}\right)\leq 224q$, which means $q^5(q-1)(2t-1)\varphi(n_1)\leq 224$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 8. When $\alpha=8$, we have $q^7(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{128}\right)$.

When $q=2$, for Equation (4), we have $128(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{128}\right)=2\times 130$, which means $32(2t-1)\varphi(n_1)=65$. This equation has no solution for Equation (1).

When $q\geq 3$, for Equation (4), we have $q^7(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{128}\right)\leq 256q$, which means $q^6(q-1)(2t-1)\varphi(n_1)\leq 256$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 9. When $\alpha=9$, we have $q^8(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{144}\right)$.

When $q=2$, for Equation (4), we have $256(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{144}\right)=2\times 148$, which means $32(2t-1)\varphi(n_1)=37$. This equation has no solution for Equation (1).

When $q\geq 3$, for Equation (4), we have $q^8(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{144}\right)\leq 288q$, which means $q^7(q-1)(2t-1)\varphi(n_1)\leq 288$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 10. When $\alpha=10$, we have $q^9(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{160}\right)$.

When $q=2$, for Equation (4), we have $512(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(2^{160}\right)=2\times 164$, which means $64(2t-1)\varphi(n_1)=41$. This equation has no solution for Equation (1).

When $q\geq 3$, for Equation (4), we have $q^9(q-1)(2t-1)\varphi(n_1)=2S\left(n^{16}\right)=2S\left(q^{160}\right)\leq 320q$, which means $q^8(q-1)(2t-1)\varphi(n_1)\leq 320$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Case 11. When $\alpha\geq 11$, for $q\geq 2$, we have $q^{\alpha-1}(q-1)(2t-1)\varphi(n_1)=2S\left(q^{16\alpha}\right)\leq 32\alpha q$, which means $q^{\alpha-2}(q-1)(2t-1)\varphi(n_1)\leq 32\alpha$. It can be easily seen that this equation is not satisfied. Therefore, Equation (1) has no solution in this case.

Combining the discussions in the 11 cases above, we can conclude Theorem 1. The proof is completed.

4. Conclusion

In this study, we investigated equations of the form tφ(n)=φ2(n)+S(n), where the Euler function φ(n), Smarandache function S(n), and generalized Euler function φe(n) are involved, with k and n belonging to the set of positive integers (Z+). By employing piecewise classification and combining the properties of these three functions with elementary methods, we determined the values of k that yield positive integer solutions for the corresponding equations when n=16, and provided all positive integer solutions for these equations. Furthermore, we demonstrated that the solvability problem of equations of the form tφ(n)=φ2(n)+S(n) for other positive integer values of n∈Z+ can also be addressed using the methodology described in this paper. This research contributes to the understanding of the relationships between the Smarandache function and the Euler function, as well as the generalized Euler function, in terms of their properties and behavior. Further investigations can be conducted to explore additional aspects and applications of these equations in number theory.

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