Properties of a Hexagon Circumscribed to a Circle

Prof. Ion Pătrașcu
Frații Buzești College, Craiova, Romania

Dr. Florentin Smarandache
University of New Mexico, Gallup, NM 87301, USA

In this paper we analyze and prove two properties of a hexagon circumscribed to a circle:

Property 1.
If $ABCDEF$ is a hexagon circumscribed to a circle with the center in O, tangent to the sides AB, BC, CD, DE, EF, FA respectively in A', B', C', D', E', F', and if the lines of the triplet formed from two lines that belong to the set $\{AD, BE, CF\}$ and a line that belongs to the set $\{A'D', B'E', C'F'\}$ are concurrent, then the lines $AD, BE, CF, A'D', B'E', C'F'$ are concurrent.

Property 2.
If $ABCDEF$ is a hexagon circumscribed to a circle with the center in O, tangent to the sides AB, BC, CD, DE, EF, FA respectively in A', B', C', D', E', F', such that the hexagon $A'B'C'D'E'F'$ is circumscribable, then the lines $AD, BE, CF, A'D', B'E', C'F'$ are concurrent.
To prove these propositions we'll use:

Lemma 1 (Brianchon's Theorem)
If $ABCDEF$ is a hexagon circumscribable then the lines AD, BE, CF are concurrent.

Lemma 2
If $ABCDEF$ is a hexagon circumscribed to a circle tangent to the sides AB, BC, CD, DE, EF, FA respectively in A', B', C', D', E', F', such that $A'D' \cap C'F' = \{A_o\}$, $B'E' \cap A'D' = \{B_o\}$, $C'F' \cap B'E' = \{C_o\}$, then $A_o \in AD,\ B_o \in BE,\ C_o \in CF$.

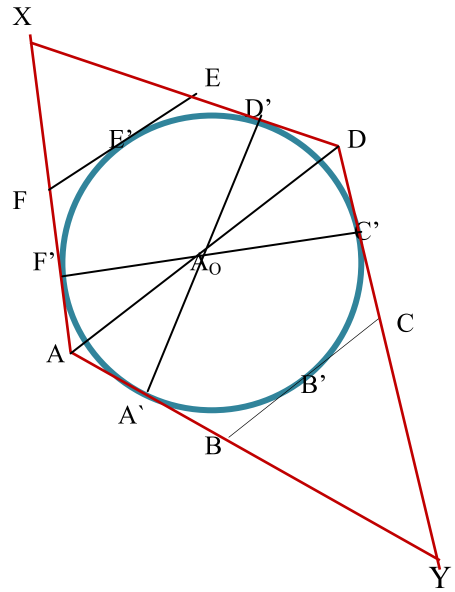

Fig. 1

Proof of Lemma 2

We note $\{X\} = AF \cap DE$ and $\{Y\} = AB \cap DC$ (see figure 1).

In the quadrilateral $XAYD$ circumscribed, the Newton’s theorem gives that the lines AD, $A'D'$, $C'F'$ and XY are concurrent, therefore $A_O \in AD$.

Similarly, is proven that $B_O \in BE$ and that $C_O \in CF$

Proof of Property 1

We suppose that AD, BE and $A'D'$ are concurrent in the point I (see fig. 2).

We denote $\{X\} = AF \cap DE$ and $\{Y\} = AB \cap DC$, we apply Newton’s theorem in the quadrilateral $XAYD$, it results that the line $C'F'$ also passes through I.

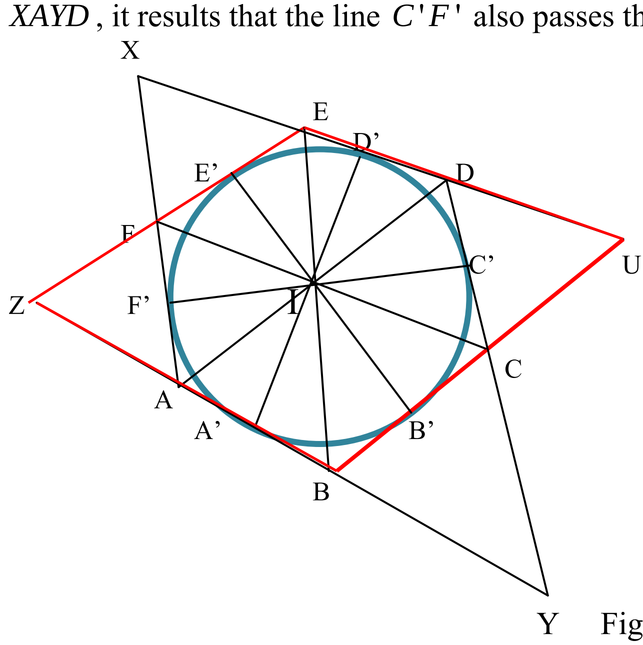

Fig. 2

On the other side from Lemma 1 it results that CF passes through I.

We note $\{Z\} = EF \cap AB$ and $\{U\} = BC \cap ED$ in the circumscribed quadrilateral $EZBU$. Newton's theorem shows that the lines BE, ZU, $B'E'$ and $A'D'$ are concurrent. Because BE and $A'D'$ pass through I, it results that also $B'E'$ passes through I, and the proof is complete.

Observation

There exist circumscribable hexagons $ABCDEF$ in which the six lines from above are concurrent (a banal example is the regular hexagon).

Proof of Property 2

From Lemma 1 we obtain that $AD \cap BE \cap CF = \{I\}$ and $A'D' \cap B'E' \cap C'F' = \{I'\}$. From Lemma 2 it results that $I' \in AD$ and $I' \in BE$, because $AD \cap BE = \{I\}$, we obtain that $I = I'$ and consequently all six lines are concurrent.

Reference:

Florentin Smarandache, "Problems with and without…problems!", Somipress, Fés, Morocco, 1983.