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Quantum-treesoft set and quantum-forestsoft set

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Abstract. Fuzzy Sets [17, 44], Neutrosophic Sets [30], and Plithogenic Sets [33] provide powerful frameworks for modeling uncertainty across diverse domains. The Soft Set, introduced by Molodtsov [22], associates each parameter with a subset of a universe, enabling flexible approximations. Building on this idea, researchers have developed Fuzzy Soft Sets [16], Neutrosophic Soft Sets [15], HyperSoft and SuperHyperSoft Sets [34, 35], TreeSoft Sets [37], ForestSoft Sets [20], and GraphicSoft Sets [7].

A TreeSoft Set arranges parameters in a hierarchical tree, while a ForestSoft Set unites multiple tree mappings under one framework. Separately, the Quantum-Soft Set [18] represents each parameter as a normalized quantum superposition, encoding membership through amplitude coefficients and measurement probabilities. In this paper, we introduce Quantum TreeSoft Sets and Quantum ForestSoft Sets, which integrate hierarchical and forest-structured Soft Sets with the quantum superposition paradigm. These new constructs aim to enrich the theoretical foundations and practical applications of Soft Set Theory at the intersection with quantum information science.

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1. Introduction

1.1. Soft sets and their extensions

The challenge of modeling uncertainty has led to a variety of mathematical frameworks, including fuzzy sets [44], hyperfuzzy sets [32], superhyperfuzzy sets [32], rough sets [15], hyperrough sets [6], intuitionistic fuzzy sets [2], picture fuzzy sets [4], hesitant fuzzy sets [39], neutrosophic sets [30], and plithogenic sets [33].

Among these, soft sets [22, 21] are notable for associating each parameter with a subset of a universal set, thereby offering a flexible tool for decision-making under imprecision. Building on this idea, researchers have proposed numerous generalizations:

- *Hypersoft Sets* [34] and *SuperHypersoft Sets* [35] , which add layers of parameter interactions;
- *Expert Soft Sets* [40] and *Hypersoft Expert Sets* [10, 12], integrating expert judgments;
- *TreeSoft Sets* [37, 24], organizing parameters into hierarchies, and *ForestSoft Sets* [7], which combine multiple trees into a cohesive framework.

TreeSoft and ForestSoft Sets have proven useful across decision analysis [11], topology [23], graph theory [29], risk assessment [10], and algebraic structures. TreeSoft Sets handle hierarchically structured parameters to enable multi-level condition filtering and attribute-relationship visualization, while ForestSoft Sets integrate multiple trees to support comprehensive evaluation across diverse attribute domains. Consequently, advancing soft set theory and its variants remains an important endeavor.

In addition, the *Quantum-Soft Set* [18] enriches the classical soft set by mapping each parameter to a normalized quantum superposition on a Hilbert space (cf.[43]), capturing membership via amplitude coefficients and measurement probabilities. This quantum perspective opens new avenues at the intersection of soft set theory and quantum information science. Moreover, given the growing prominence of fields such as quantum physics and quantum information science, research on quantum soft sets is both timely and critically important.

1.2. Our Contributions: Quantum Treesoft Set and the Quantum Forestsoft Set

Although the quantum-soft set has been defined, its integration into richer soft-set frameworks remains unexplored. In this paper, we introduce and formalize the *Quantum Treesoft Set* and the *Quantum Forestsoft Set*, deriving their fundamental properties and structural features. By combining the quantum-soft paradigm with Treesoft and Forestsoft architectures, we aim to drive further advances in set theory, decision-making methodologies, topological analysis, algebraic modeling, and quantum applications.

2. Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. Throughout this paper, all sets and structures are assumed to be finite. Unless otherwise stated, the symbol n denotes a non-negative integer. The empty set is considered to be a subset of any set. For the basic operations related to each concept, the reader is referred to the corresponding references.

2.1. Soft set and treeSoft set

A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U. This provides a flexible framework for approximating objects within U [14, 22, 21]. A *TreeSoft Set* is a mapping from subsets of a hierarchical, tree-like parameter structure $\mathrm{Tree}(A)$ to subsets of a universal set U. This structure supports multi-level attributes for more refined and detailed analyses [28, 1, 26].

The definitions of Soft Set and TreeSoft Set are provided below.

Definition 2.1 (Soft Set) [21]. Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Definition 2.2 (Treesoft) [36]. Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geqslant 1$, where each attribute A_i (for $1 \leqslant i \leqslant n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as Tree(A), with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where $\text{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

Example 2.3 (Thermoelectric Materials via TreeSoft Set). Let

$$U = \{\text{Mat}_1, \ldots, \text{Mat}_N\},$$
$$H = \{\text{Mat}_i \in U \mid \text{Mat}_i \text{ has both electrical and thermal data}\}.$$

Define a two-level tree

$$A = \{A_1, A_2\}, \quad A_1 = \{\text{Res}, \text{Car}\},\ A_2 = \{\text{Therm}, \text{Seebeck}\},$$

each refined into $\{\text{ Low}, \text{Med}, \text{High}\}$. Then

$$\begin{aligned}\text{Tree}(A) = \{A_1, A_2\} &\cup \{\text{Res}, \text{Car}, \text{Therm}, \text{Seebeck}\} \\ &\cup \{\text{their Low/Med/High nodes}\}.\end{aligned}$$

For $X \subseteq \text{Tree}(A)$, set

$$F(X) = \{\text{Mat}_i \in H \mid \text{Mat}_i \text{ meets every attribute in } X\}.$$

E.g.,

$$F(\{\text{Res}_{\text{Low}}\}) = \{\text{Mat}_i \mid \rho(\text{Mat}_i) \text{ is low}\},$$
$$F(\{\text{Res}_{\text{Low}}, \text{Seebeck}_{\text{High}}\}) = \{\text{Mat}_i \mid \rho(\text{Mat}_i) \text{ low and } S(\text{Mat}_i) \text{ high}\}.$$

In general $F(\bigcup_k X_k) = \bigcap_k F(X_k)$, since all chosen criteria must hold.

2.2. ForestSoft set

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets (cf. [20, 38, 19, 8, 13]).

Definition 2.4 (ForestSoft Set) [7]. Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\{\, F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H) \,\}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree $\mathrm{Tree}(A^{(t)})$ of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\mathrm{Forest}\big(\{A^{(t)}\}_{t \in T}\big) \;=\; \bigsqcup_{t \in T} \mathrm{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\mathrm{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\mathrm{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} F_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\mathrm{Tree}(A^{(t)})$.

Example 2.5 (Multifunctional Materials with ForestSoft Set). Let

$$U = \{\mathrm{Mat}_1, \ldots, \mathrm{Mat}_N\},$$
$$H = \{\mathrm{Mat}_i \in U \mid \text{thermoelectric and mechanical data exist}\}.$$

Two attribute trees:

$$\mathrm{Tree}(A^{(1)}) = \{E, S\},\; E = \{\mathrm{Res}, \mathrm{Car}\},\; S = \{\mathrm{Therm}, \mathrm{Seebeck}\},$$

$$\text{Tree}(A^{(2)}) = \{M, T\},$$
$$M = \{\text{YoungMod}, \text{Poisson}\},$$
$$T = \{\text{Hardness}, \text{Toughness}\}.$$

Form the forest $\mathrm{F} = \text{Tree}(A^{(1)}) \sqcup \text{Tree}(A^{(2)})$. Any $X \subseteq \mathrm{F}$ splits $X = X_1 \cup X_2$. Define

$$F_1 : P(\text{Tree}(A^{(1)})) \to P(H), \quad F_2 : P(\text{Tree}(A^{(2)})) \to P(H),$$

and then

$$\mathbf{F}(X) = F_1(X_1) \ \cup \ F_2(X_2).$$

For instance,

$$\mathbf{F}(\{\text{Res}_{\text{Low}}, \text{Seebeck}_{\text{Med}}\} \cup \{\text{YoungMod}_{\text{High}}, \text{Toughness}_{\text{Med}}\})$$

$$= F_1(\{\text{Res}_{\text{Low}}, \text{Seebeck}_{\text{Med}}\}) \cup F_2(\{\text{YoungMod}_{\text{High}}, \text{Toughness}_{\text{Med}}\}).$$

This $\mathbf{F}$ is a ForestSoft Set on F.

2.3. Quantum-soft set

A Hilbert space is a complete inner product vector space over real or complex numbers, enabling geometric analysis of functions [43, 3, 5]. Hilbert spaces are used in various contexts such as quantum theory. A Quantum-Soft Set maps each classical attribute to a normalized quantum superposition over elements [18]. The definition is stated as follows.

Definition 2.6 (Real Hilbert Space) (cf.[43, 3, 5]. A *real Hilbert space* is a pair $(H, \langle\cdot,\cdot\rangle)$ satisfying the following conditions:

1. H is a real vector space.

2. $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ is a function (called an *inner product*) fulfilling:

$$\text{(Positive-definiteness)} \quad \langle x, x\rangle > 0$$
$$\text{for all } x \in H\backslash\{0\}, \langle x, x\rangle = 0 \iff x = 0,$$
$$\text{(Symmetry)} \quad \langle x, y\rangle = \langle y, x\rangle$$
$$\text{for all } x, y \in H,$$

(Linearity in the first argument)

$$\langle \alpha x + \beta y,\ z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$$
$$\text{for all } \alpha, \beta \in \mathbb{R},\ x, y, z \in H.$$

3. The norm $\|\cdot\|$ induced by the inner product, defined by

$$\|x\| = \sqrt{\langle x, x\rangle} \quad \text{for } x \in H,$$

makes $(H, \|\cdot\|)$ into a complete metric space; that is, every Cauchy sequence in $(H, \|\cdot\|)$ converges to some limit in H.

Definition 2.7 (Complex Hilbert Space) (cf. [41]). A *complex Hilbert space* is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ satisfying the following conditions:

1. $\mathcal{H}$ is a vector space over $\mathbb{C}$.

2. $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a function (called a *Hermitian inner product*) fulfilling:

$$\text{(Positive-definiteness)} \quad \langle x, x\rangle > 0$$
$$\text{for all } x \in \mathcal{H}\backslash\{0\}, \langle x, x\rangle = 0 \iff x = 0,$$
$$\text{(Conjugate symmetry)} \quad \langle x, y\rangle = \overline{\langle y, x\rangle}$$
$$\text{for all } x, y \in \mathcal{H},$$
$$\text{(Sesquilinearity)} \quad \langle \alpha x + \beta y,\ z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$$
$$\text{for all } \alpha, \beta \in \mathbb{C},\ x, y, z \in \mathcal{H},$$

$$\langle x,\ \alpha\, y + \beta\, z \rangle \ = \ \overline{\alpha}\,\langle x, y\rangle \ + \ \overline{\beta}\,\langle x, z\rangle$$

for all $\alpha, \beta \in \mathbb{C},\ x, y, z \in \mathcal{H}$.

3. The norm $\|\cdot\|$ induced by the inner product, defined by

$$\|x\| \ = \ \sqrt{\langle x, x\rangle} \quad \text{for } x \in \mathcal{H},$$

makes $(\mathcal{H}, \|\cdot\|)$ into a complete metric space; that is, every Cauchy sequence in $(\mathcal{H}, \|\cdot\|)$ converges to some limit in $\mathcal{H}$.

Definition 2.8 (Quantum-Soft Set) [18]. Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe of discourse, and let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of parameters (attributes). Denote by $\mathcal{H}(U)$ the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A *Quantum-Soft Set* over (U, A) is a mapping

$$F : A \ \longrightarrow \ \mathcal{H}(U), \quad a_j \ \mapsto \ |\psi_j\rangle = \sum_{i=1}^{n} \alpha_{ij}\ |u_i\rangle\,,$$

subject to the normalization condition

$$\sum_{i=1}^{n} \left|\alpha_{ij}\right|^2 \ = \ 1, \quad j = 1, 2, \ldots, m.$$

Here, $\alpha_{ij} \in \mathbb{C}$ is the amplitude of element u_i under attribute a_j. Measuring the quantum state $|\psi_j\rangle$ in the basis $\{|u_i\rangle\}$ yields $|u_i\rangle$ with probability

$$P\big(u_i \mid a_j\big) \ = \ \left|\alpha_{ij}\right|^2.$$

Example 2.9 (Photon Polarization under Filter Angles). Let

$$U = \{|H\rangle\,, |V\rangle\}, \quad A = \{a_1, a_2, a_3\}, \quad \theta_1 = 0°,\ \theta_2 = 45°,\ \theta_3 = 30°.$$

Define

$$F : A \longrightarrow \mathcal{H}(U), \quad F(a_j) = |\psi_j\rangle = \cos\theta_j\ |H\rangle + \sin\theta_j\ |V\rangle\,,$$

so each $|\psi_j\rangle$ is normalized. Measuring $|\psi_j\rangle$ in $\{|H\rangle\,, |V\rangle\}$ yields

$$P(H|a_j) = \cos^2\theta_j, \quad P(V|a_j) = \sin^2\theta_j.$$

Thus F encodes the photon's polarization state for polarizer angles $0°, 45°, 30°$.

3. Result: Quantum-treesoft set

We now introduce the notion of a *Quantum-TreeSoft Set*, which simultaneously generalizes the concepts of a Quantum-Soft Set and a TreeSoft Set.

Definition 3.1 (Quantum-TreeSoft Set). Let $U = \{u_1, u_2, \ldots, u_n\}$ be a finite universe, and let

$$A = \{A_1, A_2, \ldots, A_m\}$$

be a finite set of first-level attributes. As in Definition 2.2, let $\text{Tree}(A)$ denote the finite rooted tree of attributes obtained by iterating sub-attribute expansions

$$A_i = \{\, A_{i,1}, A_{i,2}, \ldots \}, \quad A_{i,j} = \{\, A_{i,j,1}, A_{i,j,2}, \ldots \}, \quad \ldots$$

up to depth m. Denote by $P\big(\text{Tree}(A)\big)$ the power set of all nodes (including leaves) in the attribute-tree.

Let $\mathcal{H}(U)$ be the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A *Quantum-TreeSoft Set* over $(U, \text{Tree}(A))$ is a mapping

$$F \;:\; P\big(\text{Tree}(A)\big) \;\longrightarrow\; \mathcal{H}(U),$$

subject to the following normalization condition:

For each $X \in P\big(\text{Tree}(A)\big)$,

$$\begin{aligned} F(X) &= |\psi_X\rangle \\ &= \sum_{i=1}^{n} \alpha_i(X)\,|u_i\rangle, \end{aligned}$$

$$\sum_{i=1}^{n} \left|\alpha_i(X)\right|^2 \;=\; 1.$$

Here, $\alpha_i(X) \in \mathbb{C}$ is the amplitude associated with element u_i under the tree-node subset X. Measuring the state $|\psi_X\rangle$ in the basis $\{|u_i\rangle\}$ yields outcome $|u_i\rangle$ with probability

$$P\big(u_i \;\big|\; X\big) = \big|\alpha_i(X)\big|^2.$$

Remark 3.2.

(a) If $\mathrm{Tree}(A)$ has depth 1, so that $\mathrm{Tree}(A) \cong A$ (i.e., no sub-attributes beyond level 1), then

$$P\big(\mathrm{Tree}(A)\big) \;=\; P(A),$$

and a Quantum-TreeSoft Set F restricts to a mapping $\widetilde{F} : P(A) \to \mathcal{H}(U)$. In particular, if one only evaluates F on singleton sets $\{\, A_j\} \subseteq A$, one recovers a Quantum-Soft Set as in Definition 3.1 by setting $|\psi_{A_j}\rangle = F(\{\, A_j\})$.

(b) Conversely, if for every $X \subseteq \mathrm{Tree}(A)$ the amplitude vector $(\alpha_1(X), \ldots, \alpha_n(X))$ has support exactly equal to a classical subset $H(X) \subseteq U$ (i.e., $\alpha_i(X) \neq 0$ if and only if $u_i \in H(X)$, and we choose $\alpha_i(X) = 1/\sqrt{|H(X)|}$ for each $u_i \in H(X)$), then measuring $|\psi_X\rangle$ yields uniformly distributed membership in $H(X)$. In that case, one can identify $\widetilde{H} : P\big(\mathrm{Tree}(A)\big) \to P(U)$ via

$$\widetilde{H}(X) \;=\; \Big\{ u_i \in U \;\Big|\; \alpha_i(X) \neq 0 \Big\}.$$

This $\widetilde{H}$ is exactly a TreeSoft Set as in Definition 2.2, and thus a Quantum-TreeSoft Set whose amplitudes are restricted to such uniform-support states recovers a classical TreeSoft Set.

Example 3.3 (Photon Polarization and Spatial Mode). Let

$$U = \{|H,0\rangle, |H,1\rangle, |V,0\rangle, |V,1\rangle\},$$
$$\mathrm{Tree}(A) = \{\mathsf{Pol}, H, V\} \sqcup \{\mathrm{Mode}, 0, 1\}.$$

Any $X \subseteq \mathrm{Tree}(A)$ splits as $X = X_P \cup X_M$ with $X_P \subseteq \{\mathsf{Pol}, H, V\}$, $X_M \subseteq \{\mathrm{Mode}, 0, 1\}$. Define

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{4} \alpha_i(X)\,|u_i\rangle, \quad \sum_i |\alpha_i(X)|^2 = 1,$$

where nonzero $\alpha_i(X)$ reflect the chosen nodes. For instance,

$$|\psi_{\{H,1\}}\rangle = |H,1\rangle,$$
$$|\psi_{\{H\}}\rangle = \tfrac{1}{\sqrt{2}}(|H,0\rangle + |H,1\rangle),$$
$$|\psi_{\varnothing}\rangle = \tfrac{1}{2}\sum_{u\in U} |u\rangle.$$

Measuring $|\psi_X\rangle$ in $\{\,|P,m\rangle\}$ yields $|u_i\rangle$ with probability $|\alpha_i(X)|^2$. This construction defines a valid Quantum-TreeSoft Set on the two-level attribute tree.

Theorem 3.4 (Generalization Property). *Let $F : P\big(\mathrm{Tree}(A)\big) \to \mathcal{H}(U)$ be a Quantum-TreeSoft Set as in Definition 3.1. Then:*

(i) **(Reduction to Quantum-Soft Set).** *If $\mathrm{Tree}(A)$ has no proper sub-attributes (i.e., $\mathrm{Tree}(A) = A$ at depth 1), then the restriction*

$$F_{\mathrm{flat}} : P(A) \longrightarrow \mathcal{H}(U), \quad X \mapsto F(X)$$

satisfies that, upon further restricting to singleton subsets $\{\,a_j\} \subseteq A$, the mapping $\{\,a_j\} \mapsto F(\{\,a_j\})$ is exactly the definition of a Quantum-Soft Set over (U, A).

(ii) **(Recovery of TreeSoft Set).** *Suppose that for each $X \subseteq \mathrm{Tree}(A)$, the state $|\psi_X\rangle = F(X)$ satisfies*

$$\begin{aligned} \alpha_i(X) \neq 0 \quad &\Longleftrightarrow \quad u_i \in H(X), \text{and} \\ \alpha_i(X) \; &= \; \frac{1}{\sqrt{|H(X)|}} \\ &\text{for } u_i \in H(X), \end{aligned}$$

where $H(X) \subseteq U$ *is some classical subset of* U. *Define* $\widetilde{H} : P\big(\text{Tree}(A)\big) \to P(U)$ *by* $\widetilde{H}(X) = H(X)$. *Then* $\widetilde{H}$ *is a TreeSoft Set in the sense of Definition 2.2.*

Proof.

(i) If Tree(A) has depth 1, then Tree$(A) = A$ and $P\big(\text{Tree}(A)\big) = P(A)$. By Definition 3.1, F is a map $P(A) \to \mathcal{H}(U)$. In particular, for each singleton $\{ A_j\} \subseteq A$,

$$F\big(\{ A_j\}\big) = \; |\psi_{\{A_j\}}\rangle \; = \; \sum_{i=1}^{n} \alpha_i\big(\{A_j\}\big) \; |u_i\rangle ,$$

$$\sum_{i=1}^{n} \big|\alpha_i(\{A_j\})\big|^2 = 1.$$

Setting $|\psi_j\rangle = F(\{ A_j\})$ for $j = 1, 2, \ldots, m$ recovers exactly a Quantum-Soft Set $\widetilde{F} : A \to \mathcal{H}(U)$, $\widetilde{F}(A_j) = |\psi_j\rangle$, as in Definition ..... Hence F restricted to singletons is a Quantum-Soft Set.

(ii) Suppose for each $X \subseteq$ Tree(A) we have

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X) \; |u_i\rangle , \quad \sum_{i=1}^{n} \big|\alpha_i(X)\big|^2 = 1,$$

and the nonzero amplitudes $\alpha_i(X)$ occur precisely on indices i such that u_i lies in some classical subset $H(X) \subseteq U$. Further assume $\alpha_i(X) = 1/\sqrt{|H(X)|}$ whenever $u_i \in H(X)$. Define $\widetilde{H} : P\big(\text{Tree}(A)\big) \to P(U)$ by $\widetilde{H}(X) = H(X)$.

We must check that $\widetilde{H}$ satisfies the axioms of a TreeSoft Set (Definition 2.2). Namely:

(a) $\widetilde{H}$ is well-defined: for each $X \subseteq \text{Tree}(A)$, $H(X)$ is determined uniquely as the support of $|\psi_X\rangle$.

(b) For each X, $\widetilde{H}(X) \subseteq U$ is some classical subset. Thus $\widetilde{H}$ indeed takes values in $P(U)$.

(c) If $X \cap \text{Tree}(A^{(j)}) \neq \varnothing$ for finitely many sub-trees $\text{Tree}(A^{(j)})$, then measuring $|\psi_X\rangle$ uniformly yields elements in the corresponding union of the classical images. Concretely, if X decomposes as $X = \bigcup_j X_j$ with $X_j \subseteq \text{Tree}(A^{(j)})$ across disjoint sub-trees, one checks

$$H(X) \;=\; \bigcup_j H\bigl(X_j\bigr),$$

by construction of amplitudes: the nonzero amplitudes of $|\psi_X\rangle$ must coincide with those coming from each $|\psi_{X_j}\rangle$. Thus $\widetilde{H}$ behaves exactly as the union of the corresponding TreeSoft Set mappings on each component sub-tree.

(d) Therefore $\widetilde{H}$ satisfies the requirement of being the (point-wise) union of the images under each TreeSoft component, as in Definition 2.4. In particular, if $\text{Tree}(A)$ itself is a single tree, then this reduces exactly to the axioms for a TreeSoft Set.

Hence $\widetilde{H}$ is a TreeSoft Set. □

Theorem 3.5 (Normalization Preservation). *For every subset $X \subseteq \text{Tree}(A)$, the state $|\psi_X\rangle = F(X)$ satisfies*

$$\bigl\||\psi_X\rangle\bigr\| \;=\; 1.$$

Proof. By Definition 3.1, each image $F(X)$ is given by

$$|\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle, \quad \sum_{i=1}^{n} \big|\alpha_i(X)\big|^2 = 1.$$

Hence

$$\|\,|\psi_X\rangle\,\| = \sqrt{\langle \psi_X \mid \psi_X \rangle} = \sqrt{\sum_{i=1}^{n} \big|\alpha_i(X)\big|^2} = \sqrt{1} = 1,$$

as required. □

Theorem 3.6 (Conjugate Symmetry of Inner Product). *For any two subsets $X, Y \subseteq \mathrm{Tree}(A)$,*

$$\langle \psi_X \mid \psi_Y \rangle = \overline{\langle \psi_Y \mid \psi_X \rangle}.$$

Proof. Write

$$|\psi_X\rangle = \sum_i \alpha_i(X)\,|u_i\rangle, \quad |\psi_Y\rangle = \sum_i \alpha_i(Y)\,|u_i\rangle.$$

Then by linearity of the inner product in the first slot and conjugate-linearity in the second,

$$\langle \psi_X \mid \psi_Y \rangle = \sum_{i=1}^{n} \overline{\alpha_i(X)}\,\alpha_i(Y),$$

while

$$\langle \psi_Y \mid \psi_X \rangle = \sum_{i=1}^{n} \overline{\alpha_i(Y)}\,\alpha_i(X) = \overline{\sum_{i=1}^{n} \overline{\alpha_i(X)}\,\alpha_i(Y)} = \overline{\langle \psi_X \mid \psi_Y \rangle},$$

establishing the conjugate-symmetry property. □

Theorem 3.7 (Orthogonality of Disjoint Supports). *Suppose that for two subsets $X, Y \subseteq \mathrm{Tree}(A)$ the supports of their amplitude vectors are disjoint:*

$$\{\, i \mid \alpha_i(X) \neq 0\} \ \cap\ \{\, i \mid \alpha_i(Y) \neq 0\} = \varnothing.$$

Then

$$\langle \psi_X \mid \psi_Y \rangle = 0,$$

i.e. $|\psi_X\rangle$ *and* $|\psi_Y\rangle$ *are orthogonal.*

Proof. Write

$$|\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\,|u_i\rangle, \quad |\psi_Y\rangle = \sum_{i=1}^{n} \alpha_i(Y)\,|u_i\rangle.$$

By definition of support, $\alpha_i(X)\,\alpha_i(Y) = 0$ for every $i = 1, \ldots, n$. Therefore

$$\langle \psi_X \mid \psi_Y \rangle = \sum_{i=1}^{n} \overline{\alpha_i(X)}\,\alpha_i(Y) = \sum_{i=1}^{n} 0 = 0.$$

Thus the two states are orthogonal whenever their nonzero-amplitude indices do not overlap. □

Theorem 3.8 (Inner–Product Bound). *For any* $X, Y \subseteq \mathrm{Tree}(A)$*,*

$$\left|\langle \psi_X \mid \psi_Y \rangle\right| \;\leqslant\; 1.$$

Proof. Using the Cauchy–Schwarz inequality in the Hilbert space $\mathcal{H}(U)$, we have

$$\left|\langle \psi_X \mid \psi_Y \rangle\right| \;\leqslant\; \|\psi_X\|\,\|\psi_Y\| =\; 1 \cdot 1 =\; 1,$$

where we used normalization theorem. Hence the inner product is always bounded by unity. □

4. Result: Quantum-forestsoft set

We now introduce the concept of a *Quantum-ForestSoft Set*, which simultaneously generalizes the notions of a Quantum-Soft Set, a Quantum-TreeSoft Set, and a ForestSoft Set.

Definition 4.1 (Attribute Forest). Let $\{\,\mathrm{Tree}(A^{(t)})\,\}_{t\in T}$ be a (finite or countable) collection of rooted attribute-trees, each constructed as in Definition 2.2. The *forest* of these trees is the disjoint union

$$\mathrm{Forest}\big(\{A^{(t)}\}_{t\in T}\big) \;=\; \bigsqcup_{t\in T} \mathrm{Tree}\big(A^{(t)}\big).$$

Denote by $P\big(\text{Forest}(\{A^{(t)}\})\big)$ the power set of all nodes (including leaves) in this forest.

Definition 4.2 (Quantum-ForestSoft Set). Let $U = \{u_1, u_2, \dots, u_n\}$ be a finite universe of discourse. Let

$$\mathcal{F} \;=\; \text{Forest}\big(\{A^{(t)}\}_{t\in T}\big)$$

be an attribute forest comprising the disjoint union of trees $\text{Tree}(A^{(t)})$ (for each $t \in T$), as in the preceding definition. We write

$$P(\mathcal{F}) \;=\; P\big(\text{Forest}(\{A^{(t)}\})\big)$$

for its power set.

Let $\mathcal{H}(U)$ be the n-dimensional complex Hilbert space with orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$. A *Quantum-ForestSoft Set* over $(U, \mathcal{F})$ is a mapping

$$F \;:\; P(\mathcal{F}) \;\longrightarrow\; \mathcal{H}(U),$$

satisfying the following normalization condition:

For each $X \in P(\mathcal{F})$,

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle\,,$$
$$\sum_{i=1}^{n} \big|\alpha_i(X)\big|^2 = 1.$$

Here, $\alpha_i(X) \in \mathbb{C}$ is the amplitude associated with element u_i under the forest-subset X. Measuring $|\psi_X\rangle$ in the basis $\{|u_i\rangle\}$ yields outcome $|u_i\rangle$ with probability

$$P\big(u_i \;\big|\; X\big) = \big|\alpha_i(X)\big|^2.$$

Remark 4.3.

(a) If the forest $\mathcal{F}$ consists of a single tree of depth 1, so that $\mathcal{F} = A$ is merely a flat set of first-level attributes, then $P(\mathcal{F}) = P(A)$. Restricting F to singleton subsets $\{ a_j\} \subseteq A$ recovers exactly a *Quantum-Soft Set* (Definition 2.8) via $|\psi_{a_j}\rangle = F(\{ a_j\})$.

(b) If each tree in the forest has arbitrary finite depth but we choose each vector $F(X)$ to have support equal to a classical subset $H_t(X) \subseteq U$ whenever $X \subseteq \mathrm{Tree}(A^{(t)})$, with uniform amplitudes $\frac{1}{\sqrt{|H_t(X)|}}$ on that support, then F restricted to each individual tree recovers a *TreeSoft Set* (Definition 2.2).

In particular, if $X \in P(\mathcal{F})$ decomposes into disjoint pieces $X = \bigsqcup_{j\in J} X_j$ with $X_j \subseteq \mathrm{Tree}(A^{(t_j)})$, then the support of $|\psi_X\rangle$ is the union $\bigcup_{j\in J} H_{t_j}(X_j) \subseteq U$, recovering the union property of a *ForestSoft Set* (Definition 2.4).

Example 4.4 (Photon Polarization and Path Encoding). Let

$$U = \{|H, A\rangle, |H, B\rangle, |V, A\rangle, |V, B\rangle\},$$
$$\mathcal{F} = \{\mathsf{Pol}, H, V\} \sqcup \{\mathrm{Path}, A, B\}.$$

Each subset $X \subseteq \mathcal{F}$ is $X = X_P \cup X_T$ with $X_P \subseteq \{\mathsf{Pol}, H, V\}$, $X_T \subseteq \{\mathrm{Path}, A, B\}$. We define

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{4} \alpha_i(X)\,|u_i\rangle, \quad \sum_i |\alpha_i(X)|^2 = 1,$$

where the nonzero amplitudes reflect the chosen attributes. For instance:

$$|\psi_{\{H,A\}}\rangle = |H,A\rangle\,,$$
$$|\psi_{\{H\}}\rangle = \tfrac{1}{\sqrt{2}}(|H,A\rangle + |H,B\rangle),$$
$$|\psi_{\{\mathrm{Path}\}}\rangle = \tfrac{1}{2}\sum_{u\in U}|u\rangle\,.$$

Measuring $|\psi_X\rangle$ yields $|u_i\rangle$ with probability $|\alpha_i(X)|^2$, realizing a Quantum-ForestSoft Set over $\mathcal{F}$.

Theorem 4.5 (Generalization of Quantum-Soft, Quantum-TreeSoft, and ForestSoft). *Let $F : P(\mathcal{F}) \to \mathcal{H}(U)$ be a Quantum-ForestSoft Set as in Definition 4.2. Then:*

(i) **(Reduction to Quantum-Soft Set).** *If every tree in $\mathcal{F}$ has depth 1, so that $\mathcal{F} = A$ is a single flat set of attributes, then $P(\mathcal{F}) = P(A)$, and restricting F to singleton subsets $\{a_j\} \subseteq A$ yields a Quantum-Soft Set*

$$\widetilde{F} : A \longrightarrow \mathcal{H}(U), \quad a_j \mapsto F(\{a_j\}).$$

(ii) **(Recovery of Quantum-TreeSoft Set).** *Suppose $\mathcal{F}$ consists of exactly one rooted tree* Tree(A) *(so $\mathcal{F} =$* Tree(A)*), and assume that for each $X \subseteq$* Tree(A)*, the state $|\psi_X\rangle = F(X)$ has support equal to a classical tree-soft set image $H(X) \subseteq U$ with uniform amplitudes $\frac{1}{\sqrt{|H(X)|}}$. Then the restricted mapping*

$$\widetilde{H} : P\big(\mathrm{Tree}(A)\big) \longrightarrow P(U), \quad X \mapsto H(X),$$

is a TreeSoft Set *(Definition 2.2), and F recovers the corresponding Quantum-TreeSoft Set of Definition 3.1.*

(iii) **(Recovery of ForestSoft Set).** *Suppose each tree $t \in T$ has an associated classical TreeSoft Set*

$$H_t : P\big(\mathrm{Tree}(A^{(t)})\big) \longrightarrow P(U),$$

and that for every $X \in P(\mathcal{F})$ *(where* $\mathcal{F} = \bigsqcup_{t \in T} \mathrm{Tree}(A^{(t)})$*) we have*

$$F(X) = |\psi_X\rangle \;=\; \sum_{i=1}^{n} \alpha_i(X)\; |u_i\rangle, \quad \sum_{i=1}^{n} \left|\alpha_i(X)\right|^2 = 1,$$

with the property that

$$\alpha_i(X) \neq 0 \quad \Longleftrightarrow \quad u_i \in \bigsqcup_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} H_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

and whenever u_i *belongs to that union,*

$$\alpha_i(X) = 1/\sqrt{\Big|\bigcup_t H_t(X \cap \mathrm{Tree}(A^{(t)}))\Big|}.$$

Then defining

$$\widetilde{H} : P(\mathcal{F}) \;\longrightarrow\; P(U), \quad X \;\mapsto\; \bigcup_{\substack{t \in T \\ X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} H_t\big(X \cap \mathrm{Tree}(A^{(t)})\big)$$

produces a ForestSoft Set *(Definition 2.4). In particular, F recovers the classical ForestSoft Set under this uniform-amplitude support assumption.*

Proof.

(i) If each tree in $\mathcal{F}$ has depth 1, then $\mathcal{F}$ is a single flat set A of first-level attributes. Hence $P(\mathcal{F}) = P(A)$. By Definition 4.2, F is a map $P(A) \to \mathcal{H}(U)$. For each singleton subset $\{a_j\} \subseteq A$, we have

$$F\big(\{a_j\}\big) \;=\; |\psi_{\{a_j\}}\rangle \;=\; \sum_{i=1}^{n} \alpha_i\big(\{a_j\}\big)\; |u_i\rangle, \quad \sum_{i=1}^{n} \left|\alpha_i(\{a_j\})\right|^2 = 1.$$

Defining $\widetilde{F} : A \to \mathcal{H}(U)$ by $\widetilde{F}(a_j) = F(\{a_j\})$ yields exactly a *QuantumSoft Set* as in Definition 2.8.

(ii) Suppose $\mathcal{F} = \mathrm{Tree}(A)$ is a single rooted tree. For each $X \subseteq \mathrm{Tree}(A)$, let

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\; |u_i\rangle, \quad \sum_{i=1}^{n} \left|\alpha_i(X)\right|^2 = 1,$$

and assume that the nonzero amplitudes $\alpha_i(X)$ occur if and only if $u_i \in H(X) \subseteq U$, where H is a classical TreeSoft Set on Tree(A). Furthermore, assume $\alpha_i(X) = 1/\sqrt{|H(X)|}$ for $u_i \in H(X)$. Then define

$$\widetilde{H} : P\big(\mathrm{Tree}(A)\big) \longrightarrow P(U), \quad X \mapsto H(X).$$

We verify that $\widetilde{H}$ satisfies the axioms of a TreeSoft Set (Definition 2.2):

- For each $X \subseteq \mathrm{Tree}(A)$, $\widetilde{H}(X) = H(X) \subseteq U$ is well-defined, since $H(X)$ is exactly the support of $|\psi_X\rangle$.
- The mapping $\widetilde{H}$ takes values in $P(U)$ by hypothesis.
- If $X = \bigcup_{j=1}^{k} X_j$ is a disjoint union of subsets each contained in some sub-tree of Tree(A), then the support of $|\psi_X\rangle$ is the union $\bigcup_{j=1}^{k} H(X_j)$. This matches the union property required of TreeSoft Sets.

Hence $\widetilde{H}$ is a TreeSoft Set, and F restricted to uniform-support states recovers the *Quantum-TreeSoft Set* of Definition 3.1.

(iii) Now let $\mathcal{F} = \bigsqcup_{t \in T} \mathrm{Tree}(A^{(t)})$ be a disjoint union of several trees. For each tree $t \in T$, suppose there is a classical TreeSoft Set

$$H_t : P\big(\mathrm{Tree}(A^{(t)})\big) \longrightarrow P(U).$$

For any $X \in P(\mathcal{F})$, write

$$X = \bigsqcup_{t \in T_0} \big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

where $T_0 = \{\, t \in T \mid X \cap \mathrm{Tree}(A^{(t)}) \neq \varnothing \,\}$. By hypothesis,

$$F(X) = |\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle, \quad \sum_{i=1}^{n} \big|\alpha_i(X)\big|^2 = 1,$$

with

$$\alpha_i(X) \neq 0 \quad \Longleftrightarrow \quad u_i \in \bigcup_{t \in T_0} H_t\big(X \cap \mathrm{Tree}(A^{(t)})\big),$$

and whenever u_i lies in that union, $\alpha_i(X) = \big|\bigcup_{t \in T_0} H_t(X \cap \mathrm{Tree}(A^{(t)}))\big|^{-1/2}$.

Define

$$\widetilde{H} : P(\mathcal{F}) \longrightarrow P(U), \quad X \mapsto \bigcup_{t \in T_0} H_t\big(X \cap \mathrm{Tree}(A^{(t)})\big).$$

We must check that $\widetilde{H}$ satisfies the axioms of a *ForestSoft Set* (Definition 2.4):

- For each X, $\widetilde{H}(X) \subseteq U$ is well-defined, since it is the union of the supports of the quantum states $|\psi_{X \cap \mathrm{Tree}(A^{(t)})}\rangle$ over $t \in T_0$.
- If $X \cap \mathrm{Tree}(A^{(t)}) = \varnothing$, then $H_t(X \cap \mathrm{Tree}(A^{(t)})) = \varnothing$ by the TreeSoft Set property, so such t does not contribute to the union.
- If $X = \bigcup_{j=1}^{k} X_j$ with each $X_j \subseteq \mathrm{Tree}(A^{(t_j)})$, then

$$\widetilde{H}(X) = \bigcup_{j=1}^{k} H_{t_j}(X_j),$$

  matching exactly the union rule in Definition 2.4.
- Thus $\widetilde{H}$ is a ForestSoft Set, and F recovers it under the uniform-amplitude support assumption.

Hence all three reductions are verified. □

Theorem 4.6 (Normalization Preservation). *For every forest-subset $X \subseteq \mathcal{F}$, the state $|\psi_X\rangle = F(X)$ satisfies*

$$\big\||\psi_X\rangle\big\| = 1.$$

Proof. By Definition 4.2, for each $X \in P(\mathcal{F})$,

$$|\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\ |u_i\rangle, \quad \sum_{i=1}^{n} \left|\alpha_i(X)\right|^2 = 1.$$

Hence

$$\|\,|\psi_X\rangle\| = \sqrt{\langle \psi_X \mid \psi_X \rangle} = \sqrt{\sum_{i=1}^{n} \left|\alpha_i(X)\right|^2} = \sqrt{1} = 1,$$

as required. □

Theorem 4.7 (Conjugate Symmetry). *For any two forest-subsets $X, Y \subseteq \mathcal{F}$,*

$$\langle \psi_X \mid \psi_Y \rangle = \overline{\langle \psi_Y \mid \psi_X \rangle}.$$

Proof. Write

$$|\psi_X\rangle = \sum_{i=1}^{n} \alpha_i(X)\,|u_i\rangle, \quad |\psi_Y\rangle = \sum_{i=1}^{n} \alpha_i(Y)\,|u_i\rangle.$$

Then by the sesquilinearity of the inner product,

$$\begin{aligned} \langle \psi_X \mid \psi_Y \rangle &= \sum_{i=1}^{n} \overline{\alpha_i(X)}\,\alpha_i(Y) \\ \implies \quad \langle \psi_Y \mid \psi_X \rangle &= \sum_{i=1}^{n} \overline{\alpha_i(Y)}\,\alpha_i(X). \end{aligned}$$

Since complex conjugation reverses the product,

$$\begin{aligned} \overline{\langle \psi_Y \mid \psi_X \rangle} &= \overline{\sum_i \overline{\alpha_i(Y)}\,\alpha_i(X)} \\ &= \sum_i \overline{\alpha_i(X)}\,\alpha_i(Y) = \langle \psi_X \mid \psi_Y \rangle. \end{aligned}$$

□

Theorem 4.8 (Orthogonality of Disjoint Support). *If two forest-subsets $X, Y \subseteq \mathcal{F}$ have disjoint amplitude supports,*

$$\{\, i \mid \alpha_i(X) \neq 0\} \ \cap\ \{\, i \mid \alpha_i(Y) \neq 0\} = \varnothing,$$

then

$$\langle \psi_X \mid \psi_Y \rangle = 0.$$

Proof. Again write

$$|\psi_X\rangle = \sum_i \alpha_i(X)\,|u_i\rangle, \quad |\psi_Y\rangle = \sum_i \alpha_i(Y)\,|u_i\rangle.$$

By definition of disjoint support, for each basis index i, either $\alpha_i(X) = 0$ or $\alpha_i(Y) = 0$. Hence every term in $\overline{\alpha_i(X)}\,\alpha_i(Y)$ vanishes, so

$$\langle \psi_X \mid \psi_Y \rangle = \sum_{i=1}^{n} \overline{\alpha_i(X)}\,\alpha_i(Y) = 0. \qquad \square$$

Theorem 4.9 (Inner-Product Bound). *For any forest-subsets $X, Y \subseteq \mathcal{F}$,*

$$\left|\langle \psi_X \mid \psi_Y \rangle\right| \leqslant 1.$$

Proof. The Cauchy–Schwarz inequality in $\mathcal{H}(U)$ gives

$$\left|\langle \psi_X \mid \psi_Y \rangle\right| \leqslant \|\psi_X\|\,\|\psi_Y\| = 1 \cdot 1 = 1,$$

where we used normalization theorem. $\square$

5. Conclusion and future work

In this paper, we have introduced and rigorously formalized the notions of the Quantum TreeSoft Set and the Quantum ForestSoft Set, thereby broadening the framework of Quantum Soft Sets. In future work, we plan to investigate further extensions of these structures by incorporating additional uncertainty models such as Fuzzy Sets [44], Neutrosophic Sets [31], QuadriPartitioned Neutrosophic Sets [42], Heptapartitioned Neutrosophic Sets [25], and Vague Sets [9].

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