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The Reciprocal of The Butterfly Theorem

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In this paper, we present two proofs of the *reciprocal butterfly theorem*.

The statement of the *butterfly theorem* is:

> Let us consider a chord PQ of midpoint M in the circle $\Omega(O)$. Through M, two other chords AB and CD are drawn, such that A and C are on the same side of PQ. We denote by X and U the intersection of AD respectively CB with PQ. Consequently, $XM = YM$.

For the proof of this theorem, see [1].

The *reciprocal of the butterfly theorem* has the following statement:

> In the circle $\Omega(O)$, let us consider the chords PQ, AB and CD which are concurrent in the point $M \neq O$, such as the points A and C are on the same side of the line PQ. Let X and Y respectively be the intersections of the chord PQ with AD and BC respectively. If $XM = YM$, then M is the middle of the chord PQ.

Proof 1.

We construct the circumscribed circle of the isosceles triangle *BOD* and denote by *E* and *F* the points where *AB* and *CD* cut again the circle (see *Fig. 1*).

The quadrilateral *DBEF* being inscribed, we have that $\sphericalangle CDB \equiv \sphericalangle BEF$. But $\sphericalangle CDB \equiv \sphericalangle BAC$, therefore we obtain that $\sphericalangle BAC \equiv \sphericalangle BEF$, with the consequence $AC \parallel EF$ (1).

We denote by *N* the second point of intersection of the circumscribed circles of the triangles *AXM* and *CYM*.

The quadrilaterals *AXMN* and *CYMN* being inscribed, we have that $\sphericalangle XAM \equiv \sphericalangle XNM$ and $\sphericalangle YCM \equiv \sphericalangle YNM$. Because $\sphericalangle XAM \equiv \sphericalangle YCM$ (*ADBC* being an inscribed quadrilateral), previous relations lead to $\sphericalangle XNM \equiv \sphericalangle YNM$. This relation, along with the condition from the hypothesis *XM*=*YM*, shows that, in the triangle *NXY*, *NM* is both median and bisector, therefore this triangle is isosceles, and $NM \perp XY$. (2)

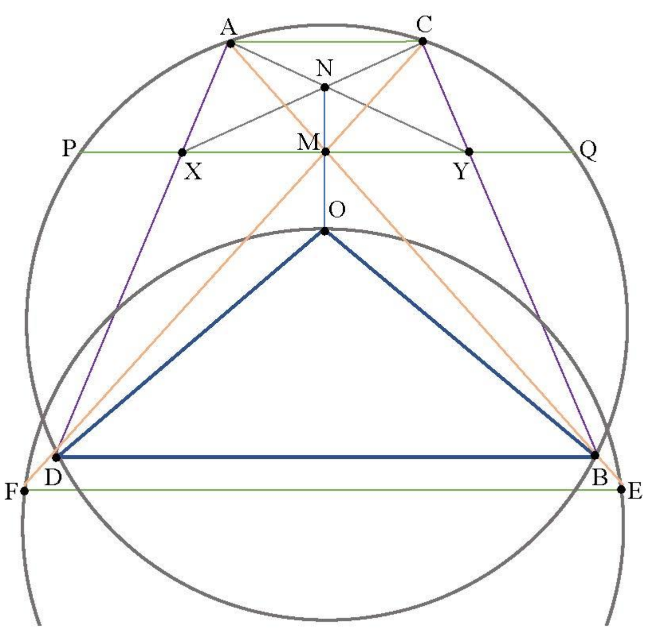

Figure 1

The relation (2) implies m($\widehat{NCB}$)=90^0 and m($\widehat{NAX}$)=90^0. But m($\widehat{NCB}$)= m($\widehat{NCM}$)+ m($\widehat{DCB}$) =90^0.

On the other hand, m($\widehat{DCB}$)+ m($\widehat{OBD}$) =90^0, because m($\widehat{DCB}$)=$\frac{1}{2}$m($\widehat{DOB}$).

We also have that m($\widehat{ODB}$)=m($\widehat{OFD}$), because the quadrilateral *FDOB* is inscribed.

These relations lead to $\sphericalangle NCM \equiv \sphericalangle OFD$, which further implies $NC \parallel OF$ (3).

Analogously it is shown that $NA \parallel OE$ (4).

Relations (1), (3) and (4) show that the triangles *NAC* and *OEF* have respectively parallel sides, therefore they are homothetic, the center of homothety being the point $\{M\}=CF \cap AE$.

Then the homothetic points *N* and *O* are collinear with *M*, having $NM \perp PQ$, it follows as well that $OM \perp PQ$, consequently *M* is the middle of the chord *PQ*.

The relation (2) implies $m(\widehat{NCB})=90^0$ and $m(\widehat{NAX})=90^0$.

But $m(\widehat{NCB})=m(\widehat{NCM})+m(\widehat{DCB})=90^0$.

On the other hand, $m(\widehat{DCB})+m(\widehat{OBD})=90^0$, because $m(\widehat{DCB})=\frac{1}{2}m(\widehat{DOB})$.

Proof 2.

Assuming the opposite, $PM \neq QM$, therefore *OM* is not perpendicular on *PQ*.

We construct the perpendicular in *M* on *OM* and denote by *U* and *V* its intersections with the circle *Ω(O)*.

We denote by *R* and *S* the intersections of the chord *UV* with *AD* and *CB* respectively (see *Fig. 2*).

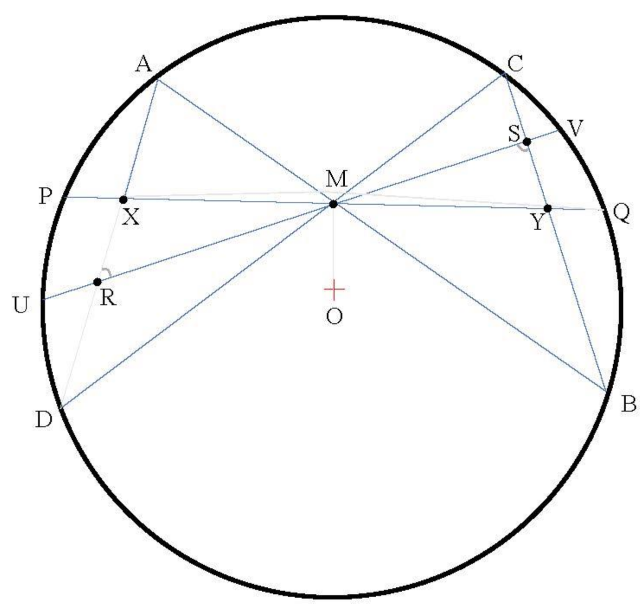

Figure 2

Because M is the middle of the chord UV, applying *the butterfly theorem*, we have that $MR=MS$.

We obtain that $\Delta MXR \equiv \Delta MYS$ (side-angle-side), and consequently $\sphericalangle XRM \equiv \sphericalangle YSM$, therefore $AD \parallel BC$.

The condition $AD \parallel BC$ leads to two possibilities for the quadrilateral $ADBC$. This can be an isosceles trapezoid if $AD \neq BC$, or rectangle if $AD=BC$.

We eliminate the possibility $ADBC$ - rectangle, because this rectangle would have the center M and it should be that $M=O$.

Let us consider $ADBC$ - isosceles trapezoid with AD the small base. In this case, we observe that M - the intersection of the diagonals of the trapezoid, and O are on the axis of symmetry of the trapezoid, and $UV \perp OM$ contradicts the fact that the points A and C must be on the same side of the right UV.

The contradictions show that M must be the middle of the chord PQ.

Bibliography

[1] Nguyen Tien Dung. Three Syntetic Proofs of the Butterfly Theory. *Forum Geometricorum*, vol. 17 (2017), 355-358.

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