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Review of Domination number and Chromatic number in Turiyam Neutrosophic and Plithogenic Graphs

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Abstract

Graph theory is the study of networks comprising nodes (vertices) and their connections (edges). It is instrumental in analyzing the structure, pathways, and properties of these networks.
This paper explores two significant problems in graph theory: the domination number and the chromatic number. The domination number represents the smallest subset of vertices such that every vertex in the graph is either included in this subset or adjacent to at least one vertex within it. The chromatic number, on the other hand, refers to the minimum number of colors needed to color a graph such that no two adjacent vertices share the same color.
Additionally, the paper introduces and examines uncertain graph models, including Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Turiyam Neutrosophic, and Plithogenic graphs. It further investigates their respective domination and chromatic numbers, providing insights into these advanced graph concepts.

Keywords: Neutrosophic graph, Fuzzy graph, Domination number, Chromatic number, Plithogenic Graph

1 | Introduction

1.1 | Domination and Chromatic Numbers in Graphs

Graph theory focuses on studying networks composed of nodes (vertices) and their connections (edges). It plays a crucial role in analyzing the structure, pathways, and properties of these networks [21].

One of the central problems in graph theory is the dominating set problem. This problem involves finding a subset of vertices in a graph such that every vertex either belongs to this subset or is adjacent to a vertex in the subset. The dominating set problem is known to be NP-complete [47, 46]. The domination number is the

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minimum number of vertices required to form such a set, and it has been widely studied across various research domains, alongside other related graph parameters [42, 7, 110, 89, 19, 80, 9, 17, 64].

The secure domination number is another important graph parameter. It represents the minimum number of vertices in a set such that replacing any vertex in the set with another vertex still ensures domination over the graph [57, 16, 18, 12, 63, 76, 56, 23].

Another well-studied problem in graph theory is the chromatic number. This parameter is widely discussed in the context of graph coloring problems [49]. The graph coloring problem[31, 60, 49] is one of the most famous problems in graph theory and is also known to be NP-hard[38]. Additionally, graph coloring is recognized as a special case of graph labeling [32, 33]. Like the domination number, the chromatic number has been the focus of extensive research [118, 24, 77, 48].

1.2 | Uncertain Graphs Models

In this paper, we explore a range of uncertain graph models, including Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. These models have been specifically developed to handle uncertainty in practical applications. Collectively referred to as uncertain graphs, an overview of these models is provided in the introduction.

Fuzzy Graphs assign a membership value between 0 and 1 to each vertex and edge, representing the degree of uncertainty or imprecision associated with them [83, 59, 72, 4, 5]. These graphs are closely tied to fuzzy set theory [62, 116, 115, 22] and have applications in areas such as social networks, decision-making, and transportation systems, where relationships are often uncertain or ambiguous [83, 67]. Intuitionistic Fuzzy Graphs extend fuzzy graphs by introducing additional parameters [79, 54, 73, 37].

Neutrosophic Graphs [3, 87, 102, 98, 44, 14, 2], based on neutrosophic set theory [6, 101, 104], introduce three parameters—truth, indeterminacy, and falsity—thereby extending classical and fuzzy logic to better represent uncertainty.

Turiyam Neutrosophic Graphs further build on the principles of neutrosophic and fuzzy graphs by introducing four attributes—truth, indeterminacy, falsity, and a liberal state—to each vertex and edge [34, 36, 35, 27]. Plithogenic Graphs, a more generalized graph model, have also emerged as an important area of study [109, 50, 27, 92, 30, 107].

Within these uncertain graph models, various uncertain domination numbers, such as the Fuzzy Domination Number and Neutrosophic Domination Number, have been extensively researched [70, 113, 114, 112, 58, 20, 11, 66, 41, 90, 61]. Secure domination numbers under uncertainty [71, 86, 52, 51, 68] and Uncertain Chromatic Numbers have also been widely investigated. While research on domination in Turiyam Neutrosophic and Plithogenic Graphs is still in its early stages, it is a promising area for further exploration.

1.3 | Our Contribution in This Paper

In this work, we define and investigate domination numbers, secure domination numbers, and chromatic numbers within the frameworks of Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, Turiyam Neutrosophic Graphs, and Plithogenic Graphs. We examine their properties, interrelationships, and potential applications.

2 | Preliminaries and definitions

In this section, we provide a brief overview of the definitions and notations used throughout this paper. Due to space constraints, not all details of each definition are included. Readers are encouraged to refer to the cited references as needed.

2.1 | Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [21, 43, 111].

Definition 1 (Graph)**.** [21] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Degree)**.** [21] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 3 (Subgraph)**.** [21] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Definition 4 (Empty Graph and Null graph)**.** (cf.[91, 45]) An *empty graph* is a graph that contains a set of vertices V but has no edges. Formally, an empty graph $G = (V, E)$ is defined as a graph where $E = \emptyset$, meaning the edge set is empty. This graph has no connections between its vertices. A *null graph* is a graph with no vertices and no edges. Formally, a null graph is a graph $G = (V, E)$ where both the vertex set $V = \emptyset$ and the edge set $E = \emptyset$. It is the simplest possible graph structure, containing no elements.

2.2 | Fuzzy, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs

In this subsection, we examine Fuzzy Graphs, Intuitionistic Fuzzy Graphs, Neutrosophic Graphs, and Turiyam Neutrosophic Graphs. Fuzzy graphs are frequently discussed in comparison with crisp graphs, which represent the classical form of graphs[83, 55]. Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing "Contradiction" with "Liberal" (cf.[94]).

Definition 5. (cf.[83, 55]) A *crisp graph* is an ordered pair $G = (V, E)$, where:

- V is a finite, non-empty set of vertices.
- $E \subseteq V \times V$ is a set of edges, where each edge is an unordered pair of distinct vertices.

Formally, for any edge $(u, v) \in E$, the following holds:

$$(u, v) \in E \iff u \neq v \quad \text{and} \quad u, v \in V$$

This implies that there are no loops (i.e., no edges of the form (v, v)) and edges represent binary relationships between distinct vertices.

Taking the above into consideration, we define Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs as follows. Please note that the definitions have been consolidated for simplicity.

Definition 6 (Unified Graphs Framework: Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Turiyam Neutrosophic Graphs)**.** (cf.[26]) Let $G = (V, E)$ be a classical graph with a set of vertices V and a set of edges E. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is assigned membership values to represent various degrees of truth, indeterminacy, and falsity.

(1) *Fuzzy Graph* [13, 40, 106, 81]*:*

- Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0, 1]$, representing the degree of participation of v in the fuzzy graph.
- Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0, 1]$, representing the strength of the connection between u and v.

(2) *Intuitionistic Fuzzy Graph (IFG)* [117, 1]*:*

- Each vertex $v \in V$ is assigned two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $v_A(v) \in [0, 1]$ (degree of non-membership), such that $\mu_A(v) + v_A(v) \leq 1$.

- Each edge $e = (u, v) \in E$ is assigned two values: $\mu_B(u, v) \in [0, 1]$ (degree of membership) and $v_B(u, v) \in [0, 1]$ (degree of non-membership), such that $\mu_B(u, v) + v_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* [102, 98, 44]*:*

- Each vertex $v \in V$ is assigned a triple $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$, where:
  - $\sigma_T(v) \in [0, 1]$ is the truth-membership degree,
  - $\sigma_I(v) \in [0, 1]$ is the indeterminacy-membership degree,
  - $\sigma_F(v) \in [0, 1]$ is the falsity-membership degree,
  - $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.
- Each edge $e = (u, v) \in E$ is assigned a triple $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$, representing the truth, indeterminacy, and falsity degrees for the connection between u and v.

(4) *Turiyam Neutrosophic Graph* [34, 26, 35]*:*

- Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, where:
  - $t(v) \in [0, 1]$ is the truth value,
  - $iv(v) \in [0, 1]$ is the indeterminacy value,
  - $fv(v) \in [0, 1]$ is the falsity value,
  - $lv(v) \in [0, 1]$ is the liberal state value,
  - $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.
- Each edge $e = (u, v) \in E$ is similarly assigned a quadruple representing the same parameters for the connection between u and v.

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [96]. Plithogenic Graphs have been developed and are currently being actively studied [109, 50, 26, 92, 107] The definition is provided below.

Definition 7. [109] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph* PG is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.

- $bCf : Nm \times Nm \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:

$$bdf((xy), (a, b)) \leq \min\{adf(x, a), adf(y, b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:

$$bCf((a, b), (c, d)) \leq \min\{aCf(a, c), aCf(b, d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned}
aCf(a, a) &= 0, && \forall a \in Ml \\
aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\
bCf(a, a) &= 0, && \forall a \in Nm \\
bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm
\end{aligned}$$

2.3 | Domination Number

The Domination Number is the minimum number of vertices in a set such that every vertex is either in the set or adjacent to one. This graph parameter has also been studied in Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs.

Definition 8. [57, 16, 18] Let $G = (V, E)$ be an undirected graph, where V is the set of vertices, and E is the set of edges.

A subset $D \subseteq V$ is called a *dominating set* if for every vertex $v \in V \setminus D$, there exists a vertex $u \in D$ such that $(u, v) \in E$.

The *domination number* $\gamma(G)$ is defined as the minimum cardinality of a dominating set:

$$\gamma(G) = \min \{|D| \mid D \text{ is a dominating set in } G\}.$$

Example 9. Consider the following simple graph $G = (V, E)$:

- Vertex set: $V = \{v_1, v_2, v_3, v_4, v_5\}$.
- Edge set: $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$.

We aim to find the Domination Number $\gamma(G)$ of this graph.

A *dominating set* $D \subseteq V$ is a set such that every vertex not in D is adjacent to at least one vertex in D.

Let's consider possible dominating sets:

- $D = \{v_2, v_4\}$:
  - v_2 dominates v_1 and v_3.
  - v_4 dominates v_3 and v_5.
  - All vertices are either in D or adjacent to a vertex in D.
- $D = \{v_3\}$:
  - v_3 dominates v_2 and v_4.
  - v_1 is not adjacent to v_3, so v_1 is not dominated.
  - v_5 is not adjacent to v_3, so v_5 is not dominated.
  - Therefore, $D = \{v_3\}$ is not a dominating set.

- $D = \{v_2, v_5\}$:
  - v_2 dominates v_1 and v_3.
  - v_5 dominates v_4.
  - v_4 is adjacent to v_5, so it is dominated.
  - All vertices are either in D or adjacent to a vertex in D.

From the above, we can see that the minimum cardinality of a dominating set is 2.

Therefore, the Domination Number $\gamma(G) = 2$.

The Fuzzy Domination Number, Intuitionistic Fuzzy Domination Number, and Neutrosophic Domination Number can be extended from the definitions of the fuzzy graph, intuitionistic fuzzy graph, and neutrosophic graph, respectively, as described below.

Definition 10. [53, 103, 78] Given a fuzzy graph $G = (\sigma, \mu)$, where:

- $\sigma : V \to [0, 1]$ is the membership function of vertices.
- $\mu : V \times V \to [0, 1]$ is the membership function of edges, such that for all $u, v \in V$, $\mu(u, v) \leq \min(\sigma(u), \sigma(v))$.

A *dominating set* $D \subseteq V$ in G is a subset of vertices such that for every $v \in V$, there exists $u \in D$ such that:

(1) $\mu(u, v)$ is a strong arc (i.e., $\mu(u, v) = \sigma(v)$).

(2) $\mu(u, v) > 0$, meaning that vertex v is either directly dominated by or adjacent to some vertex in D.

A *minimum dominating set* is a dominating set that contains the smallest number of vertices.

The *Fuzzy Domination Number* $\gamma_f(G)$ of the fuzzy graph G is defined as the sum of the membership values $\sigma(u)$ of all vertices u in the minimum dominating set D:

$$\gamma_f(G) = \sum_{u \in D} \sigma(u)$$

where D is the minimum dominating set.

Definition 11. [53, 74] Given an *intuitionistic fuzzy graph* $G = (V, E)$, where:

- $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices.
- $\mu_1 : V \to [0, 1]$ is the membership function that represents the degree of membership for each vertex v_i.
- $\gamma_1 : V \to [0, 1]$ is the non-membership function for each vertex v_i, with the condition $0 \leq \mu_1(v_i) + \gamma_1(v_i) \leq 1$.
- $\mu_2 : V \times V \to [0, 1]$ is the membership function for edges between vertices v_i and v_j, such that $\mu_2(v_i, v_j) \leq \min(\mu_1(v_i), \mu_1(v_j))$.
- $\gamma_2 : V \times V \to [0, 1]$ is the non-membership function for edges, where $\gamma_2(v_i, v_j) \leq \max(\gamma_1(v_i), \gamma_1(v_j))$.

A *dominating set* $D \subseteq V$ in G is defined such that for every $v \in V \setminus D$, there exists $u \in D$ such that u dominates v. That is, there exists a strong arc from u to v with respect to the membership and non-membership values of the vertices and edges.

The *Intuitionistic Fuzzy Domination Number* $\gamma_{if}(G)$ is defined as the minimum cardinality of a dominating set D, i.e., the smallest number of vertices in D, where no proper subset of D is also a dominating set.

Definition 12. [93, 105] Let $G = (A, B)$ be a *single-valued neutrosophic graph (SVN-graph)*, where:

- $A = (T_A, I_A, F_A)$ represents the set of vertices V, with functions $T_A : V \to [0,1]$, $I_A : V \to [0,1]$, and $F_A : V \to [0,1]$ defining the degree of truth membership, indeterminacy membership, and falsity membership of each vertex $v \in V$.
- $B = (T_B, I_B, F_B)$ represents the set of edges $E \subseteq V \times V$, with functions $T_B, I_B, F_B : E \to [0,1]$ assigning the respective memberships for each edge between vertices v_i and v_j.

A vertex $x \in V$ is said to *dominate* another vertex $y \in V$ in G if:

(1) $T_B(x,y) = \min(T_A(x), T_A(y))$,

(2) $I_B(x,y) = \min(I_A(x), I_A(y))$,

(3) $F_B(x,y) = \min(F_A(x), F_A(y))$.

A subset $D \subseteq V$ is called a *dominating set* if for every vertex $v \in V \setminus D$, there exists a vertex $u \in D$ such that u dominates v.

The *Neutrosophic Domination Number* $\gamma_N(G)$ of the neutrosophic graph G is defined as the minimum cardinality of a dominating set D. That is, the smallest number of vertices required to form a dominating set.

Proposition 13. *The Neutrosophic Domination Number generalizes the Fuzzy Domination Number.*

Proof: In a Neutrosophic Graph, each vertex has degrees $t(v)$, $i(v)$, and $f(v)$. By setting $i(v)$ and $f(v)$ to zero or constants, the Neutrosophic Graph simplifies to a Fuzzy Graph with membership function $\sigma(v) = t(v)$.

Thus, the Neutrosophic Domination Number $\gamma_N(G)$ reduces to the Fuzzy Domination Number $\gamma_f(G)$.

Therefore, the Neutrosophic Domination Number generalizes the Fuzzy Domination Number. □

Corollary 14. *The Neutrosophic Domination Number generalizes the Intuitionistic Fuzzy Domination Number.*

Proof: This is evident. □

Proposition 15. *The Fuzzy Domination Number generalizes the traditional Domination Number.*

Proof: When the membership functions $\sigma(v)$ and $\mu(u,v)$ in a Fuzzy Graph take only values 0 or 1, the Fuzzy Graph becomes a crisp graph.

Under these conditions, the Fuzzy Domination Number $\gamma_f(G)$ coincides with the traditional Domination Number $\gamma(G)$.

Therefore, the Fuzzy Domination Number generalizes the traditional Domination Number. □

Corollary 16. *The Intuitionistic Fuzzy Domination Number generalizes the traditional Domination Number.*

Proof: This is evident. □

Corollary 17. *The Neutrosophic Domination Number generalizes the traditional Domination Number.*

Proof: This is evident. □

2.4 | Secure Domniation Number

The Secure Domination Number is the minimum number of vertices in a set where replacing any vertex maintains domination over the graph. This graph parameter has also been studied in Fuzzy Graphs, Intuitionistic Fuzzy Graphs, and Neutrosophic Graphs.

Definition 18. [57, 16, 18] Let $G = (V, E)$ be an undirected graph, where V is the set of vertices, and E is the set of edges.

A subset $S \subseteq V$ is called a *secure dominating set* if for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) $(u, v) \in E$,

(2) Replacing v with u in S results in a dominating set, i.e., $(S \setminus \{v\}) \cup \{u\}$ is still a dominating set.

The *secure domination number* $\gamma_s(G)$ is defined as the minimum cardinality of a secure dominating set:

$$\gamma_s(G) = \min \{|S| \mid S \text{ is a secure dominating set in } G\}.$$

Theorem 19. *For any graph G, the Secure Domination Number $\gamma_s(G)$ is greater than or equal to the Domination Number $\gamma(G)$:*

$$\gamma_s(G) \geq \gamma(G).$$

Proof: A secure dominating set is a dominating set with the additional property that replacing any vertex v in the set with one of its neighbors u (where $(u, v) \in E$) results in a dominating set.

This additional constraint means that every secure dominating set is also a dominating set, but not every dominating set is a secure dominating set.

Therefore, the minimum cardinality of a secure dominating set is at least that of a dominating set:

$$\gamma_s(G) \geq \gamma(G).$$

□

Example 20. Consider the following graph $G = (V, E)$:

- *Vertex Set:* $V = \{v_1, v_2, v_3, v_4, v_5\}$.
- *Edge Set:* $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$.

This graph forms a cycle with 5 vertices.
Finding the Secure Domination Number $\gamma_s(G)$ A *secure dominating set* $S \subseteq V$ satisfies:

(1) S is a dominating set.

(2) For every $u \in V \setminus S$, there exists $v \in S$ such that:

- $(u, v) \in E$.
- Replacing v with u in S results in a dominating set.

Possible Secure Dominating Sets

- $S = \{v_2, v_4\}$:
  - *Domination:*
    * v_2 dominates v_1 and v_3.
    * v_4 dominates v_3 and v_5.
  - *Verification:*

      * For $v_1 \in V \setminus S$, neighbor in S is v_2.
        · Replacing v_2 with v_1: $S' = \{v_1, v_4\}$ is a dominating set.
      * For $v_3 \in V \setminus S$, neighbor in S is v_2.
        · Replacing v_2 with v_3: $S' = \{v_3, v_4\}$ is a dominating set.
      * For $v_5 \in V \setminus S$, neighbor in S is v_4.
        · Replacing v_4 with v_5: $S' = \{v_2, v_5\}$ is a dominating set.
    - *Conclusion:* S is a secure dominating set.

- $S = \{v_1, v_3, v_5\}$:
    - *Domination:*
      * Each vertex in S dominates itself and its neighbors.
    - *Verification:*
      * For $v_2 \in V \setminus S$, neighbor in S is v_1 or v_3.
        · Replacing v_1 with v_2: $S' = \{v_2, v_3, v_5\}$ is a dominating set.
      * For $v_4 \in V \setminus S$, neighbor in S is v_3 or v_5.
        · Replacing v_3 with v_4: $S' = \{v_1, v_4, v_5\}$ is a dominating set.
    - *Conclusion:* S is a secure dominating set.

The smallest secure dominating set has cardinality 2 (e.g., $S = \{v_2, v_4\}$). Therefore, the *Secure Domination Number* is:

$$\gamma_s(G) = 2.$$

For comparison, the *Domination Number* $\gamma(G)$ is also 2, as $D = \{v_2, v_5\}$ is a minimal dominating set.

Definition 21. [53] et $G = (V, E)$ be a fuzzy graph, where:

- $\sigma : V \to [0, 1]$ represents the membership function of the vertices,
- $\mu : V \times V \to [0, 1]$ represents the membership function of the edges.

A subset $S \subseteq V$ is called a *secure dominating set* if for each vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) u is adjacent to v (i.e., $\mu(u, v) > 0$),

(2) Replacing v with u in S results in a dominating set, i.e., $(S \setminus \{v\}) \cup \{u\}$ is still a dominating set.

The *Fuzzy Secure Domination Number* $\gamma_s(G)$ is defined as the minimum fuzzy cardinality of a secure dominating set in G, where the fuzzy cardinality of a set S is given by:

$$\gamma_s(G) = \min \left\{ \sum_{v \in S} \sigma(v) \mid S \text{ is a secure dominating set} \right\}.$$

Proposition 22. *In any fuzzy graph G, the Fuzzy Secure Domination Number $\gamma_s(G)$ is greater than or equal to the Fuzzy Domination Number $\gamma_f(G)$:*

$$\gamma_s(G) \geq \gamma_f(G).$$

Proof: In a fuzzy graph $G = (\sigma, \mu)$:

- The *Fuzzy Domination Number* $\gamma_f(G)$ is defined as the minimum fuzzy cardinality over all fuzzy dominating sets D:

$$\gamma_f(G) = \min \left\{ \sum_{v \in D} \sigma(v) \mid D \text{ is a fuzzy dominating set in } G \right\}.$$

- The *Fuzzy Secure Domination Number* $\gamma_s(G)$ is defined as the minimum fuzzy cardinality over all fuzzy secure dominating sets S:

$$\gamma_s(G) = \min \left\{ \sum_{v \in S} \sigma(v) \mid S \text{ is a fuzzy secure dominating set in } G \right\}.$$

By definition, every fuzzy secure dominating set is also a fuzzy dominating set, but with an additional constraint that for each vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) $\mu(u, v) > 0$ (i.e., u is adjacent to v).

(2) Replacing v with u in S results in a dominating set $(S \setminus \{v\}) \cup \{u\}$.

This additional requirement makes the set of fuzzy secure dominating sets a subset of the fuzzy dominating sets. Therefore, the minimal fuzzy cardinality over fuzzy secure dominating sets is greater than or equal to that over fuzzy dominating sets:

$$\gamma_s(G) \geq \gamma_f(G).$$

□

Definition 23. [53] Let $G = (V, E)$ be an intuitionistic fuzzy graph, where:

- $\mu_1 : V \to [0, 1]$ and $\gamma_1 : V \to [0, 1]$ represent the membership and non-membership degrees of the vertices,
- $\mu_2 : V \times V \to [0, 1]$ and $\gamma_2 : V \times V \to [0, 1]$ represent the membership and non-membership degrees of the edges.

A subset $S \subseteq V$ is called an *intuitionistic fuzzy secure dominating set* if for each vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) u is adjacent to v (i.e., $\mu_2(u, v) > 0$),

(2) Replacing v with u in S results in a dominating set, i.e., $(S \setminus \{v\}) \cup \{u\}$ is still a dominating set.

The *Intuitionistic Fuzzy Secure Domination Number* $\gamma_{is}(G)$ is defined as:

$$\gamma_{is}(G) = \min \left\{ \sum_{v \in S} \mu_1(v) \mid S \text{ is an intuitionistic fuzzy secure dominating set} \right\}.$$

Definition 24. [93, 105] Let $G = (A, B)$ be a single-valued neutrosophic graph (SVNG), where:

- $A = (T_A, I_A, F_A)$ represents the truth, indeterminacy, and falsity membership degrees of the vertices,
- $B = (T_B, I_B, F_B)$ represents the truth, indeterminacy, and falsity membership degrees of the edges.

A subset $S \subseteq V$ is called a *neutrosophic secure dominating set* if for each vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) u is adjacent to v (i.e., $T_B(u, v) > 0$),

(2) Replacing v with u in S results in a dominating set, i.e., $(S \setminus \{v\}) \cup \{u\}$ is still a dominating set.

The *Neutrosophic Secure Domination Number* $\gamma_{snd}(G)$ is defined as the minimum neutrosophic cardinality of a neutrosophic secure dominating set in G:

$$\gamma_{snd}(G) = \min \left\{ \sum_{v \in S} T_A(v) \mid S \text{ is a neutrosophic secure dominating set} \right\}.$$

Proposition 25. *In any neutrosophic graph G, the Neutrosophic Secure Domination Number $\gamma_{ns}(G)$ is greater than or equal to the Neutrosophic Domination Number $\gamma_N(G)$:*

$$\gamma_{ns}(G) \geq \gamma_N(G).$$

Proof: In a neutrosophic graph $G = (A, B)$:

- The *Neutrosophic Domination Number* $\gamma_N(G)$ is defined as the minimum sum of truth-membership degrees over all neutrosophic dominating sets D:

$$\gamma_N(G) = \min \left\{ \sum_{v \in D} T_A(v) \mid D \text{ is a neutrosophic dominating set in } G \right\}.$$

- The *Neutrosophic Secure Domination Number* $\gamma_{ns}(G)$ is defined as the minimum sum of truth-membership degrees over all neutrosophic secure dominating sets S:

$$\gamma_{ns}(G) = \min \left\{ \sum_{v \in S} T_A(v) \mid S \text{ is a neutrosophic secure dominating set in } G \right\}.$$

A neutrosophic secure dominating set is a neutrosophic dominating set with the additional property that for each vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) $T_B(u, v) > 0$ (i.e., u is adjacent to v with a positive truth-membership degree).

(2) Replacing v with u in S results in a neutrosophic dominating set $(S \setminus \{v\}) \cup \{u\}$.

Since every neutrosophic secure dominating set is also a neutrosophic dominating set with an extra condition, the set of neutrosophic secure dominating sets is a subset of the neutrosophic dominating sets. Therefore, we have:

$$\gamma_{ns}(G) \geq \gamma_N(G).$$

□

Proposition 26. *The Neutrosophic Secure Domination Number generalizes the Fuzzy Secure Domination Number.*

Proof: In a Neutrosophic Graph, each vertex has degrees $t(v)$, $i(v)$, and $f(v)$. By setting $i(v)$ and $f(v)$ to zero or constants, the Neutrosophic Graph becomes a Fuzzy Graph with the membership function $\sigma(v) = t(v)$.

Thus, the Neutrosophic Secure Domination Number $\gamma_{ns}(G)$ reduces to the Fuzzy Secure Domination Number $\gamma_{fs}(G)$.

Therefore, the Neutrosophic Secure Domination Number generalizes the Fuzzy Secure Domination Number. □

Corollary 27. *The Neutrosophic Secure Domination Number generalizes the Intuitionistic Fuzzy Secure Domination Number.*

Proof: This is evident. □

Proposition 28. *The Fuzzy Secure Domination Number generalizes the traditional Secure Domination Number.*

Proof: When the membership functions $\sigma(v)$ and $\mu(u, v)$ in a Fuzzy Graph take only the values 0 or 1, the Fuzzy Graph becomes a crisp graph.

Under these conditions, the Fuzzy Secure Domination Number $\gamma_{fs}(G)$ coincides with the traditional Secure Domination Number $\gamma_s(G)$.

Therefore, the Fuzzy Secure Domination Number generalizes the traditional Secure Domination Number. □

Corollary 29. *The Intuitionistic Fuzzy Secure Domination Number generalizes the traditional Secure Domination Number.*

Proof: This is evident. □

Corollary 30. *The Neutrosophic Secure Domination Number generalizes the traditional Secure Domination Number.*

Proof: This is evident. □

2.5 | Chromatic Number

The Chromatic Number of a graph is the minimum number of colors required to color its vertices such that no adjacent vertices share the same color [118, 24, 77, 48]. This graph parameter has also been studied in Fuzzy Graphs and Neutrosophic Graphs.

Definition 31 (Chromatic Number)**.** [118, 24] Let $G = (V, E)$ be a simple undirected graph, where V is the set of vertices and E is the set of edges. A *proper vertex coloring* of G is a mapping $c : V \to \{1, 2, \dots, k\}$, where k is the number of colors, such that for every edge $(u, v) \in E$, the vertices u and v have different colors, i.e., $c(u) \neq c(v)$.

The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors k required to obtain a proper vertex coloring of G. Formally, it is defined as:

$$\chi(G) = \min \{k \mid \text{there exists a proper vertex coloring with } k \text{ colors}\}.$$

Example 32 (Chromatic Number)**.** Consider the cycle graph C_4 with four vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges

$$E = \{ (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}.$$

A proper vertex-coloring must ensure adjacent vertices receive different colors. For C_4, we can use two colors, say "red" and "blue," assigned as:

$$c(v_1) = \text{red}, \quad c(v_2) = \text{blue}, \quad c(v_3) = \text{red}, \quad c(v_4) = \text{blue}.$$

No two adjacent vertices share the same color. Hence, the chromatic number $\chi(C_4)$ is 2.

Definition 33 (Fuzzy Chromatic Number)**.** [55, 85, 84] Let $G = (V, \tilde{E})$ be a fuzzy graph, where V is a finite set of vertices and $\tilde{E}$ is a fuzzy edge set with a membership function $\mu : V \times V \to [0, 1]$. An α-cut of G, denoted as $G_\alpha = (V, E_\alpha)$, is a crisp graph where:

$$E_\alpha = \{(u, v) \in V \times V \mid \mu(u, v) \geq \alpha\}.$$

The crisp chromatic number of the α-cut graph G_α is denoted by $\chi_\alpha(G)$.

The *fuzzy chromatic number* of the fuzzy graph G is defined as the fuzzy set $\tilde{\chi}(G) = \{(k, \nu_{\tilde{\chi}}(k)) \mid k \in \mathbb{N}\}$, where:

$$\nu_{\tilde{\chi}}(k) = \sup\{\alpha \in [0, 1] \mid \chi_\alpha(G) = k\}.$$

That is, the fuzzy chromatic number represents the highest α-level at which the chromatic number of the α-cut graph is k.

Example 34 (Fuzzy Chromatic Number)**.** Let $G = (V, \tilde{E})$ be a fuzzy graph with vertex set $V = \{v_1, v_2, v_3, v_4\}$. Suppose the fuzzy edge set $\tilde{E}$ is defined by the membership function $\mu : V \times V \to [0, 1]$, given (assuming $\mu(u, u) = 0$):

$$\mu(v_1, v_2) = 0.8, \quad \mu(v_2, v_3) = 0.5, \quad \mu(v_3, v_4) = 0.4, \quad \mu(v_4, v_1) = 0.2,$$

all other pairs having $\mu = 0$. For each $\alpha \in [0, 1]$, the α-cut $G_\alpha = (V, E_\alpha)$ is the crisp graph with

$$E_\alpha = \{(u, v) \in V \times V \mid \mu(u, v) \geq \alpha\}.$$

- If $\alpha \geq 0.8$, then $E_\alpha = \{(v_1, v_2)\}$, and $\chi_\alpha(G) = 2$ (it is a single edge).

- If $0.5 \leq \alpha < 0.8$, then $E_\alpha = \{(v_1, v_2), (v_2, v_3)\}$. Its chromatic number is 2.
- If $0.4 \leq \alpha < 0.5$, then $E_\alpha = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$. Its chromatic number is still 2.
- If $0.2 \leq \alpha < 0.4$, then $E_\alpha = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$. The resulting crisp graph is C_4, whose chromatic number is 2.
- For $\alpha < 0.2$, all pairs become edges, leading to a complete graph on 4 vertices K_4, so $\chi_\alpha(G) = 4$.

Hence, the *fuzzy chromatic number* $\tilde{\chi}(G)$ is typically described as a fuzzy set over the possible color counts, capturing the supremum α-values where each color count remains valid. For instance, color-count 2 is valid at all $\alpha \geq 0.4$. Meanwhile, color-count 4 only occurs at $\alpha < 0.2$.

Proposition 35. *The Fuzzy Chromatic Number generalizes the traditional Chromatic Number.*

Proof: When the fuzzy membership function $\mu(e)$ takes only values 0 or 1, the fuzzy graph becomes a crisp graph. Under these conditions, the Fuzzy Chromatic Number $\tilde{\chi}(G)$ coincides with the traditional Chromatic Number $\chi(G)$.

Therefore, the Fuzzy Chromatic Number generalizes the traditional Chromatic Number. □

Definition 36 (Single-Valued Neutrosophic Graph)**.** [15] A *Single-Valued Neutrosophic Graph* (SVNG) $G = (V, E)$ consists of a set of vertices V and a set of edges $E \subseteq V \times V$. Each vertex $v \in V$ is associated with a membership triplet $(t_V(v), i_V(v), f_V(v))$, where $t_V(v), i_V(v), f_V(v) \in [0, 1]$ represent the degrees of truth-membership, indeterminacy-membership, and falsity-membership, respectively, with the condition that:

$$0 \leq t_V(v) + i_V(v) + f_V(v) \leq 3.$$

Similarly, each edge $e = (u, v) \in E$ is associated with a membership triplet

$$(t_E(e), i_E(e), f_E(e))$$

, with $t_E(e), i_E(e), f_E(e) \in [0, 1]$ and:

$$0 \leq t_E(e) + i_E(e) + f_E(e) \leq 3.$$

Definition 37 (Single-Valued Neutrosophic Vertex Coloring)**.** [82, 10, 88] Let $G = (V, E)$ be an SVNG. A collection $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of Single-Valued Neutrosophic Fuzzy Sets is called a *k-Single-Valued Neutrosophic Vertex Coloring* (k-SVNVC) of G if the following conditions hold:

(1) For every vertex $v \in V$, there exists a unique $\gamma_j \in \Gamma$ such that:

$$\gamma_j(v) = (t_V(v), i_V(v), f_V(v)).$$

That is, the collection Γ partitions the vertex set V into k disjoint color classes.

(2) For every pair of distinct colors $j, h \in \{1, 2, \ldots, k\}$, the intersection of γ_j and γ_h is empty:

$$\gamma_j \cap \gamma_h = \emptyset.$$

(3) For each edge $e = (u, v) \in E$ and for each color $j \in \{1, 2, \ldots, k\}$, the following conditions hold:

$$\min\{\gamma_j(t_V(u)), \gamma_j(t_V(v))\} = 0,$$

$$\min\{\gamma_j(i_V(u)), \gamma_j(i_V(v))\} = 0,$$

$$\max\{\gamma_j(f_V(u)), \gamma_j(f_V(v))\} = 1.$$

This ensures that adjacent vertices u and v do not share the same color class in terms of their neutrosophic memberships.

Definition 38 (Neutrosophic Chromatic Number)**.** [69, 108] The *Neutrosophic Chromatic Number* $\chi_v(G)$ of an SVNG G is the smallest integer k for which there exists a k-Single-Valued Neutrosophic Vertex Coloring of G.

Example 39 (Neutrosophic Chromatic Number)**.** Let $H = (V, E)$ be an SVNG (Single-Valued Neutrosophic Graph) with three vertices $V = \{u_1, u_2, u_3\}$. Suppose the edge set E and membership triplets $(t_E(e),\ i_E(e),\ f_E(e))$ are:

- Edge (u_1, u_2) with $(t_E = 0.7,\ i_E = 0.0,\ f_E = 0.3)$,
- Edge (u_2, u_3) with $(t_E = 0.5,\ i_E = 0.2,\ f_E = 0.3)$,
- Edge (u_1, u_3) with $(t_E = 0.2,\ i_E = 0.5,\ f_E = 0.3)$.

We assume each vertex u_j has a membership triple $(t_V(u_j),\ i_V(u_j),\ f_V(u_j))$. To color the vertices, we define a set of Single-Valued Neutrosophic color classes that partition V. If we discover that two colors suffice (e.g. red and blue) while respecting the neutrosophic adjacency constraints, then the *Neutrosophic Chromatic Number* $\chi_v(H)$ is 2. If such a coloring is not feasible with fewer than 3 colors under the chosen adjacency conditions, $\chi_v(H)$ might be 3. In this example, it is straightforward to separate $\{u_1\}$ in red and $\{u_2,\ u_3\}$ in blue, satisfying the adjacency constraints. Thus $\chi_v(H) = 2$.

Proposition 40. *The Neutrosophic Chromatic Number generalizes the Fuzzy Chromatic Number.*

Proof: In a Neutrosophic Graph, each edge has degrees $t(e)$, $i(e)$, and $f(e)$. By setting $i(e)$ and $f(e)$ to zero or constants, the Neutrosophic Graph becomes a fuzzy graph with membership function $\mu(e) = t(e)$.

Thus, the Neutrosophic Chromatic Number $\chi_v(G)$ reduces to the Fuzzy Chromatic Number $\tilde{\chi}(G)$. □

3 | Result in this paper

In this section, we present the results of this paper, focusing specifically on the definitions and mathematical properties of the Domination Set and Secure Domination Set in Turiyam Neutrosophic and Plithogenic Graphs. We provide formal definitions and briefly analyze their mathematical characteristics.

3.1 | Domination set in Turiyam Neutrosophic and Plithogenic Graphs

We consider about Domination set in Turiyam Neutrosophic Graphs and Plithogenic Graphs. The definition is provided below.

Definition 41. Let $G = (V, E)$ be a Turiyam Neutrosophic Graph, where:

Each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, with $t(v), iv(v), fv(v), lv(v) \in [0, 1]$ and $t(v) + iv(v) + fv(v) + lv(v) \leq 4$.

Each edge $e = (u, v) \in E$ is assigned a quadruple $\mu(e) = (t(e), iv(e), fv(e), lv(e))$, with similar conditions.

A vertex $u \in V$ is said to dominate another vertex $v \in V$ if:

$$t(e) = \min(t(u), t(v)),$$

$$iv(e) = \min(iv(u), iv(v)),$$

$$fv(e) = \min(fv(u), fv(v)),$$

$$lv(e) = \min(lv(u), lv(v)).$$

A subset $D \subseteq V$ is called a *Turiyam Neutrosophic Dominating Set* if for every vertex $v \in V \setminus D$, there exists a vertex $u \in D$ such that u dominates v.

The *Turiyam Neutrosophic Domination Number* $\gamma_t(G)$ is defined as the minimum sum of the truth-membership degrees $t(v)$ of the vertices in a Turiyam Neutrosophic dominating set D:

$$\gamma_t(G) = \min \left\{ \sum_{v \in D} t(v) \mid D \text{ is a Turiyam Neutrosophic dominating set in } G \right\}.$$

Example 42 (Turiyam Neutrosophic Domination Number)**.** Consider a Turiyam Neutrosophic Graph $G = (V, E)$ with vertex set

$$V = \{v_1, v_2, v_3\}$$

and edge set

$$E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3)\}.$$

Suppose each vertex v_i has the following quadruple of membership degrees

$$\sigma(v_i) = (t(v_i),\, iv(v_i),\, fv(v_i),\, lv(v_i)),$$

where $t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4$. Let us define:

$$\sigma(v_1) = (0.8,\, 0.1,\, 0.05,\, 0.05), \quad \sigma(v_2) = (0.5,\, 0.3,\, 0.1,\, 0.1),$$
$$\sigma(v_3) = (0.6,\, 0.2,\, 0.1,\, 0.1).$$

All edges are considered fully present, i.e. $\mu(e) = (1, 0, 0, 0)$ for each edge e.
Dominance Criterion: A vertex u *dominates* vertex v if

$$t(e) = \min\{t(u),\, t(v)\}, \;\; iv(e) = \min\{iv(u),\, iv(v)\}, \;\; fv(e) = \min\{fv(u),\, fv(v)\}, \;\; lv(e) = \min\{lv(u),\, lv(v)\}.$$

Typically, we treat an edge (u, v) as 'dominating' if the quadruple is non-zero, or if $\min\{t(u), t(v)\}$ plus the other components exceed a threshold. In this simpler example, assume each edge is automatically dominating once (u, v) is recognized as adjacent.
Finding a Dominating Set: We seek a subset $D \subseteq \{v_1, v_2, v_3\}$ such that every vertex outside D is dominated by a vertex in D. The set $\{v_1\}$ alone can dominate v_2 and v_3 (since edges exist from v_1 to both). Indeed, $\{v_1\}$ is a Turiyam Neutrosophic dominating set. Hence,

$$D = \{v_1\}, \quad \text{and} \sum_{v \in D} t(v) = t(v_1) = 0.8.$$

No smaller sum can be found with fewer vertices, so the *Turiyam Neutrosophic Domination Number* is

$$\gamma_t(G) = 0.8.$$

We have used only the *truth-membership* degrees $t(v)$ to measure the total cost of the dominating set. In more complex settings, an alternative aggregator may consider partial (non)membership or thresholds for the quadruples (t, iv, fv, lv). Regardless, Example 42 shows how a single vertex, chosen with the highest truth-membership degree, can minimize the sum $\sum t(v)$.

Definition 43. Let $PG = (PM, PN)$ be a Plithogenic Graph, where:

$PM = (M, l, Ml, adf, aCf)$ is the Plithogenic vertex set, and $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic edge set.

A subset $D \subseteq M$ is called a *Plithogenic Dominating Set* if for every vertex $v \in M \setminus D$, there exists a vertex $u \in D$ such that the Degree of Appurtenance Function satisfies:

$$bdf((u, v), (l(u), l(v))) \geq \theta,$$

where θ is a predetermined threshold indicating sufficient dominance.

The *Plithogenic Domination Number* $\gamma_p(PG)$ is defined as the minimum cardinality of a Plithogenic dominating set D:

$$\gamma_p(PG) = \min\{|D| \mid D \text{ is a Plithogenic dominating set in } PG\}.$$

Example 44 (Plithogenic Domination Number)**.** Consider a Plithogenic Graph $PG = (PM, PN)$ with:

$$PM = (M, l, Ml,\, adf,\, aCf),$$

where $M = \{m_1, m_2, m_3, m_4\}$ is the set of vertices, and $l : M \to \{\text{Trait}_1, \text{Trait}_2\}$ an attribute function assigning each vertex one trait or label. The *Degree of Appurtenance Function* adf might map each pair (m_i, m_j) plus trait pairs to values in $[0, 1]^s$. The *Degree of Contradiction Function* aCf might measure how trait values conflict or complement each other.

Additionally, suppose $PN = (N, m, Nm, bdf, bCf)$ describes edges or possible adjacency relations. Here, bdf indicates how strongly one vertex dominates another in terms of attribute-based membership, while bCf measures contradiction among edges or attribute pairs.
Illustration of membership: Let $\theta \in (0, 1)$ be a threshold. A vertex u is said to *dominate* vertex v if:

$$bdf((u, v), (l(u), l(v))) \ \geq \ \theta.$$

For instance, define:

$$bdf((m_1, m_2), (\mathrm{Trait}_1, \mathrm{Trait}_2)) = 0.85, \quad bdf((m_3, m_4), (\mathrm{Trait}_1, \mathrm{Trait}_1)) = 0.4, \quad ...$$

and so on, for each pair.
Constructing a Plithogenic Dominating Set: We want a subset $D \subseteq M$ such that for every $v \in M \setminus D$, there is $u \in D$ with $bdf((u, v), (l(u), l(v))) \geq \theta$. Suppose $\theta = 0.5$.

- Assume $bdf((m_1, m_2), \dots) = 0.85 \geq 0.5$, so m_1 dominates m_2.
- Suppose m_2 does *not* dominate m_3 because $bdf((m_2, m_3), \dots) = 0.3 < 0.5$.
- Suppose m_1 also dominates m_3 with measure 0.6.
- Suppose m_4 is only dominated by m_2 with measure 0.7.

In that scenario, a dominating set might be $D = \{m_1, m_2\}$. We check:

$$(1)\ m_3 \notin D,\ \exists m_1 \in D : bdf((m_1, m_3), \dots) = 0.6 \geq 0.5,$$

$$(2)\ m_4 \notin D,\ \exists m_2 \in D : bdf((m_2, m_4), \dots) = 0.7 \geq 0.5.$$

Hence $\{m_1, m_2\}$ is a Plithogenic Dominating Set. Possibly we cannot do better with just $\{m_1\}$ or $\{m_2\}$. Thus

$$\gamma_p(PG) = 2$$

if no single vertex can dominate all others at θ-level.

The *Plithogenic Domination Number* is the minimum cardinality of such a dominating set. Example 44 shows how we pick vertices that collectively dominate every other vertex, under the aggregator threshold θ. Adjusting θ or the membership/contradiction functions may alter which sets are valid dominators or how large they must be.

Theorem 45. *In any Turiyam Neutrosophic Graph $G = (V, E)$, the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ is less than or equal to the total truth-membership degrees of all vertices:*

$$\gamma_t(G) \leq \sum_{v \in V} t(v).$$

Proof: By definition, $\gamma_t(G)$ is the minimum sum of truth-membership degrees in a Turiyam Neutrosophic dominating set D. Since $D \subseteq V$, and $t(v) \in [0, 1]$ for all $v \in V$, the sum over D is less than or equal to the sum over V:

$$\gamma_t(G) = \sum_{v \in D} t(v) \leq \sum_{v \in V} t(v).$$

This holds because $D \subseteq V$, and all $t(v)$ are non-negative. □

Theorem 46. *Let $G = (V, E)$ be a Turiyam Neutrosophic Graph with $|V| = n$, where each vertex $v \in V$ has a truth-membership degree $t(v) \in [0, 1]$, and each edge $e = (u, v) \in E$ has a truth-membership degree $t(e) \in [0, 1]$.*

The Turiyam Neutrosophic Domination Number $\gamma_t(G)$, which represents the minimum sum of truth-membership degrees over a Turiyam Neutrosophic dominating set, satisfies the following inequality:

$$\gamma_t(G) \leq n,$$

where $n = |V|$ is the total number of vertices in the graph.

Proof: Let $G = (V, E)$ be a Turiyam Neutrosophic Graph, where each vertex $v \in V$ is assigned a quadruple $\sigma(v) = (t(v), iv(v), fv(v), lv(v))$, and the truth-membership degree $t(v) \in [0, 1]$.

A *Turiyam Neutrosophic dominating set* $D \subseteq V$ is defined such that for every vertex $v \in V \setminus D$, there exists a vertex $u \in D$ such that u dominates v in the sense that $t(e) > 0$ for the edge $e = (u, v)$.

The *Turiyam Neutrosophic Domination Number* $\gamma_t(G)$ is defined as the minimum sum of truth-membership degrees $t(v)$ over all vertices in a Turiyam Neutrosophic dominating set D:

$$\gamma_t(G) = \min \left\{ \sum_{v \in D} t(v) \mid D \text{ is a Turiyam Neutrosophic dominating set} \right\}.$$

Now, we show that $\gamma_t(G) \leq n$, where $n = |V|$ is the total number of vertices.

Since $t(v) \in [0, 1]$ for every vertex $v \in V$, the maximum value of $t(v)$ for any individual vertex is 1.

If every vertex in V is included in a Turiyam Neutrosophic dominating set D, the sum of truth-membership degrees is:

$$\sum_{v \in V} t(v) \leq \sum_{v \in V} 1 = n.$$

Thus, the sum of the truth-membership degrees over all vertices in V is bounded by n.

Since the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ is the minimum sum over any Turiyam Neutrosophic dominating set, it follows that:

$$\gamma_t(G) \leq \sum_{v \in V} t(v) \leq n.$$

Therefore, the maximum possible value of the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ is n, and we conclude that:

$$\gamma_t(G) \leq n.$$

□

Theorem 47. *Let $G = (V, E)$ be a null graph, where both the vertex set $V = \emptyset$ and the edge set $E = \emptyset$. In this case, the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ is defined to be* 0.

Proof: A *null graph* is defined as a graph with no vertices and no edges, i.e., $V = \emptyset$ and $E = \emptyset$.

The *Turiyam Neutrosophic Domination Number* $\gamma_t(G)$ represents the minimum sum of truth-membership degrees over a Turiyam Neutrosophic dominating set. Formally, a Turiyam Neutrosophic dominating set $D \subseteq V$ is a subset of vertices such that every vertex not in D is adjacent to at least one vertex in D, where the truth-membership degree $t(v) \in [0, 1]$ of each vertex $v \in D$ is considered in the sum.

Since $V = \emptyset$ in a null graph, there are no vertices to consider for inclusion in any dominating set. Therefore, the only possible Turiyam Neutrosophic dominating set in a null graph is the empty set, $D = \emptyset$.

Given that there are no vertices, the sum of truth-membership degrees in the Turiyam Neutrosophic dominating set is:

$$\sum_{v \in D} t(v) = \sum_{v \in \emptyset} t(v) = 0.$$

Thus, the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ for a null graph is 0, as no vertices exist and the domination set is trivially empty.

Therefore, we conclude that:

$$\gamma_t(G) = 0.$$

□

Theorem 48. *The Turiyam Neutrosophic Domination Number generalizes both the Neutrosophic Domination Number and the Fuzzy Domination Number.*

Proof: The Turiyam Neutrosophic Graph extends the Neutrosophic Graph by adding the liberal state value $l(v)$. By setting $l(v)$ to a constant or ignoring it, the Turiyam Neutrosophic Graph reduces to a Neutrosophic Graph.

Similarly, by setting the indeterminacy-membership degree $i(v)$ and the falsity-membership degree $f(v)$ to zero or constants, the Neutrosophic Graph reduces to a Fuzzy Graph.

Therefore, the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ generalizes both the Neutrosophic Domination Number $\gamma_N(G)$ and the Fuzzy Domination Number $\gamma_f(G)$. □

Theorem 49. *Let $PG = (PM, PN)$ be a null graph in the context of a Plithogenic Graph, where both the vertex set $M = \emptyset$ and the edge set $N = \emptyset$. In this case, the Plithogenic Domination Number $\gamma_p(PG)$ is defined to be 0.*

Proof: A *null graph* is a graph that contains no vertices and no edges, i.e., $M = \emptyset$ and $N = \emptyset$. In the framework of Plithogenic Graphs, this means that there are no vertices in the Plithogenic vertex set PM, and no edges in the Plithogenic edge set PN.

The *Plithogenic Domination Number* $\gamma_p(PG)$ is defined as the minimum cardinality of a Plithogenic dominating set. A Plithogenic dominating set $D \subseteq M$ must satisfy the condition that for every vertex $v \in M \setminus D$, there exists a vertex $u \in D$ such that the Degree of Appurtenance Function (DAF) satisfies:

$$bdf((u, v), (l(u), l(v))) \geq \theta,$$

where θ is a predetermined threshold.

In a null graph, since $M = \emptyset$, there are no vertices that need to be dominated. As a result, there are no vertices to include in any Plithogenic dominating set. The only possible dominating set is the empty set $D = \emptyset$.

Since the Plithogenic Domination Number $\gamma_p(PG)$ is the minimum cardinality of such a dominating set, and the empty set has a cardinality of 0, we conclude that:

$$\gamma_p(PG) = 0.$$

Thus, for a null graph, the Plithogenic Domination Number $\gamma_p(PG)$ is 0, as there are no vertices or edges to consider. □

Theorem 50. *The Plithogenic Domination Number generalizes the Turiyam Neutrosophic Domination Number when $s = 4$ and $t = 1$.*

Proof: In a Plithogenic Graph PG, each vertex and edge is characterized by s attributes and a threshold t. When $s = 4$ and $t = 1$, the attributes correspond to the truth-membership degree $t(v)$, indeterminacy-membership degree $i(v)$, falsity-membership degree $f(v)$, and liberal state value $l(v)$ in Turiyam Neutrosophic Graphs.

The Plithogenic Domination Number $\gamma_p(PG)$ is defined based on the Degree of Appurtenance Function (DAF) and the Degree of Contradiction Function (DCF). With $s = 4$ and $t = 1$, these functions incorporate the four components of Turiyam Neutrosophic logic.

Therefore, the Plithogenic Domination Number $\gamma_p(PG)$ reduces to the Turiyam Neutrosophic Domination Number $\gamma_t(G)$ when $s = 4$ and $t = 1$. □

Theorem 51. *The Plithogenic Domination Number generalizes the Neutrosophic Domination Number when $s = 3$ and $t = 1$.*

Proof: When $s = 3$ and $t = 1$, the attributes in the Plithogenic Graph correspond to the truth-membership degree $t(v)$, indeterminacy-membership degree $i(v)$, and falsity-membership degree $f(v)$ of a Neutrosophic Graph.

The Plithogenic Domination Number $\gamma_p(PG)$ aligns with the Neutrosophic Domination Number $\gamma_N(G)$ under these parameters.

Therefore, the Plithogenic Domination Number generalizes the Neutrosophic Domination Number when $s = 3$ and $t = 1$. □

Theorem 52. *The Plithogenic Domination Number generalizes the Fuzzy Domination Number when $s = 1$ and $t = 1$.*

Proof: For $s = 1$ and $t = 1$, the Plithogenic Graph simplifies to a Fuzzy Graph with a single attribute representing the vertex membership function $\sigma(v)$.

Under these conditions, the Plithogenic Domination Number $\gamma_p(PG)$ becomes equivalent to the Fuzzy Domination Number $\gamma_f(G)$.

Therefore, the Plithogenic Domination Number generalizes the Fuzzy Domination Number when $s = 1$ and $t = 1$. □

3.2 | Secure Domination set in Turiyam Neutrosophic and Plithogenic Graphs

We consider about Domination set in Secure Domination set in Turiyam Neutrosophic and Plithogenic Graphs. The definition is provided below.

Definition 53 (Turiyam Neutrosophic Secure Dominating Set)**.** In a Turiyam Neutrosophic Graph $G = (V, E)$, a subset $S \subseteq V$ is called a *Turiyam Neutrosophic Secure Dominating Set* if for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

$$t(e) > 0 \quad \text{for edge } e = (u, v),$$

and replacing v with u in S results in a Turiyam Neutrosophic dominating set, i.e.,

$$(S \setminus \{v\}) \cup \{u\} \quad \text{is still a Turiyam Neutrosophic dominating set.}$$

The *Turiyam Neutrosophic Secure Domination Number* $\gamma_{ts}(G)$ is defined as the minimum sum of the truth-membership degrees $t(v)$ in a Turiyam Neutrosophic secure dominating set S:

$$\gamma_{ts}(G) = \min\left\{\sum_{v\in S} t(v) \mid S \text{ is a Turiyam Neutrosophic secure dominating set in } G\right\}.$$

Example 54 (Turiyam Neutrosophic Secure Domination Number)**.** Consider a Turiyam Neutrosophic Graph $G = (V, E)$ with four vertices:

$$V = \{v_1, v_2, v_3, v_4\},$$

and edges E such that every pair of vertices is connected by an edge (v_i, v_j). We assign each vertex v_i a quadruple

$$\sigma(v_i) = (t(v_i),\ iv(v_i),\ fv(v_i),\ lv(v_i)),$$

where $t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4$. Let us define:

$$\sigma(v_1) = (0.8,\ 0.1,\ 0.05,\ 0.05),$$
$$\sigma(v_2) = (0.6,\ 0.2,\ 0.1,\ 0.1),$$
$$\sigma(v_3) = (0.5,\ 0.3,\ 0.1,\ 0.1),$$
$$\sigma(v_4) = (0.4,\ 0.3,\ 0.2,\ 0.1).$$

For simplicity, assume each edge (v_i, v_j) has $t(e) = 0.9$ and $(iv, fv, lv) =$ (small positive). In effect, every vertex can dominate every other vertex, at least in terms of the truth-membership degree.
Constructing a Turiyam Neutrosophic Secure Dominating Set. A dominating set $S \subseteq V$ must ensure that each vertex outside S is dominated by some $v \in S$. Because all edges are strong enough, any vertex in S can dominate all others.

However, *secure* dominance requires that if $u \in V \setminus S$ is dominated by $v \in S$, we can replace v with u and still have a dominating set.

- Suppose we choose $S = \{v_1\}$. Then indeed v_1 dominates v_2, v_3, v_4. But is it *secure*? If we try to replace v_1 with, say, v_2, the new set is $\{v_2\}$. Vertex v_2 is also able to dominate v_3 and v_4. This works for any single vertex since the graph is quite complete. So $\{v_1\}$ is indeed a *Turiyam Neutrosophic Secure Dominating Set*, and so is $\{v_2\}$, or $\{v_3\}$, etc.

- Minimizing $\sum_{v \in S} t(v)$ suggests picking the vertex with the smallest truth-membership might not be best if we want the *sum*. Actually we want the minimal sum, so we pick the vertex with the *lowest* truth-membership or we pick a single vertex with the *lowest* $t(\cdot)$. Suppose

$$t(v_1) = 0.8, \quad t(v_2) = 0.6, \quad t(v_3) = 0.5, \quad t(v_4) = 0.4.$$

Then the minimal sum is achieved by picking $\{v_4\}$. That yields $\sum t(v) = 0.4$.

Hence the *Turiyam Neutrosophic Secure Domination Number* is

$$\gamma_{ts}(G) = 0.4.$$

This example demonstrates both the correctness and typical usage: a single vertex with minimal t-value that can still (securely) dominate the entire graph is indeed the best choice.

Definition 55 (Plithogenic Secure Dominating Set)**.** In a Plithogenic Graph $PG = (PM, PN)$, a subset $S \subseteq M$ is called a *Plithogenic Secure Dominating Set* if for every vertex $u \in M \setminus S$, there exists a vertex $v \in S$ such that:

$$bdf((u, v), (l(u), l(v))) \geq \theta,$$

and replacing v with u in S results in a Plithogenic dominating set, i.e.,

$$(S \setminus \{v\}) \cup \{u\} \quad \text{is still a Plithogenic dominating set.}$$

The *Plithogenic Secure Domination Number* $\gamma_{ps}(PG)$ is defined as the minimum cardinality of a Plithogenic secure dominating set S:

$$\gamma_{ps}(PG) = \min \{|S| \mid S \text{ is a Plithogenic secure dominating set in } PG\}.$$

Example 56 (Plithogenic Secure Domination Number)**.** Consider a Plithogenic Graph $PG = (PM, PN)$ with:

$$PM = (M, l, Ml, adf, aCf), \quad \text{and} \quad PN = (N, m, Nm, bdf, bCf).$$

Let $M = \{u_1, u_2, u_3, u_4\}$. Suppose the aggregator threshold is $\theta = 0.5$. We interpret $bdf((u_i, u_j), (l(u_i), l(u_j)))$ as the measure of how strongly u_i dominates u_j.
Membership and Contradiction Data. Let us assume the following partial data:

$$bdf((u_1, u_2), (\text{Trait}_1, \text{Trait}_2)) = 0.7, \quad bdf((u_1, u_3), (\text{Trait}_1, \text{Trait}_1)) = 0.4, \quad bdf((u_2, u_4), (\text{Trait}_2, \text{Trait}_2)) = 0.9, \ldots$$

We treat edges or adjacency as valid whenever $bdf \geq \theta$. Contradiction function bCf is present but might not be used directly here except to handle multi-attribute conflicts.
Dominating Set. A set $S \subseteq M$ is dominating if for every $u \in M \setminus S$, there is some $v \in S$ with $bdf((v, u), (l(v), l(u))) \geq \theta$.

- Suppose u_1 dominates u_2 because $0.7 \geq 0.5$.
- Suppose u_1 does not dominate u_3 since $0.4 < 0.5$.
- Suppose u_2 dominates u_4 with measure 0.9.

Hence no single vertex can dominate all. For instance, $\{u_1\}$ fails to dominate u_3. But consider $S = \{u_1, u_3\}$. Then

$$\text{(i) } u_2 \notin S \implies bdf((u_1, u_2), \ldots) = 0.7 \geq 0.5 \quad \checkmark$$

$$\text{(ii) } u_4 \notin S \implies \text{maybe } u_3 \text{ or } u_1 \text{ dominates it?}$$

If, e.g., u_3 has measure $bdf((u_3, u_4), \ldots) = 0.6 \geq 0.5$, then $\{u_1, u_3\}$ is indeed a dominating set.

Security Condition. We also need that if a vertex u outside S is dominated by some $v \in S$, replacing v with u in S still yields a dominating set.

- E.g. if u_2 is dominated by u_1, we can form $(S \setminus \{u_1\}) \cup \{u_2\} = \{u_2, u_3\}$. We must check that $\{u_2, u_3\}$ continues to dominate the entire set $\{u_1, u_4\}$. If that holds, the condition is satisfied.
- Similarly for u_4.

If we confirm these replacements work for each dominated vertex, $\{u_1, u_3\}$ is a *Plithogenic Secure Dominating Set*. Then

$$\gamma_{ps}(PG) \leq 2.$$

Potentially $\{u_2, u_3\}$ might also be secure, or a single vertex might not be feasible given the adjacency thresholds. The minimal cardinality could be 2.

Hence in a consistent aggregator scenario, we have $\gamma_{ps}(PG) = 2$. This matches how we define secure dominations in a multi-attribute environment.

The example underscores that (1) the aggregator threshold θ strongly influences which sets are dominating, and (2) the 'secure' condition requires we can swap dominator/dominated pairs. All these steps demonstrate the correctness of the Plithogenic secure domination definition and how it is used in practice.

Theorem 57. *In any Plithogenic Graph PG, the Plithogenic Secure Domination Number $\gamma_{ps}(PG)$ is greater than or equal to the Plithogenic Domination Number $\gamma_p(PG)$:*

$$\gamma_{ps}(PG) \geq \gamma_p(PG).$$

Proof: In a Plithogenic Graph $PG = (PM, PN)$:

- The *Plithogenic Domination Number* $\gamma_p(PG)$ is defined as the minimum cardinality of a Plithogenic dominating set D:

$$\gamma_p(PG) = \min \{|D| \mid D \text{ is a Plithogenic dominating set in } PG\}.$$

- The *Plithogenic Secure Domination Number* $\gamma_{ps}(PG)$ is defined as the minimum cardinality of a Plithogenic secure dominating set S:

$$\gamma_{ps}(PG) = \min \{|S| \mid S \text{ is a Plithogenic secure dominating set in } PG\}.$$

A Plithogenic secure dominating set is a Plithogenic dominating set with an additional property that for every vertex $u \in M \setminus S$, there exists a vertex $v \in S$ such that:

(1) The Degree of Appurtenance Function satisfies:

$$bdf((u,v),(l(u),l(v))) \geq \theta,$$

where θ is a predetermined threshold.

(2) Replacing v with u in S results in a Plithogenic dominating set:

$$(S \setminus \{v\}) \cup \{u\} \quad \text{is a Plithogenic dominating set.}$$

Since every Plithogenic secure dominating set is also a Plithogenic dominating set with extra conditions, the set of Plithogenic secure dominating sets is a subset of the Plithogenic dominating sets. Therefore, the minimal cardinality over Plithogenic secure dominating sets is greater than or equal to that over Plithogenic dominating sets:

$$\gamma_{ps}(PG) \geq \gamma_p(PG).$$

□

Theorem 58. *In any Turiyam Neutrosophic Graph G, the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ is greater than or equal to the Turiyam Neutrosophic Domination Number $\gamma_t(G)$:*

$$\gamma_{ts}(G) \geq \gamma_t(G).$$

Proof: In a Turiyam Neutrosophic Graph $G = (V, E)$:

- The *Turiyam Neutrosophic Domination Number* $\gamma_t(G)$ is defined as the minimum sum of truth-membership degrees over all Turiyam Neutrosophic dominating sets D:

$$\gamma_t(G) = \min\left\{\sum_{v\in D} t(v) \mid D \text{ is a Turiyam Neutrosophic dominating set in } G\right\}.$$

- The *Turiyam Neutrosophic Secure Domination Number* $\gamma_{ts}(G)$ is defined as the minimum sum of truth-membership degrees over all Turiyam Neutrosophic secure dominating sets S:

$$\gamma_{ts}(G) = \min\left\{\sum_{v\in S} t(v) \mid S \text{ is a Turiyam Neutrosophic secure dominating set in } G\right\}.$$

A Turiyam Neutrosophic secure dominating set is a Turiyam Neutrosophic dominating set with the additional property that for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) The truth-membership degree of the edge $e = (u, v)$ satisfies $t(e) > 0$.

(2) Replacing v with u in S results in a Turiyam Neutrosophic dominating set:

$$(S \setminus \{v\}) \cup \{u\} \quad \text{is a Turiyam Neutrosophic dominating set.}$$

Since every Turiyam Neutrosophic secure dominating set is also a Turiyam Neutrosophic dominating set with extra conditions, the minimal sum of truth-membership degrees over Turiyam Neutrosophic secure dominating sets is greater than or equal to that over Turiyam Neutrosophic dominating sets:

$$\gamma_{ts}(G) \geq \gamma_t(G).$$

□

Theorem 59. *For any Turiyam Neutrosophic Graph $G = (V, E)$, the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ is greater than or equal to the Turiyam Neutrosophic Domination Number $\gamma_t(G)$:*

$$\gamma_{ts}(G) \geq \gamma_t(G).$$

Proof: Every Turiyam Neutrosophic secure dominating set is also a Turiyam Neutrosophic dominating set, but with an additional constraint. Therefore, the minimal sum of truth-membership degrees for secure dominating sets is at least as large as that for dominating sets:

$$\gamma_{ts}(G) = \min\left\{\sum_{v\in S} t(v) \mid S \text{ is a Turiyam Neutrosophic secure dominating set}\right\} \geq \gamma_t(G).$$

□

Theorem 60. *Let $G = (V, E)$ be a Turiyam Neutrosophic Graph with $|V| = n$, where each vertex $v \in V$ has a truth-membership degree $t(v) \in [0, 1]$, and each edge $e = (u, v) \in E$ has a truth-membership degree $t(e) \in [0, 1]$.*

The Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$, which represents the minimum sum of truth-membership degrees over a Turiyam Neutrosophic secure dominating set, satisfies the following inequality:

$$\gamma_{ts}(G) \leq n,$$

where $n = |V|$ is the total number of vertices in the graph.

Proof: Let $G = (V, E)$ be a Turiyam Neutrosophic Graph, where each vertex $v \in V$ is assigned a quadruple

$$\sigma(v) = (t(v), iv(v), fv(v), lv(v))$$

, and the truth-membership degree $t(v) \in [0, 1]$.

A *Turiyam Neutrosophic secure dominating set* $S \subseteq V$ is defined such that for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that:

(1) $t(e) > 0$ for the edge $e = (u, v)$,

(2) Replacing v with u in S results in a Turiyam Neutrosophic dominating set, i.e., $(S \setminus \{v\}) \cup \{u\}$ is still a Turiyam Neutrosophic dominating set.

The *Turiyam Neutrosophic Secure Domination Number* $\gamma_{ts}(G)$ is defined as the minimum sum of the truth-membership degrees $t(v)$ in a Turiyam Neutrosophic secure dominating set:

$$\gamma_{ts}(G) = \min \left\{ \sum_{v \in S} t(v) \mid S \text{ is a Turiyam Neutrosophic secure dominating set in } G \right\}.$$

Now, we show that $\gamma_{ts}(G) \leq n$, where $n = |V|$ is the total number of vertices in the graph.

Since $t(v) \in [0, 1]$ for every vertex $v \in V$, the maximum possible value of $t(v)$ is 1.

If every vertex in V is included in a Turiyam Neutrosophic secure dominating set S, the sum of truth-membership degrees is:

$$\sum_{v \in V} t(v) \leq \sum_{v \in V} 1 = n.$$

Thus, the sum of truth-membership degrees over all vertices in V is bounded by n.

Since the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ is the minimum sum of truth-membership degrees over a Turiyam Neutrosophic secure dominating set, it follows that:

$$\gamma_{ts}(G) \leq \sum_{v \in V} t(v) \leq n.$$

Therefore, the maximum possible value of the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ is n. □

Theorem 61. *In a null graph $G = (V, E)$, where both the vertex set $V = \emptyset$ and the edge set $E = \emptyset$, the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ is defined to be* 0.

Proof: A null graph contains no vertices and no edges, i.e., $V = \emptyset$ and $E = \emptyset$. In the context of a Turiyam Neutrosophic Graph, where each vertex is assigned a truth-membership degree $t(v) \in [0, 1]$, a Turiyam Neutrosophic Secure Dominating Set $S \subseteq V$ is required to satisfy that for every vertex $u \in V \setminus S$, there exists a vertex $v \in S$ such that $t(e) > 0$ for the edge $e = (u, v)$, and replacing v with u results in a Turiyam Neutrosophic dominating set.

Since $V = \emptyset$, there are no vertices to dominate, and thus the only possible dominating set is the empty set $S = \emptyset$.

As the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ is defined as the minimum sum of the truth-membership degrees over all vertices in a secure dominating set, we have:

$$\gamma_{ts}(G) = 0.$$

Thus, for a null graph, the Turiyam Neutrosophic Secure Domination Number is 0. □

Theorem 62. *In a null graph $PG = (PM, PN)$, where both the vertex set $M = \emptyset$ and the edge set $N = \emptyset$, the Plithogenic Secure Domination Number $\gamma_{ps}(PG)$ is defined to be* 0.

Proof: A null graph in the context of a Plithogenic Graph means that both the vertex set $M = \emptyset$ and the edge set $N = \emptyset$. In a Plithogenic Graph, a Plithogenic Secure Dominating Set $S \subseteq M$ must satisfy that for every vertex $u \in M \setminus S$, there exists a vertex $v \in S$ such that the Degree of Appurtenance Function (DAF) satisfies:

$$bdf((u, v), (l(u), l(v))) \geq \theta,$$

and replacing v with u results in a Plithogenic dominating set.

Since $M = \emptyset$, there are no vertices to dominate, and hence the only possible dominating set is the empty set $S = \emptyset$.

As the Plithogenic Secure Domination Number $\gamma_{ps}(PG)$ is defined as the minimum cardinality of a Plithogenic secure dominating set, we have:

$$\gamma_{ps}(PG) = 0.$$

Thus, for a null graph, the Plithogenic Secure Domination Number is 0. □

Theorem 63. *The Turiyam Neutrosophic Secure Domination Number generalizes both the Neutrosophic Secure Domination Number and the Fuzzy Secure Domination Number.*

Proof: The Turiyam Neutrosophic Graph extends the Neutrosophic Graph by incorporating the liberal state value $l(v)$. By setting $l(v)$ to a constant or ignoring it, the Turiyam Neutrosophic Graph reduces to a Neutrosophic Graph.

Similarly, by setting the indeterminacy-membership degree $i(v)$ and the falsity-membership degree $f(v)$ to zero or constants, the Neutrosophic Graph simplifies to a Fuzzy Graph.

Therefore, the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$ generalizes both the Neutrosophic Secure Domination Number $\gamma_{ns}(G)$ and the Fuzzy Secure Domination Number $\gamma_{fs}(G)$. □

Theorem 64. *The Plithogenic Secure Domination Number generalizes the Turiyam Neutrosophic Secure Domination Number when $s = 4$ and $t = 1$.*

Proof: In a Plithogenic Graph PG, each vertex and edge are characterized by s attributes and a threshold t. When $s = 4$ and $t = 1$, these attributes correspond to the truth-membership degree $t(v)$, indeterminacy-membership degree $i(v)$, falsity-membership degree $f(v)$, and liberal state value $l(v)$ in a Turiyam Neutrosophic Graph.

The Plithogenic Secure Domination Number $\gamma_{ps}(PG)$ is defined based on the Degree of Appurtenance Function (DAF) and the Degree of Contradiction Function (DCF). With these parameters, the definitions and conditions of the Plithogenic Secure Domination Number align with those of the Turiyam Neutrosophic Secure Domination Number $\gamma_{ts}(G)$.

Therefore, the Plithogenic Secure Domination Number generalizes the Turiyam Neutrosophic Secure Domination Number when $s = 4$ and $t = 1$. □

Theorem 65. *The Plithogenic Secure Domination Number generalizes the Neutrosophic Secure Domination Number when $s = 3$ and $t = 1$.*

Proof: When $s = 3$ and $t = 1$, the attributes in the Plithogenic Graph correspond to the truth-membership degree $t(v)$, indeterminacy-membership degree $i(v)$, and falsity-membership degree $f(v)$ of a Neutrosophic Graph.

Under these parameters, the Plithogenic Secure Domination Number $\gamma_{ps}(PG)$ reduces to the Neutrosophic Secure Domination Number $\gamma_{ns}(G)$, as the definitions and conditions coincide.

Therefore, the Plithogenic Secure Domination Number generalizes the Neutrosophic Secure Domination Number when $s = 3$ and $t = 1$. □

Theorem 66. *The Plithogenic Secure Domination Number generalizes the Fuzzy Secure Domination Number when $s = 1$ and $t = 1$.*

Proof: For $s = 1$ and $t = 1$, the Plithogenic Graph simplifies to a Fuzzy Graph with a single attribute representing the membership function $\sigma(v)$.

In this case, the Plithogenic Secure Domination Number $\gamma_{ps}(PG)$ becomes equivalent to the Fuzzy Secure Domination Number $\gamma_{fs}(G)$.

Therefore, the Plithogenic Secure Domination Number generalizes the Fuzzy Secure Domination Number when $s = 1$ and $t = 1$. □

3.3 | Turiyam Neutrosophic and Plithogenic Chromatic Number

In this section, we consider the Turiyam Neutrosophic Chromatic Number and the Plithogenic Chromatic Number. These concepts extend the classical chromatic number by incorporating the specific conditions and properties of Turiyam Neutrosophic graphs and Plithogenic graphs. We provide the definitions below.

Definition 67 (Turiyam Neutrosophic Vertex Coloring)**.** Let $G = (V, E)$ be a Turiyam Neutrosophic Graph. A *k-Turiyam Neutrosophic Vertex Coloring* is a mapping $c : V \to \{1, 2, \dots, k\}$ that assigns one of k colors to each vertex such that the following conditions are satisfied for every edge $e = (u, v) \in E$:

(1) If the truth-membership degree $t(e)$ is significant (i.e., $t(e) \geq \alpha$, where $\alpha \in (0, 1]$ is a predefined threshold), then u and v must be assigned *different colors*:
$$t(e) \geq \alpha \implies c(u) \neq c(v).$$

(2) If the falsity-membership degree $fv(e)$ is significant (i.e., $fv(e) \geq \beta$, with $\beta \in (0, 1]$), then u and v may be assigned the *same color*:
$$fv(e) \geq \beta \implies c(u) = c(v).$$

(3) If the indeterminacy-membership degree $iv(e)$ is significant (i.e., $iv(e) \geq \gamma$, with $\gamma \in (0, 1]$), then the coloring of u and v is *indeterminate*; they may be assigned the same or different colors.

(4) If the liberal state value $lv(e)$ is significant (i.e., $lv(e) \geq \delta$, with $\delta \in (0, 1]$), then u and v can be assigned any colors without restrictions.

Definition 68 (Turiyam Neutrosophic Chromatic Number)**.** The *Turiyam Neutrosophic Chromatic Number* of a Turiyam Neutrosophic Graph G, denoted $\chi_t(G)$, is the smallest integer k such that there exists a k-Turiyam Neutrosophic Vertex Coloring of G.

Theorem 69. *The Turiyam Neutrosophic Chromatic Number generalizes both the Neutrosophic Chromatic Number and the Fuzzy Chromatic Number.*

Proof: The Turiyam Neutrosophic Graph extends the Neutrosophic Graph by adding a liberal state value $l(e)$. By setting $l(e)$ to a constant or ignoring it, the Turiyam Neutrosophic Graph reduces to a Neutrosophic Graph.

Further, by setting the indeterminacy-membership degree $i(e)$ and falsity-membership degree $f(e)$ to zero or constants, the Neutrosophic Graph simplifies to a fuzzy graph. Therefore, the Turiyam Neutrosophic Chromatic Number $\chi_t(G)$ generalizes both $\chi_v(G)$ and $\tilde{\chi}(G)$. □

Definition 70 (Plithogenic Vertex Coloring)**.** Let $PG = (PM, PN)$ be a Plithogenic Graph. A *k-Plithogenic Vertex Coloring* is a mapping $c : M \to \{1, 2, \dots, k\}$ that assigns one of k colors to each vertex such that for every edge $e = (u, v) \in N$, the following condition holds:

$$aCf(l(u), l(v)) \geq \theta \implies c(u) \neq c(v),$$

where:

- $l(u)$ and $l(v)$ are the attribute values of vertices u and v, respectively.
- $aCf(l(u), l(v))$ is the degree of contradiction between the attributes of u and v.
- $\theta \in [0, 1]$ is a predetermined threshold indicating the level of contradiction that requires adjacent vertices to be colored differently.

In other words, if the contradiction between the attributes of two adjacent vertices exceeds the threshold θ, then those vertices must be assigned different colors.

Definition 71 (Plithogenic Chromatic Number)**.** The *Plithogenic Chromatic Number* of a Plithogenic Graph PG, denoted $\chi_p(PG)$, is the smallest integer k such that there exists a k-Plithogenic Vertex Coloring of PG.

Theorem 72. *In a null graph $G = (V, E)$, where both the vertex set $V = \emptyset$ and the edge set $E = \emptyset$, the Turiyam Neutrosophic Chromatic Number $\chi_t(G)$ is defined to be* 0*.*

Proof: A null graph contains no vertices and no edges, i.e., $V = \emptyset$ and $E = \emptyset$. In the context of a Turiyam Neutrosophic Graph, a *k-Turiyam Neutrosophic Vertex Coloring* is defined by assigning one of k colors to each vertex such that for every edge $e = (u, v) \in E$, the truth-membership degree $t(e)$ governs whether u and v should have different colors. Since $V = \emptyset$, there are no vertices to color, and thus no need for any colors.

By convention, the chromatic number of a graph with no vertices is defined as 0. Hence, the Turiyam Neutrosophic Chromatic Number for a null graph is 0. □

Theorem 73. *In a null graph $PG = (PM, PN)$, where both the vertex set $M = \emptyset$ and the edge set $N = \emptyset$, the Plithogenic Chromatic Number $\chi_p(PG)$ is defined to be* 0*.*

Proof: A null graph in the context of a Plithogenic Graph means that both the vertex set $M = \emptyset$ and the edge set $N = \emptyset$. In a Plithogenic Graph, a *k-Plithogenic Vertex Coloring* assigns colors to the vertices in such a way that if the contradiction between the attributes of two adjacent vertices exceeds a given threshold θ, those vertices must be colored differently.

Since $M = \emptyset$, there are no vertices to color, and thus no need for any colors. Therefore, the Plithogenic Chromatic Number, which represents the minimum number of colors needed for a valid coloring, is 0.

Thus, for a null graph, the Plithogenic Chromatic Number is 0. □

Theorem 74. *The Plithogenic Chromatic Number generalizes the Turiyam Neutrosophic Chromatic Number when $s = 4$ and $t = 1$.*

Proof: In a Plithogenic Graph PG, each edge is characterized by s attributes and a threshold t. When $s = 4$ and $t = 1$, the attributes correspond to the truth-membership degree $t(e)$, indeterminacy-membership degree $i(e)$, falsity-membership degree $f(e)$, and liberal state value $l(e)$ in Turiyam Neutrosophic Graphs.

The Plithogenic Vertex Coloring condition is:

$$aCf(l(u), l(v)) \geq \theta \implies c(u) \neq c(v),$$

where aCf is the attribute contradiction function, $l(u)$ and $l(v)$ are the attribute values of vertices u and v, and θ is a predefined threshold.

When $s = 4$ and $t = 1$, this condition becomes identical to the Turiyam Neutrosophic Vertex Coloring condition, which accounts for the four components of the Turiyam Neutrosophic logic. Thus, the Plithogenic Chromatic Number $\chi_p(PG)$ reduces to the Turiyam Neutrosophic Chromatic Number $\chi_t(G)$. □

Theorem 75. *The Plithogenic Chromatic Number generalizes the Neutrosophic Chromatic Number when $s = 3$ and $t = 1$.*

Proof: For $s = 3$ and $t = 1$, the Plithogenic Graph's attributes correspond to the truth-membership degree $t(e)$, indeterminacy-membership degree $i(e)$, and falsity-membership degree $f(e)$ of a Neutrosophic Graph.

The Plithogenic coloring condition simplifies to:

$$aCf(l(u), l(v)) \geq \theta \implies c(u) \neq c(v).$$

With $s = 3$, this condition matches the Neutrosophic Vertex Coloring criteria. Therefore, the Plithogenic Chromatic Number $\chi_p(PG)$ becomes equivalent to the Neutrosophic Chromatic Number $\chi_v(G)$. □

Theorem 76. *The Plithogenic Chromatic Number generalizes the Fuzzy Chromatic Number when $s = 1$ and $t = 1$.*

Proof: When $s = 1$ and $t = 1$, the Plithogenic Graph reduces to a fuzzy graph with a single attribute representing the edge membership function $\mu(e)$.

The Plithogenic coloring condition reduces to:

$$aCf(l(u), l(v)) \geq \theta \implies c(u) \neq c(v),$$

which aligns with the Fuzzy Vertex Coloring condition based on α-cuts. Thus, the Plithogenic Chromatic Number $\chi_p(PG)$ reduces to the Fuzzy Chromatic Number $\tilde{\chi}(G)$. □

4 | Future tasks

In the future, we will explore the domination number and chromatic number in Regular Turiyam Neutrosophic Graphs. Although still in the conceptual stage, the description of the Regular Turiyam Neutrosophic Graph is as follows.

Definition 77 (Regular Turiyam Neutrosophic Graph)**.** A *Regular Turiyam Neutrosophic Graph* $G = (V, E)$ is a Turiyam Neutrosophic graph where for every vertex $u \in V$, the sum of the truth-membership degrees, indeterminacy-membership degrees, falsity-membership degrees, and liberal-state degrees between u and all other distinct vertices $v \in V \setminus \{u\}$ is constant. Formally, the graph satisfies the following conditions:

$$\sum_{u \neq v} t(u, v) = \text{Constant},$$
$$\sum_{u \neq v} iv(u, v) = \text{Constant},$$
$$\sum_{u \neq v} fv(u, v) = \text{Constant},$$
$$\sum_{u \neq v} lv(u, v) = \text{Constant},$$

where $t(u, v)$, $iv(u, v)$, $fv(u, v)$, and $lv(u, v)$ represent the truth-membership, indeterminacy, falsity, and liberal-state degrees, respectively, between the vertices u and v.

In other words, a Regular Turiyam Neutrosophic Graph has constant aggregate membership values across all vertices for these four degrees, making it regular in the Turiyam Neutrosophic sense.

Furthermore, other well-known concepts such as rough graphs [29], soft graphs [65, 25], Neutrosophic Offset (Offgraph)[95, 99, 100], Treesoft sets(Treesoft Graph)[8, 75, 39], and superhypergraphs [97, 28, 98] are also recognized in the field. We aim to advance research on properties like the domination number and chromatic number within these frameworks.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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