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Smarandache Curved Ruled Surfaces and Their Characterizations According to Modified Orthogonal Frame in E^3

Gülnur Şaffak Atalay

Department of Mathematics and Science Education
Faculty of Education
Ondokuz Mayıs University
55270, Samsun, Turkey
gulnur.saffak@omu.edu.tr

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Abstract

In this study, the ruled surfaces generated by Smarandache curves are expressed according to the modified orthogonal frame defined according to both curvature and torsion of the given unit speed curve in 3-dimensional Euclidean space; some special characterizations of these surfaces such as developability, striction curves, distribution parameters are given. In addition, some examples of these surfaces are given in graphical form. We use Maple to draw the graphs of the examples.

Keywords: Modified orthogonal frame with curvature and torsion; Smarandache curve; Striction curve; Ruled surface; Developable

MSC2010: 53A04, 53A05

1. Introduction

Curves and surfaces are one of the most important basic subjects of differential geometry, and we encounter this subject in almost every differential geometry book. Recently, there have been studies on characterising surfaces, including characterising curves defined according to different frame and special surfaces such as ruled and tubular surfaces (for details, see Altın (2020), Li (2022), Eren (2022)). In contrast, a ruled surface is a surface that can be traced by moving a line in space and can be used in various fields such as architecture, CAD, electric discharge machining and etc. (Brauner (1977), Ravani (1991), Hanbay (2016)). Examples of ruled surfaces are cylinders, cones, conical surfaces, right conics and helicoids. As it is known, there are various frames that can be installed on a curve, and the one that is most frequently studied is the Frenet frame. Although the Frenet frame is a frame that characterizes the curve, one of its disadvantages is that this frame cannot be defined if the curvature of the curve is zero. Bishop (1975) eliminated this disadvantage and defined a new frame, the Bishop frame. In addition, Sasai (1984) defined the modified orthogonal frame at points where the curvature is different from zero. Bükçü and Karacan (2016) expressed Sasai's work in 3-dimensional Minkowski space. Additionally, they gave a new version of the modified orthogonal frame with torsion in three dimensional Euclidean and Minkowski space. Also, there have been various studies involving special curve pairs and special surfaces based on the modified orthogonal frame (for details, see Lone (2019), Baş (2019), Azak (2021), Akyiğit (2021), Atalay (2024) and Yakut (2024)). Smarandache curves are a family of geometric curves in Smarandache geometry. These curves have many applications in various fields of mathematics and science and are often discussed in the context of differential geometry. In the field of robotics, Smarandache curves could be applied to path planning for autonomous systems, where the curvature and torsional properties of a path are important to minimise the energy consumption or to maximise the smoothness of the motion.

In mechanical design, especially the design of gears, cams and linkages, Smarandache curves can be used to define the shape of moving parts that must follow specific motion constraints. The geometric properties of the curves can be useful in the design of systems that require smooth, continuous motion with no abrupt changes in speed or direction. Smarandache curves in Euclidean space, and more specifically curves whose curvature and torsion follow certain relations or constraints, are the most commonly studied types of Smarandache curves. Turgut and Yılmaz (2008) introduced Smarandache curves in Minkowski space E_1^4; Frenet frame vectors on another regular curve form its position vector. They also identified a special case of this type, called Smarandache TB_2. Later, Ali (2010), introduced special curves of Frenet-Serret vector fields in Euclidean space, known as Smarandache TN, NB, and TNB curves. Smarandache curves related to Darboux frames in Minkowski space of 3-dimensions were studied in Abdel-Aziz and Saad (2017). In addition, in the 3-dimensional Euclidean space, Smarandache curved ruled surfaces with respect to the Frenet frame, the Darboux frame and the alternative frame and their characterisations have been studied by Ouarab (see references Ouarab (2021a, 2021b, 2021c)). Recently, Nurkan and Güven (2022) defined the adjoint curves of Smarandache curves defined according to the Frenet frame in 3-dimensional Euclidean space and obtained necessary and sufficient conditions for these curves to be general helices and slant helices. The studies of these types of curves are also available in the literature (Kahraman (2014), Taşköprü (2014), Atalay (2016), Elzawy (2017), Solouma (2017)). Şenyurt et al. (2023) studied the special Smarandache ruled surfaces and their characterisations obtained according to the Flc frame in 3-dimensional Euclidean space. Very recently, Atalay (2025) defined the pairs of ruled surfaces of the curve given by the Sabban frame defined on the unit sphere and investigate various characterisations such as developability and minimality in 3-dimensional Euclidean space.

In this paper, we construct a ruled surface using Smarandache curves defined with respect to a modified orthogonal frame having both curvature and torsion in 3-dimensional Euclidean space and give characterizations of this surface. Finally, we give an example and illustrate the corresponding graphs of the ruled surfaces with the help of Maple 17.

2. Ruled Surfaces Related to Smarandache Curves and Their Characterizations in E^3

2.1. Ruled surfaces related to Smarandache curves according to modified orthogonal frame with curvature in E^3

In this subsection, we first construct ruled surfaces by means of the curve $\alpha(s)$ defined with respect to the modified orthogonal frame with curvature and its Smarandache curves. We also obtain the distribution parameters and striction curves for each of these ruled surfaces and give some characterisations for them. Throughout this subsection, base curves of ruled surfaces are taken as curve (s).

Definition 2.1.

Let $\alpha = \alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\kappa, N_\kappa, B_\kappa\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $T_\kappa N_\kappa$ Smarandache curve (expressed with $\beta_\kappa(s)$), we define $T_\kappa N_\kappa$ Smarandache ruled surface as following:

$$\phi_{\beta_\kappa}(s,v) = \alpha(s) + v\beta_\kappa(s) = \alpha(s) + v\left(\frac{1}{\sqrt{1+\kappa^2}}(T_\kappa + N_\kappa)\right). \quad (1)$$

Theorem 2.1.

(i.) The ruled surface $\varphi_{\beta_\kappa}(s,v)$ is developable if and only if $\alpha(s)$ is a plane curve.
(ii.) The base curve and the striction curve of $\varphi_{\beta_\kappa}(s,v)$ never intersect.

Proof :

(i.) From the definition (see B. O'Neill (1966)), the distribution parameter of $\varphi_{\beta_\kappa}(s,v)$ is

$$\delta_{\varphi_{\beta_\kappa}} = \frac{\tau(1+\kappa^2)^2}{(\kappa')^2(\kappa^2+\kappa^4) - 2(\kappa')^2\kappa^2(1+\kappa^2) + (1+\kappa^2)^2(\kappa^2+\kappa^4+2\kappa\kappa' + (\kappa')^2 + \kappa^2\tau^4)}.$$

The necessary and sufficient condition for the distribution parameter to be 0 is that $\tau = 0$. Therefore, the curve $\alpha(s)$ must be planar for the ruled surface $\varphi_{\beta_\kappa}(s,v)$ to be developable.

(ii.) From the definition (see B. O'Neill (1966)), the striction curve $\mu_{\varphi_{\beta_\kappa}}(s)$ on $\varphi_{\beta_\kappa}(s,v)$ is

$$\mu_{\varphi_{\beta_\kappa}}(s) = \alpha(s) + \frac{(1+\kappa^2)^{3/2}\left(\kappa\kappa' + \kappa^2(1+\kappa^2)^3\right)}{(\kappa')^2(\kappa^2+\kappa^4) - 2(\kappa')^2\kappa^2(1+\kappa^2) + (1+\kappa^2)^2(\kappa^2+\kappa^4+2\kappa\kappa' + (\kappa')^2 + \kappa^2\tau^4)}\beta_\kappa(s).$$

From this it is clear that the striction curve will not coincide with the base curve. So, the proof is complete. ■

Definition 2.2.

Let $\alpha = \alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\kappa, N_\kappa, B_\kappa\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $T_\kappa B_\kappa$ Smarandache curve (expressed with $\gamma_\kappa(s)$), we define $T_\kappa B_\kappa$ Smarandache ruled surface as following:

$$\begin{aligned}\varphi_{\gamma_\kappa}(s,v) &= \alpha(s) + v\gamma_\kappa(s)\\ &= \alpha(s) + v\left(\frac{1}{\sqrt{1+\kappa^2}}\left(T_\kappa + B_\kappa\right)\right).\end{aligned} \tag{2}$$

Theorem 2.2.

(i.) The ruled surface $\varphi_{\gamma_\kappa}(s,v)$ is developable if and only if to be is that $\tau = 1$.
(ii.) The base curve and the striction curve of $\varphi_{\gamma_\kappa}(s,v)$ do not intersect. In particular, if the curvature κ of the alpha curve is constant, the striction curve and the base curve coincide.

Proof:

(i.) The distribution parameter of $\varphi_{\gamma_\kappa}(s,v)$ is

$$\delta_{\varphi_{\gamma_\kappa}} = \frac{(\tau-1)(1+\kappa^2)^2}{(\kappa')^2(\kappa^4+\kappa^2)+(1+\kappa^2)^2(\kappa^2(1-\tau^2)+(\kappa')^2)-2(\kappa')^2\kappa^2(1+\kappa^2)}.$$

The necessary and sufficient condition for the distribution parameter to be 0 is that $\tau = 1$.

(ii.) The striction curve $\mu_{\varphi_{\gamma_\kappa}}(s)$ on $\varphi_{\gamma_\kappa}(s,v)$ is

$$\mu_{\varphi_{\gamma_\kappa}}(s) = \alpha(s) + \frac{\kappa\kappa'(1+\kappa^2)^{3/2}}{(\kappa')^2(\kappa^4+\kappa^2)+(1+\kappa^2)^2(\kappa^2(1-\tau^2)+(\kappa')^2)-2(\kappa')^2\kappa^2(1+\kappa^2)}\gamma_\kappa(s).$$

As can be seen from this equation, if the curvature K of the alpha is not constant, the striction curve and the base curve do not coincide. So, the proof is complete. ■

Definition 2.3.

Let $\alpha = \alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\kappa, N_\kappa, B_\kappa\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $N_\kappa B_\kappa$ Smarandache curve (expressed with $\varsigma_\kappa(s)$), we define $N_\kappa B_\kappa$ Smarandache ruled surface as following:

$$\varphi_{\varsigma_\kappa}(s,v) = \alpha(s) + v\varsigma_\kappa(s)$$
$$= \alpha(s) + v\left(\frac{1}{\kappa\sqrt{2}}\left(N_\kappa + B_\kappa\right)\right). \quad (3)$$

Theorem 2.3.

(i.) The ruled surface $\varphi_{\varsigma_\kappa}(s,v)$ is developable if and only if $\alpha(s)$ is a plane curve.
(ii.) The base curve and the striction curve of $\varphi_{\varsigma_\kappa}(s,v)$ never intersect.

***Proof*:**

(i.) The distribution parameter of $\varphi_{\varsigma_\kappa}(s,v)$ is

$$\delta_{\varphi_{\varsigma_\kappa}} = \frac{2\tau}{\kappa^2(2\tau^2 + \kappa^2)}.$$

The necessary and sufficient condition for the distribution parameter to be 0 is that $\tau = 0$. Therefore, the curve $\alpha(s)$ must be planar for the ruled surface $\varphi_{\varsigma_\kappa}(s,v)$ to be developable.

(ii.) The striction curve $\mu_{\varphi_{\varsigma_\kappa}}(s)$ on $\varphi_{\varsigma_\kappa}(s,v)$ is

$$\mu_{\varphi_{\varsigma_\kappa}}(s) = \alpha(s) + \sqrt{2}\,\frac{\kappa}{2\tau^2 + \kappa^2}\,\varsigma_\kappa(s).$$

Since $\kappa \neq 0$, (ii) is obvious. So, the proof is complete. ■

Definition 2.4.

Let $\alpha = \alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\kappa, N_\kappa, B_\kappa\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $T_\kappa N_\kappa B_\kappa$ Smarandache curve (expressed with $\varsigma_\kappa(s)$), we define $T_\kappa N_\kappa B_\kappa$ Smarandache ruled surface as following:

$$\varphi_{\xi_\kappa}(s,v) = \alpha(s) + v\xi_\kappa(s)$$
$$= \alpha(s) + v\left(\frac{1}{\sqrt{1+2\kappa^2}}\left(T_\kappa + N_\kappa + B_\kappa\right)\right). \quad (4)$$

Theorem 2.4.

(i.) The ruled surface $\varphi_{\xi_\kappa}(s,v)$ is developable if and only if $\tau = 1/2$.
(ii.) The base curve and the striction curve of $\varphi_{\xi_\kappa}(s,v)$ never intersect.

Proof :

(i.) The distribution parameter of $\varphi_{\xi_\kappa}(s,v)$ is

$$\delta\varphi_{\xi_\kappa} = \frac{\left(2\tau-1\right)(1+2\kappa^2)^2}{\begin{array}{l}(4\left(\kappa'\right)^2\kappa^2+5\kappa^4+\kappa^2+2\kappa\kappa'-2\kappa^2\tau+2\kappa^2\tau^2+2\left(\kappa'\right)^2+8\kappa^6+8\kappa^3\kappa' \\ +4\kappa^8+8\kappa^5\kappa'-8\kappa^4\tau+8\kappa^6\tau+8\kappa^4\tau^2-8\kappa^6\tau^2)\end{array}}.$$

The necessary and sufficient condition for the distribution parameter to be 0 is that $\tau = 1/2$.

(ii.) The striction curve $\mu_{\varphi_{\xi_\kappa}}(s)$ on $\varphi_{\xi_\kappa}(s,v)$ is

$$\mu_{\varphi_{\xi_\kappa}}(s) = \alpha(s) + \frac{2\kappa\kappa'(1+2\kappa^2)^{3/2}+\kappa^2(1+2\kappa^2)^{5/2}}{\begin{array}{l}(4\left(\kappa'\right)^2\kappa^2+5\kappa^4+\kappa^2+2\kappa\kappa'-2\kappa^2\tau+2\kappa^2\tau^2+2\left(\kappa'\right)^2+8\kappa^6 \\ +8\kappa^3\kappa'+4\kappa^8+8\kappa^5\kappa'-8\kappa^4\tau+8\kappa^6\tau+8\kappa^4\tau^2-8\kappa^6\tau^2)\end{array}}\xi_\kappa(s).$$

Since $\kappa \neq 0$, (ii) is obvious. So, the proof is complete. ■

2.2. Ruled surfaces related to Smarandache curves according to modified orthogonal frame with torsion in E³

In this subsection, we first construct ruled surfaces by means of the curve $\alpha(s)$ defined with respect to the modified orthogonal frame with torsion and its Smarandache curves. We also obtain the distribution parameters and striction curves for each of these ruled surfaces and give some characterisations for them. Throughout this subsection, base curves of ruled surfaces are taken as curve $\alpha(s)$.

Definition 2.5.

Let $\alpha = \alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\tau, N_\tau, B_\tau\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $T_\tau N_\tau$ Smarandache curve (expressed with $\beta_\tau(s)$), we define $T_\tau N_\tau$ Smarandache ruled surface as following:

$$\begin{aligned}\varphi_{\beta_\tau}(s,v) &= \alpha(s) + v\beta_\tau(s) \\ &= \alpha(s) + v\left(\frac{1}{\sqrt{1+\tau^2}}\left(T_\tau + N_\tau\right)\right).\end{aligned} \tag{5}$$

Theorem 2.5.

(i.) The ruled surface $\varphi_{\beta_\tau}(s,v)$ is not developable.

(ii.) The base curve and the striction curve of $\varphi_{\beta_\tau}(s,v)$ never intersect.

Proof :

(i.) From the definition (see B. O'Neill (1966)), the distribution parameter of $\varphi_{\beta_\tau}(s,v)$ is

$$\delta_{\varphi_{\beta_\tau}} = \frac{\tau\left(1+\tau^2\right)^2}{(\tau')^2\left(\tau^2+\tau^4\right)-2\tau'\tau^2(1+\tau^2)+(1+\tau^2)^2\left(\kappa^2\tau^2+\kappa^2+2\kappa\tau'+(\tau')^2+\tau^4\right)}.$$

Since $\tau\neq 0$, (i) is obvious.

(ii.) From the definition (see B. O'Neill (1966)), the striction curve $\mu_{\varphi_{\beta_\tau}}(s)$ on $\varphi_{\beta_\tau}(s,v)$ is

$$\mu_{\varphi_{\beta_\tau}}(s)=\alpha(s)+\frac{\tau\tau'+(1+\tau^2)\kappa\tau}{(\tau')^2\left(\tau^2+\tau^4\right)-2\tau'\tau^2(1+\tau^2)+(1+\tau^2)^2\left(\kappa^2\tau^2+\kappa^2+2\kappa\tau'+(\tau')^2+\tau^4\right)}(1+\tau^2)^{3/2}\beta_\tau(s).$$

Since $\kappa\neq 0$ and $\tau\neq 0$, (ii) is obvious. So, the proof is complete. ■

Definition 2.6.

Let $\alpha=\alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\tau, N_\tau, B_\tau\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $T_\tau B_\tau$ Smarandache curve (expressed with $\gamma_\tau(s)$), we define $T_\tau B_\tau$ Smarandache ruled surface as following:

$$\begin{aligned}\varphi_{\gamma_\tau}(s,v)&=\alpha(s)+v\gamma_\tau(s)\\ &=\alpha(s)+v\left(\frac{1}{\sqrt{1+\tau^2}}\left(T_\tau+B_\tau\right)\right).\end{aligned} \tag{6}$$

Theorem 2.6.

(i.) The ruled surface $\varphi_{\gamma_\tau}(s,v)$ is developable if and only if $\kappa=\tau^2$ and $\tau\neq$ constant *t*.
(ii.) The base curve and the striction curve of $\varphi_{\gamma_\tau}(s,v)$ is not coincide. In particular, if the curvature τ of the alpha curve is constant and $\kappa\neq\tau^2$, the striction curve and the base curve coincide.

Proof :

(i.) The distribution parameter of $\varphi_{\gamma_\tau}(s,v)$ is

$$\delta_{\varphi_{\gamma_\tau}} = \frac{(\tau^2-\kappa)(1+\tau^2)^{3/2}}{\tau\left(\left(\tau'\right)^2\left(\tau^2+\tau^4\right)-2\tau^2\left(\tau'\right)^2(1+\tau^2)+(1+\tau^2)^2\left(\left(\tau'\right)^2+(\kappa-\tau^2)^2\right)\right)}.$$

The necessary and sufficient condition for the distribution parameter to be 0 is that $\kappa=\tau^2$ and $\tau\neq$

constant t.

(ii.) The striction curve $\mu_{\varphi_{\gamma_\tau}}(s)$ on $\varphi_{\gamma_\tau}(s,v)$ is

$$\mu_{\varphi_{\gamma_\tau}}(s)=\alpha(s)+\frac{\tau\tau'(1+\tau^2)^{3/2}}{(\tau')^2\left(\tau^2+\tau^4\right)-2\tau^2(\tau')^2(1+\tau^2)+(1+\tau^2)^2\left((\tau')^2+(\kappa-\tau^2)^2\right)}\gamma_\tau(s).$$

Here, if $\tau=$ constant t and $\kappa\neq\tau^2$, the striction curve and the base curve coincide. So, the proof is complete. ■

Definition 2.7.

Let $\alpha=\alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\tau, N_\tau, B_\tau\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $N_\tau B_\tau$ Smarandache curve (expressed with $\varsigma_\tau(s)$), we define $N_\tau B_\tau$ Smarandache ruled surface as following:

$$\begin{aligned}\varphi_{\varsigma_\tau}(s,v)&=\alpha(s)+v\varsigma_\tau(s)\\&=\alpha(s)+v\left(\frac{1}{\tau\sqrt{2}}\left(N_\tau+B_\tau\right)\right).\end{aligned}\tag{7}$$

Theorem 2.7.

(i.) The ruled surface $\varphi_{\varsigma_\tau}(s,v)$ is not developable.
(ii.) The base curve and the striction curve of $\varphi_{\varsigma_\tau}(s,v)$ never intersect.

Proof :

(i.) The distribution parameter of $\varphi_{\varsigma_\tau}(s,v)$ is

$$\delta_{\varphi_{\varsigma_\tau}}=\frac{2}{\tau(\kappa^2+2\tau^2)}.$$

As can be seen, this equality is always different from 0. So, (i) is obvious.

(ii.) The striction curve $\mu_{\varphi_{\varsigma_\tau}}(s)$ on $\varphi_{\varsigma_\tau}(s,v)$ is

$$\mu_{\varphi_{\varsigma_\tau}}(s)=\alpha(s)+\sqrt{2}\frac{\kappa}{\kappa^2+2\tau^2}\varsigma_\tau(s).$$

Since $\kappa\neq 0$, (ii) is obvious. So, the proof is complete. ■

Definition 2.8.

Let $\alpha=\alpha(s)$ be a unit speed curve in E^3 and denote $\{T_\tau, N_\tau, B_\tau\}$ as the modified orthogonal frame with curvature of α. By taking the base curve as α and the ruling vector as $T_\tau N_\tau B_\tau$ Smarandache curve (expressed with $\xi_\tau(s)$), we define $T_\tau N_\tau B_\tau$ Smarandache ruled surface as following:

$$\begin{aligned}\varphi_{\xi_\tau}(s,v) &= \alpha(s) + v\xi_\tau(s)\\ &= \alpha(s) + v\left(\frac{1}{\sqrt{1+2\tau^2}}\left(T_\tau + N_\tau + B_\tau\right)\right).\end{aligned} \tag{8}$$

Theorem 2.8.

(i.) The ruled surface $\varphi_{\xi_\tau}(s,v)$ is developable if and only if $\kappa = 2\tau^2$.

(ii.) The base curve and the striction curve of $\varphi_{\xi_\tau}(s,v)$ never intersect.

Proof :

(i.) The distribution parameter of $\varphi_{\xi_\tau}(s,v)$ is

$$\delta_{\varphi_{\xi_\tau}} = \frac{(2\tau^2-\kappa)(1+2\tau^2)^2}{\begin{array}{c}\tau(\kappa^2+(\tau')^2+\tau^4+2\kappa\tau'-2\kappa\kappa^2-2\tau'\tau^2+5\kappa^2\tau^2+4\tau^2(\tau')^2+4\tau^6+8\kappa\tau'\tau^2-8\kappa\tau^4\\ -8\tau'\tau^4+8\kappa^2\tau^4+4\tau^8+8\kappa\tau'\tau^4-8\kappa\tau^6-4\tau'\tau^6+4\kappa^2\tau^6+8\tau'\tau^8)\end{array}}.$$

The necessary and sufficient condition for the distribution parameter to be 0 is that $\kappa = 2\tau^2$.

(ii.) The striction curve $\mu_{\varphi_{\xi_\tau}}(s)$ on $\varphi_{\xi_\tau}(s,v)$ is

$$\mu_{\varphi_{\xi_\tau}}(s) = \alpha(s) + \frac{2\tau'\tau(1+2\tau^2)^{3/2}+\kappa\tau(1+2\tau^2)^{5/2}}{\begin{array}{c}(\kappa^2+(\tau')^2+\tau^4+2\kappa\tau'-2\kappa'\kappa^2-2\tau'\tau^2+5\kappa^2\tau^2+4\tau^2(\tau')^2+4\tau^6+8\kappa\tau'\tau^2-8\kappa\tau^4\\ -8\tau'\tau^4+8\kappa^2\tau^4+4\tau^8+8\kappa\tau'\tau^4-8\kappa\tau^6-4\tau'\tau^6+4\kappa^2\tau^6+8\tau'\tau^8)\end{array}}\xi_\tau(s).$$

As can be seen from this equation, since $\kappa \neq 0$ and $\tau \neq 0$, the striction curve does not coincide with the base curve. So, the proof is complete. ■

3. Example of Generating Smarandache Curved Ruled Surfaces According to Modified Orthogonal Frame

Let $\alpha(s) = \left(\frac{3}{5}\sin s, \frac{3}{5}\cos s, \frac{4}{5}s\right)$ unit speed curve. The Frenet apparatus of α are

$$\begin{cases} t(s)=\left(\frac{3}{5}\cos s,-\frac{3}{5}\sin s,\frac{4}{5}\right), \\ n(s)=\left(-\sin s,-\cos s,0\right), \\ b(s)=\left(\frac{4}{5}\cos s,-\frac{4}{5}\sin s,-\frac{3}{5}\right), \\ \kappa(s)=\frac{3}{5}\ ,\ \tau(s)=-\frac{4}{5}. \end{cases}$$

The modified orthogonal frame with curvature and torsion of the unit speed curve $\alpha(s)$ the derived elements as follows, respectively:

$$\begin{cases} T_\kappa(s)=\left(\frac{3}{5}\cos s,-\frac{3}{5}\sin s,\frac{4}{5}\right) \\ N_\kappa(s)=\left(-\frac{3}{5}\sin s,-\frac{3}{5}\cos s,0\right) \\ B_\kappa(s)=\left(\frac{12}{25}\cos s,-\frac{12}{25}\sin s,-\frac{12}{25}\right) \end{cases} \quad , \quad \begin{cases} T_\tau(s)=\left(\frac{3}{5}\cos s,-\frac{3}{5}\sin s,\frac{4}{5}\right) \\ N_\tau(s)=\left(\frac{4}{5}\sin s,\frac{4}{5}\cos s,0\right) \\ B_\tau(s)=\left(-\frac{16}{25}\cos s,\frac{16}{25}\sin s,\frac{12}{25}\right) \end{cases}$$

So, the $T_\kappa N_\kappa$ Smarandache curve $\beta_\kappa(s)$, $T_\kappa B_\kappa$ Smarandache curve $\gamma_\kappa(s)$, $N_\kappa B_\kappa$ Smarandache curve $\varsigma_\kappa(s)$ and $T_\kappa N_\kappa B_\kappa$ Smarandache curve $\xi_\kappa(s)$ of the curve $\alpha(s)$ given by the modified orthogonal frame with curvature are written as

$$\begin{aligned} \beta_\kappa(s) &= \frac{1}{\sqrt{34}}(3(\cos s-\sin s),-3(\sin s+\cos s),4), \\ \gamma_\kappa(s) &= \frac{1}{5\sqrt{34}}(27\cos s,-27\sin s,8), \\ \varsigma_\kappa(s) &= \frac{1}{\sqrt{2}}\left(\frac{4}{5}\cos s-\sin s,-\frac{4}{5}\sin s-\cos s,-\frac{4}{5}\right), \\ \xi_\kappa(s) &= \frac{1}{\sqrt{43}}\left(\frac{27}{5}\cos s-3\sin s,-\frac{27}{5}\sin s-3\cos s,\frac{8}{5}\right), \end{aligned}$$

respectively and the $T_\tau N_\tau$ Smarandache curve $\beta_\tau(s)$, $T_\tau B_\tau$ Smarandache curve $\gamma_\tau(s)$, $N_\tau B_\tau$ Smarandache curve $\varsigma_\tau(s)$ and $T_\tau N_\tau B_\tau$ Smarandache curve $\xi_\tau(s)$ of the curve $\alpha(s)$ given by the modified orthogonal frame with torsion are written as

$$\beta_\tau(s)=\frac{1}{\sqrt{41}}\left(3\cos s+4\sin s,-3\sin s+4\cos s,4\right),$$
$$\gamma_\tau(s)=\frac{1}{5\sqrt{41}}\left(-\cos s,\sin s,32\right),$$
$$\varsigma_\tau(s)=\frac{1}{\sqrt{2}}\left(-\sin s+\frac{4}{5}\cos s,\ -\cos s-\frac{4}{5}\sin s,\ -\frac{3}{5}\right),$$
$$\xi_\tau(s)=\frac{1}{\sqrt{57}}\left(-\frac{\cos s}{5}+4\sin s,\frac{\sin s}{5}+4\cos s,\frac{32}{5}\right).$$

The curve α and $T_\kappa N_\kappa$, $T_\kappa B_\kappa$, $N_\kappa B_\kappa$, $T_\kappa N_\kappa B_\kappa$, $T_\tau N_\tau$ $T_\tau B_\tau$, $N_\tau B_\tau$, $T_\tau N_\tau B_\tau$ Smarandache curves of α are shown in the following computer generated graphs. In the graphs, red, gray, blue, green and yellow curves indicate α, $T_\kappa N_\kappa$ and $T_\tau N_\tau$, $T_\kappa B_\kappa$ and $T_\tau B_\tau$, $N_\kappa B_\kappa$ and $N_\tau B_\tau$, $T_\kappa N_\kappa B_\kappa$ and $T_\tau N_\tau B_\tau$ Smarandache curves of α, respectively. In Figure 1 and 2, we have shown α and the Smarandache curves of α given by the modified orthogonal frame with curvature and torsion, respectively.

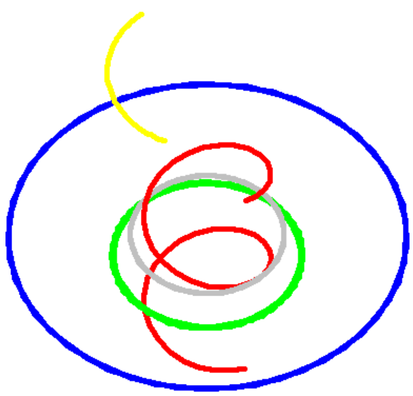

Figure 1. α and Smarandache curves of α given by modified orthogonal frame with curvature

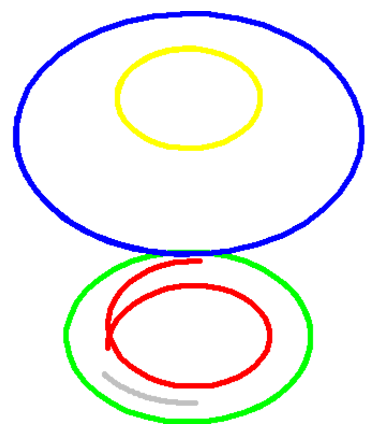

Figure 2. α and Smarandache curves of α given by modified orthogonal frame with torsion

The ruled surfaces $\varphi_{\beta_\kappa}(s,v)$, $\varphi_{\gamma_\kappa}(s,v)$, $\varphi_{\varsigma_\kappa}(s,v)$, $\varphi_{\xi_\kappa}(s,v)$, $\varphi_{\beta_\tau}(s,v)$, $\varphi_{\gamma_\tau}(s,v)$, $\varphi_{\varsigma_\tau}(s,v)$ and $\varphi_{\xi_\tau}(s,v)$ we obtain that

$$\varphi_{\beta_\kappa}(s,v)=\begin{pmatrix}\frac{3}{5}\sin s+\frac{v}{\sqrt{34}}(3\cos s-3\sin s),\ \frac{3}{5}\cos s+\frac{v}{\sqrt{34}}(-3\sin s-3\cos s),\\ \frac{4}{5}s+\frac{4v}{\sqrt{34}}\end{pmatrix},$$

$$\varphi_{\gamma_\kappa}(s,v)=\left(\frac{3}{5}\sin s+v\frac{27}{5\sqrt{34}}\cos s,\frac{3}{5}\cos s-v\frac{27}{5\sqrt{34}}\sin s,\frac{4}{5}s+v\frac{8}{5\sqrt{34}}\right),$$

$$\varphi_{\varsigma_\kappa}(s,v)=\begin{pmatrix}\frac{3}{5}\sin s+v\frac{1}{\sqrt{2}}\left(\frac{4}{5}\cos s-\sin s\right),\ \frac{3}{5}\cos s+v\frac{1}{\sqrt{2}}\left(-\frac{4}{5}\sin s-\cos s\right),\\ \frac{4}{5}s-v\frac{4}{5\sqrt{2}}\end{pmatrix},$$

$$\varphi_{\xi_\kappa}(s,v)=\begin{pmatrix}\frac{3}{5}\sin s+v\frac{1}{\sqrt{43}}\left(\frac{27}{5}\cos s-3\sin s\right),\frac{3}{5}\cos s+v\frac{1}{\sqrt{43}}\left(-\frac{27}{5}\sin s-3\cos s\right),\\ \frac{4}{5}s+v\frac{8}{5\sqrt{43}}\end{pmatrix},$$

$$\varphi_{\beta_\tau}(s,v)=\begin{pmatrix}\frac{3}{5}\sin s+\frac{v}{\sqrt{41}}(3\cos s+4\sin s),\ \frac{3}{5}\cos s+\frac{v}{\sqrt{41}}(-3\sin s+4\cos s),\\ \frac{4}{5}s+\frac{4v}{\sqrt{41}}\end{pmatrix},$$

$$\varphi_{\gamma_\tau}(s,v)=\left(\frac{3}{5}\sin s-v\frac{1}{5\sqrt{41}}\cos s,\frac{3}{5}\cos s+v\frac{1}{5\sqrt{41}}\sin s,\frac{4}{5}s+v\frac{32}{5\sqrt{41}}\right),$$

$$\varphi_{\varsigma_\tau}(s,v)=\begin{pmatrix}\frac{3}{5}\sin s+v\frac{1}{\sqrt{2}}\left(-\sin s+\frac{4}{5}\cos s\right),\ \frac{3}{5}\cos s+v\frac{1}{\sqrt{2}}\left(-\cos s-\frac{4}{5}\sin s\right),\\ \frac{4}{5}s-v\frac{3}{5\sqrt{2}}\end{pmatrix},$$

$$\varphi_{\xi_\tau}(s,v)=\begin{pmatrix}\frac{3}{5}\sin s+v\frac{1}{\sqrt{57}}\left(\frac{-\cos s}{5}+4\sin s\right),\frac{3}{5}\cos s+v\frac{1}{\sqrt{57}}\left(\frac{\sin s}{5}+4\cos s\right),\\ \frac{4}{5}s+v\frac{32}{5\sqrt{57}}\end{pmatrix}.$$

The following Figures 3 through 10 show the graphic of this ruled surfaces for $-2\pi\le s\le 2\pi,\ -2\le v\le 2$.

The given alpha curve and the equations and the shapes of the striction curves of the Smarandache curved ruled surfaces defined according to the modified orthogonal frame of this alpha curve are as follows:

$$\mu_{\varphi_{\beta_\kappa}}(s)=\left(\frac{3}{5}\sin s+\frac{1734}{2765}(\cos s-\sin s),\ \frac{3}{5}\cos s-\frac{1734}{2765}(\sin s+\cos s),\frac{4}{5}s+\frac{2312}{2765}\right),$$

$$\mu_{\varphi_{\xi_\kappa}}(s)=\begin{pmatrix}\frac{3}{5}\sin s+\frac{15}{41}\left(\frac{4}{5}\cos s-\sin s\right),\ \frac{3}{5}\cos s+\frac{15}{41}\left(-\frac{4}{5}\sin s-\cos s\right),\\ \frac{4}{5}s-\frac{12}{41}\end{pmatrix},$$

For this example, since $\tau(s)=-\frac{4}{5}=cons\tan t$ and $\kappa\neq\tau^2$, the striction curve of the ruled surface with TB Smarandache curve (according to modified orthogonal frame with curvature) and the alpha base curve coincide.

$$\mu_{\varphi_{\gamma_\kappa}}(s)=\alpha(s)=\left(\frac{3}{5}\sin s,\frac{3}{5}\cos s,\frac{4}{5}s\right),$$

$$\mu_{\varphi_{\xi_\kappa}}(s)=\left(\frac{3}{5}\sin s-\frac{9245}{114494}(\frac{27}{5}\cos s-3\sin s),\ \frac{3}{5}\cos s+\frac{9245}{114494}(\frac{27}{5}\sin s+3\cos s),\frac{4}{5}s+\frac{14792}{114494}\right),$$

$$\mu_{\varphi_{\beta_\tau}}(s)=\left(\frac{3}{5}\sin s-\frac{12}{25}\left(3\cos s+4\sin s\right),\ \frac{3}{5}\cos s-\frac{12}{25}\left(4\cos s-3\sin s\right),\frac{4}{5}s-\frac{48}{25}\right),$$

$$\mu_{\varphi_{\xi_\tau}}(s)=\left(\frac{3}{5}\sin s+\frac{15}{41}\left(\frac{4}{5}\cos s-\sin s\right),\ \frac{3}{5}\cos s-\frac{15}{41}\left(\cos s+\frac{4}{5}\sin s\right),\frac{4}{5}s-\frac{9}{41}\right),$$

For this example, since $\tau(s)=-\frac{4}{5}=cons\tan t$ and $\kappa\neq\tau^2$, the striction curve of the ruled surface with TB Smarandache curve (according to modified orthogonal frame with torsion) and the alpha base curve coincide.

$$\mu_{\varphi_{\gamma_\tau}}(s)=\alpha(s)=\left(\frac{3}{5}\sin s,\frac{3}{5}\cos s,\frac{4}{5}s\right),$$

$$\mu_{\varphi_{\xi_\tau}}(s)=\begin{pmatrix}\frac{3}{5}\sin s+\frac{1979250187500}{127430740191}\left(\frac{-\cos s}{5}+4\sin s\right),\ \frac{3}{5}\cos s+\frac{1979250187500}{127430740191}\left(4\cos s+\frac{\sin s}{5}\right),\\ \frac{4}{5}s+\frac{395850037500}{407778368612}\end{pmatrix}.$$

The following figure 11-18 shows the graphic of this curves for $-4\leq s\leq 4$, respectively.

4. Conclusions

In this study, some important results are obtained for the characterisation of ruled surfaces containing Smarandache curves given by the modified orthogonal frame in 3-dimensional Euclidean space. Smarandache curves are interesting from the point of view of differential geometry, where they can be studied in terms of properties such as curvature and torsion. The study of such curves helps to understand the behavior of curves in space and their applications can range from general relativity to the study of cosmic structure in physics. In this sense, it is thought that our study can help the fields of study in the aforementioned application areas by obtaining the relations between the curvature and torsion of the curve given for the Smarandache curved ruled surface to be developable.

This work can be considered in Minkowski space, Galilean space or various higher dimensional spaces.

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Appendix

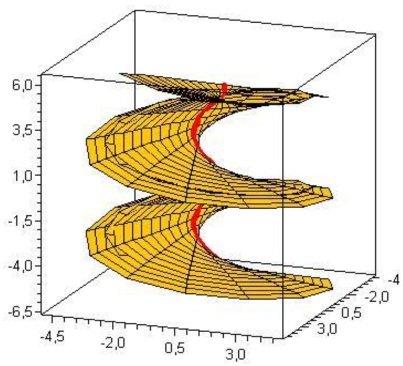

Figure 3. Non-developable ruled surface with $T_\kappa N_\kappa$ Smarandache curve

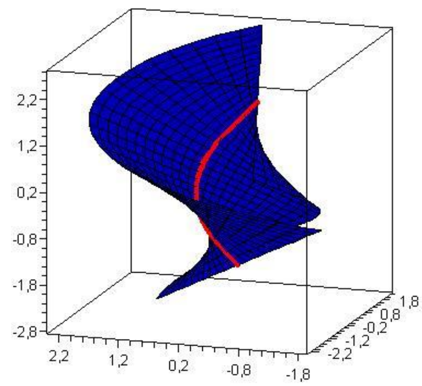

Figure 4. Non-developable ruled surface with $T_\kappa B_\kappa$ Smarandache curve

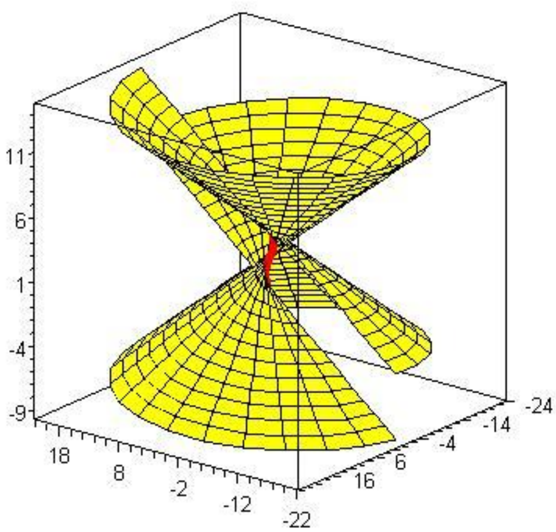

Figure 5. Non-developable ruled surface with $N_\kappa B_\kappa$ Smarandache curve

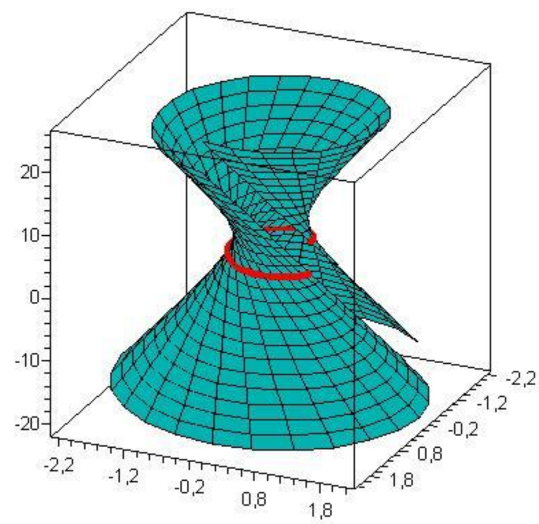

Figure 6. Non-developable ruled surface with $T_\kappa N_\kappa B_\kappa$ Smarandache curve

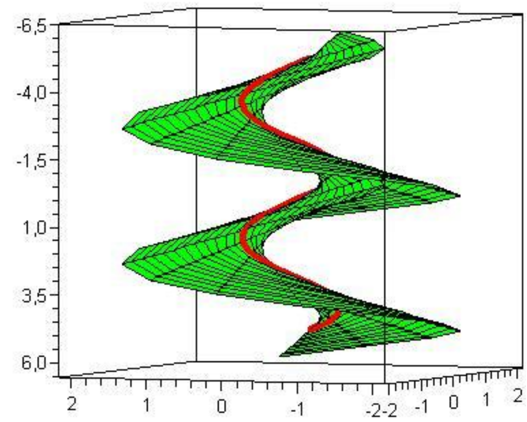

Figure 7. Non-developable ruled surface with $T_\tau N_\tau$ Smarandache curve

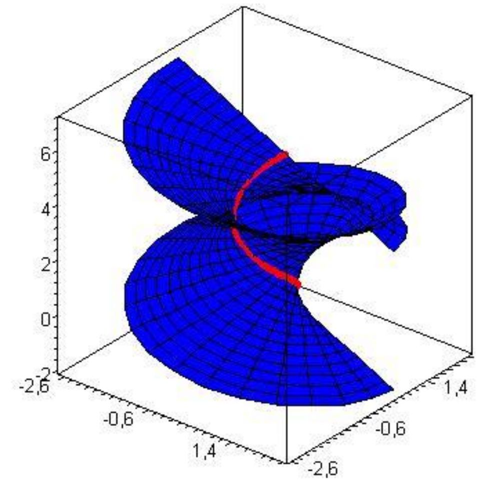

Figure 8. Non-developable ruled surface with $T_\tau B_\tau$ Smarandache curve

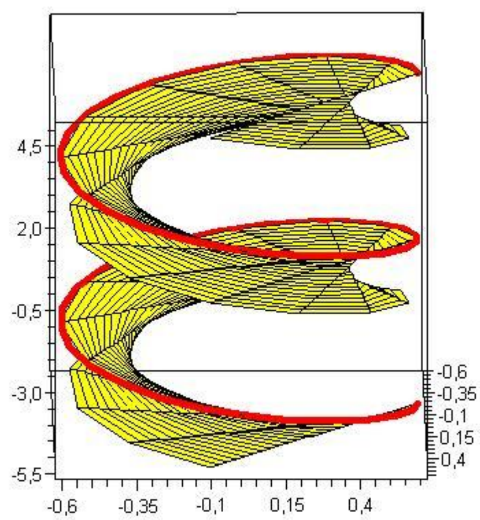

Figure 9. Non-developable ruled surface with $T_\tau N_\tau$ Smarandache curve

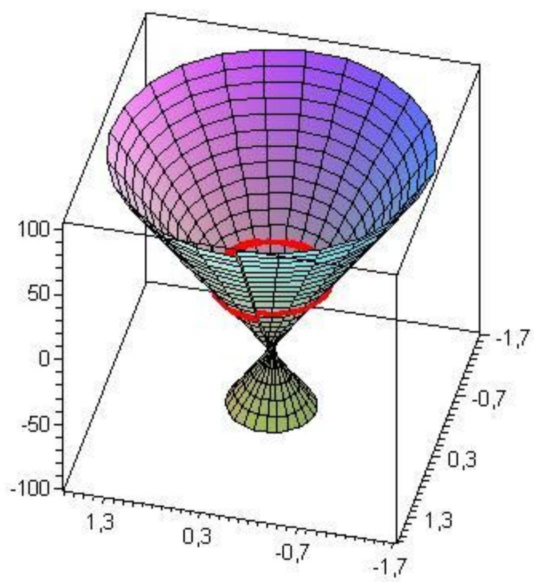

Figure 10. Non-developable ruled surface with $T_\tau N_\tau B_\tau$ Smarandache curve

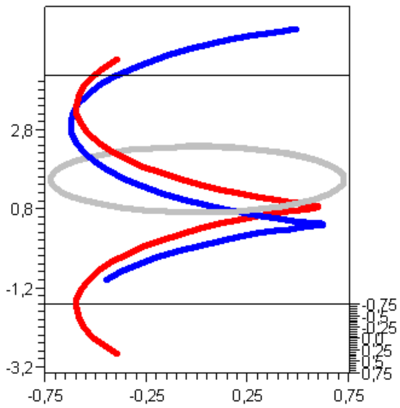

Figure 11. α (in red) and $\mathrm{T}_\kappa \mathrm{N}_\kappa$ Smarandache curve (in gray) and $\mu_{\varphi_{\beta_\kappa}}$ striction curve (in blue)

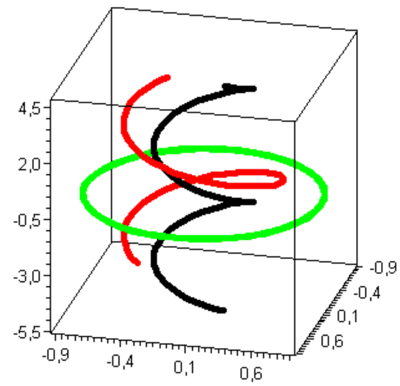

Figure 12. α (in red) and $\mathrm{N}_\kappa \mathrm{B}_\kappa$ Smarandache curve (in green) and $\mu_{\varphi_{\varsigma_\kappa}}$ striction curve (in black)

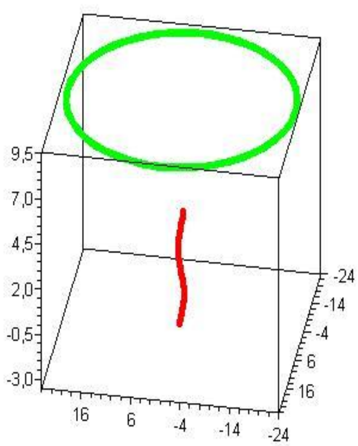

Figure 13. $\alpha = \mu_{\varphi_{\varsigma_\kappa}}$ (in red) and $\mathrm{T}_\kappa \mathrm{B}_\kappa$ Smarandache curve (in green)

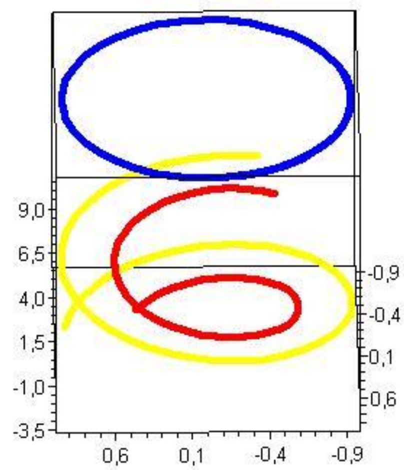

Figure 14. α (in red) and $\mathrm{T}_\kappa \mathrm{N}_\kappa \mathrm{B}_\kappa$ Smarandache curve (in green) and $\mu_{\varphi_{\xi_\kappa}}$ striction curve (in yellow)

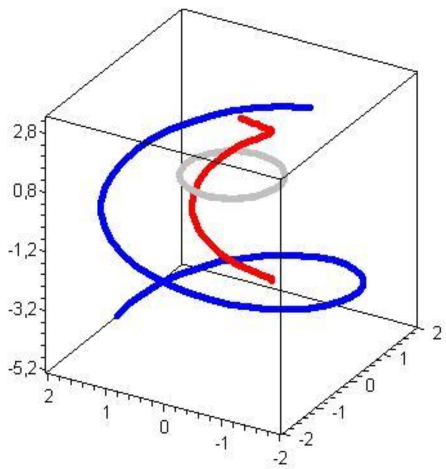

Figure 15. α (in red) and $T_\tau N_\tau$ Smarandache curve (in gray) and $\mu_{\varphi_{\beta_\tau}}$ striction curve (in blue)

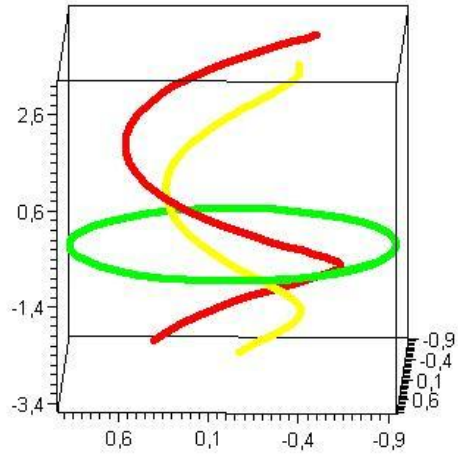

Figure 16. α (in red) and $N_\tau B_\tau$ Smarandache curve (in green) and $\mu_{\varphi_{\varsigma_\tau}}$ striction curve (in yellow)

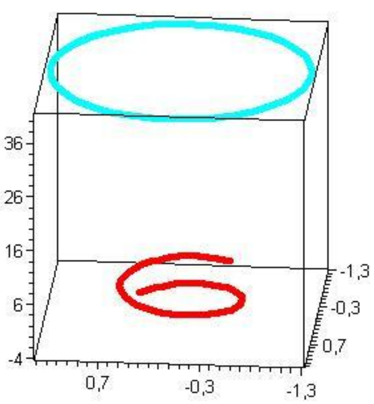

Figure 17. $\alpha = \mu_{\varphi_{\gamma_\tau}}$ (in red) and $T_\tau B_\tau$ Smarandache curve (in blue)

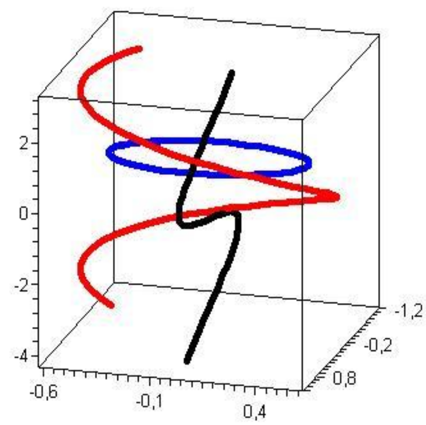

Figure 18. α (in red) and $T_\tau N_\tau B_\tau$ Smarandache curve (in blue) and $\mu_{\varphi_{\xi_\tau}}$ striction curve (in black)