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Soft Directed n-SuperHyperGraphs with Some Real-World Applications

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Abstract. Graph theory offers a powerful language for modeling pairwise relationships via vertices and edges [1]. Hypergraphs extend this framework by allowing hyperedges to join arbitrary subsets of vertices [2], and SuperHyperGraphs generalize further through iterative powerset constructions that capture multi-level hierarchical connectivity [3]. To address uncertainty and imprecision, a variety of set-based frameworks—fuzzy sets [4], soft sets [5], intuitionistic fuzzy sets [6], rough sets [7], neutrosophic sets [8], and plithogenic sets [9]—have been developed. In directed contexts, these ideas yield directed graphs [10], directed hypergraphs [11], directed SuperHyperGraphs [12], and soft directed graphs [13].
While Soft SuperHyperGraphs and Directed SuperHyperGraphs each capture important aspects of uncertainty and directionality in hierarchical networks, their integration remains largely unexplored. In this paper, we introduce the *Directed Soft SuperHyperGraph*, a unified framework that combines directionality, recursive hyperstructure, and soft-set parameterization. We present formal definitions and core operations, demonstrate the model through real-world examples, and offer a qualitative comparison with existing approaches. Our framework is especially well suited to representing complex, multi-layered directed systems—such as urban infrastructure, transportation networks, and information-flow architectures—where deep hierarchies and uncertain relationships must be managed simultaneously.
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1. Introduction

1.1. From Graphs to SuperHyperGraphs

Simple graphs depict pairwise relationships by representing items as nodes and their connections as edges [14]. To describe interactions among multiple entities at once, *hypergraphs* permit each hyperedge to join any nonempty subset of vertices, thus capturing

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higher-order associations [15, 16]. Building upon this, a *SuperHyperGraph* is formed by iteratively taking the powerset of the vertex set to create successive layers of links, embedding nested hierarchies within the network [17, 18].

1.2. Capturing Uncertainty in Networks

Real-world networks frequently include incomplete, imprecise, or even contradictory information. To model such ambiguity, a range of set-based frameworks—fuzzy sets [4], intuitionistic fuzzy sets [6], hyperfuzzy sets [19, 20, 21], hesitant fuzzy sets [22, 23], picture fuzzy sets[24], neutrosophic sets [25, 26], soft sets [27, 5], hypersoft sets[28, 29, 30, 31], and plithogenic sets [32, 33, 34]—have been adapted for graph modeling. These adaptations give rise to structures such as fuzzy graphs [35], intuitionistic fuzzy graphs [36], neutrosophic graphs [25], plithogenic graphs [37], and soft graphs [38], each integrating graded membership, indeterminacy, and conflict measures to better reflect real-world uncertainty. Moreover, fuzzy graphs, soft graphs, and neutrosophic graphs have been generalized to hypergraphs and SuperHyperGraphs, with notable recent developments [39, 40].

1.3. Introducing Directionality

Many real-world systems exhibit directional or asymmetric ties. In a *directed graph* (digraph), each edge carries an orientation to indicate the direction of flow—be it information, resources, or influence—between nodes [41, 42]. This concept extends to *directed hypergraphs*, where hyperarcs distinguish between tail and head subsets, and further to *directed superhypergraphs*, which layer these directed connections across multiple hierarchical strata [43, 44, 45, 46, 12].

1.4. Contributions of This Paper

From the foregoing discussion, research on Soft SuperHyperGraphs and Directed SuperHyperGraphs is critically important, as it enables intuitive modeling of hierarchical real-world concepts. However, the integrated fusion of Soft SuperHyperGraphs and Directed SuperHyperGraphs has received little attention to date. To bridge this gap, in this paper we introduce the *Directed Soft SuperHyperGraph*, a novel framework that unifies directed relationships, hierarchical superhyperstructures, and soft-set parameterization. We demonstrate its effectiveness in modeling complex, multi-layered directed networks—such as urban transit systems, logistics networks, and information pipelines—where deep structural hierarchies and uncertainty must be managed concurrently.

1.5. Structure of This Paper

The remainder of this paper is organized as follows. In Section 2, we review the existing definitions of HyperGraphs, SuperHyperGraphs, the n-th iterated powerset, Soft Directed Hypergraphs, and related concepts. Section 3 presents the definition of the Directed Soft SuperHyperGraph and examines its fundamental properties. Section 4 provides a brief discussion on the effectiveness of the Directed Soft SuperHyperGraph introduced in this paper. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Preliminaries

Throughout this paper, we work with finite, simple graphs unless stated otherwise. This section reviews the key definitions and notation that will be used in later sections; for more in-depth discussions, please refer to the cited sources.

2.1. SuperHyperGraphs

Classical graphs model binary relations between vertices, but many applications require connecting larger subsets at once. A *hypergraph* achieves this by replacing edges with *hyperedges*, each of which may join any nonempty collection of vertices [2, 47]. To capture multi-level or recursive structures, one can iterate the subset construction: a *SuperHyperGraph* applies the powerset operation successively, creating "supervertices" and "superedges" at higher abstraction layers [3, 48, 49, 50].

Definition 1 (Base Set)**.** *[51] Let S be a* base set*, the foundational domain from which all subsequent constructions are drawn:*

$$S = \{ x : x \text{ is an element of the universe of discourse} \}.$$

Definition 2 (Powerset)**.** *[52] The* powerset *of a set S, denoted* $\mathrm{POWS}(S)$*, is the collection of all subsets of S, including $\varnothing$ and S itself:*

$$\mathrm{POWS}(S) = \{ A : A \subseteq S \}.$$

Definition 3 (Hypergraph)**.** *[2, 47] A* hypergraph $H = (V, E)$ *consists of*

- *a finite vertex set V,*
- *a family E of nonempty subsets of V, each called a* hyperedge.

Definition 4 (n-th Iterated Powerset)**.** *[53, 54, 55] Define the iterated powerset of a set X by*

$$\mathrm{POWS}_1(X) = \mathrm{POWS}(X), \quad \mathrm{POWS}_{k+1}(X) = \mathrm{POWS}\big(\mathrm{POWS}_k(X)\big), \quad k \geq 1.$$

The corresponding nonempty *iterated powerset is*

$$\mathrm{POWS}^*_1(X) = \mathrm{POWS}(X) \setminus \{\varnothing\}, \quad \mathrm{POWS}^*_{k+1}(X) = \mathrm{POWS}\big(\mathrm{POWS}^*_k(X)\big) \setminus \{\varnothing\}.$$

Example 1 (Organizational Hierarchy via Iterated Powersets)**.** *Let the base set of employees be*

$$X = \{\text{Keisuke, Bob, Sakura}\}.$$

- $\mathrm{POWS}_1(X) = \mathrm{POWS}(X)$ *is the set of all possible* teams*:*

$$\{\varnothing, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}.$$

- $\mathrm{POWS}_2(X) = \mathrm{POWS}(\mathrm{POWS}_1(X))$ *is the set of all* committees of teams*, for example*

$$\{\{\{A, B\}, \{B, C\}\},\ \{\{A\}, \{C\}\},\ \dots\}.$$

- $\mathrm{POWS}_3(X) = \mathrm{POWS}(\mathrm{POWS}_2(X))$ *is the set of all* councils of committees*, such as*

$$\{\{\{\{A, B\}, \{B, C\}\},\ \{\{A\}, \{C\}\}\}, \dots\}.$$

In practice:

(i) A team *is any subset of employees (level-1 grouping).*

(ii) A committee *is a collection of teams (level-2 grouping).*

(iii) A council *is a set of committees (level-3 grouping).*

Thus iterated powersets naturally model successive layers of organizational structure.

Definition 5 (n-SuperHyperGraph)**.** *[56, 57, 58] Let V_0 be a finite base set and define $\mathrm{POWS}^k(V_0)$ by iterating the powerset k times. An* n-SuperHyperGraph *is a pair*

$$\mathrm{SuHG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathrm{POWS}^n(V_0), \quad E \subseteq \mathrm{POWS}^n(V_0).$$

Members of V are called n-supervertices *and members of E are* n-superedges*.*

Example 2 (Software Architecture as a 2-SuperHyperGraph)**.** *Consider a software system whose lowest-level components are*

$$S = \{\texttt{Auth},\ \texttt{DataAccess},\ \texttt{UI}\}.$$

At the first level, these components form modules:

$$M_1 = \{\texttt{Auth}\}, \quad M_2 = \{\texttt{DataAccess}\}, \quad M_3 = \{\texttt{UI}\},$$

so that $\mathrm{POWS}^1(S) \supseteq V_1 = \{M_1, M_2, M_3\}$.

At the second level, modules combine into subsystems:

$$\mathrm{Sub}_A = \{M_1, M_2\}, \quad \mathrm{Sub}_B = \{M_2, M_3\}, \quad \mathrm{Sub}_{\mathrm{All}} = \{M_1, M_2, M_3\},$$

so that $\mathrm{POWS}^2(S) \supseteq V = \{\mathrm{Sub}_A, \mathrm{Sub}_B, \mathrm{Sub}_{\mathrm{All}}\}$.

We choose the set of 2*-superedges to capture key integration points:*

$$E = \{e_1 = \{\,\mathrm{Sub}_A, \mathrm{Sub}_{\mathrm{All}}\,\}, \quad e_2 = \{\,\mathrm{Sub}_B, \mathrm{Sub}_{\mathrm{All}}\,\}\},$$

where:

- e_1 *models the integration between the Auth/Data subsystem and the full system.*
- e_2 *models the integration between the Data/UI subsystem and the full system.*

Hence

$$\mathrm{SuHG}^{(2)} = (\, V,\, E\,)$$

is a 2-SuperHyperGraph *representing the two–tier module-subsystem structure of the software architecture.*

2.2. Soft SuperHypergraph

Soft graphs capture families of subgraphs indexed by a parameter set, where each parameter selects a subgraph on a fixed vertex and edge base [59, 38, 60]. Equivalently, a soft graph realizes the concept of a soft set—a parameterized collection of subsets that models uncertainty or preference without requiring numeric membership or probability assignments [5, 27]. A *soft hypergraph* extends this notion by associating each parameter with a subset of vertices and the induced hyperedges within a hypergraph [61]. Finally, a *soft n-superhypergraph* applies these parameterized substructures to each layer of an n-SuperHyperGraph, yielding a hierarchically organized, uncertainty-aware model (cf.[12]).

Definition 6 (Soft Hypergraph)**.** *[61, 62] Let $H = (V, E)$ be a hypergraph with $E \subseteq \mathrm{POWS}(V)$, and let C be a nonempty set of parameters. A* soft hypergraph *over H with parameters C is a quadruple*

$$\bigl(H,\, C,\, A,\, B\bigr),$$

where

$$A : C \longrightarrow \mathrm{POWS}(V), \qquad B : C \longrightarrow \mathrm{POWS}(E),$$

and for each $c \in C$,

$$B(c) \;\subseteq\; \{\, e \in E \mid e \subseteq A(c)\}.$$

The pair $\bigl(A(c), B(c)\bigr)$ is called the soft subhypergraph *of H at parameter c.*

Example 3 (University Curriculum as a Soft Hypergraph)**.** *Consider the hypergraph $H = (V, E)$ where*

$$V = \{\textit{Calculus},\ \textit{Linear Algebra},\ \textit{Data Structures},\ \textit{Operating Systems},\ \textit{Networks}\},$$

$$E = \bigl\{e_{\mathsf{math}} = \{\textit{Calculus},\ \textit{Linear Algebra}\},\quad e_{\mathsf{CS}} = \{\textit{Data Structures},\ \textit{Operating Systems},\ \textit{Networks}\},\quad e_{\mathsf{comp}} =$$

Let $C = \{\mathsf{Math},\ \mathsf{CS},\ \mathsf{Inter}\}$ be the set of program-type parameters, and define

$$A(\mathsf{Math}) = \{\textit{Calculus},\ \textit{Linear Algebra}\},\quad B(\mathsf{Math}) = \{e_{\mathsf{math}}\},$$

$$A(\mathsf{CS}) = \{\textit{Data Structures},\ \textit{Operating Systems},\ \textit{Networks}\},\quad B(\mathsf{CS}) = \{e_{\mathsf{CS}}\},$$

$$A(\mathsf{Inter}) = \{\textit{Linear Algebra},\ \textit{Data Structures}\},\quad B(\mathsf{Inter}) = \{e_{\mathsf{comp}}\}.$$

Then $\bigl(H,\, C,\, A,\, B\bigr)$ is a soft hypergraph in which each parameter $c \in C$ induces the subhypergraph $\bigl(A(c), B(c)\bigr)$ corresponding to the courses and curriculum bundles for that program type.

Definition 7 (Soft n-Superhypergraph)**.** *[12] Let* $\mathrm{SuHG}^{(n)} = (V, E)$ *be an* n*-SuperHyperGraph, and let* C *be a nonempty parameter set. A* soft n-superhypergraph *is a quintuple*

$$\big(V,\, E,\, C,\, A,\, B\big),$$

where

$$A : C \longrightarrow \mathrm{POWS}(V), \qquad B : C \longrightarrow \mathrm{POWS}(E),$$

such that for every $c \in C$*,*

$$A(c) \subseteq V, \quad B(c) \subseteq \{\, e \in E \mid e \subseteq A(c)\}.$$

Thus $\big(A(c), B(c)\big)$ *is a sub-superhypergraph of* $\mathrm{SuHG}^{(n)}$ *at parameter* c*.*

Example 4 (Personalized Curriculum Planning as a Soft 2-Superhypergraph)**.** *Imagine a university offering a modular curriculum. The ground set of all courses is*

$$S = \{\mathit{Calculus},\ \mathit{Programming},\ \mathit{Statistics},\ \mathit{Logic}\}.$$

At the first level, courses are grouped into subject areas (first-level supervertices):

$$V_1 = \big\{M_{Math} = \{\mathit{Calculus}, \mathit{Statistics}\},\ M_{CS} = \{\mathit{Programming}, \mathit{Logic}\}\big\} \ \subseteq\ \mathrm{POWS}^1(S).$$

At the second level, these areas form broader tracks (second-level supervertices):

$$V_2 = \big\{T_1 = \{M_{Math}\},\ T_2 = \{M_{CS}\},\ T_{All} = \{M_{Math}, M_{CS}\}\big\} \ \subseteq\ \mathrm{POWS}^2(S).$$

We define the set of second-level "superedges" to indicate available program pathways:

$$E = \big\{e_1 = \{M_{Math}, T_{All}\}, \quad e_2 = \{M_{CS}, T_{All}\}\big\},$$

so that e_1 *and* e_2 *represent progression from a subject area into the comprehensive track.*

Next, let the parameter set C *capture two different student profiles:*

$$C = \{\mathit{Math\text{-}Focused},\ \mathit{CS\text{-}Focused}\}.$$

Define soft selection maps

$$A : C \ \longrightarrow\ \mathrm{POWS}(V_2), \qquad B : C \ \longrightarrow\ \mathrm{POWS}(E),$$

by

$$A(\mathit{Math\text{-}Focused}) = \{T_1, T_{All}\}, \quad B(\mathit{Math\text{-}Focused}) = \{e_1\},$$
$$A(\mathit{CS\text{-}Focused}) = \{T_2, T_{All}\}, \qquad B(\mathit{CS\text{-}Focused}) = \{e_2\}.$$

- *Under the* ***Math-Focused*** *profile, a student selects the mathematics area first* (T_1) *and then the joint track* (T_{All})*, following edge* e_1*.*
- *Under the* ***CS-Focused*** *profile, the programming–logic area* (T_2) *leads into the joint track, via edge* e_2*.*

Thus the quintuple

$$\big(V_2,\, E,\, C,\, A,\, B\big)$$

is a soft 2-superhypergraph *modeling how different student goals select distinct sub-paths through the curriculum hierarchy.*

Example 5 (Curriculum Program Selection as a Soft 3-Superhypergraph)**.** *Consider the ground set of courses*

$$S = \{\text{Calculus, Programming, Statistics, Logic}\}.$$

First-level "modules" are

$$M_{\text{Math}} = \{\text{Calculus, Statistics}\}, \quad M_{\text{CS}} = \{\text{Programming, Logic}\}, \quad \text{POWS}^1(S) \supseteq \{M_{\text{Math}},\, M_{\text{CS}}\}.$$

Second-level "tracks" are

$$T_1 = \{M_{\text{Math}}\}, \quad T_2 = \{M_{\text{CS}}\}, \quad T_{\text{All}} = \{M_{\text{Math}},\, M_{\text{CS}}\}, \quad \text{POWS}^2(S) \supseteq \{T_1,\, T_2,\, T_{\text{All}}\}.$$

Third-level "programs" (supervertices) are

$$P_1 = \{T_1\}, \quad P_2 = \{T_2\}, \quad P_{\text{All}} = \{T_1,\, T_2\}, \quad V = \{P_1,\, P_2,\, P_{\text{All}}\} \subseteq \text{POWS}^3(S).$$

Define the set of program hyperedges

$$E = \big\{e_1 = \{P_1,\, P_{\text{All}}\}, \quad e_2 = \{P_2,\, P_{\text{All}}\}\big\} \subseteq \text{POWS}(V).$$

Let the parameter set be

$$C = \{\text{Math-Track, CS-Track}\},$$

and define the soft-selection maps

$$A : C \to \text{POWS}(V), \quad B : C \to \text{POWS}(E)$$

by

$$A(\text{Math-Track}) = \{P_1,\, P_{\text{All}}\}, \quad B(\text{Math-Track}) = \{e_1\},$$
$$A(\text{CS-Track}) = \{P_2,\, P_{\text{All}}\}, \quad B(\text{CS-Track}) = \{e_2\}.$$

Then

$$\big(V,\, E,\, C,\, A,\, B\big)$$

is a soft 3-superhypergraph *modeling how students select and transition through programs in a three-tier curriculum.*

2.3. Soft Directed Hypergraphs

A soft directed graph generalizes directed graphs by using parameters to select vertex subsets and their corresponding directed edge subsets [63, 64, 65]. A *soft directed hypergraph* generalizes both directed hypergraphs and soft sets by selecting substructures via parameters (cf.[66, 67]).

Definition 8 (Soft Directed Hypergraph). *(cf.[66, 67]) Let* $\mathrm{SoDHG}^* = (\Gamma, \Xi)$ *be a simple directed hypergraph, where* Γ *is the vertex set and* $\Xi \subseteq \mathrm{POWS}(\Gamma) \times \mathrm{POWS}(\Gamma)$ *is the set of hyperarcs (each an ordered pair of nonempty tail and head). Let* C *be a nonempty parameter set. A* soft directed hypergraph *is the quadruple*

$$\big(\mathrm{SoDHG}^*,\, C,\, A,\, B\big),$$

where

$$A : C \longrightarrow \mathrm{POWS}(\Gamma), \qquad B : C \longrightarrow \mathrm{POWS}(\Xi),$$

and for every $c \in C$*,*

$$A(c) \subseteq \Gamma, \quad B(c) \subseteq \big\{\, (T, H) \in \Xi : T \subseteq A(c),\ H \subseteq A(c)\big\}.$$

Each pair $\big(A(c), B(c)\big)$ *is called the* soft induced subhypergraph *of* SoDHG^* *at parameter* c*.*

Example 6 (Sensor Selection in a Directed Hypergraph). *Consider a directed hypergraph* $\mathrm{SoDHG}^* = (\{s_1, s_2, s_3\}, \Xi)$ *where*

$$\Xi = \big\{(\{s_1\}, \{s_2, s_3\}),\ (\{s_2\}, \{s_3\})\big\}.$$

Let parameters $C = \{Day, Night\}$*. Define*

$$A(Day) = \{s_1, s_2\}, \quad B(Day) = \{(\{s_1\}, \{s_2\})\},$$

$$A(Night) = \{s_2, s_3\}, \quad B(Night) = \{(\{s_2\}, \{s_3\})\}.$$

Then $\big(A(c), B(c)\big)$ *for each* $c \in C$ *is a directed subhypergraph on the active sensors, illustrating how soft parameters select relevant hyperarcs.*

2.4. Directed SuperHyperGraph

Directed SuperHyperGraphs are graph classes that extend SuperHyperGraphs, respectively, in a manner analogous to Directed Graphs(cf.[68, 3]) . Below, we present their formal definitions and illustrative examples.

Definition 9 (Directed n-SuperHyperGraph). *(cf.[68, 3]) Let* S *be a nonempty* base set *and let* $n \geq 0$ *be an integer. Define iterated powersets by*

$$\mathrm{POWS}^0(S) = S, \quad \mathrm{POWS}^{k+1}(S) = \mathrm{POWS}\big(\mathrm{POWS}^k(S)\big) \quad (k \geq 0).$$

A directed n-SuperHyperGraph *is a pair*

$$\mathrm{DSuHG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathrm{POWS}^n(S), \quad E \subseteq \mathrm{POWS}^n(S) \times \mathrm{POWS}^n(S),$$

and each directed n-superedge $e \in E$ is an ordered pair

$$e = \big(\mathrm{Tail}(e), \mathrm{Head}(e)\big), \quad \mathrm{Tail}(e), \mathrm{Head}(e) \subseteq \mathrm{POWS}^n(S),$$

typically both nonempty. Such an e carries "flow" from the entire set $\mathrm{Tail}(e)$ *of n-supervertices into* $\mathrm{Head}(e)$*.*

Example 7 (Corporate Reporting as a Directed 2-SuperHyperGraph)**.** *Consider a small company with three employees:*

$$S = \{\text{Keisuke, Bob, Sakura}\}.$$

They form two teams:

$$T_A = \{\text{Keisuke}, \text{Bob}\}, \quad T_B = \{\text{Bob}, \text{Sakura}\},$$

so that $\mathrm{POWS}^1(S) \supseteq \{T_A, T_B\}$*. At the next level, these teams report to departments:*

$$D_1 = \{T_A\}, \quad D_2 = \{T_B\}, \quad D_{\mathrm{All}} = \{T_A, T_B\},$$

so that $\mathrm{POWS}^2(S) \supseteq V = \{D_1, D_2, D_{\mathrm{All}}\}$*. We define the directed* 2*-superhyperedges*

$$E = \big\{e_1 = \big(\{D_1\}, \{D_{\mathrm{All}}\}\big), \quad e_2 = \big(\{D_2\}, \{D_{\mathrm{All}}\}\big)\big\}.$$

Here:

- *e_1 models Team T_A reporting up into the combined department D_{All}.*
- *e_2 models Team T_B reporting up into D_{All}.*

Hence

$$\mathrm{DSuHG}^{(2)} = \big(V, E\big)$$

is a directed 2-SuperHyperGraph *that captures the two-tier reporting structure in this organization.*

Example 8 (Corporate Hierarchy as a Directed 3-SuperHyperGraph)**.** *Consider a multinational company with four employees:*

$$S = \{\mathit{Keisuke}, \mathit{Bob}, \mathit{Sakura}, \mathit{Dave}\}.$$

They form two project teams:

$$T_1 = \{\mathit{Keisuke}, \mathit{Bob}\}, \quad T_2 = \{\mathit{Sakura}, \mathit{Dave}\},$$

so that $\mathrm{POWS}^1(S) \supseteq \{T_1, T_2\}$. *At the next level, these teams belong to departments:*

$$D_1 = \{T_1\}, \quad D_2 = \{T_2\},$$

so that $\mathrm{POWS}^2(S) \supseteq \{D_1, D_2\}$. *At the third level, departments are grouped into divisions:*

$$\mathrm{Div}_1 = \{D_1\}, \quad \mathrm{Div}_2 = \{D_2\}, \quad \mathrm{Div}_{\mathrm{All}} = \{D_1, D_2\},$$

so that $\mathrm{POWS}^3(S) \supseteq V = \{\mathrm{Div}_1, \mathrm{Div}_2, \mathrm{Div}_{\mathrm{All}}\}$. *We then define the directed* 3*-superhyperedges*

$$E = \big\{e_1 = \big(\{\mathrm{Div}_1\}, \{\mathrm{Div}_{\mathrm{All}}\}\big), \quad e_2 = \big(\{\mathrm{Div}_2\}, \{\mathrm{Div}_{\mathrm{All}}\}\big)\big\}.$$

Here:

- e_1 *models Division 1 reporting into the overall corporate division.*
- e_2 *models Division 2 reporting into the overall corporate division.*

Thus,

$$\mathrm{DSuHG}^{(3)} = (V, E)$$

is a directed 3*-SuperHyperGraph capturing the four-tier hierarchy—employees, teams, departments, and divisions—of the organization.*

3. Main Results

In this section, we present the main contributions of this paper, focusing on the structure and properties of Soft Directed n-SuperHyperGraphs.

3.1. Soft Directed n-SuperHyperGraphs

We now introduce *Soft Directed n-SuperHyperGraphs*, which unite the ideas of directed n-SuperHyperGraphs and soft n-SuperHyperGraphs under a single framework.

Definition 10 (Soft Directed n-SuperHyperGraph)**.** *Let S be a finite base set and $n \geq 0$. Define iterated powersets by*

$$\mathrm{POWS}^0(S) = S, \quad \mathrm{POWS}^{k+1}(S) = \mathrm{POWS}\big(\mathrm{POWS}^k(S)\big).$$

A directed n-SuperHyperGraph *is a pair* $\mathrm{DSuHG}^{(n)} = (V, E)$ *with*

$$V \subseteq \mathrm{POWS}^n(S), \quad E \subseteq \mathrm{POWS}^n(S) \times \mathrm{POWS}^n(S),$$

where each $e \in E$ is an ordered pair $(\mathrm{Tail}(e), \mathrm{Head}(e))$.

Let C be a nonempty parameter set. A soft directed n-SuperHyperGraph *is the quadruple*

$$\big(\mathrm{DSuHG}^{(n)}, C, A, B\big),$$

where

$$A : C \longrightarrow \mathrm{POWS}(V), \qquad B : C \longrightarrow \mathrm{POWS}(E),$$

and for each $c \in C$*,*

$$A(c) \subseteq V, \quad B(c) \subseteq \{\, e \in E : \mathrm{Tail}(e) \subseteq A(c),\ \mathrm{Head}(e) \subseteq A(c)\}.$$

Each induced pair $\big(A(c), B(c)\big)$ *is called the* soft directed sub-n-superhypergraph *at parameter* c*.*

Example 9 (Emergency Response Network as a Soft Directed 2-SuperHyperGraph)**.** *Consider a country-wide emergency response system. The base set of ground units is*

$$S = \{\, Team_A,\ Team_B,\ Team_C\}.$$

They form regional clusters (first-level supervertices):

$$V_1 = \big\{R_1 = \{\, Team_A, Team_B\},\ R_2 = \{\, Team_B, Team_C\}\big\}$$
$$\subseteq\ \mathrm{POWS}^1(S).$$

At the second level, these regions coordinate under a national command:

$$V_2 = \big\{N = \{R_1, R_2\}\big\}\ \subseteq\ \mathrm{POWS}^2(S).$$

We define directed 2*-superedges to model information flow:*

$$E = \big\{e_1 = (\{R_1\}, \{N\}), \quad e_2 = (\{R_2\}, \{N\})\big\}.$$

Next, let the parameter set capture the system's operational mode:

$$C = \{Routine, Emergency\}.$$

Define soft selection maps

$$A : C \longrightarrow \mathrm{POWS}(V_2), \qquad B : C \longrightarrow \mathrm{POWS}(E),$$

by

$$A(Routine) = \{R_1\}, \qquad B(Routine) = \{e_1\},$$
$$A(Emergency) = \{R_1, R_2, N\}, \quad B(Emergency) = \{e_1, e_2\}.$$

Here:

- *Under* ***Routine*** *mode, only region* R_1 *reports upward, so we restrict to* $\{R_1\}$ *and edge* e_1*.*
- *Under* ***Emergency*** *mode, all regions and the national command are active, so we include* R_1, R_2, N *and both edges* e_1, e_2*.*

Thus the quadruple

$$\big(\mathrm{DSuHG}^{(2)},\, C,\, A,\, B\big)$$

is a soft directed 2-SuperHyperGraph*, illustrating how parameter settings select which supervertices and superedges are active in different operational scenarios.*

Example 10 (Shift-Based Communication in a Global Corporation as a Soft Directed 3-SuperHyperGraph)**.** *Consider a multinational company whose employees form teams, departments, and divisions. We model information flow through three hierarchical levels of "supervertices" and allow soft selection by work shift.*

Base set of employees:

$$S = \{\text{Keisuke, Bob, Sakura}\}.$$

Level 1 supervertices (teams):

$$T_1 = \{\text{Keisuke}, \text{Bob}\}, \quad T_2 = \{\text{Bob}, \text{Sakura}\},$$

so $\mathrm{POWS}^1(S) \supseteq V_1 = \{T_1, T_2\}$.

Level 2 supervertices (departments):

$$D_1 = \{T_1\}, \quad D_2 = \{T_2\},$$

so $\mathrm{POWS}^2(S) \supseteq V_2 = \{D_1, D_2\}$.

Level 3 supervertices (divisions):

$$U_1 = \{D_1\}, \quad U_2 = \{D_2\}, \quad U_3 = \{D_1, D_2\},$$

so

$$\mathrm{POWS}^3(S) \supseteq V_3 = \{U_1, U_2, U_3\}.$$

Directed **3*****-superedges (communication channels):***

$$E_3 = \big\{e_1 = (\{U_1\},\, \{U_3\}),\; e_2 = (\{U_2\},\, \{U_3\})\big\}.$$

Here e_1 *represents Division 1 reporting to the Corporate Division* U_3*, and* e_2 *likewise for Division 2.*

Soft selection by work shift: *Let the parameter set be*

$$C = \{\mathit{Morning},\, \mathit{Evening}\}.$$

Define

$$A : C \longrightarrow \mathrm{POWS}(V_3), \qquad B : C \longrightarrow \mathrm{POWS}(E_3),$$

by

$$\begin{aligned} A(\mathit{Morning}) &= \{U_1,\, U_2,\, U_3\}, & B(\mathit{Morning}) &= \{e_1,\, e_2\}, \\ A(\mathit{Evening}) &= \{U_2,\, U_3\}, & B(\mathit{Evening}) &= \{e_2\}. \end{aligned}$$

- ***Morning shift:*** *All divisions (U_1, U_2) and corporate (U_3) are active; both channels e_1, e_2 operate.*
- ***Evening shift:*** *Only Division 2 (U_2) reports to corporate (U_3), so only e_2 is active.*

Hence

$$\left(\mathrm{DSuHG}^{(3)},\, C,\, A,\, B\right)$$

is a soft directed 3-SuperHyperGraph *capturing how shift schedules select which hierarchical communication channels are used in different operational contexts.*

Example 11 (Logistics Network as a Soft Directed 4-SuperHyperGraph)**.** *We model a four-level logistics hierarchy—from individual vehicles up to global command—under different operational modes.*

Level 0 (vehicles):

$$S_0 = \{\mathrm{Truck}_A,\ \mathrm{Truck}_B,\ \mathrm{Truck}_C\}.$$

Level 1 (regional hubs):

$$H_1 = \{\mathrm{Truck}_A, \mathrm{Truck}_B\}, \quad H_2 = \{\mathrm{Truck}_B, \mathrm{Truck}_C\},$$

so $\mathrm{POWS}^1(S_0) \supseteq V_1 = \{H_1, H_2\}$.

Level 2 (national centers):

$$N_1 = \{H_1\}, \quad N_2 = \{H_2\},$$

so $\mathrm{POWS}^2(S_0) \supseteq V_2 = \{N_1, N_2\}$.

Level 3 (continental divisions):

$$C_1 = \{N_1\}, \quad C_2 = \{N_2\}, \quad C_3 = \{N_1, N_2\},$$

so $\mathrm{POWS}^3(S_0) \supseteq V_3 = \{C_1, C_2, C_3\}$.

Level 4 (global command):

$$G_1 = \{C_1, C_2\}, \quad G_2 = \{C_3\},$$

so $\mathrm{POWS}^4(S_0) \supseteq V_4 = \{G_1, G_2\}$.

Directed* 4*-superedges (reporting channels):

$$E_4 = \big\{e_1 = (\,\{G_1\}, \{G_2\}\,), \quad e_2 = (\,\{G_2\}, \{G_1\}\,)\big\}.$$

Here e_1 models reports from the union of divisions to the consolidated command G_2, and e_2 allows feedback.

Operational modes (parameters):

$$C = \{Standard,\ NightShift,\ Weekend\}.$$

Define soft selections

$$A : C \longrightarrow \mathrm{POWS}(V_4), \quad B : C \longrightarrow \mathrm{POWS}(E_4),$$

by

$$\begin{aligned} A(Standard) &= \{G_1,\, G_2\}, & B(Standard) &= \{e_1,\, e_2\}, \\ A(NightShift) &= \{G_1\}, & B(NightShift) &= \{e_1\}, \\ A(Weekend) &= \{G_2\}, & B(Weekend) &= \{e_2\}. \end{aligned}$$

- ***Standard:*** *Both global nodes active; both reporting and feedback channels open.*
- ***NightShift:*** *Only the primary command node G_1 operates, sending data upward via e_1.*
- ***Weekend:*** *Only the consolidated command node G_2 remains active, issuing feedback via e_2.*

Therefore, the quadruple

$$\big(\mathrm{DSuHG}^{(4)},\, C,\, A,\, B\big)$$

is a soft directed 4-SuperHyperGraph *that captures how different operational modes select which global reporting channels are in use.*

Theorem 1 (Generalization of Directed and Soft n-SuperHyperGraphs)**.** *The notion of soft directed n-SuperHyperGraph simultaneously extends:*

(a) Every directed n-SuperHyperGraph $\mathrm{DSuHG}^{(n)} = (V, E)$ *by taking a singleton parameter set* $C = \{*\}$ *with* $A(*) = V$ *and* $B(*) = E$*.*

(b) Every soft n-SuperHyperGraph $\big(V, E,\, C,\, A,\, B\big)$ *by viewing each undirected superedge* $e \in E$ *as a degenerate directed edge* $(e, e) \in E'$*, and setting* $E' = \{(e, e) : e \in E\}$*.*

Proof. **(a) Recovering directed n-SuHG.** Given $\mathrm{DSuHG}^{(n)} = (V, E)$, set $C = \{*\}$, $A(*) = V$, and $B(*) = E$. Then by definition each soft directed sub-n-superhypergraph at $*$ is (V, E), so we exactly recover the original directed structure.

(b) Recovering soft n-SuHG. Let $\big(V, E, C, A, B\big)$ be a soft n-SuperHyperGraph. Define a directed n-SuperHyperGraph $\mathrm{DSuHG}'^{(n)}$ with vertex set V and directed edge set

$$E' = \{\,(e, e) : e \in E\},$$

i.e. each undirected superedge is treated as a looped directed edge. Retain the same parameter set C and mappings A, B. Then for each $c \in C$,

$$B(c) \;\subseteq\; \{\, e \in E : e \subseteq A(c)\} \iff \{\,(e, e) : e \in B(c)\} \;\subseteq\; \{\,(e, e) \in E' : e \subseteq A(c)\},$$

so $\big(A(c), B(c)\big)$ coincides with the soft directed substructure at c. Hence the soft directed n-SuperHyperGraph reduces to the original soft n-SuperHyperGraph when all edges are looped.

Theorem 2. *Let*

$$\big(\Delta^*,\, C,\, R,\, S\big)$$

be a soft directed hypergraph on a simple directed hypergraph $\Delta^* = (\Gamma, \Xi)$*, with*

$$R : C \to \mathrm{POWS}(\Gamma), \quad S : C \to \mathrm{POWS}(\Xi).$$

Define $n = 1$ *and form the directed* 1*-SuperHyperGraph* $\mathrm{DSuHG}^{(1)} = (V, E)$ *by*

$$V = \big\{\{v\} : v \in \Gamma\big\} \subseteq \mathrm{POWS}^1(\Gamma), \quad E = \big\{\, \big(\{T(e)\}, \{H(e)\}\big) : e = (T(e), H(e)) \in \Xi\big\},$$

where each original hyperarc e *becomes a directed* 1*-superedge between the singleton sets of its tail and head. Then the maps*

$$A : C \;\longrightarrow\; \mathrm{POWS}(V), \quad A(c) \;=\; \{\{v\} : v \in R(c)\},$$

$$B : C \;\longrightarrow\; \mathrm{POWS}(E), \quad B(c) \;=\; \big\{(\{T(e)\}, \{H(e)\}) : e \in S(c)\big\}$$

turn $\big(\mathrm{DSuHG}^{(1)},\, C,\, A,\, B\big)$ *into a soft directed* 1*-SuperHyperGraph that is isomorphic to the original soft directed hypergraph* $\big(\Delta^*, C, R, S\big)$*.*

Proof. We verify each requirement in turn.

1. Construction of $\mathrm{DSuHG}^{(1)}$**.** Since $\mathrm{POWS}^1(\Gamma) = \mathrm{POWS}(\Gamma)$, the set of all singleton subsets $\{\{v\} : v \in \Gamma\}$ is a valid choice for V. Each hyperarc $e = (T(e), H(e)) \in \Xi$ becomes a directed 1-superedge $(\{T(e)\}, \{H(e)\})$ in $E \subseteq \mathrm{POWS}^1(\Gamma) \times \mathrm{POWS}^1(\Gamma)$. By simplicity of Δ^*, no two hyperarcs share both the same tail and head, so no parallel superedges occur.

2. Definition of soft selections. For each parameter $c \in C$, set

$$A(c) = \{\{v\} : v \in R(c)\} \subseteq V,$$

$$B(c) = \{(\{T(e)\}, \{H(e)\}) : e \in S(c)\} \subseteq E.$$

Because $S(c) \subseteq \Xi$ and $R(c) \subseteq \Gamma$, these indeed select subsets of V and E.

3. Validity of the induced substructure. By the soft directed hypergraph definition, every hyperarc $e \in S(c)$ has $T(e) \subseteq R(c)$ and $H(e) \subseteq R(c)$. Therefore

$$\{T(e)\} \subseteq \{\{v\} : v \in R(c)\} = A(c), \quad \{H(e)\} \subseteq A(c),$$

so each directed 1-superedge $(\{T(e)\}, \{H(e)\})$ lies in $\{\,(\tau, h) \in E : \tau,\ h \subseteq A(c)\}$. Thus $(A(c), B(c))$ is a valid soft directed 1-SuperHyperGraph substructure.

4. Isomorphism with the original. Define

$$\Phi_V :\ \Gamma \;\longrightarrow\; V, \quad \Phi_V(v) = \{v\},$$

$$\Phi_E : \Xi \longrightarrow E, \quad \Phi_E\big((T, H)\big) = (\{T\}, \{H\}).$$

Together, (Φ_V, Φ_E) is a bijection of directed hypergraphs that carries the soft selection $(R(c), S(c))$ precisely to $(A(c), B(c))$. In particular

$$\Phi_V\big(R(c)\big) = A(c), \quad \Phi_E\big(S(c)\big) = B(c),$$

for all $c \in C$. Hence the two soft structures are isomorphic, completing the proof.

Theorem 3 (Isomorphism Invariance)**.** *Let* $\mathrm{DSuHG}^{(n)} = (V, E)$ *and* $\mathrm{DSuHG}'^{(n)} = (V', E')$ *be two directed* n*-SuperHyperGraphs. Suppose* $\varphi : V \to V'$ *is a bijection such that for every directed superedge* $e = (T, H) \in E$*,*

$$(\varphi(T), \varphi(H)) \in E',$$

where $\varphi(T) = \{\varphi(v) : v \in T\}$*. Given a soft structure* (C, A, B) *on the first graph, define*

$$A'(c) = \varphi\big(A(c)\big), \quad B'(c) = \{\, (\varphi(T), \varphi(H)) : (T, H) \in B(c)\}.$$

Then

$$\big(\mathrm{DSuHG}^{(n)}, C, A, B\big) \;\cong\; \big(\mathrm{DSuHG}'^{(n)}, C, A', B'\big)$$

as soft directed n*-SuperHyperGraphs.*

Proof. We break the proof into three parts:

1. Extend φ to supervertices. Since $V \subseteq \mathrm{POWS}^n(S)$, define

$$\varphi_n : \mathrm{POWS}^n(S) \longrightarrow \mathrm{POWS}^n(S'), \quad \varphi_n(X) = \{\varphi(v) : v \in X\}.$$

By bijectivity of φ, φ_n is also a bijection on n-supervertices.

2. Preserve directed edges. Take any directed superedge $e = (T, H) \in E$. By hypothesis, $(\varphi(T), \varphi(H)) \in E'$. Thus φ_n carries the entire relation E onto E'.

3. Map soft selections. For each parameter $c \in C$:

$$A'(c) = \varphi_n\big(A(c)\big) \quad \text{and} \quad B'(c) = \{(\varphi_n(T), \varphi_n(H)) : (T, H) \in B(c)\}.$$

Because φ_n is bijective and respects inclusion,

$$T \subseteq A(c) \iff \varphi_n(T) \subseteq \varphi_n\big(A(c)\big) = A'(c),$$

and similarly for heads. Hence $(A(c), B(c))$ is carried exactly to $(A'(c), B'(c))$, verifying the isomorphism of soft structures.

Theorem 4 (Recovery of Directed n-SuperHyperGraphs)**.** *If* $C = \{*\}$ *and we define*

$$A(*) = V, \quad B(*) = E,$$

then the soft directed n*-SuperHyperGraph* $\big(\mathrm{DSuHG}^{(n)}, C, A, B\big)$ *reduces to the underlying directed* n*-SuperHyperGraph* $\mathrm{DSuHG}^{(n)}$*.*

Proof. With exactly one parameter, the only soft sub-n-superhypergraph is $(A(*), B(*)) = (V, E)$. There is no restriction or filtering: all supervertices and superedges are included. Therefore the soft structure coincides with the original directed graph.

Theorem 5 (Recovery of Soft n-SuperHyperGraphs)**.** *Let $\big(V, E, C, A, B\big)$ be a soft (undirected) n-SuperHyperGraph. Construct a directed looped version $\mathrm{DSuHG}'^{(n)} = (V, E')$ by*

$$E' = \{(e, e) : e \in E\}.$$

Then $\big(\mathrm{DSuHG}'^{(n)}, C, A, B\big)$ as a soft directed n-SuperHyperGraph recovers $\big(V, E, C, A, B\big)$.

Proof. Each undirected superedge $e \in E$ becomes a directed hyperedge (e, e) that loops from e back to e. For each $c \in C$:

$$B(c) \subseteq \{\, e \in E : e \subseteq A(c)\}$$

is equivalent to

$$B(c) \subseteq \{\, (e, e) \in E' : \mathrm{Tail}(e, e) = e \subseteq A(c)\},$$

since $\mathrm{Tail}(e, e) = \mathrm{Head}(e, e) = e$. Hence the soft substructure $(A(c), B(c))$ is unchanged, demonstrating exact recovery of the original soft n-SuperHyperGraph.

Theorem 6 (Monotonicity under Parameter Inclusion)**.** *If $c_1, c_2 \in C$ satisfy*

$$A(c_1) \subseteq A(c_2) \quad \textit{and} \quad B(c_1) \subseteq B(c_2),$$

then for every directed superedge $(T, H) \in B(c_1)$, we have $(T, H) \in B(c_2)$, and every supervertex in $A(c_1)$ belongs to $A(c_2)$. In other words, the soft sub-n-superhypergraph at c_1 is contained in that at c_2.

Proof. By hypothesis: 1. Every $v \in A(c_1)$ also lies in $A(c_2)$. 2. Every $(T, H) \in B(c_1)$ satisfies $T, H \subseteq A(c_1) \subseteq A(c_2)$, so (T, H) is a valid element of $B(c_2)$.

Therefore the inclusion of soft selections is preserved, and the theorem follows immediately.

Theorem 7 (Intersection Closure)**.** *Let*

$$\big(\mathrm{DSuHG}^{(n)},\, C,\, A,\, B\big)$$

be a soft directed n-SuperHyperGraph. For any two parameters $c_1, c_2 \in C$, define

$$A_\wedge = A(c_1) \,\cap\, A(c_2), \quad B_\wedge = B(c_1) \,\cap\, B(c_2).$$

Then the pair $(A_\wedge, B_\wedge)$ satisfies

$$B_\wedge \;\subseteq\; \big\{\, e \in E : \mathrm{Tail}(e) \subseteq A_\wedge,\; \mathrm{Head}(e) \subseteq A_\wedge \big\},$$

so $\big(A_\wedge, B_\wedge\big)$ is itself a valid soft induced sub–n-superhypergraph.

Proof. By definition of a soft directed n-SuperHyperGraph, for each $i = 1, 2$,

$$B(c_i) \subseteq \{\, e \in E : \text{Tail}(e) \subseteq A(c_i),\ \text{Head}(e) \subseteq A(c_i)\}.$$

Hence any $e \in B_\wedge = B(c_1) \cap B(c_2)$ satisfies

$$\text{Tail}(e) \subseteq A(c_1) \quad \text{and} \quad \text{Tail}(e) \subseteq A(c_2),$$

so $\text{Tail}(e) \subseteq A(c_1) \cap A(c_2) = A_\wedge$. An identical argument applies to Head(e). Therefore e lies in $\{\, e \in E : \text{Tail}(e) \subseteq A_\wedge, \text{Head}(e) \subseteq A_\wedge\}$, establishing the claimed inclusion.

Theorem 8 (Union Closure)**.** *With the same notation, define*

$$A_\vee = A(c_1) \cup A(c_2), \quad B_\vee = B(c_1) \cup B(c_2).$$

Then every $e \in B_\vee$ *satisfies*

$$\text{Tail}(e) \subseteq A_\vee, \quad \text{Head}(e) \subseteq A_\vee,$$

so $\big(A_\vee, B_\vee\big)$ *also defines a soft induced sub–n-superhypergraph.*

Proof. Take any hyperedge $e \in B_\vee$. By definition, e belongs either to $B(c_1)$ or to $B(c_2)$.

- If $e \in B(c_1)$, then $\text{Tail}(e) \subseteq A(c_1) \subseteq A(c_1) \cup A(c_2) = A_\vee$, and similarly $\text{Head}(e) \subseteq A_\vee$.
- If $e \in B(c_2)$, the same argument shows $\text{Tail}(e) \subseteq A_\vee$ and $\text{Head}(e) \subseteq A_\vee$.

In either case, e lies in the set of directed n-superedges whose tail and head both lie inside $A_\vee$. Consequently,

$$B_\vee \subseteq \{\, e \in E : \text{Tail}(e) \subseteq A_\vee,\ \text{Head}(e) \subseteq A_\vee\},$$

as required.

Theorem 9 (Lattice Structure of Soft Selections)**.** *The collection of pairs* $\{(A(c), B(c)) : c \in C\}$*, ordered by*

$$(A_1, B_1) \preceq (A_2, B_2) \iff A_1 \subseteq A_2 \text{ and } B_1 \subseteq B_2,$$

forms a bounded lattice under the meet $(\cap, \cap)$ *and join* $(\cup, \cup)$ *operations defined above.*

Proof. We must show that any two elements $(A(c_1), B(c_1))$ and $(A(c_2), B(c_2))$ have a greatest lower bound (meet) and a least upper bound (join) with respect to $\preceq$.

Meet. By Theorem 7,

$$\big(A(c_1) \cap A(c_2),\ B(c_1) \cap B(c_2)\big)$$

is a valid soft substructure, and clearly it is the largest pair below both $(A(c_1), B(c_1))$ and $(A(c_2), B(c_2))$.

Join. By Theorem 8,

$$\big(A(c_1) \cup A(c_2),\, B(c_1) \cup B(c_2)\big)$$

is valid, and it is the smallest pair above both initial elements.

Bounds. The minimum element is $(\varnothing, \varnothing)$, since no edges or vertices lie outside the empty set. The maximum element is (V, E), the full directed n-superhypergraph. Hence the family forms a bounded lattice.

Definition 11 (Acyclic Directed n-SuperHyperGraph)**.** *A directed n-SuperHyperGraph* $\mathrm{DSuHG}^{(n)} = (V, E)$ *is* acyclic *if there exists no finite sequence of distinct hyperedges*

$$e_1 = (T_1, H_1),\, e_2 = (T_2, H_2),\, \ldots,\, e_k = (T_k, H_k) \in E$$

and vertices $v_0, v_1, \ldots, v_k = v_0 \in V$ such that for each $i = 1, \ldots, k$:

$$v_{i-1} \in T_i \quad \textit{and} \quad v_i \in H_i.$$

Theorem 10 (Acyclicity Preservation)**.** *Let $\big(\mathrm{DSuHG}^{(n)}, C, A, B\big)$ be a soft directed n-SuperHyperGraph whose underlying* $\mathrm{DSuHG}^{(n)}$ *is acyclic. Then for any parameter $c \in C$, the induced soft sub-n-superhypergraph $\big(A(c), B(c)\big)$ is also acyclic.*

Proof. Assume, for contradiction, that some soft sub-n-superhypergraph $(A(c), B(c))$ contains a directed cycle. Then there exist distinct edges

$$e_1, \ldots, e_k \in B(c)$$

and vertices $v_0, v_1, \ldots, v_k = v_0 \in A(c)$ with

$$v_{i-1} \in \mathrm{Tail}(e_i), \quad v_i \in \mathrm{Head}(e_i), \quad i = 1, \ldots, k.$$

Since $B(c) \subseteq E$ and $A(c) \subseteq V$, the same sequence of edges and vertices witnesses a cycle in the full graph $\mathrm{DSuHG}^{(n)}$. This contradicts its acyclicity. Hence no such cycle can occur in any soft substructure.

Definition 12 (Reachability)**.** *In a directed n-SuperHyperGraph (V, E), a vertex $u \in V$ is* reachable *from $v \in V$ if there exists a finite sequence of hyperedges*

$$e_1 = (T_1, H_1),\, e_2 = (T_2, H_2),\, \ldots,\, e_m = (T_m, H_m)$$

and vertices $v = v_0, v_1, \ldots, v_m = u$ such that for each i:

$$v_{i-1} \in T_i, \quad v_i \in H_i.$$

Theorem 11 (Reachability Restriction)**.** *Under the same hypothesis, if u is reachable from v in the soft sub-n-superhypergraph $\big(A(c), B(c)\big)$, then u is also reachable from v in the original directed n-SuperHyperGraph.*

Proof. Suppose in $(A(c), B(c))$ there is a path

$$v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_m} v_m = u,$$

with each $e_i \in B(c)$ and $v_{i-1} \in \mathrm{Tail}(e_i)$, $v_i \in \mathrm{Head}(e_i)$. Since $B(c) \subseteq E$, every e_i also belongs to the full edge set E. Moreover, all $v_i \in A(c) \subseteq V$. Thus the same sequence of edges and vertices forms a valid reachability path in the underlying $\mathrm{DSuHG}^{(n)}$. Therefore reachability in any soft induced subgraph implies reachability in the original graph.

4. Discussion

In this section, we examine Soft Directed n-SuperHyperGraphs in relation to established models. Table 1 compares Soft Directed Graphs, Soft Directed HyperGraphs, and Soft Directed n-SuperHyperGraphs, while Table 2 contrasts Soft n-SuperHyperGraphs with Soft Directed n-SuperHyperGraphs.

Concept	**Base Structure**	**Selection Maps**
Soft Directed Graph	Directed graph (V, E) where $E \subseteq V \times V$	Each parameter chooses a subset of vertices in V and a subset of directed edges in E.
Soft Directed HyperGraph	Directed hypergraph (Γ, Ξ) where $\Xi \subseteq \mathcal{P}(\Gamma) \times \mathcal{P}(\Gamma)$	Each parameter chooses a subset of nodes in Γ and a subset of hyperarcs in Ξ.
Soft Directed n-SuperHyperGraph	Directed n-SuperHyperGraph (V, E) with $V \subseteq \mathcal{P}^n(S)$, $E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S)$	Each parameter chooses a subset of n-supervertices in V and a subset of directed n-superedges in E.

Table 1: Overview of soft-parameterized directed network models

Concept	**Base Structure**	**Selection Maps**
Soft n-SuperHyperGraph	n-SuperHyperGraph (V, E) with $V \subseteq \mathcal{P}^n(S)$, $E \subseteq \mathcal{P}^n(S)$	Each parameter chooses a subset of n-supervertices in V and a subset of n-superedges in E.
Soft Directed n-SuperHyperGraph	Directed n-SuperHyperGraph (V, E) with $V \subseteq \mathcal{P}^n(S)$, $E \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S)$	Each parameter chooses a subset of directed n-supervertices in V and a subset of directed n-superedges in E.

Table 2: Overview of Soft n-SuperHyperGraphs versus Soft Directed n-SuperHyperGraphs

From these comparisons, it is clear that Soft Directed n-SuperHyperGraphs excel at modeling highly hierarchical, complex, and inherently directional phenomena. Future work involving computational experiments and algorithm development for Soft Directed n-SuperHyperGraphs will yield both qualitative insights and detailed quantitative evaluations, paving the way for practical applications.

5. Conclusion and Future Tasks

In this paper, we explored the mathematical properties and real-world applications of Soft Directed n-SuperHyperGraphs. These structures provide a flexible and expressive framework for modeling hierarchical, directional, and parameterized relationships under uncertainty.

As directions for future work, we aim to investigate further generalizations and extensions by incorporating advanced uncertainty-based structures such as HyperSoft Sets[31], Soft Expert Sets [69], Hypersoft Expert Set[70, 71, 72], SuperHyperSoft Sets[73, 74], and Soft Rough Sets [75]. These developments have the potential to enhance modeling capabilities in complex and imprecise domains. Furthermore, several more refined notions of directed graphs have been proposed in the literature, such as bidirected graphs [76, 77], bunch graphs [78], pangene graphs[79], and multidirected graphs [80]. As part of future work, we intend to explore potential extensions of our framework using these advanced graph structures. Furthermore, we also hope that future research will explore applications of Soft Directed n-SuperHyperGraphs in areas such as decision-making, network theory, and graph neural networks.

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Data Availability

This work is purely theoretical and does not involve original datasets. We invite future empirical research to validate and expand upon the ideas presented here.

Ethical Approval

No ethical review was required, as this study did not involve human participants or animal subjects.

Conflicts of Interest

The authors declare that they have no competing interests related to this publication.

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Research Integrity

We confirm that this manuscript reports original work by the authors, has not been published elsewhere, and is not under consideration by another journal.

Disclaimer on Computational Tools

All proofs and analyses were performed manually. We did not employ any computer algebra systems, automated theorem provers, or proof assistants (e.g., Mathematica, SageMath, Coq).

Code Availability

No software or source code was produced for this theoretical investigation.

Clinical Trial

This study did not include any clinical trials.

Consent to Participate

Not applicable.

Disclaimer on Scope and Accuracy

The concepts and models introduced here have yet to undergo empirical testing. Future work may explore their practical applications and rigorously assess their validity. While we have strived for accuracy and proper citation, inadvertent errors may remain; readers are encouraged to consult original sources and contact us with any questions. The conclusions are drawn within the specific theoretical framework and assumptions described, and may not generalize beyond those boundaries. All viewpoints expressed are those of the authors alone.

Consent to Publish

All authors have given their consent for submission of this manuscript to the journal.

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