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Solving Problems by Using a Function in The Number Theory

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SOLVING PROBLEMS BY USING A FUNCTION IN THE NUMBER THEORY

Let $n \geq 1$, $h \geq 1$,and $a \geq 2$ be integers. For which values of a and n is $(n+h)!$ a multiple of a^n ? (A generalization of the problem $n^0 = 1270$, Mathematics Magazine, Vol. 60, No. 3, June 1987, p. 179, proposed by Roger B. Eggleton, The University of Newcastle, Australia.)

Solution (For $h = 1$ the problem $n^0 = 1270$ is obtained.)

§1. Introduction

We have constructed a function η (see [1]) having the fallowing properties:

(a) For each non-null integer n,$\eta(n)!$ is multiple of n;

(b) $\eta(n)$ is the smallest natural number with the property (a).

It is easy to prove:

Lemma 1. $(\forall)k, p \in N^*$, $p \neq 1, k$ *is uniquely written in the form:*

$$k = t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)},$$

where $a_{n_i}^{(p)} = (p^{n_i} - 1)/(p-1), i = 1, 2, \ldots, l$, $n_1 > n_2 > \ldots > n_l > 0$ *and* $1 \leq t_j \leq (p-1)$, $j = 1,\ 2, \ldots, l-1, 1 \leq t_l \leq p$, $n_j, t_j \in N, i = 1, 2, \ldots, l$, $l \in N^*$.

We have constructed the function η_p, p *prime* > 0, $\eta_p : N^* \to N^*$, *thus :*

$(\forall)n \in N^*$, $\eta_p(a_n^{(p)}) = p^n$, *and* $\eta_p(t_1 a_{n_1}^{(p)} + \ldots + t_l a_{n_l}^{(p)}) = t_1 \eta_p(a_{n_1}^{(p)}) + \ldots + t_l \eta_p(a_{n_l}^{(p)})$.

Of course:

Lemma 2. (a) $(\forall)k \in N^*$, $\eta_p(k)! = Mp^k$.

(b) $\eta_p(k)$ *is the smallest number with the property* (a). *Now, we construct another function:* $\eta : Z \setminus 0 \to N$ *defined is follows:*

$$\begin{cases} \eta(\pm 1) = 0, \\ (\forall)n = \epsilon p_1^{\alpha_1} \ldots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1, p_i \text{ prime and } p_i \neq p_j \text{ for } i \neq j, \text{ all} \\ \alpha_i \in N^*, \eta(n) = \max\limits_{1 \leq i \leq s} \{\eta_p(\alpha_i)\} \end{cases}$$

It is not difficult to prove η has the demanded properties of §1.

§2. Now, let $a = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$, with all $\alpha_i \in N^*$ and all p_i distinct primes. By the previous theory we have:

$\eta(a) = \max\limits_{1 \leq i \leq s} \{n_{p_i}(\alpha_i)\} = \eta_p(\alpha)$ (by notation).

Hance $\eta(a) = \eta(p^\alpha), \eta(p^\alpha)! = Mp^\alpha$.

We know:

$$(t_1p^{n_1}+\ldots+t_lp^{n_l})! = Mp^{t_1\frac{p^{n_1}-1}{p-1}+\ldots+t_l\frac{p^{n_l}-1}{p-1}}$$

We put:

$t_1p^{n_1}+\ldots+t_lp^{n_l} = n+h$ and $t_1\frac{p^{n_1}-1}{p-1}+\ldots+t_l\frac{p^{n_l}-1}{p-1} = \alpha n$.

Whence

$\frac{1}{\alpha}[\frac{p^{n_1}-1}{p-1}+\ldots+t_l\frac{p^{n_l}-1}{p-1}] \geq t_1p^{n_1}+\ldots+t_lp^{n_l}-h$ or

(1) $\alpha(p-1)h \geq (\alpha p-\alpha-1)[t_1p^{n_1}+\ldots+t_lp^{n_l}]+(t_1+\ldots+t_l)$.

On this condition we take $n_0 = t_1p^{n_1}+\ldots+t_lp^{n_l}-h$ (see Lemma 1), hence $n = \begin{cases} n_0, n_0>0; \\ 1, n_0\leq 0 \end{cases}$

Consider giving $a\neq 2$, we have a finite number of n. There is an infinite number of n if and only if $\alpha p-\alpha-1=0$ i.e., $\alpha=1$ and $p=2$, i.e.,$a=2$

§3 Particular Case

If $h=1$ and $a\neq 2$, bacause $t_1p^{n_1}+\ldots+t_lp^{n_l}\geq p^{n_l}>1$

and $t_1+\ldots+t_l\geq 1$, it follows from (1) that :

(1′) $(\alpha p-\alpha) > (\alpha p-\alpha-1)\cdot 1+1 = \alpha p-\alpha$,

which is impossible. If $h=1$ and $a=2$ then $\alpha=1, p=2$, or

(1″) $1\leq t_1+\ldots+t_l$,

hance $l=1, t_1=1$ whence $n = t_1p^{n_1}+\ldots+t_lp^{n_l}-h = 2^{n_1}-1, n_1\in N^*$ (the solution to problem 1270).

Example 1. *Let $h=16$ and $a=3^4\cdot 5^2$. Find all n such that*

$$(n+16)! = M2025^n.$$

Solution

$\eta(2025) = \max\{\eta_3(4),\eta_5(2)\} = \max\{9,10\} = 10 = \eta_5(2) = \eta(5^2)$. Whence $\alpha=2, p=5$. From (1) we have:

$$128\geq 7[t_15^{n_1}+\ldots t_l5^{n_l}]+t_1+\ldots+t_l.$$

Because $5^4>128$ and $7[t_15^{n_1}+\ldots t_l5^{n_l}]<128$ we find $l=1$,

$$128\geq 7t_15^{n_1}+t_1,$$

whence $n_1\leq 1$, i.e. $n_1=1$, and $t_1=1,2,3$. Then $n_0=t_15-16<0$, hence we take $n=1$.

Example 2. $(n+7)! = M3^n$ *when* $n = 1, 2, 3, 4, 5.$

$(n+7)! = M5^n$ *when* $n = 1.$

$(n+7)! = M7^n$ *when* $n = 1.$

But $(n+7)! \neq Mp^n$ *for p prime* $> 7, (\forall) n \in N^*.$

$(n+7)! \neq M2^n$ *when*

$$n_0 = t_1 2^{n_1} + \ldots + t_l 2^{n_l} - 7,$$

$$t_1, \ldots, t_{l-1} = 1,$$

$$1 \leq t_l \leq 2,\ t_1 + \ldots + t_l \leq 7$$

$$\text{and } n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0. \end{cases} \quad \text{etc.}$$

Exercise for Readers

If $n \in N^*, a \in N^* \setminus \{1\}$, find all values of a and n such that:

$(n+7)!$ is a multiple of a^n.

Some Unsolved Problems (see [2])

Solve the diophantine equations:

(1) $\eta(x) \cdot \eta(y) = \eta(x+y).$

(2) $\eta(x) = y!$ (A solution: $x = 9, y = 3$).

(3) Conjecture: the equation $\eta(x) = \eta(x+1)$ has no solution.

References

[1] Florentine Smaramndache, "A Function in the Number Theory", Analele Univ. Timisoara, Fasc. 1, Vol. XVIII, pp. 79-88, 1980, MR: 83c: 10008.

[2] Idem, Un Infinity of Unsolved Problems Concerning a Function in Number Theory, International Congress of Mathematicians, Univ. of Berkeley, CA, August 3-11, 1986.

[A comment about this generalization was published in "Mathematics Magazine"], Vol. 61, No. 3, June 1988, p. 202: "Smarandache considered the general problem of finding positive integers n, a and k, so that $(n+k)!$ should be a multiple of a^n. Also, for positive integers p and k, with p prime, he found a formula for determining the smallest integer $f(k)$ with the property that $(f(k))!$ is a multiple of p^k."]