SOME RATIO TYPE ESTIMATORS UNDER MEASUREMENT ERRORS

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Abstract: This article addresses the problem of estimating the population mean using auxiliary information in the presence of measurement errors. A comparative study is made among the proposed estimators, the mean per unit estimator and the ratio estimator in the presence of measurement errors.

Key words: Auxiliary variate · Measurement error · Observational error · Mean square error · Efficiency

INTRODUCTION

In survey sampling, the properties of the estimators based on data usually presuppose that the observations are the correct measurements on characteristics being studied. Unfortunately, this ideal is not met in practice for a variety of reasons, such as non response errors, reporting errors and computing errors. When the measurement errors are negligible small, the statistical inferences based on observed data continue to remain valid. On the contrary when they are not appreciably small and negligible, the inferences may not be simply invalid and inaccurate but may often lead to unexpected, undesirable and unfortunate consequences [1]. Some authors including [2-7] have paid their attention towards the estimation of population mean μ_y of the study variable y using auxiliary information in the presence of measurement errors.

For a simple random sampling scheme, let (x_i, y_i) be observed values instead of the true values (X_i, Y_i) on two characteristics (x,y) respectively for the i^{th} (i=1.2….n) unit in the sample of size n. Let the measurement errors be

$$u_i = y_i - Y_i \quad (1.1)$$
$$v_i = x_i - X_i \quad (1.2)$$

which are stochastic in nature with mean zero and variances σ_u^2 and σ_v^2 respectively and are independent. Further, let the population means of (x, y) be (μ_X, μ_Y) population variances of (x,y) be $\left(\sigma_x^2, \sigma_y^2\right)$ and σ_{xy} and ρ be the population covariance and the population correlation coefficient between x and y respectively [4].

In this paper we study the behaviour of some estimators in presence of measurement error.

Exponential Ratio-type Estimator under Measurement Error: [8], suggested an exponential ratio type estimator for estimating $\bar{Y}$ as

$$t_1 = \bar{y}\exp\left(\frac{\mu_x - \bar{x}}{\mu_x + \bar{x}}\right) \quad (2.1)$$

Let $w_u = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} u_i,\ w_y = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(y_i - \mu_y\right)$

$$w_v = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} v_i,\ w_x = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(x_i - \mu_x\right)$$

$$C_x = \frac{\sigma_x}{\mu_x} \text{ and } C_y = \frac{\sigma_y}{\mu_y}$$

Expression (2.1) can be written as

$$t_1 = \left[\mu_y + \left(\bar{y} - \mu_y\right)\right]\exp\left[\frac{\mu_x - \left(\mu_x + \bar{x} - \mu_x\right)}{\mu_x + \left(\mu_x + \bar{x} - \mu_x\right)}\right]$$

$$\left(\mu_y + k_1\right)\exp\left[\frac{-k_2}{2\mu_x + k_2}\right] \quad (2.2)$$

where,

$$k_1 = \bar{y} - \mu_y = \frac{1}{\sqrt{n}}\left(w_y - w_u\right) \text{ and } k_2 = \bar{x} - \mu_x = \frac{1}{\sqrt{n}}\left(w_x - w_v\right) \quad (2.3)$$

On simplifying expression (2.2), we have

$$\left(t_1-\mu_y\right)=\mu_y\left[-\frac{1}{2}\left(\frac{k_2}{\mu_x}\right)+\frac{3}{8}\left(\frac{k_2}{\mu_x}\right)^2+..\right]$$
$$+k_1\left[1-\frac{1}{2}\left(\frac{k_2}{\mu_x}\right)+\frac{3}{8}\left(\frac{k_2}{\mu_x}\right)^2+..\right] \quad (2.4)$$

On taking the expectations and using the results

$$E\left(k_1\right)=E\left(k_2\right)=0$$

$$E\left(k_1^2\right)=\frac{\sigma_y^2}{n}\left(1+\frac{\sigma_u^2}{\sigma_y^2}\right)=V_{ym} \quad (2.5)$$

$$E\left(k_2^2\right)=\frac{\sigma_x^2}{n}\left(1+\frac{\sigma_v^2}{\sigma_x^2}\right)=V_{xm} \quad (2.6)$$

$$E\left(k_1k_2\right)=\frac{\rho\sigma_y\sigma_x}{n}=V_{yxm} \quad (2.7)$$

Taking expectation on both side of (2.4) the bias of t_1, to first order of approximation is

$$Bias\left(t_1\right)=\frac{1}{\mu_x}\left(\frac{3}{8}R_mV_{xm}-\frac{1}{2}V_{yxm}\right) \quad (2.8)$$

where $R_m=\frac{\mu_y}{\mu_x}$.

Squaring both side of (2.4) and taking expectation, the mean square error of $t_1$up to the first order of approximation, is

$$MSE=E\left(t_1-\mu_y\right)^2$$
$$=\frac{\mu_y^2}{4\mu_x^2}E\left(k_2^2\right)+E\left(k_1^2\right)-\frac{\mu_y}{\mu_x}E\left(k_1k_2\right)$$
$$=\frac{1}{4}R_m^2\left(\frac{\sigma_x^2}{n}\left(1+\frac{\sigma_v^2}{\sigma_x^2}\right)\right)+\frac{\sigma_y^2}{n}\left(1+\frac{\sigma_u^2}{\sigma_y^2}\right)-\frac{\rho\sigma_y\sigma_x}{n}R_m$$
$$=\frac{\sigma_y^2}{n}\left[1-\frac{C_x}{C_y}\left(\rho-\frac{C_x}{4C_y}\right)\right]+\frac{1}{n}\left[\frac{\mu_y^2}{4\mu_x^2}\sigma_v^2+\sigma_u^2\right]=M_{t_1}^*+M_{t_1} \quad (2.9)$$

where $M_{t_1}^*=\frac{\sigma_y^2}{n}\left[1-\frac{C_x}{C_y}\left(\rho-\frac{C_x}{4C_y}\right)\right]$ is the mean squared error of t_1 without measurement error and $M_{t_1}=\frac{1}{n}\left[\frac{\mu_y^2}{4\mu_x^2}\sigma_v^2+\sigma_u^2\right]$ is the contribution of measurement errors in t_1

Another Estimator under Measurement Error: [9], suggested a regression type estimator t_2 as-

$$t_2=\omega_1\bar{y}+\omega_2\left(\mu_x-\bar{x}\right) \quad (3.1)$$

where ω_1 are ω_2 constants that have no restriction.

Expression (3.1) can be written as

$$t_2-\mu_y=\omega_1\ k_1(\omega_1-1)-\omega_2\ \omega_2 \quad (3.2)$$

Taking expectation of both side of (3.2), we get the bias of the estimator t_2 to order $O(n^{-1})$ as

$$Bias(t_2)=\mu_y\,(\omega_1-1)$$

Squaring both side of (3.2) and taking expectation, the MSE of t_2 to the to order $O(n^{-1})$ is

$$MSE\left(t_2\right)=\mu_y^2\left(\omega_1-1\right)^2+\frac{1}{n}\omega_1^2\sigma_y^2+\frac{1}{n}\omega_2^2\sigma_x^2-\frac{2}{n}\omega_1\omega_2\rho\sigma_y\sigma_x+\frac{1}{n}\left(\omega_1^2\sigma_u^2+\omega_2^2\sigma_v^2\right) \quad (3.4)$$

$$=M_{t_2}^*+M_{t_2} \quad (3.5)$$

Where,

$M_{t_2}^*=\mu_y^2\left(\omega_1-1\right)^2+\frac{1}{n}\omega_1^2\sigma_y^2+\frac{1}{n}\omega_2^2\sigma_x^2-\frac{2}{n}\omega_1\omega_2\rho\sigma_y\sigma_x$, is the MSE of t_2 without measurement error and $M_{t_2}=\frac{1}{n}\left(\omega_1^2\sigma_u^2+\omega_2^2\sigma_v^2\right)$ is the contribution of measurement error in t_2.

Expressing MSE of t_2 as

$$MSE\left(t_2\right)=\left(\omega_1-1\right)^2\mu_y^2+\omega_1^2\left(a_1\right)+\omega_2^2a_2+2\omega_1\omega_2\left(-a_3\right) \quad (3.6)$$

where,

$a_1=(V_{ym})$, $a_2=(V_{xm})$, and $a_3=(V_{yxm})$,

Now, optimising MSE of the estimator t_2 with respect to ω_1 and ω_2 we get

where,

$$b_1 = \mu_y^2 + a_1, \quad b_2 = -a_3, \quad b_3 = a_2, \quad and \quad b_4 = \mu_y^2.$$

Using the values of ω_1^* and ω_2^* from equation (3.7) into equation (3.6), we get the minimum MSE of the estimator t_2 as

$$MSE(t_2)_{\min} = \left[\mu_y^2 - \frac{b_3 b_4^2}{b_1 b_3 - b_2^2}\right] \tag{3.8}$$

A General Class of Estimators: We propose a general class of estimator t_3 as

$$t_3 = \left[m_1 \bar{y} + m_2(\mu_x - \bar{x})\right]\left(\frac{\mu_x}{\bar{x}}\right)^{\alpha} \exp\left[\frac{\mu_x - \bar{x}}{\mu_x + \bar{x}}\right]^{\beta} \tag{4.1}$$

Expanding equation (4.1) and subtracting μ_y from both side, we have

$$(t_3 - \mu_y) = \left\{(m_1 - 1)\mu_y + m_1\mu_y\left\{-B\frac{k_2}{\mu_x} + \frac{k_2^2 A}{\mu_x^2 8}\right\} + m_1 k_1\left\{1 - \frac{Bk_2}{\mu_x} + \frac{k_2^2 A}{\mu_x^2 8}\right\}\right\} \tag{4.2}$$

Taking expectation of both side of (4.2), we get the bias of the estimator t_3 to the order $O(n^{-1})$

$$Bias(t_3) = (m_1 - 1)\mu_y + m_1\mu_y\left\{\frac{V_{xm}A}{8\mu_x^2}\right\} - m_1\left\{\frac{B}{\mu_x}V_{yxm}\right\} + m_2\left\{\frac{B}{\mu_x}V_{xm}\right\} \tag{4.3}$$

where,

$$A = \frac{1}{8}\left[4\alpha(\alpha+1) + \beta(\beta+2) + 4\alpha\beta\right], \text{ and } B = \left(\alpha + \frac{\beta}{2}\right)$$

Squaring both side of (4.2) and taking expectation, the MSE of t_3 to the to order $O(n^{-1})$ is

$$\begin{aligned} MSE(t_3) &= E(t_3 - \mu_y)^2 \\ &= (m_1 - 1)^2\mu_y^2 + m_1^2(V_{ym}) + (m_1 BR_m + m_2)^2(V_{xm}) - 2m_1(m_1 BR_m + m_2)(V_{yxm}) \\ &\quad - 2(m_1 - 1)\mu_y\left\{\frac{m_1 B}{\mu_x}V_{yxm} + \frac{1}{\mu_x}\left(\frac{m_1 R_m A}{8} + m_2 B\right)V_{xm}\right\} \end{aligned}$$

$$\begin{aligned} \mathrm{MSE}(t_3) &= m_1^2\left(\mu_y^2 + V_{xm} + B^2 R_m^2 - 2BR_m V_{yxm}\right) + m_2^2(V_{xm}) + 2m_1 m_2\left(BR_m V_{xm} - V_{yxm}\right) \\ &\quad - 2m_1\mu_y^2 + \mu_y^2 - 2m_1^2 BR_m Vyxm + 2m_1 R_m BV_{yxm} - \frac{m_1^2 R_m^2 A}{4}V_{yxm} \\ &\quad - 2m_2 R_m BV_{yxm} + \frac{R_m^2 A m_1}{4}V_{xm} + 2m_2 R_m BV_{xm} \end{aligned} \tag{4.4}$$

Writing MSE of the estimator t_3 as

$$MSE(t_3) = (m_1 - 1)^2\mu_y^2 + m_1^2(A_1 + 2A_3) + m_2^2 A_2 + 2m_1 m_2(-A_4 - A_5) - 2m_1 A_3 + 2m_2 A_5 \tag{4.5}$$

where,

$$A_1 = \left(V_{ym} + B^2 R_m^2 V_{xm} - 2R_m BV_{yxm}\right), \quad A_2 = (V_{xm}),$$

$$A_3 = \left(AR_m^2 V_{xm} - BR_m V_{yxm}\right), \quad A_4 = \left(V_{yxm} - BV_{xm}R_m\right),$$

$$A_5 = (-BR_m V_{xm}).$$

Now, optimising MSE t_3 with respect to m_1 we get the optimum values of m_2 as-

$$m_1^* = \frac{B_3 B_4 - B_2 B_5}{B_1 B_3 - B_2^2} \quad and \quad m_2^* = \frac{B_1 B_5 - B_2 B_4}{B_1 B_3 - B_2^2} \tag{4.6}$$

where,

$$B_1 = \mu_y^2 + A_1 + 2A_3, \quad B_2 = -A_4 - A_5, \quad B_3 = A_2, \quad B_4 = \mu_y^2 + A_3 \quad \text{and} \quad B_6 = -A_5.$$

Minimum MSE of the estimator t_3 is given by

$$MSE\left(t_3\right)_{\min} = \left[\mu_y^2 - \frac{B_1 B_5^2 + B_3 B_4^2 - 2B_2 B_4 B_5}{B_1 B_3 - B_2^2}\right] \tag{4.7}$$

Theoretical Efficiency Comparison: The MSE of the proposed estimator t_1 will be smaller than usual estimator under measurement error case, if the following condition is satisfied by the data set

$$\frac{\sigma_y^2}{n}\left[1 - \frac{C_x}{C_y}\left(\rho - \frac{C_x}{4C_y}\right)\right] + \frac{1}{n}\left[\frac{\mu_y^2}{4\mu_x^2}\sigma_v^2 + \sigma_u^2\right] \le \frac{\sigma_y^2}{n}\left(1 + \frac{\sigma_u^2}{\sigma_y^2}\right)$$

or

$$R_m^2 \frac{V_{xm}}{V_{yxm}} \le 4 \tag{5.1}$$

As the estimator t_3 defined in (3.1) is the particular member of the generalised estimator t_3 defined in (4.1), if the condition (5.2) is satisfied for different values of α,β the m_1, $m_2$estimator t_3(or t_2) will be better than usual estimator under measurement errors.

$MSE\left(t_3\right)_{\min} \le V\left(\bar{y}_m\right)$
if,

$$\left[\mu_y^2 - \frac{B_1 B_5^2 + B_3 B_4^2 - 2B_2 B_4 B_5}{B_1 B_3 - B_2^2}\right] \le V\left(\bar{y}_m\right) \tag{5.2}$$

Empirical Study

Data Statistics: The data used for empirical study has been taken from [10].

Where, Y_i = True consumption expenditure,
X_i = True income,
y_i = Measured consumption expenditure,
x_i = Measured income.

n	μ_y	μ_y	σ_y^2	σ_x^2	ρ	σ_u^2	σ_v^2
10	127	170	1278	3300	0.964	36.00	36.00

Table 6.1: Showing the MSE of the estimators with and without measurement errors.

Estimators	MSE without measurement error	Contribution of measurement error in MSE	MSE with measurement error
Usual estimator $(\bar{y})$	319.500	9.0005	328.5005
t_1	64.8134	10.2558	75.0700
t_{reg}	22.5000	12.2401	34.7050
t_{2min}(α = 0, β = 0)	22.4687	12.1619	34.6306
t_{3min}(α = 1, β = 0)	22.4678	12.1619	34.6306
(α = 0, β = 1)	22.1033	12.0680	34.1713
(α = 1, β = 1)	20.9734	12.4075	33.3809

CONCLUSION

From Table 6.1, we observe that the MSE of the estimator t_3 (for $\alpha = 1$, $\beta = 1$) is minimum. From third column of the Table we observe that the usual estimator () is least affected by measurement errors. We also observe from that Table that, if due care of observational error is not given, the estimate of the variances that we get gives an under estimate of the true variances.

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