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SPECIAL SMARANDACH CURVES ACCORDING TO THE QUASI FRAME IN 4-DIMENSIONAL EUCLIDEAN SPACE $\mathbb{E}^4$

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Abstract. The first purpose of this paper is to investigate the quasi frame, the quasi equations and show the relations between the Frenet and the quasi curvatures in $\mathbb{E}^4$. The second purpose is to study Smarandach curves in $\mathbb{E}^4$. We obtain the Frenet and the quasi invariants for Smarandach curves in $\mathbb{E}^4$ and by obtaining the Frenet's curvatures we deduce the quasi's curvatures.

1. Introduction

If we have a differentiable curve in an open interval, we can construct a set of mutually orthonormal vectors, these vectors are known as a moving frame or a Frenet frame. The amount by which a curve deviates from being a straight line is called curvature of the curve. Elements of a curve (vectors and curvatures) are called the Frenet apparatus [3]. In recent years, researches deal with the theory of degenerate submanifolds and differential geometry topics have been extended to other spaces such as Minkowski [9] and Galilean [8] spaces.

Smarandach curves are regular curves whose position vectors are composed of Frenet frame vectors on another regular curve. Special Smarandach curves, according to the Frenet frame, are studied in three-dimensional Euclidean space [1].

The parallel transport frame of Bishop [2] is an alternative to the Frenet frame which is well defined on a smooth curve even at points where the second derivative vanishes. The concept of parallel transport frame based on the observation of the tangent vector which is unchanged and the other derivatives are considered in a plane perpendicular to the tangent vector and their derivatives take the same

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direction of tangent vector. The parallel transport frame in 4-dimensional space was studied in [4]. Also, Smarandach curves in 4-dimensional space, from a suitable Frenet frame of a regular curve in 4-dimensional space were studied in [6].

The quasi frame is another alternative to the Frenet frame but, it is easier in computations, has the same accuracy, and can be considered as a generalization of parallel transport frame. The concept of a quasi frame is based on a fixed *projection vector* and a Euclidean angle between the principal normal and quasi-normal vector field [5]. When the second derivative vanishes the frame rotates by a Euclidean angle and the quasi-normal is given as the unit vector perpendicular to the tangent and the projection vectors.

This paper is organized as follows: In section 2, we define some basic definitions of Euclidean 4-space $\mathbb{E}^4$ and the quasi frame in 3-space $\mathbb{E}^3$. In section 3, we investigate the quasi frame and the quasi equation in Euclidean 4-space $\mathbb{E}^4$ and show the relation between the quasi and the parallel transport frames. Furthermore, the relations between the Frenet and the quasi curvatures. In section 4, we study special Smarandach curves in Euclidean 4-space. We obtain the Frenet and the quasi invariants for Smarandach curves in $\mathbb{E}^4$ and by obtaining the Frenet's curvatures we deduce the quasi's curvatures.

2. Preliminaries

Let $\alpha(s) = \alpha : I \subset \mathbb{R} \to \mathbb{E}^4$ be any space curve in Euclidean 4-space. Let $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$ and $\vec{z} = (z_1, z_2, z_3, z_4)$ be three vectors in $\mathbb{E}^4$, with the standard inner product as $< \vec{x}, \vec{y} >= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. The norm of vector of $\mathbb{E}^4$ is given by $\| \vec{x} \| = \sqrt{g(\vec{x}, \vec{x})}$. The curve α is said to be parameterized by arc length s if $g(\alpha', \alpha') = 1$. The vector product of $\vec{x}$, $\vec{y}$, $\vec{z}$ is given by the determinant as follows

$$\vec{x} \times \vec{y} \times \vec{z} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix},$$

where $e_1 \times e_2 \times e_3 = e_4,\ e_2 \times e_3 \times e_4 = e_1,\ e_3 \times e_4 \times e_1 = e_2$ *and* $e_4 \times e_1 \times e_2 = e_3$.

The Frenet equations $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s),\ \mathbf{B}_2(s)\}$ of a unit speed curve $\alpha(s)$ is given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix},$$

where $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are called the tangent, normal, first and second binormals vector fields, respectively. The functions κ_1, κ_2 and κ_3 are called the first, second and third curvatures, respectively.

Let $\alpha = \alpha(t)$ be any curve in $\mathbb{E}^4$ then, the Frenet apparatus can be obtained by the following

$$\begin{aligned}
\mathbf{T} &= \frac{\alpha'}{\| \alpha' \|}, \\
\mathbf{N} &= \frac{\| \alpha' \|^2 \alpha'' - < \alpha', \alpha'' > \alpha}{\| \| \alpha' \|^2 \alpha'' - < \alpha', \alpha'' > \alpha \|}, \\
\mathbf{B}_1 &= \eta \mathbf{B}_2 \times \mathbf{T} \times \mathbf{N}, \\
\mathbf{B}_2 &= \eta \frac{\mathbf{T} \times \mathbf{N} \times \alpha'''}{\| \mathbf{T} \times \mathbf{N} \times \alpha''' \|}, \\
\kappa_1 &= \frac{\| \| \alpha' \|^2 \alpha'' - < \alpha', \alpha'' > \alpha' \|}{\| \alpha' \|^4}, \\
\kappa_2 &= \frac{\| \mathbf{T} \times \mathbf{N} \times \alpha''' \| \| \alpha' \|}{\| \| \alpha' \|^2 \alpha'' - < \alpha', \alpha'' > \alpha' \|}, \\
\kappa_3 &= \frac{< \alpha^{iv}, \mathbf{B}_2 >}{\| \mathbf{T} \times \mathbf{N} \times \alpha''' \| \| \alpha' \|},
\end{aligned}$$

where $\eta = \pm 1$ such that the the Frenet matrix $\{T, N, B_1, B_2\}$ has determinant $+1$.

The quasi frame is an alternative to the Frenet frame, and involves a fixed unit vector $\mathbf{k}$. For a curve $\alpha(t)$ in E^3, the quasi-frame consists of three orthogonal vectors called the unit tangent $\mathbf{T}$, the quasi-normal $\mathbf{N}_q$ and the quasi-binormal $\mathbf{B}_q$ with a Euclidean angle θ between the principal normal and quasi-normal. The quasi frame $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$ is defined by

$$\mathbf{T} = \frac{\alpha'}{\| \alpha' \|},\ \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{k}}{\| \mathbf{T} \times \mathbf{k} \|} \ \text{ and } \ \mathbf{B}_q = \mathbf{T} \times \mathbf{N}_q,$$

where $\mathbf{k}$ is the projection vector. The quasi frame becomes singular in all cases where $\mathbf{T}$ and $\mathbf{k}$ are parallel and in these cases we change the projection i.e. near a point where $\mathbf{T} = (0, 0, 1)$ we could choose $\mathbf{k} = (0, 1, 0)$ or $(1, 0, 0)$ but not $(0, 0, 1)$. The quasi equations $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_q\}$ of a unit speed curve α is given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where the quasi curvatures are: $K_1 = \kappa_1\ cos\theta$, $K_2 = -\kappa_1\ sin\theta$ and $K_3 = d\theta + \kappa_2$.

3. Quasi frame and quasi equations in $\mathbb{E}^4$

In this section, we investigate the quasi frame as an adapted frame along a space curve in $\mathbb{E}^4$. Let $\alpha = \alpha(s)$ be a space curve, the quasi frame in $\mathbb{E}^4$ consists of four orthonormal vectors $\{\mathbf{T}, \mathbf{N}_q, \mathbf{B}_{1q}, \mathbf{B}_{2q}\}$, where $\mathbf{T}$ is the unit tangent vector, $\mathbf{N}_q$ is the quasi-normal vector field, $\mathbf{B}_{1q}$ and $\mathbf{B}_{2q}$ are the first and second quasi-binormals, respectively. The frame is given by

$$\mathbf{T} = \frac{\alpha'}{\parallel \alpha' \parallel}, \quad \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{k}_1 \times \mathbf{k}_2}{\parallel \mathbf{T} \times \mathbf{k}_1 \times \mathbf{k}_2 \parallel}, \quad \mathbf{B}_{2q} = \zeta \frac{\mathbf{T} \times \mathbf{N}_q \times \alpha'''}{\parallel \mathbf{T} \times \mathbf{N}_q \times \alpha''' \parallel}$$
$$\text{and} \qquad \mathbf{B}_{1q} = \zeta \mathbf{B}_{2q} \times \mathbf{T} \times \mathbf{N}_q,$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the projection vectors and ζ is ± 1, where the determinant of matrix is equal to 1. For simplicity, we choose $\mathbf{k}_1 = (0, 0, 0, 1)$ and $\mathbf{k}_2 = (0, 0, 1, 0)$ in our calculations. It is also singular whenever $\mathbf{T}$ lies in the plane spanned by $\mathbf{k}_1$ and $\mathbf{k}_2$. In those cases we may change our projection vectors.

The transformation matrix should be chosen to keep the tangent vector $\mathbf{T}$ unchanged. Then, we consider three possible planes of rotations for the Frenet vectors $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$. The first rotation is in the space spanned by $\mathbf{B}_1$ and $\mathbf{B}_2$ with an angle ϕ. The second rotation in the space plane spanned by $\mathbf{N}$ and $\mathbf{B}_2$ with an angle θ. The third rotation in the space plane spanned by $\mathbf{N}$ and $\mathbf{B}_1$ with an angle ψ. The transformation matrix M is of the form

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\phi & sin\phi \\ 0 & 0 & -sin\phi & cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & 0 & -sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\psi & sin\psi & 0 \\ 0 & -sin\psi & cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta\ cos\psi & cos\theta\ sin\psi & -sin\theta \\ 0 & cos\psi\ sin\theta\ sin\phi - cos\phi\ sin\psi & cos\phi\ cos\psi + sin\theta\ sin\phi\ sin\psi & cos\theta\ sin\phi \\ 0 & cos\phi\ cos\psi\ sin\theta + sin\phi\ sin\psi & -cos\psi\ sin\phi + cos\phi\ sin\theta\ sin\psi & cos\theta\ cos\phi \end{bmatrix}.$$

Theorem 3.1. *Let $\alpha : I \to \mathbb{E}^4$ be a unit speed space curve then, the quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_{1q}' \\ \boldsymbol{B}_{2q}' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & K_4 & K_5 \\ -K_2 & -K_4 & 0 & K_6 \\ -K_3 & -K_5 & -K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix},$$

where
$K_1 = \kappa_1\ cos\theta\ cos\psi$,
$K_2 = \kappa_1\ (cos\psi\ sin\theta\ sin\phi - cos\phi\ sin\psi)$,
$K_3 = \kappa_1\ (cos\phi\ cos\psi\ sin\theta + sin\phi\ sin\psi)$,
$K_4 = sin\phi\ (\kappa_3\ sin\psi - \theta') + cos\phi\ \Big(\kappa_3\ cos\psi\ sin\theta + cos\theta\ (\kappa_2 + \psi')\Big)$,
$K_5 = -\kappa_3\ cos\psi\ sin\theta\ sin\phi + cos^2\theta\ cos\phi\ (\kappa_3\ sin\psi - \theta') + cos\phi\ sin^2\theta\ sin\psi\ (\kappa_3 - sin\psi\ \theta')$
$- cos\theta\ sin\phi\ sin^2\psi\ (\kappa_2 + \psi') - cos^2\psi\ \Big(cos\phi\ sin^2\theta\ \theta' + cos\theta\ sin\phi\ (\kappa_2 + \psi')\Big)$,
$K_6 = \kappa_3\ cos\theta\ cos\psi + cos^2\theta\ cos^2\phi\ \phi' + cos^2\phi\ sin\theta\ (-\kappa_2 + sin\theta\ \phi' - \psi')$
$+ sin^2\phi\Big(\phi' - sin\theta\ (\kappa_2 + \psi')\Big)$.

PROOF. Since $\mathbf{N}_q = [cos\theta\ cos\psi]\ \mathbf{N} + [cos\theta\ sin\psi]\ \mathbf{B}_1 + [-sin\theta]\ \mathbf{B}_2$. Hence,
$\mathbf{N}_q' = [-\kappa_1\ cos\theta\ cos\psi]\ \mathbf{T} + [-cos\psi\ sin\theta\ \theta' - cos\theta\ sin\psi\ (\kappa_2 + \psi')]\ \mathbf{N} + [sin\theta\ (\kappa_3 - sin\psi\ \theta') + cos\theta\ cos\psi\ (\kappa_2 + \psi')]\ \mathbf{B}_1 + [cos\theta\ (sin\psi\ \kappa_3 - \theta')]\ \mathbf{B}_2$.

Since $\mathbf{B}_{1q} = [cos\psi\ sin\theta\ sin\phi - cos\phi\ sin\psi]\ \mathbf{N} + [cos\phi\ cos\psi + sin\theta\ sin\phi\ sin\psi]\ \mathbf{B}_1 + [cos\theta\ sin\phi]\ \mathbf{B}_2$. Hence,
$\mathbf{B}_{1q}' = [\kappa_1\ (-cos\psi\ sin\theta\ sin\phi + cos\phi\ sin\psi)]\ \mathbf{T} + \Big[cos\phi\ cos\psi(\kappa_2 - sin\theta\ \phi' + \psi') + sin\phi\Big(\theta' cos\theta cos\psi + sin\psi\ [\phi' - sin\theta\ (\kappa_2 + \psi')]\Big)\Big]\ \mathbf{N} + \Big[cos\theta\ sin\phi\ (-\kappa_3 + sin\psi\ \theta') - cos\phi\ sin\psi\ (\kappa_2 - sin\theta\ \phi' + \psi') + cos\psi\ sin\phi\Big(-\phi' + sin\theta(\kappa_2 + \psi')\Big)\Big]\mathbf{B}_1 + [sin\theta\ sin\phi\ (\kappa_3\ sin\psi - \theta') + cos\phi\ (\kappa_3\ cos\psi + cos\theta\ \phi')]\ \mathbf{B}_2$.

Since $\mathbf{B}_{2q} = [cos\phi\ cos\psi\ sin\theta + sin\phi\ sin\psi]\ \mathbf{N} + [-cos\psi\ sin\phi + cos\phi\ sin\theta\ sin\psi]\ \mathbf{B}_1 + [cos\theta\ cos\phi]\ \mathbf{B}_2$. Hence,
$\mathbf{B}_{2q}' = [-\kappa_1\ (cos\phi\ cos\psi\ sin\theta + sin\phi\ sin\psi)]\ \mathbf{T} + \Big[cos\phi\ sin\psi\Big(\phi' - sin\theta\ (\kappa_2 + \psi')\Big) + cos\psi\Big(cos\theta$
$cos\phi\ \theta' + sin\phi(\kappa_2 - sin\theta\ \phi' + \psi')\Big)\Big]\ \mathbf{N} + \Big[cos\theta\ cos\phi\ (-\kappa_3 + sin\psi\ \theta') + sin\phi\ sin\psi(\kappa_2 -$

$sin\theta\ \phi' + \psi') + cos\phi\ cos\psi\ \Big(- \phi' + sin\theta(\kappa_2 + \psi')\Big)\Big]\mathbf{B}_1 + [-\kappa_3\ cos\psi\ sin\phi + cos\phi\ sin\theta\ (\kappa_3\ sin\psi - \theta') - cos\theta\ sin\phi\ \phi']\ \mathbf{B}_2$.

Therefore,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}_q' \\ \mathbf{B}_{1q}' \\ \mathbf{B}_{2q}' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & K_4 & K_5 \\ -K_2 & -K_4 & 0 & K_6 \\ -K_3 & -K_5 & -K_6 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix},$$

□

Corollary 3.1. *If we put* $\theta = 0 = \phi$, *we get* $K_1 = \kappa_1 cos\psi$, $K_2 = -\kappa_1 sin\psi$, $K_4 = \kappa_2 + \psi'$ *and* $K_3 = 0 = K_5 = K_6$ *which are the same results of the quasi frame in* $\mathbb{E}^3$.

Corollary 3.2. *If we put* ($\kappa_2 = \phi'\ sin\theta - \psi'$, $\kappa_3 = \frac{\theta'}{sin\psi}$, $\phi'\ cos\theta + \theta'\ cot\psi = 0$), *we get the same results of parallel transport frame in* $\mathbb{E}^4$.

Corollary 3.3. *The quasi curvatures* $\{K_i | i = 1, 2, 3, 4, 5, 6\}$ *are given by*

$K_1 = g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}_q', \boldsymbol{T}), K_2 = g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}_{1q}', \boldsymbol{T}), K_3 = g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}_{2q}', \boldsymbol{T}), K_4 = g(\boldsymbol{N}_q', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}_{1q}', \boldsymbol{N}_q), K_5 = g(\boldsymbol{N}_q', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}_{2q}', \boldsymbol{N}_q)$ *and* $K_6 = g(\boldsymbol{B}_{1q}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}_{2q}', \boldsymbol{B}_{1q})$.

4. Smarandach curves with respect to quasi frame in $\mathbb{E}^4$

Definition 4.1. *Let* $\alpha = \alpha(s)$ *be a unit speed curve in* $\mathbb{E}^4$ *and* $\{\boldsymbol{T}_\alpha, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{1q\alpha}, \boldsymbol{B}_{2q\alpha}\}$ *be it's moving quasi frame. The* $\boldsymbol{TN}_q$ *Smarandach curves are defined by*

$$\beta(s_\beta) = \frac{1}{\sqrt{2}}(\boldsymbol{T}_\alpha + \boldsymbol{N}_{q\alpha}).$$

In this section, we define some types of Smarandach curves in $\mathbb{E}^4$. We show that if a curve $\alpha = \alpha(s)$ be a unit speed curve with constant principal curvatures $\{K_{i\alpha} | i = 1, 2, 3, 4, 5, 6\}$ and a curve $\beta(s_\beta)$ be a type of Smarandach curves in $\mathbb{E}^4$ defined by quasi frame vectors of $\alpha = \alpha(s)$. Then, the quasi frame of β can be formed by the quasi frame of α and principal curvatures of β, $\{K_{i\beta} | i = 1, 2, 3, 4, 5, 6\}$, can be obtained by the principal curvatures of α.

Theorem 4.1. *Let* $\alpha = \alpha(s)$ *be a unit speed curve with constant principal curvatures* $\{K_{i\alpha} | i = 1, 2, 3, 4, 5, 6\}$ *and* $\beta(s_\beta)$ *be* $\boldsymbol{TN}_q$ *Smarandach curves in* $\mathbb{E}^4$ *defined by quasi frame vectors of* $\alpha = \alpha(s)$. *Then, the quasi frame of* β *can be formed from quasi frame of* α *and the principal curvatures of* β, $\{K_{i\beta} | i = 1, 2, 3, 4, 5, 6\}$, *can be obtained from the principal curvatures of* α.

PROOF. We investigate the quasi frame of $\mathbf{TN}_q$ Smarandach curves according to $\alpha(s)$. Since $\beta(s_\beta) = \frac{1}{\sqrt{2}}(\mathbf{T}_\alpha + \mathbf{N}_{q\alpha})$, by differentiating $\beta(s)$ with respect to s, we get

$$\mathbf{T}_\beta \frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}(-K_{1\alpha}\mathbf{T}_\alpha + K_{1\alpha}\mathbf{N}_{q\alpha} + (K_{2\alpha} + K_{4\alpha})\mathbf{B}_{1q\alpha} + (K_{3\alpha} + K_{5\alpha})\mathbf{B}_{2q\alpha}).$$

so, $\mathbf{T}_\beta$ can be written as

$$\mathbf{T}_\beta = \frac{-K_{1\alpha}\mathbf{T}_\alpha + K_{1\alpha}\mathbf{N}_{q\alpha} + (K_{2\alpha} + K_{4\alpha})\mathbf{B}_{1q\alpha} + (K_{3\alpha} + K_{5\alpha})\mathbf{B}_{2q\alpha}}{\sqrt{2K_{1\alpha}^2 + (K_{2\alpha} + K_{4\alpha})^2 + (K_{3\alpha} + K_{5\alpha})^2}},$$

where

$$\frac{ds_\beta}{ds} = \frac{1}{\sqrt{2}}\sqrt{2K_{1\alpha}^2 + (K_{2\alpha} + K_{4\alpha})^2 + (K_{3\alpha} + K_{5\alpha})^2}.$$

By differentiating $\mathbf{T}_\beta$ with respect to s, we get

$$\mathbf{T}'_\beta = \frac{d\mathbf{T}_\beta}{ds} = \lambda_0\mathbf{T}_\alpha + \lambda_1\mathbf{N}_{q\alpha} + \lambda_2\mathbf{B}_{1q\alpha} + \lambda_3\mathbf{B}_{2q\alpha},$$

where

$\lambda_0 = \frac{-\sqrt{2}(K_{1\alpha}^2 + K_{2\alpha}(K_{2\alpha}+K_{4\alpha}) + K_{3\alpha}(K_{3\alpha}+K_{5\alpha}))}{2K_{1\alpha}^2 + (K_{2\alpha}+K_{4\alpha})^2 + (K_{3\alpha}+K_{5\alpha})^2}$,

$\lambda_1 = \frac{-\sqrt{2}(K_{1\alpha}^2 + K_{4\alpha}(K_{2\alpha}+K_{4\alpha}) + K_{5\alpha}(K_{3\alpha}+K_{5\alpha}))}{2K_{1\alpha}^2 + (K_{2\alpha}+K_{4\alpha})^2 + (K_{3\alpha}+K_{5\alpha})^2}$,

$\lambda_2 = \frac{-\sqrt{2}(K_{1\alpha}(K_{2\alpha}-K_{4\alpha}) + K_{6\alpha}(K_{3\alpha}+K_{5\alpha}))}{2K_{1\alpha}^2 + (K_{2\alpha}+K_{4\alpha})^2 + (K_{3\alpha}+K_{5\alpha})^2}$,

$\lambda_3 = \frac{-\sqrt{2}(K_{1\alpha}(K_{3\alpha}-K_{5\alpha}) - K_{6\alpha}(K_{2\alpha}+K_{4\alpha}))}{2K_{1\alpha}^2 + (K_{2\alpha}+K_{4\alpha})^2 + (K_{3\alpha}+K_{5\alpha})^2}$.

The first curvature of β according to Frenet frame is

$$\kappa_{1\beta} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}.$$

The principal normal of the curve β is give by

$$\mathbf{N}_{q\beta} = \frac{\lambda_0\mathbf{T}_\alpha + \lambda_1\mathbf{N}_{q\alpha} + \lambda_2\mathbf{B}_{1q\alpha} + \lambda_3\mathbf{B}_{2q\alpha}}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}}.$$

The third derivative of β is

$\beta''' = \frac{\sqrt{2}}{\sqrt{2K_{1\alpha}^2 + (K_{2\alpha}+K_{4\alpha})^2 + (K_{3\alpha}+K_{5\alpha})^2}}[-(\lambda_1K_{1\alpha} + \lambda_2K_{2\alpha} + \lambda_3K_{3\alpha})\mathbf{T}_\alpha + (\lambda_0K_{1\alpha} - \lambda_2K_{4\alpha} - \lambda_3K_{5\alpha})\mathbf{N}_{q\alpha} + (\lambda_0K_{2\alpha} + \lambda_1K_{4\alpha} - \lambda_3K_{6\alpha})\mathbf{B}_{1q\alpha} + (\lambda_0K_{3\alpha} + \lambda_1K_{5\alpha} + \lambda_2K_{6\alpha})\mathbf{B}_{2q\alpha}]$.

And now,

$\mathbf{T}_\beta \times \mathbf{N}_\beta \times \beta''' = C_0\mathbf{T} + C_1\mathbf{N}_{q\alpha} + C_2\mathbf{B}_{1q\alpha} + C_3\mathbf{B}_{2q\alpha}$,

where

$C_0 = \frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{1\alpha}^2 + (K_{2\alpha}+K_{4\alpha})^2 + (K_{3\alpha}+K_{5\alpha})^2)}[-(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)((K_{2\alpha}$

$+K_{4\alpha})\lambda_1 - K_{1\alpha}\lambda_2) + (K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)((K_{3\alpha} + K_{5\alpha})\lambda_1 - K_{1\alpha}\lambda_3) + (K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)(-(K_{3\alpha} + K_{5\alpha})\lambda_2 + (K_{2\alpha} + K_{4\alpha})\lambda_3)],$

$C_1 = -\frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2(2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2)}}[-(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)((K_{2\alpha} + K_{4\alpha})\lambda_0 + K_{1\alpha}\lambda_2) + (K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)((K_{3\alpha} + K_{5\alpha})\lambda_0 + K_{1\alpha}\lambda_3) + (K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)((K_{3\alpha} + K_{5\alpha})\lambda_2 - (K_{2\alpha} + K_{4\alpha})\lambda_3)],$

$C_2 = \frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2(2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2)}}[-K_{1\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)(\lambda_0 + \lambda_1) + (K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)((K_{3\alpha} + K_{5\alpha})\lambda_1 - K_{1\alpha}\lambda_3) + (K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)((K_{3\alpha} + K_{5\alpha})\lambda_0 + K_{1\alpha}\lambda_3)],$

$C_3 = -\frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2(2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2)}}[-K_{1\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)(\lambda_0 + \lambda_1) + (K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)((K_{2\alpha} + K_{4\alpha})\lambda_1 - K_{1\alpha}\lambda_2) + (K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)((K_{2\alpha} + K_{4\alpha})\lambda_0 + K_{1\alpha}\lambda_2)].$

The second binormal of the curve β is given by

$$\mathbf{B}_{2q\beta} = \frac{C_0\mathbf{T}_\alpha + C_1\mathbf{N}_{q\alpha} + C_2\mathbf{B}_{1q\alpha} + C_3\mathbf{B}_{2q\alpha}}{\sqrt{C_0^2 + C_1^2 + C_2^2 + C_3^2}}.$$

The first binormal of the curve β is given by

$$\mathbf{B}_{1q\beta} = \mathbf{B}_{2q\beta} \times \mathbf{T}_\beta \times \mathbf{N}_{q\beta} = \gamma_0\mathbf{T}_\alpha + \gamma_1\mathbf{N}_{q\alpha} + \gamma_2\mathbf{B}_{1q\alpha} + \gamma_3\mathbf{B}_{2q\alpha},$$

where

$\gamma_0 = \frac{1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2}}[-C_3(K_{2\alpha} + K_{4\alpha})\lambda_1 + C_3K_{1\alpha}\lambda_2 + (K_{3\alpha} + K_{5\alpha})(C_2\lambda_1 - C_1\lambda_2) - C_2K_{1\alpha}\lambda_3 + C_1(K_{2\alpha} + K_{4\alpha})\lambda_3],$

$\gamma_1 = -\frac{1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2}}\Big[(K_{3\alpha} + K_{5\alpha})(C_2\lambda_0 - C_0\lambda_2) - C_3\Big(K_{4\alpha}\lambda_0 + K_{1\alpha}(\lambda_0 + \lambda_2)\Big) + C_2K_{1\alpha}\lambda_3 + C_0(K_{1\alpha} + K_{4\alpha})\lambda_3\Big],$

$\gamma_2 = \frac{1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2}}[-C_3K_{1\alpha}(\lambda_0 + \lambda_1) + (K_{3\alpha} + K_{5\alpha})(C_1\lambda_0 - C_0\lambda_1) + (C_0 + C_1)K_{1\alpha}\lambda_3],$

$\gamma_3 = -\frac{1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{1\alpha}^2+(K_{2\alpha}+K_{4\alpha})^2+(K_{3\alpha}+K_{5\alpha})^2}}[-C_2K_{1\alpha}(\lambda_0 + \lambda_1) + (K_{2\alpha} + K_{4\alpha})(C_1\lambda_0 - C_0\lambda_1) + (C_0 + C_1)K_{1\alpha}\lambda_2].$

For simplicity, we assume
$X = \sqrt{2K_{1\alpha}^2 + (K_{2\alpha} + K_{4\alpha})^2 + (K_{3\alpha} + K_{5\alpha})^2}$, $Y = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}$ and
$Z = \sqrt{C_0^2 + C_1^2 + C_2^2 + C_3^2}$.

The quasi frame for the curve β has the form

$\mathbf{N}_{q\beta} = [-\frac{C_0\ sin\theta_\beta}{Z} + cos\theta_\beta\ (\frac{\lambda_0\ cos\psi_\beta}{Y} + \gamma_0\ sin\psi_\beta)]\ \mathbf{T}_\alpha + [-\frac{C_1\ sin\theta_\beta}{Z} + cos\theta_\beta\ (\frac{\lambda_1\ cos\psi_\beta}{Y} + \gamma_1\ sin\psi_\beta)]\ \mathbf{N}_{q\alpha} + [-\frac{C_2\ sin\theta_\beta}{Z} + cos\theta_\beta\ (\frac{\lambda_2\ cos\psi_\beta}{Y} + \gamma_2\ sin\psi_\beta)]\ \mathbf{B}_{1q\alpha} + [-\frac{C_3\ sin\theta_\beta}{Z} + cos\theta_\beta\ (\frac{\lambda_3\ cos\psi_\beta}{Y} + \gamma_3\ sin\psi_\beta)]\ \mathbf{B}_{2q\alpha},$

$$\mathbf{B}_{1q\beta} =$$
$$\Big[\tfrac{1}{YZ}\Big(Zcos\phi_\beta(Y\gamma_0 cos\psi_\beta - \lambda_0 sin\psi_\beta)$$
$$+ sin\phi_\beta\Big(C_0 Y cos\theta_\beta + Zsin\theta_\beta(\lambda_0 cos\psi_\beta + Y\gamma_0 sin\psi_\beta)\Big)\Big)\Big]\mathbf{T}_\alpha$$
$$+ \Big[\tfrac{1}{YZ}\Big(Zcos\phi_\beta(Y\gamma_1 cos\psi_\beta - \lambda_1 sin\psi_\beta) + sin\phi_\beta\Big(C_1 Y cos\theta_\beta + Zsin\theta_\beta(\lambda_1 cos\psi_\beta +$$
$$Y\gamma_1 sin\psi_\beta)\Big)\Big)\Big]\mathbf{N}_{q\alpha}$$
$$+ \Big[\tfrac{1}{YZ}\Big(Zcos\phi_\beta(Y\gamma_2 cos\psi_\beta - \lambda_2 sin\psi_\beta)$$
$$+ sin\phi_\beta\Big(C_2 Y cos\theta_\beta + Zsin\theta_\beta(\lambda_2 cos\psi_\beta + Y\gamma_2 sin\psi_\beta)\Big)\Big)\Big]\mathbf{B}_{1q\alpha}$$
$$+ \Big[\tfrac{1}{YZ}(Zcos\phi_\beta(Y\gamma_3 cos\psi_\beta - \lambda_3 sin\psi_\beta)$$
$$+ sin\phi_\beta\Big(C_3 Y cos\theta_\beta + Zsin\theta_\beta(\lambda_3 cos\psi_\beta + Y\gamma_3 sin\psi_\beta)\Big)\Big)\Big]\mathbf{B}_{2q\alpha},$$

$$\mathbf{B}_{2q\beta} = \Big[\tfrac{1}{YZ}\Big(C_0 Y cos\theta_\beta cos\phi_\beta$$
$$+ Zcos\phi_\beta sin\theta_\beta(\lambda_0 cos\psi_\beta + Y\gamma_0 sin\psi_\beta)$$
$$+ Zsin\phi_\beta(-Y\gamma_0 cos\psi_\beta + \lambda_0 sin\psi_\beta)\Big)\Big]\mathbf{T}_\alpha$$
$$+ \Big[\tfrac{1}{YZ}\Big(C_1 Y cos\theta_\beta cos\phi_\beta + Zcos\phi_\beta sin\theta_\beta(\lambda_1 cos\psi_\beta + Y\gamma_1 sin\psi_\beta)$$
$$+ Zsin\phi_\beta(-Y\gamma_1 cos\psi_\beta + \lambda_1 sin\psi_\beta)\Big)\Big]\mathbf{N}_{q\alpha}$$
$$+ \Big[\tfrac{1}{YZ}\Big(C_2 Y cos\theta_\beta cos\phi_\beta + Zcos\phi_\beta sin\theta_\beta(\lambda_2 cos\psi_\beta + Y\gamma_2 sin\psi_\beta)$$
$$+ Zsin\phi_\beta(-Y\gamma_2 cos\psi_\beta + \lambda_2 sin\psi_\beta)\Big)\Big]\mathbf{B}_{1q\alpha}$$
$$+ \Big[\tfrac{1}{YZ}\Big(C_3 Y cos\theta_\beta cos\phi_\beta + Zcos\phi_\beta sin\theta_\beta(\lambda_3 cos\psi_\beta + Y\gamma_3 sin\psi_\beta)$$
$$+ Zsin\phi_\beta(-Y\gamma_3 cos\psi_\beta + \lambda_3 sin\psi_\beta)\Big)\Big]\mathbf{B}_{2q\alpha}.$$

The fourth derivative of the curve β is

$$\beta^{iv} = \tfrac{2}{X^2}\Big(\Big[K_{1\alpha}(-K_{1\alpha}\lambda_0 + K_{4\alpha}\lambda_2 + K_{5\alpha}\lambda_3) - K_{3\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2) -$$
$$K_{2\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)\Big]T_\alpha + \Big[-K_{1\alpha}(K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3) - K_{5\alpha}(K_{3\alpha}\lambda_0$$
$$+K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2) - K_{4\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)\Big]N_{q\alpha} + \Big[- K_{2\alpha}(K_{1\alpha}\lambda_1 +$$
$$K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3) + K_{1\alpha}\lambda_0 K_{4\alpha} - K_{4\alpha}(K_{4\alpha}\lambda_2 + K_{5\alpha}\lambda_3) - K_{6\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 +$$
$$K_{6\alpha}\lambda_2)\Big]B_{1q\alpha} + \Big[- K_{3\alpha}(K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3) + K_{1\alpha}\lambda_0 K_{5\alpha} - K_{5\alpha}(K_{4\alpha}\lambda_2 +$$
$$K_{5\alpha}\lambda_3) + K_{6\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)\Big]B_{2q\alpha}\Big).$$

Note that: $\theta_\beta = \int \frac{\kappa_{3\beta}}{\sqrt{\kappa_{1\beta}^2+\kappa_{2\beta}^2}} ds_\beta, \psi_\beta = -\int \left[\kappa_{2\beta} + \kappa_{3\beta}\frac{\sqrt{\kappa_{3\beta}^2-\theta_\beta'^2}}{\sqrt{\kappa_{1\beta}^2+\kappa_{2\beta}^2}}\right] ds_\beta$ and $\phi_\beta = -\int \frac{\sqrt{\kappa_{3\beta}^2+\theta_\beta'^2}}{cos\theta_\beta} ds_\beta$.

The second curvature of the curve β according to Frenet frame is given by $\kappa_{2\beta} = \frac{Z}{Y} = \frac{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}}$. For simplicity, put $\kappa_{2\beta} = \mu$.

The third curvature of the curve β according to Frenet frame is given by $\kappa_{3\beta} = \frac{2}{X^2Z^2}\Big[C_0\Big(K_{1\alpha}(-K_{1\alpha}\lambda_0 + K_{4\alpha}\lambda_2 + K_{5\alpha}\lambda_3) - K_{3\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2) - K_{2\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)\Big) + C_1\Big(-K_{1\alpha}(K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3) - K_{5\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2) - K_{4\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)\Big) + C_2\Big(-K_{2\alpha}(K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3) + K_{1\alpha}\lambda_0 K_{4\alpha} - K_{4\alpha}(K_{4\alpha}\lambda_2 + K_{5\alpha}\lambda_3) - K_{6\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)\Big) + C_3\Big(-K_{3\alpha}(K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3) + K_{1\alpha}\lambda_0 K_{5\alpha} - K_{5\alpha}(K_{4\alpha}\lambda_2 + K_{5\alpha}\lambda_3) + K_{6\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)\Big)\Big]$. For simplicity, put $\kappa_{3\beta} = \eta$.

The first curvature of the curve β according to the quasi frame is given by

$$K_{1q\beta} = Y cos\theta_\beta\ cos\psi_\beta.$$

The second curvature of the curve β according to the quasi frame is given by

$$K_{2q\beta} = Y(-cos\phi_\beta\ sin\psi_\beta + sin\phi_\beta\ sin\theta_\beta\ cos\psi_\beta).$$

The third curvature of the curve β according to the quasi frame is given by

$$K_{3q\beta} = Y(sin\phi_\beta\ sin\psi_\beta + cos\phi_\beta\ sin\theta_\beta\ cos\psi_\beta).$$

The fourth curvature of the curve β according to the quasi frame is given by

$$K_{4q\beta} = sin\phi_\beta(\eta\ sin\psi_\beta - \theta_\beta') + cos\phi_\beta\Big(\eta\ cos\psi_\beta\ sin\theta_\beta + cos\theta_\beta(\mu + \psi_\beta')\Big).$$

The fifth curvature of the curve β according to the quasi frame is given by

$$K_{5q\beta} = -\eta\ cos\psi_\beta\ sin\theta_\beta\ sin\phi_\beta + cos^2\theta_\beta\ cos\phi_\beta(\eta\ sin\psi_\beta - \theta_\beta') + cos\phi_\beta\ sin^2\theta_\beta\ sin\psi_\beta(\eta - sin\psi_\beta\ \theta_\beta') - cos\theta_\beta\ sin\phi_\beta\ sin^2\psi_\beta(\mu + \psi_\beta') - cos^2\psi_\beta\Big(cos\phi_\beta\ sin^2\theta_\beta\ \theta_\beta' + cos\theta_\beta\ sin\phi_\beta(\eta + \psi_\beta')\Big).$$

The sixth curvature of the curve β according to the quasi frame is given by

$$K_{6q\beta} = \eta\ cos\theta_\beta\ cos\psi_\beta + sin^2\theta_\beta\ cos^2\phi_\beta\ \phi_\beta' + cos^2\phi_\beta\ sin\theta_\beta(-\mu + sin\theta_\beta\ \phi_\beta' - \psi_\beta) + sin^2\phi_\beta\Big(\phi_\beta' - sin\theta_\beta(\mu + \psi_\beta')\Big).$$

□

Corollary 4.1. *In the above proof, if we put the principal curvatures* $\{K_{i\alpha} = 0|i = 4,5,6\}$*, we get the same results of* $\boldsymbol{TN}_q$ *Smarandach curves that computed in* [6] *according to parallel transport frame.*

Remark 4.1. *In the next two theorems, we will use the same assumptions of theorem (4.1). So, the results would look like (4.1)'s results but different in values. In other words, the changes will be found in* $\{T_\beta, \{\lambda_i|i = 0,1,2,3\}, \{C_i|i = 0,1,2,3\}, \{\gamma_i|i = 0,1,2,3\}\}$ *while,* $\{N_\beta, B_{1\beta}, B_{2\beta}, N_{q\beta}, B_{1\beta}, B2\beta, \{\kappa_i|i = 1,2,3\}$ $, \{K_{i\beta}|i = 1,2,3,4,5,6\}\}$ *would look like (4.1)'s results but different in values because of their dependency on* $\{\{\lambda_i|i = 0,1,2,3\}, \{C_i|i = 0,1,2,3\}, \{\gamma_i|i = 0,1,2,3\}\}$.

Definition 4.2. *Let* $\alpha = \alpha(s)$ *be a unit speed curve in* $\mathbb{E}^4$ *and* $\{\boldsymbol{T}_\alpha, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{1q\alpha}, \boldsymbol{B}_{2q\alpha}\}$ *be it's moving quasi frame. The* $\boldsymbol{TB}_{1q}$ *Smarandach curves are defined by*

$$\beta(s_\beta) = \frac{1}{\sqrt{2}}(\boldsymbol{T}_\alpha + \boldsymbol{B}_{1q\alpha}).$$

Theorem 4.2. *Let* $\alpha = \alpha(s)$ *be a unit speed curve with constant principal curvatures* $\{K_{i\alpha}|i = 1,2,3,4,5,6\}$ *and* $\beta(s_\beta)$ *be* $\boldsymbol{TB}_{1q}$ *Smarandach curves in* $\mathbb{E}^4$ *defined by quasi frame vectors of* $\alpha = \alpha(s)$*. Then, the quasi frame of* β *can be formed by quasi frame of* α *and principal curvatures of* β*,* $\{K_{i\beta}|i = 1,2,3,4,5,6\}$*, can be obtained by the principal curvatures of* α*.*

Proof. The tangent vector of the curve β can be written as

$$\mathbf{T}_\beta = \frac{-K_{2\alpha}\mathbf{T}_\alpha + (K_{1\alpha} - K_{4\alpha})\mathbf{N}_{q\alpha} + K_{2\alpha}\mathbf{B}_{1q\alpha} + (K_{3\alpha} + K_{6\alpha})\mathbf{B}_{2q\alpha}}{\sqrt{2K_{2\alpha}^2 + (K_{1\alpha} - K_{4\alpha})^2 + (K_{3\alpha} + K_{6\alpha})^2}},$$

$\{\lambda_i|i = 0,1,2,3\}$ are calculated as follows

$$\lambda_0 = \frac{-\sqrt{2}[K_{1\alpha}(K_{1\alpha} - K_{4\alpha}) + K_{2\alpha}^2 + K_{3\alpha}(K_{3\alpha} + K_{6\alpha})]}{2K_{2\alpha}^2 + (K_{1\alpha} - K_{4\alpha})^2 + (K_{3\alpha} + K_{6\alpha})^2}.$$
$$\lambda_1 = \frac{-\sqrt{2}[K_{1\alpha}K_{2\alpha} + K_{2\alpha}K_{4\alpha} + K_{5\alpha}(K_{3\alpha} + K_{6\alpha})]}{2K_{2\alpha}^2 + (K_{1\alpha} - K_{4\alpha})^2 + (K_{3\alpha} + K_{6\alpha})^2}.$$
$$\lambda_2 = \frac{-\sqrt{2}[K_{2\alpha}^2 - K_{4\alpha}(K_{1\alpha} - K_{4\alpha}) + K_{6\alpha}(K_{3\alpha} + K_{6\alpha})]}{2K_{2\alpha}^2 + (K_{1\alpha} - K_{4\alpha})^2 + (K_{3\alpha} + K_{6\alpha})^2}.$$
$$\lambda_3 = \frac{-\sqrt{2}[K_{2\alpha}K_{3\alpha} - K_{5\alpha}(K_{1\alpha} - K_{4\alpha}) - K_{2\alpha}K_{6\alpha}]}{2K_{2\alpha}^2 + (K_{1\alpha} - K_{4\alpha})^2 + (K_{3\alpha} + K_{6\alpha})^2}.$$

$\{C_i|i = 0,1,2,3\}$ are calculated as follows

$$C_0 = \frac{\sqrt{2}}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 (2K_{2\alpha}^2 + (K_{1\alpha} - K_{4\alpha})^2 + (K_{3\alpha} + K_{6\alpha})^2)}}[-(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2) (K_{2\alpha}\lambda_1 + (-K_{1\alpha} + K_{4\alpha})\lambda_2) - (K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)((K_{3\alpha} + K_{6\alpha})\lambda_2 - K_{2\alpha}\lambda_3) + (K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)((K_{3\alpha} + K_{6\alpha})\lambda_1 + (-K_{1\alpha} + K_{4\alpha})\lambda_3)].$$

$C_1 = \frac{-\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}[-K_{2\alpha}(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)(\lambda_0+\lambda_2) + (K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)((K_{3\alpha}+K_{6\alpha})\lambda_2 - K_{2\alpha}\lambda_3) + (K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)((K_{3\alpha}+K_{6\alpha})\lambda_0 + K_{2\alpha}\lambda_3)]$.

$C_2 = \frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}[-(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)(K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_0 + K_{2\alpha}\lambda_1) + (K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)((K_{3\alpha}+K_{6\alpha})\lambda_0 + K_{2\alpha}\lambda_3) + (K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)((K_{3\alpha}+K_{6\alpha})\lambda_1 + (-K_{1\alpha}+K_{4\alpha})\lambda_3)]$.

$C_3 = \frac{-\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}[-(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)(K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_0 + K_{2\alpha}\lambda_1) + K_{2\alpha}(K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)(\lambda_0+\lambda_2) + (K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)(K_{2\alpha}\lambda_1 + (-K_{1\alpha}+K_{4\alpha})\lambda_2)]$.

$\{\gamma_i | i = 0,1,2,3\}$ are calculated as follows

$\gamma_0 = \frac{1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}}[(K_{3\alpha}+K_{6\alpha})(C_2\lambda_1 - C_1\lambda_2) - C_3(K_{2\alpha}\lambda_1 + (-K_{1\alpha}+K_{4\alpha})\lambda_2) + C_1K_{2\alpha}\lambda_3 + C_2(-K_{1\alpha}+K_{4\alpha})\lambda_3]$

$\gamma_1 = \frac{-1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}}[-C_3K_{2\alpha}(\lambda_0+\lambda_2) + (K_{3\alpha}+K_{6\alpha})(C_2\lambda_0 - C_0\lambda_2) + (C_0+C_2)K_{2\alpha}\lambda_3]$

$\gamma_2 = \frac{1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}}[C_1(K_{3\alpha}+K_{6\alpha})\lambda_0 - C_0(K_{3\alpha}+K_{6\alpha})\lambda_1 - C_3(K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_0 + K_{2\alpha}\lambda_1) + C_1K_{2\alpha}\lambda_3 + C_0(K_{1\alpha}-K_{4\alpha})\lambda_3]$

$\gamma_3 = \frac{-1}{\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{2\alpha}^2+(K_{1\alpha}-K_{4\alpha})^2+(K_{3\alpha}+K_{6\alpha})^2)}}[-C_2(K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_0 + K_{2\alpha}\lambda_1) + C_1K_{2\alpha}(\lambda_0+\lambda_2) - C_0(K_{2\alpha}\lambda_1 - K_{1\alpha}\lambda_2 + K_{4\alpha}\lambda_2)]$. □

Definition 4.3. *Let $\alpha = \alpha(s)$ be a unit speed curve in $\mathbb{E}^4$ and $\{\boldsymbol{T}_\alpha, \boldsymbol{N}_{q\alpha}, \boldsymbol{B}_{1q\alpha}, \boldsymbol{B}_{2q\alpha}\}$ be it's moving quasi frame. The $\boldsymbol{TB}_{1q}$ Smarandach curves are defined by*

$$\beta(s_\beta) = \frac{1}{\sqrt{2}}(\boldsymbol{T}_\alpha + \boldsymbol{B}_{2q\alpha}).$$

Theorem 4.3. *Let $\alpha = \alpha(s)$ be a unit speed curve with constant principal curvatures $\{K_{i\alpha} | i = 1,2,3,4,5,6\}$ and $\beta(s_\beta)$ be $\boldsymbol{TB}_{2q}$ Smarandach curves in $\mathbb{E}^4$ defined by quasi frame vectors of $\alpha = \alpha(s)$. Then, the quasi frame of β can be formed by quasi frame of α and principal curvatures of β, $\{K_{i\beta} | i = 1,2,3,4,5,6\}$, can be obtained by the principal curvatures of α.*

Proof. The tangent vector of the curve β can be written as

$$\mathbf{T}_\beta = \frac{-K_{3\alpha}\mathbf{T}_\alpha + (K_{1\alpha} - K_{5\alpha})\mathbf{N}_{q\alpha} + (K_{2\alpha} - K_{6\alpha})\mathbf{B}_{1q\alpha} + K_{3\alpha}\mathbf{B}_{2q\alpha}}{\sqrt{2K_{3\alpha}^2 + (K_{1\alpha} - K_{5\alpha})^2 + (K_{2\alpha} - K_{6\alpha})^2}}.$$

$\{\lambda_i | i = 0,1,2,3\}$ are calculated as follows

$\lambda_0 = \frac{-\sqrt{2}}{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}[K_{1\alpha}(K_{1\alpha} - K_{5\alpha}) + K_{2\alpha}(K_{2\alpha} - K_{6\alpha}) + K_{3\alpha}^2]$.

$\lambda_1 = \frac{-\sqrt{2}}{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}[K_{1\alpha}K_{3\alpha} + K_{4\alpha}(K_{2\alpha} - K_{6\alpha}) + K_{3\alpha}K_{5\alpha}]$.
$\lambda_2 = \frac{-\sqrt{2}}{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}[K_{2\alpha}K_{3\alpha} - K_{4\alpha}(K_{1\alpha} - K_{5\alpha}) + K_{3\alpha}K_{6\alpha}]$.
$\lambda_3 = \frac{-\sqrt{2}}{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}[K_{3\alpha}^2 - K_{5\alpha}(K_{1\alpha} - K_{5\alpha}) - K_{6\alpha}(K_{2\alpha} - K_{6\alpha})]$.

$\{C_i|i = 0, 1, 2, 3\}$ are calculated as follows
$C_0 = \frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2)}[(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)$
$(-K_{2\alpha}\lambda_1+K_{6\alpha}\lambda_1+(K_{1\alpha}-K_{5\alpha})\lambda_2)+(K_{2\alpha}\lambda_0+K_{4\alpha}\lambda_1-K_{6\alpha}\lambda_3)(K_{3\alpha}\lambda_1 +(-K_{1\alpha}+$
$K_{5\alpha})\lambda_3) - (K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)(K_{3\alpha}\lambda_2 + (-K_{2\alpha} + K_{6\alpha})\lambda_3)]$
$C_1 = \frac{-\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2)}[-(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)$
$(K_{2\alpha}\lambda_0 - K_{6\alpha}\lambda_0 + K_{3\alpha}\lambda_2) + K_{3\alpha}(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)(\lambda_0 + \lambda_3) +(K_{1\alpha}\lambda_1 +$
$K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)(K_{3\alpha}\lambda_2 + (-K_{2\alpha} + K_{6\alpha})\lambda_3)]$.
$C_2 = \frac{\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2)}[-(K_{3\alpha}\lambda_0 + K_{5\alpha}\lambda_1 + K_{6\alpha}\lambda_2)$
$(K_{1\alpha}\lambda_0 - K_{5\alpha}\lambda_0 + K_{3\alpha}\lambda_1) + K_{3\alpha}(K_{1\alpha}\lambda_0 - K_{4\alpha}\lambda_2 - K_{5\alpha}\lambda_3)(\lambda_0 + \lambda_3) +(K_{1\alpha}\lambda_1 +$
$K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)(K_{3\alpha}\lambda_1 + (-K_{1\alpha} + K_{5\alpha})\lambda_3)]$.
$C_3 = \frac{-\sqrt{2}}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}(2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2)}[-(K_{2\alpha}\lambda_0 + K_{4\alpha}\lambda_1 - K_{6\alpha}\lambda_3)$
$(K_{1\alpha}\lambda_0-K_{5\alpha}\lambda_0+K_{3\alpha}\lambda_1)+(K_{1\alpha}\lambda_0-K_{4\alpha}\lambda_2-K_{5\alpha}\lambda_3)(K_{2\alpha}\lambda_0-K_{6\alpha}\lambda_0 +K_{3\alpha}\lambda_2)+$
$(K_{1\alpha}\lambda_1 + K_{2\alpha}\lambda_2 + K_{3\alpha}\lambda_3)(K_{2\alpha}\lambda_1 - K_{6\alpha}\lambda_1 + (-K_{1\alpha} + K_{5\alpha})\lambda_2)]$.

$\{\gamma_i|i = 0, 1, 2, 3\}$ are calculated as follows
$\gamma_0 = \frac{1}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}}[C_3\lambda_1 +K_{6\alpha}\lambda_1$
$+K_{1\alpha}\lambda_{2\alpha}-K_{5\alpha}\lambda_2)+C_2(K_{3\alpha}\lambda_1-K_{1\alpha}\lambda_3+K_{5\alpha}\lambda_3)-C_1(K_{3\alpha}\lambda_2-K_{2\alpha}\lambda_3+K_{6\alpha}\lambda_3)]$.
$\gamma_1 = \frac{1}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}}[-C_3(K_{2\alpha}\lambda_0 -$
$K_{6\alpha}\lambda_0 + K_{3\alpha}\lambda_2) + C_2K_{3\alpha}(\lambda_0 + \lambda_3) - C_0(K_{3\alpha}\lambda_2 - K_{2\alpha}\lambda_3 + K_{6\alpha}\lambda_3)]$.
$\gamma_2 = \frac{1}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}}[-C_3(K_{1\alpha}\lambda_0 -$
$K_{5\alpha}\lambda_0 + K_{3\alpha}\lambda_1) + C_1K_{3\alpha}(\lambda_0 + \lambda_3) - C_0(K_{3\alpha}\lambda_1 - K_{1\alpha}\lambda_3 + K_{5\alpha}\lambda_3)]$.
$\gamma_3 = \frac{1}{\sqrt{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}\sqrt{2K_{3\alpha}^2+(K_{1\alpha}-K_{5\alpha})^2+(K_{2\alpha}-K_{6\alpha})^2}\sqrt{C_0^2+C_1^2+C_2^2+C_3^2}}[(K_{2\alpha} - K_{6\alpha})$
$(C_1\lambda_0 - C_0\lambda_1) - C_2(K_{1\alpha}\lambda_0 - K_{5\alpha}\lambda_0 + K_{3\alpha}\lambda_1) +C_1K_{3\alpha}\lambda_2 + C_0(K_{1\alpha} - K_{5\alpha})\lambda_2]$.
□

Example 4.1. Theorem (4.1), provides an equal number of Chen curves as those of the quasi curves. We call the rectifying curves as Chen curves. The examples of Chen curves in 4-dimensional Euclidean space $\mathbb{E}^4$ through dilation of unit speed curves on the unit sphere are presented in [7] .For example consider the unit speed curve $\alpha : I \to \mathbb{E}^4$ defined by

$$\alpha(s) = (cos\sqrt{\frac{2}{3}}s, sin\sqrt{\frac{2}{3}}s, cos\sqrt{\frac{1}{3}}s, sin\sqrt{\frac{1}{3}}s).$$

The quasi frame vector fields of $\alpha(s)$ are given by

$$\begin{aligned}
\mathbf{T}_{\alpha} &= (-\sqrt{\frac{2}{3}}sin\sqrt{\frac{2}{3}}s, \sqrt{\frac{2}{3}}cos\sqrt{\frac{2}{3}}s, -\sqrt{\frac{1}{3}}sin\sqrt{\frac{1}{3}}s, \\
&\quad \sqrt{\frac{1}{3}}cos\sqrt{\frac{1}{3}}s) \\
\mathbf{N}_{q\alpha} &= (-cos\sqrt{\frac{2}{3}}s, -sin\sqrt{\frac{2}{3}}s, 0, 0), \\
\mathbf{B}_{1q\alpha} &= (-\sqrt{\frac{1}{3}}sin\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}cos\sqrt{\frac{2}{3}}s, \sqrt{\frac{2}{3}}sin\sqrt{\frac{1}{3}}s, \\
&\quad -\sqrt{\frac{2}{3}}cos\sqrt{\frac{1}{3}}s), \\
\mathbf{B}_{2q\alpha} &= (0, 0, cos\sqrt{\frac{1}{3}}s, sin\sqrt{\frac{1}{3}}s).
\end{aligned}$$

The quasi curvatures of $\alpha(s)$ are given by

$$K_{1\alpha} = \frac{2}{3}, K_{2\alpha} = 0 = K_{5\alpha}, K_{3\alpha} = \frac{1}{3}, K_{4\alpha} = -\frac{\sqrt{2}}{3} = K_{6\alpha}.$$

Thus, the corresponding Chen curve of $\beta(s_\beta)$ is given by

$$\begin{aligned}
\beta(s_\beta) = \frac{1}{\sqrt{2}}(&-\sqrt{\frac{2}{3}}sin\sqrt{\frac{2}{3}}s - cos\sqrt{\frac{2}{3}}s, \sqrt{\frac{2}{3}}cos\sqrt{\frac{2}{3}}s - sin\sqrt{\frac{2}{3}}s, \\
&-\sqrt{\frac{1}{3}}sin\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}cos\sqrt{\frac{1}{3}}).
\end{aligned}$$

A straightforward calculation gets the following
$\{\lambda_i | i = 0, 1, 2, 3\}$ are given by

$$\lambda_0 = -\frac{5\sqrt{2}}{11},\ \lambda_1 = -\frac{6\sqrt{2}}{11},\ \lambda_2 = -\frac{2}{11}, \lambda_3 = 0.$$

$\{C_i | i = 0, 1, 2, 3\}$ are given by

$$C_0 = \frac{16}{121}\sqrt{\frac{2}{7}},\ C_1 = \frac{-6}{121}\sqrt{\frac{2}{7}},\ C_2 = \frac{-4}{11\sqrt{7}},\ C_3 = 0.$$

$\{\gamma_i | i = 0, 1, 2, 3\}$ are given by

$$\gamma_0 = -\frac{3}{11}\sqrt{\frac{2}{5}},\ \gamma_1 = \frac{3}{11}\sqrt{2}5,\ \gamma_2 = -\frac{3\sqrt{10}}{11},\ \gamma_3 = -\frac{3}{11\sqrt{5}}.$$

The quasi frame vector fields of $\beta(s_\beta)$ are given by

$$\begin{aligned}
\mathbf{T}_\beta &= \frac{3}{\sqrt{11}}(-\frac{2}{3}cos\sqrt{\frac{2}{3}}s + \sqrt{\frac{2}{3}}sin\sqrt{\frac{2}{3}}s, -\frac{2}{3}sin\sqrt{\frac{2}{3}}s - \sqrt{\frac{2}{3}}cos\sqrt{\frac{2}{3}}s, -\frac{1}{3}cos\sqrt{\frac{1}{3}}s,\\
&\quad -\frac{1}{3}sin\sqrt{\frac{1}{3}}s),\\
\mathbf{N}_{q\beta} &= (\frac{2}{\sqrt{7}}cos\sqrt{\frac{2}{3}}s + \frac{2\sqrt{42}}{21}sin\sqrt{\frac{2}{3}}s, \frac{2}{\sqrt{7}}sin\sqrt{\frac{2}{3}}s - \frac{2\sqrt{42}}{21}cos\sqrt{\frac{2}{3}}s, \frac{1}{\sqrt{21}}sin\sqrt{\frac{1}{3}}s,\\
&\quad -\frac{1}{\sqrt{21}}cos\sqrt{\frac{1}{3}}s),\\
\mathbf{B}_{1q\beta} &= (\frac{2+5\sqrt{2}}{11}\sqrt{\frac{3}{5}}sin\sqrt{\frac{2}{3}}s - \frac{3}{11}\sqrt{\frac{2}{5}}cos\sqrt{\frac{2}{3}}s,\\
&\quad -\frac{2+5\sqrt{2}}{11}\sqrt{\frac{3}{5}}cos\sqrt{\frac{2}{3}}s - \frac{3}{11}\sqrt{\frac{1}{5}}sin\sqrt{\frac{2}{3}}s,\\
&\quad \frac{3}{11\sqrt{5}}cos\sqrt{\frac{1}{3}}s + \frac{-10+\sqrt{2}}{11}\sqrt{\frac{3}{5}}sin\sqrt{\frac{1}{3}}s,\\
&\quad \frac{3}{11\sqrt{5}}sin\sqrt{\frac{1}{3}}s - \frac{-10+\sqrt{2}}{11}\sqrt{\frac{3}{5}}cos\sqrt{\frac{1}{3}}s),\\
\mathbf{B}_{2q\beta} &= (\sqrt{\frac{2}{105}}sin\sqrt{\frac{2}{3}}s + \frac{1}{\sqrt{35}}cos\sqrt{\frac{2}{3}}s,\\
&\quad -\sqrt{\frac{2}{105}}cos\sqrt{\frac{2}{3}}s + \frac{1}{\sqrt{35}}sin\sqrt{\frac{2}{3}}s,\\
&\quad -2\sqrt{\frac{5}{21}}sin\sqrt{\frac{1}{3}}s, 2\sqrt{\frac{5}{21}}cos\sqrt{\frac{1}{3}}s).
\end{aligned}$$

The quasi curvatures of $\beta(s_\beta)$ are given by
$K_1 = \sqrt{\frac{7}{11}}$, $K_2 = \frac{10-\sqrt{2}}{11\sqrt{55}}$, $K_{3\beta} = K_{5q\beta} = 0$, $K_{4q\beta} = \frac{20+4\sqrt{2}-\sqrt{7}+4\sqrt{14}}{77\sqrt{5}}$,
$K_{6q\beta} = \frac{4(5+\sqrt{2})}{55\sqrt{7}}$.

5. Conclusion

In this paper, we investigated the quasi frame, the quasi equations and showed the relations between the Frenet and the quasi curvatures in $\mathbb{E}^4$. Furthermore, We obtained the Frenet and the quasi invariants for Smarandach curves in $\mathbb{E}^4$ and by obtaining the Frenet's curvatures we calculated the quasi's curvatures.

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