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The Neutrosophization of δ-Separation Axioms

Ahu Açikgöz [†], Ferhat Esenbel [*,†], Abdulhamit Maman and Seher Zorlu

Department of Mathematics, Balikesir University, Balikesir 10145, Turkey; ahuacikgoz@balikesir.edu.tr (A.A.); 202212615003@ogr.balikesir.edu.tr (A.M.); 202212615008@ogr.balikesir.edu.tr (S.Z.)
* Correspondence: 201612616002@baun.edu.tr
† These authors contributed equally to this work.

Abstract: Fuzzy topology has long been celebrated for its ability to address real-world challenges in areas such as information systems and decision making. However, with ongoing technological advancements and the increasing complexity of practical requirements, the focus has gradually shifted toward neutrosophic topology, a broader and more inclusive framework than fuzzy topology. While neutrosophic topology is primarily rooted in neutrosophic open sets, other related families, including neutrosophic pre-open sets, neutrosophic semi-open sets, and neutrosophic beta-open sets, have also proven instrumental in driving progress in this field. This study introduces neutrosophic δ-open sets as a significant enhancement to the current theoretical framework. In addition, we propose a novel category of separation axioms, termed neutrosophic δ-separation axioms, which are derived from the concept of neutrosophic δ-open sets. Moreover, we explore the interplay between these separation properties and their characteristics within subspaces. Our findings confirm that neutrosophic δ-separation axioms are reliably upheld in neutrosophic regular open subspaces.

Keywords: neutrosophic δ-open set; neutrosophic δ-interior point; neutrosophic δ-separation axioms; neutrosophic δ-continuous

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1. Introduction

Fuzzy and soft sets, as in [1–5], have long been recognized as a valuable tool for reducing vagueness in data analysis, enabling decision makers to develop more precise methods that classical mathematics cannot achieve. However, the growing complexities of human life have rendered this concept insufficient, prompting intellectuals and researchers to devise various advanced mathematical approaches for data analysis.

In this context, Smarandache revolutionized the classical concept of fuzzy sets by introducing neutrosophic sets in his seminal work [6]. The concept of neutrosophic topology was later pioneered by Salama and Alblowi [7]. Since then, various scholars, including Acikgoz and Esenbel, have made substantial contributions to advancing the theory of neutrosophic sets and exploring their practical applications across multiple disciplines, as detailed in [8–20]. Moreover, as discussed in [21–32], this theory has emerged as a crucial component of scientific research in fields such as communication, engineering, education, epidemiology, pharmacy, medicine, e-learning, banking, marketing, and geography, profoundly impacting and reshaping numerous aspects of human activity.

In this paper, we introduce the notions of neutrosophic δ-closure and neutrosophic δ-interior for a neutrosophic set within a neutrosophic topological space. Building on these definitions, we develop a framework for separation axioms within neutrosophic δ-topological spaces, ensuring their logical consistency and alignment with the concept of neutrosophic δ-compactness. To achieve this, we define a novel category of separation axioms, referred to as neutrosophic δ-separation axioms, based on the concept of neutrosophic δ-open sets. Additionally, we investigate the interplay between these separation properties and subspaces, showing that neutrosophic δ-separation axioms are hereditary in neutrosophic regular open subspaces. Interestingly, these axioms exhibit unique characteristics compared to those found in general topology, adding further depth to their study.

2. Preliminaries

In this section, we provide the fundamental definitions associated with neutrosophic set theory.

Definition 1 ([6]). *A neutrosophic set E within the universal set U is described as*

$$E = \{\langle u, V_E(u), Y_E(u), W_E(u)\rangle : u \in U\},$$

where $V, Y, W : U \to]^{-}0, 1^{+}[$, *and* $^{-}0 \leq V_E(u) + Y_E(u) + W_E(u) \leq 3^{+}$.

From a scientific perspective, the membership, indeterminacy, and non-membership functions of a neutrosophic set take values from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. *However, these subsets can sometimes be impractical for real-life applications, such as those in economics and engineering. To address this issue, we focus on neutrosophic sets whose membership, indeterminacy, and non-membership functions take values within the closed interval* $[0, 1]$.

Definition 2 ([33]). *Let U be a nonempty set. When m, n, and p are real standard or non-standard subsets of* $]^{-}0, 1^{+}[$, *the neutrosophic set* $u_{m,n,p}$ *is defined as a neutrosophic point in U, defined as*

$$u_{m,n,p}(u_j) = \begin{cases} (m, n, p), & \text{if } u = u_j \\ (0, 0, 1), & \text{if } u \neq u_j \end{cases}$$

For $u_j \in U$, *it is known as the support of* $u_{m,n,p}$, *where m stands for the degree of membership, n stands for the degree of indeterminacy, and p stands for the degree of non-membership of* $u_{m,n,p}$.

Definition 3 ([7]). *Consider a neutrosophic set E defined on the universe set U. The complement of E, denoted by* E^c, *is expressed as*

$$E^c = \{\langle u, W_E(u), 1 - Y_E(u), V_E(u)\rangle : u \in U\}.$$

It is clear that $[E^c]^c = E$.

Definition 4 ([7]). *Suppose that E and F are two neutrosophic sets over the universal set U. E is said to be a neutrosophic subset of F if, for every* $u \in U$, *the following conditions hold:* $V_E(u) \leq V_F(u)$, $Y_E(u) \leq Y_F(u)$, *and* $W_E(u) \geq W_F(u)$. *This is denoted by* $E \subseteq F$. *Additionally, E is considered neutrosophic equal to F if* $E \subseteq F$ *and* $F \subseteq E$. *This is represented by* $E = F$.

Definition 5 ([7]). *Assume that* W_1 *and* W_2 *are two neutrosophic sets defined on the universal set U. Their union, denoted by* $W_1 \cup W_2 = W_3$, *is defined as*

$$W_3 = \{\langle u, V_{W_3}(u), Y_{W_3}(u), W_{W_3}(u) : u \in U\rangle\},$$

where

$$V_{W_3(u)} = \max\{V_{W_1(u)}, V_{W_2}(u)\},$$

$$Y_{W_3(u)} = \max\{Y_{W_1(u)}, Y_{W_2}(u)\},$$

$$W_{W_3(u)} = \min\{W_{W_1(u)}, W_{W_2}(u)\}.$$

Definition 6 ([7])**.** *Assume that W_1 and W_2 are two neutrosophic sets defined on the universal set U. Their intersection, denoted by $W_1 \cap W_2 = W_4$, is defined as follows:*

$$W_4 = \{\langle u, V_{W_4}(u), Y_{W_4}(u), W_{W_4}(u) : u \in U\rangle\},$$

where

$$V_{W_4(u)} = \min\{V_{W_1(u)}, V_{W_2}(u)\},$$

$$Y_{W_4(u)} = \min\{Y_{W_1(u)}, Y_{W_2}(u)\},$$

$$W_{W_4(u)} = \max\{W_{W_1(u)}, W_{W_2}(u)\}.$$

Definition 7 ([7])**.** *A neutrosophic set W over the universal set U is considered a null neutrosophic set if for every $u \in U$, $V_W(u) = 0$, $Y_W(u) = 0$, and $W_W(u) = 1$. It is represented as 0_U.*

Definition 8 ([7])**.** *A neutrosophic set W on the universal set U is said to be an absolute neutrosophic set if for every $u \in U$, $V_W(u) = 1$, $Y_W(u) = 1$, and $W_W(u) = 0$. This is denoted as 1_U. It is evident that $0_U^c = 1_U$ and $1_U^c = 0_U$.*

Definition 9 ([7])**.** *Assume $NS(U)$ to be the collection of all neutrosophic sets defined on the set U, and let $\sigma \subset NS(U)$. Then, σ is said to form a neutrosophic topology on U if the following conditions are met:*

(1) Both 0_U and 1_U are elements of σ;
(2) The union of any finite or infinite number of neutrosophic soft sets in σ is contained within σ;
(3) The intersection of a finite number of neutrosophic soft sets in σ belongs to σ.

Thus, (U, σ) is referred to as a neutrosophic topological space on U. Every element of σ is called a neutrosophic open set [7].

Definition 10 ([7])**.** *Assume that (U, σ) is a neutrosophic topological space on U and W is a neutrosophic set defined on U. Then, W is called a neutrosophic closed set if and only if its complement is a neutrosophic open set.*

Definition 11 ([8])**.** *A neutrosophic point $u_{m,n,p}$ is referred to as neutrosophic quasi-coincident (or neutrosophic q-coincident for short) with W, symbolized as $u_{m,n,p}\ q\ W$, if and only if $u_{m,n,p}$ is not contained within W^c. If $u_{m,n,p}$ is not neutrosophic quasi-coincident with W, it is expressed as $u_{m,n,p}\ \tilde{q}\ W$.*

Definition 12 ([8])**.** *A neutrosophic set W in a neutrosophic topological space (U, σ) is called a neutrosophic q-neighborhood of a neutrosophic point $u_{m,n,p}$ if and only if there exists a neutrosophic open set Z such that $u_{m,n,p}\ q\ Z$ is contained in W.*

Definition 13 ([8])**.** *A neutrosophic set Z is considered neutrosophic quasi-coincident (or neutrosophic q-coincident for short) with W, denoted by $Z\ q\ W$, if and only if Z is not contained within W^c. If Z is not neutrosophic quasi-coincident with W, it is denoted as $Z\ \tilde{q}\ W$.*

Definition 14 ([11]). *A neutrosophic point $u_{m,n,p}$ is referred to as a neutrosophic interior point of a neutrosophic set W if and only if there exists a neutrosophic open q-neighborhood Z of $u_{m,n,p}$ such that Z is contained in W. The collection of all neutrosophic interior points of W is called the neutrosophic interior of W, denoted by W°.*

Definition 15 ([8]). *A neutrosophic point $u_{m,n,p}$ is considered a neutrosophic cluster point of a neutrosophic set W if and only if every neutrosophic open q-neighborhood Z of $u_{m,n,p}$ is q-coincident with W. The collection of all neutrosophic cluster points of W is referred to as the neutrosophic closure of W and is denoted by $\overline{W}$.*

Definition 16 ([8]). *Assume that r is a function from U to Q, and let F be a neutrosophic set in Q with membership function $V_F(y)$, indeterminacy function $Y_F(v)$, and non-membership function $W_F(v)$. The inverse image of F under r, denoted by $r^{-1}(F)$, is a neutrosophic subset of U with its membership function, indeterminacy function, and non-membership function defined as follows: $V_{r^{-1}(F)}(u) = V_F(r(u))$, $Y_{r^{-1}(F)}(u) = Y_F(r(u))$, and $W_{r^{-1}(F)}(u) = W_F(r(u))$ for every $u \in U$.*

On the other hand, suppose that E is a neutrosophic set in U with a membership function $V_E(u)$, indeterminacy function $Y_E(u)$, and non-membership function $W_E(u)$. The image of E under the function r, denoted as $r(E)$, is a neutrosophic subset of Q whose membership function, indeterminacy function, and non-membership function are defined as follows:

$$V_{r(E)}(v) = \begin{cases} sup_{z \in r^{-1}(v)}\{V_E(z)\}, & if\ r^{-1}(v)\ is\ not\ empty, \\ 0, & if\ r^{-1}(v)\ is\ empty, \end{cases}$$

$$Y_{r(E)}(v) = \begin{cases} sup_{z \in r^{-1}(v)}\{Y_E(z)\}, & if\ r^{-1}(v)\ is\ not\ empty, \\ 0, & if\ r^{-1}(v)\ is\ empty, \end{cases}$$

$$W_{r(E)}(y) = \begin{cases} inf_{z \in r^{-1}(v)}\{W_E(z)\}, & if\ r^{-1}(v)\ is\ not\ empty, \\ 1, & if\ r^{-1}(v)\ is\ empty, \end{cases}$$

for each v in Q, where $r^{-1}(v) = u : r(u) = v$, correspondingly.

3. Some Definitions

The current part presents a series of new definitions that form the basis for the discussions in the subsequent sections.

Definition 17. *A neutrosophic set W in a neutrosophic topological space (U, σ) is said to be neutrosphic regular open if and only if $W = (\overline{W})^\circ$.*

Definition 18. *A neutrosophic point $u_{m,n,p}$ is said to be a neutrosophic δ-cluster point of a neutrosophic set E if and only if every neutrosophic regular open q-neighborhood T of $u_{m,n,p}$ is q-coincident with E. The set of all neutrosophic δ-cluster points of E is called the neutrosophic δ-closure of E and is denoted by $\overline{E}_\delta$.*

Remark 1. *For any neutrosophic set E in a neutrosophic topological space (U, σ), the δ-closure of E is represented as follows: $\overline{E}_\delta = \bigcap\{\overline{T} : E \subseteq \overline{T}, U \in \sigma\}$*

Definition 19. *A neutrosophic set E is said to be a neutrosophic δ-neighborhood of a neutrosophic point $u_{m,n,p}$ if and only if there exists a neutrosophic regular open q-neighborhood N of $u_{m,n,p}$ such that $N \subseteq E$.*

Definition 20. *A neutrosophic set E is considered neutrosophic δ-closed if and only if* $E = \overline{E}_\delta$. *The complement of a neutrosophic δ-closed set is referred to as a neutrosophic δ-open set.*

As a δ-open set is the complement of a δ-closed set, Z is δ-open if and only if $Z = (Z)^\circ_\delta$. *Additionally, it is known that* $((E^c)^\circ_\delta)^c = \overline{E}_\delta$. *A neutrosophic set E is considered neutrosophic δ-open in a neutrosophic topological space U if and only if, for every neutrosophic point* $u_{m,n,p}$, *such that* $u_{m,n,p}qE$, *E is a neutrosophic δ-neighborhood of* $u_{m,n,p}$.

It can be easily demonstrated that $\overline{E} \subseteq \overline{E}_\delta \subseteq \overline{E}_\theta$ *for any neutrosophic set E in a neutrosophic topological space U. On the other hand, for a neutrosophic open set E in a neutrosophic topological space* (U, σ), *we have* $\overline{E} \subseteq \overline{E}_\delta$. *Furthermore, it is evident that every regular open set is δ-open, and every δ-open set is open.*

In any neutrosophic topological space (U, σ), *the set* $\overline{E}_\delta$ *is a neutrosophic δ-closed set for any neutrosophic set E. In other words,* $\overline{(\overline{E}_\delta)}_\delta = \overline{E}_\delta$.

Definition 21. *A function* $f : X \to Y$ *is called neutrosophic δ-continuous (abbreviated as n. δ.c.) if, for every neutrosophic point* $u_{m,n,p}$ *in X and any regular open q-neighborhood V of* $f(u_{m,n,p})$ *in Y, there exists a regular open q-neighborhood U of* $u_{m,n,p}$ *such that* $f(U)$ *is contained in V.*

Remark 2. *A function* $r : U \to Q$ *is considered neutrosophic δ-continuous if, for every neutrosophic δ-open set T in Q, the preimage* $r^{-1}(T)$ *is neutrosophic δ-open in U. Thus, the composition of two neutrosophic δ-continuous functions remains neutrosophic δ-continuous.*

Definition 22. *Suppose* $r : U \to Q$ *is a neutrosophic mapping.*

(1) *r is neutrosophic δ-open if, for every neutrosophic δ-open set E in U, the image* $r(E)$ *is neutrosophic δ-open in Q.*
(2) *r is considered neutrosophic δ-closed if, for every neutrosophic δ-closed set F in U, the image* $r(F)$ *is neutrosophic δ-closed in Q.*

Theorem 1. *Given that* $r : (U, \sigma) \to (Q, \vartheta)$ *is neutrosophic δ-continuous, the following conditions are interchangeable.*

(a) $r(\overline{E}_\delta) \subseteq \overline{(r(E))}_\delta$.
(b) $\overline{(r^{-1}(E))}_\delta \subseteq r^{-1}(\overline{E}_\delta)$.
(c) *Given any neutrosophic δ-closed set E in Q, the preimage* $r^{-1}(E)$ *is neutrosophic δ-closed in U.*
(d) *Given any neutrosophic δ-open set E in Q, the preimage* $r^{-1}(E)$ *is neutrosophic δ-open in U.*

Definition 23. *Assume that* (U, σ) *is a neutrosophic topological space and* η *is a neutrosophic set in* (U, σ). *Define* $U^\eta = \{N \cap \eta : N \text{ is a neutrosophip subset in } (U, \sigma)\}$. *The collection* $\sigma^\eta = \{N \cap \eta : N \in \sigma\}$ *is referred to as the neutrosophic η-topology induced by σ over η. The pair* (η, σ^η) *is called the subspace. The elements of* σ^η *are known as neutrosophic open sets in the subspace η. A set* $N \in U^\eta$ *is said to be neutrosophic closed in η when* $\eta \cap N \in \sigma^\eta$.

Definition 24. *Consider* $u_{m,n,p}$ *as a neutrosophic point within η, denoted by* $u_{m,n,p} \in \eta$. *A set* $H \in U^\eta$ *is called a neutrosophic neighborhood of* $u_{m,n,p}$ *in the subspace η whenever there exists a* $T \in \sigma^\eta$ *such that* $u_{m,n,p}$ *belongs to T and T is contained within H.*

Theorem 2. *Consider* $T \in U^\eta$. *The set U is a member of* σ^η *if and only if T functions as a neutrosophic neighborhood of every point* $u_{m,n,p}$ *that lies in T within η.*

Definition 25. *Consider* $N \in U^\eta$. *The interior of N in the subspace η is defined as the largest neutrosophic open set in η that is contained within N. In other words,* $N^{\eta\circ} = sup\{T : T \subseteq N, T \in \sigma^\eta\}$. *In a similar manner, the closure of N in η is defined as the smallest closed set in η that contains N. More precisely,* $\overline{N}_\eta = inf\{\eta \cap T^c : N \subseteq \eta \cap T^c, T \in \sigma^\eta\}$.

Definition 26. *Consider $E \in U^{\eta}$ and $u_{m,n,p} \in U$. The point $u_{m,n,p}$ is described as q-coincident with E in the subspace η when $m + V_E(u) > V_{\eta}(u)$ or $n + Y_E(u) > Y_{\eta}(u)$ or $p + W_E(u) < W_{\eta}(u)$. This relationship is denoted as $u_{m,n,p} q^{\eta} E$.*

Definition 27. *Consider $u_{m,n,p} \in \eta$. A set $H \in U^{\eta}$ is called a neutrosophic q-neighborhood of $u_{m,n,p}$ in η whenever there exists a $T \in \sigma^{\eta}$ such that $u_{m,n,p} q^{\eta} T$ and $T \subseteq H$.*

Remark 3. *Consider $T \in U^{\eta}$. The set T belongs to σ^{η} if and only if T serves as a q-neighborhood of every $u_{m,n,p}$ satisfying $u_{m,n,p} q^{\eta} T$ in η.*

Remark 4. *Consider (U, σ) as a neutrosophic topological space and (η, σ^{η}) as a neutrosophic subspace.*

(1) Given that $E \in U^{\eta}$, it follows that $E^{\circ} \subseteq E^{\eta\circ}$.
(2) In the case of any neutrosophic subset $E \in U^{\eta}$, E° is contained in $E^{\circ} \subseteq E^{\eta\circ} \cap \eta^{\circ}$.
(3) Given that $\eta \in \sigma$ and $E \in U^{\eta}$, it follows that $E^{\circ} = E^{\eta\circ}$.

4. Neutrosophic δ-Separation Axioms

In this section, we introduce new separation axioms by utilizing the concept of neutrosophic δ-open sets. Regarding neutrosophic disjointness, we know that $\eta_1 \cap \eta_2 = 0_U$ implies $\eta_1 \tilde{q} \eta_2$, but the converse does not necessarily hold. Based on this principle, we now define a new set of neutrosophic δ-separation axioms.

Definition 28. *A neutrosophic topological space U is called neutrosophic $\delta - T_0$ whenever, for any pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in U, there exists a neutrosophic δ-open set T such that $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$.*

The separation axiom defined here differs from the neutrosophic T_0 axiom, as demonstrated in the following example.

$$V_{T_\alpha}(u) = \begin{cases} 1 & \text{if } 0 \le u \le \alpha \\ 0 & \text{if } \alpha < u \le 1 \end{cases}, \; Y_{T_\alpha}(u) = \begin{cases} 1 & \text{if } 0 \le u \le \alpha \\ 0 & \text{if } \alpha < u \le 1 \end{cases}, \; W_{T_\alpha}(u) = \begin{cases} 0 & \text{if } 0 \le u \le \alpha \\ 1 & \text{if } \alpha < u \le 1 \end{cases}$$

Consider $\sigma = \{0_U, 1_U\} \bigcup \{T_\alpha : \alpha \in (0,1)\}$. It is clear that σ forms a neutrosophic topology on U, and the set of all neutrosophic δ-open sets in (U, σ) is $\{0_U, 1_U\}$. Therefore, for any two distinct neutrosophic points in (U, σ), there exists a neutrosophic open subset of U that contains one point but not the other. As a result, (U, σ) is neutrosophic T_0, but it is not neutrosophic $\delta - T_0$.

Theorem 3. *Given that $r : U \to Q$ is injective and neutrosophic δ-continuous, and that Q is neutrosophic $\delta - T_0$, it follows that U must also be neutrosophic $\delta - T_0$.*

Proof. Consider two points $j_{m,n,p}$ and $k_{r,s,t}$ in U with distinct supports. Since r is injective, $r(j_{m,n,p})$ and $r(k_{r,s,t})$ are two neutrosophic points with different supports in Q. Given that Q is neutrosophic $\delta - T_0$, there exists a neutrosophic δ-open set T, such that $r(j_{m,n,p}) \subseteq T \subseteq (r(k_{r,s,t}))^c$ or $r(k_{r,s,t}) \subseteq T \subseteq (r(j_{m,n,p}))^c$. Therefore, we have $j_{m,n,p} \subseteq r^{-1}(T) \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq r^{-1}(T) \subseteq j^c_{m,n,p}$. Moreover, since r is neutrosophic δ-continuous, $r^{-1}(T)$ is a neutrosophic δ-open set. Hence, U is neutrosophic $\delta - T_0$. □

Definition 29. *In the case of a neutrosophic topological space U, it is termed neutrosophic $\delta - T_1$ whenever, for any two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in U, there exist two neutrosophic δ-open sets T_1 and T_2 such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$.*

It is evident that every neutrosophic $\delta - T_1$ space is also neutrosophic $\delta - T_0$.

Example 1. *Let $U = \{s, t\}$ and $\tau = \{t_{p,p,1-p} \mid p \in [0,1]\} \cup \{0_U, 1_U\}$, where $t_{p,p,1-p}$ is the neutrosophic point with membership value p, indeterminacy value p, and non-membership value $1 - p$ at the support t. Then, clearly σ is a neutrosophic topology and all elements in σ are neutrosophic regular open, so they are neutrosophic δ-open. Take any two neutrosophic points $a_{m,m,1-m}$ and $b_{p,p,1-p}$ where r and p are nonzero. Then, there is a neutrosophic δ-open set $b_{p,p,1-p}$ such that $b_{p,p,1-p} \subseteq b_{p,p,1-p} \subseteq (a_{m,m,1-m})^c$, and 1_X is an only neutrosophic δ-open set with $a_{m,m,1-m} \subseteq 1_U$. Clearly, for any p, $1_U \nsubseteq (b_{p,p,1-p})^c$. Hence (U, σ) is neutrosophic $\delta - T_0$, but it is not neutrosophic $\delta - T_1$.*

Theorem 4. *A space U is neutrosophic $\delta - T_1$ if and only if every crisp neutrosophic point in U is neutrosophic δ-closed.*

Proof. Consider U as a neutrosophic $\delta - T_1$ space. For a crisp neutrosophic point $j_{m,n,p}$ in U, we aim to show that $j^c_{m,n,p}$ is neutrosophic δ-open. Choose a neutrosophic point $k_{r,s,t} \in j^c_{m,n,p}$ with a different support from $j_{m,n,p}$. Since U is neutrosophic $\delta - T_1$, there exists a neutrosophic δ-open set T such that $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$. Thus, we can express $j^c_{m,n,p}$ as $j^c_{m,n,p} = \bigcup_{k_{r,s,t} \in j^c_{m,n,p}} \{T : k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}, T \ is \ neutrosophic \ \delta - open \ in \ U\}$. Since the union of these neutrosophic δ-open sets is neutrosophic δ-open, it follows that $j^c_{m,n,p}$ is neutrosophic δ-open. Therefore, $j_{m,n,p}$ is neutrosophic δ-closed. □

Corollary 1. *Let U be a neutrosophic topological space. U is neutrosophic $\delta - T_1$ if and only if $u_{m,n,p} = \bigcap\{\overline{T}_\delta : u_{m,n,p} \subseteq \overline{T}_\delta\}$.*

Theorem 5. *Consider $r : U \to Q$ as an injective and neutrosophic δ-continuous function. Provided that Q is neutrosophic $\delta - T_1$, it follows that U is also neutrosophic $\delta - T_1$.*

Proof. Consider two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ in U with distinct supports. Since r is injective, $r(j_{m,n,p})$ and $r(k_{r,s,t})$ are neutrosophic points in Q that have different supports. Given that Q is neutrosophic $\delta - T_1$, there exist two neutrosophic δ-open sets T_1 and T_2, where $r(j_{m,n,p}) \subseteq T_1 \subseteq (r(k_{r,s,t}))^c$ and $r(k_{r,s,t}) \subseteq T_2 \subseteq (r(j_{m,n,p}))^c$. As a result, $r^{-1}(T_1)$ and $r^{-1}(T_2)$ are neutrosophic δ-open sets in U, and we have $j_{m,n,p} \subseteq r^{-1}(T_1) \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq r^{-1}(T_2) \subseteq j^c_{m,n,p}$. This shows that U is neutrosophic $\delta - T_1$. □

Definition 30. *Given any two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in U, a neutrosophic space U is called neutrosophic δ-Hausdorff or neutrosophic $\delta - T_2$ when there exist neutrosophic δ-open sets T_1 and T_2 such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$, and $T_1 \tilde{q} T_2$.*

It is clear that a neutrosophic $\delta - T_2$ space is also a neutrosophic $\delta - T_2$ space.

Theorem 6. *Assume that (U, σ) is a neutrosophic $\delta - T_1$ space. When the complement of each neutrosophic δ-open set is also neutrosophic δ-open, (U, σ) becomes a neutrosophic $\delta - T_2$ space.*

Proof. Consider two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ in U with different supports. Since U is neutrosophic $\delta - T_1$, there exists a neutrosophic δ-open set T satisfying $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$ or $k_{r,s,t} \subseteq T \subseteq j^c_{m,n,p}$. Suppose $j_{m,n,p} \subseteq T \subseteq k^c_{r,s,t}$. Then, it follows that $k_{r,s,t} \subseteq T^c \subseteq j^c_{m,n,p}$. By the given condition, T^c is neutrosophic δ-open. Therefore, (U, σ) is neutrosophic $\delta - T_2$. □

Theorem 7. *When $u \in U$ satisfies that the crisp neutrosophic point $u_{1,1,0}$ is neutrosophic δ-open in (U, σ), it follows that (U, σ) is neutrosophic $\delta - T_2$.*

Proof. Consider two neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports. In this case, $j_{m,n,p} \subseteq j_{1,1,0} \subseteq k^c_{r,s,t}$ and $k_{r,s,t} \subseteq k_{1,1,0} \subseteq j^c_{m,n,p}$. It is evident that $j_{1,1,0} \subseteq (k_{1,1,0})^c$, and based on this assumption, $j_{1,1,0}$ and $k_{1,1,0}$ are neutrosophic δ-open. Therefore, (U, σ) is neutrosophic $\delta - T_2$. □

Theorem 8. *When $r : U \to Q$ is injective and neutrosophic δ-continuous, and Q is neutrosophic $\delta - T_2$, it implies that U is also neutrosophic $\delta - T_2$.*

Proof. Consider two points $j_{m,n,p}$ and $k_{r,s,t}$ in U with different supports. As f is injective, $r(j_{m,n,p})$ and $r(k_{r,s,t})$ are neutrosophic points in Q with distinct supports. Given that Q is neutrosophic $\delta - T_2$, there exist neutrosophic δ-open sets T_1 and T_2 such that $r(j_{m,n,p}) \subseteq T_1 \subseteq (r(k_{r,s,t}))^c$, $r(k_{r,s,t}) \subseteq T_2 \subseteq (r(j_{m,n,p}))^c$, and $T_1 \subseteq (T_2)^c$. Consequently, $r^{-1}(T_1)$ and $r^{-1}(T_2)$ are neutrosophic δ-open sets in U satisfying $j_{m,n,p} \subseteq r^{-1}(T_1) \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq r^{-1}(T_2) \subseteq j^c_{m,n,p}$, and $r^{-1}(T_1) \subseteq (r^{-1}(T_2))^c$. Thus, U is neutrosophic $\delta - T_2$. □

Definition 31. *In the context of neutrosophic topology, a space U is described as neutrosophic δ-regular when, given a neutrosophic point $j_{m,n,p}$ in U and a neutrosophic δ-closed set K such that $j_{m,n,p} \subseteq C^c$, there exist neutrosophic δ-open sets T_1 and T_2 satisfying $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. Additionally, U is identified as neutrosophic $\delta - T_3$ when it meets the conditions of being both neutrosophic δ-regular and neutrosophic $\delta - T_1$.*

It is straightforward to demonstrate that any neutrosophic $\delta - T_3$ space is also neutrosophic $\delta - T_2$.

It is well established that for every neutrosophic closed set C, C° is a neutrosophic regular open set. Consequently, it is also neutrosophic δ-open. Hence, the following theorem holds true.

Theorem 9. *In a neutrosophic topological space (U, σ):*

(1) U is neutrosophic δ-regular.
(2) Given a neutrosophic point $j_{m,n,p}$ and a neutrosophic δ-open set N containing $j_{m,n,p}$, there exists a neutrosophic δ-open set T such that $j_{m,n,p} \subseteq T \subseteq \overline{T}_\delta \subseteq N$.
(3) Given a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ such that $j_{m,n,p} \subseteq C^c$, there exist neutrosophic δ-open sets T_1 and T_2 with $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$ and $\overline{(T_1)} \subseteq (\overline{T_2})^c$.
(4) Given a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ with $j_{m,n,p} \subseteq C^c$, there exist neutrosophic open sets T_1 and T_2 such that $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$ and $\overline{(T_1)} \subseteq (\overline{T_2})^c$.

Proof. (1) ⇒ (2) Consider a neutrosophic point set $j_{m,n,p}$ and a neutrosophic δ-open set N containing $j_{m,n,p}$. It follows that there exist neutrosophic δ-open sets T_1 and T_2 with $j_{m,n,p} \subseteq T_1$, $N^c \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. Consequently, $j_{m,n,p} \subseteq T_1 \subseteq (T_2)^c \subseteq N$. Therefore, $j_{m,n,p} \subseteq T_1 \subseteq \overline{T_1}_\delta \subseteq \overline{((T_2)^c)}_\delta = (T_2)^c \subseteq N$.
(2) ⇒ (3) Consider C, a neutrosophic δ-closed subset of U, and $j_{m,n,p}$, a neutrosophic point set satisfying $j_{m,n,p} \subseteq C^c$. It follows that C^c is a neutrosophic δ-open set containing $j_{m,n,p}$. By (2), there exists a neutrosophic δ-open set T with $p \subseteq T \subseteq \overline{(T)}_\delta \subseteq C^c$. Since T is a neutrosophic δ-open set containing $j_{m,n,p}$, there exists a neutrosophic δ-open set N with $j_{m,n,p} \subseteq N \subseteq \overline{(N)}_\delta \subseteq T \subseteq \overline{(T)}_\delta \subseteq C^c$. Define $T_1 = N$ and $T_2 = (\overline{T}_\delta)^c$. It follows that T_1 and T_2 are neutrosophic δ-open sets with $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$. Furthermore, $\overline{T_2} \subseteq \overline{(\overline{T}_\delta)^c)} \subseteq \overline{(T^c)} = T^c$. Therefore, $\overline{(T_1)} = \overline{N} \subseteq T \subseteq (\overline{T_2})^c$.
(3) ⇒ (4) It is evident.
(4) ⇒ (1) Consider C as a neutrosophic δ-closed subset of U and $j_{m,n,p}$ as a neutrosophic point set where $j_{m,n,p} \subseteq c^c$. By (4), there exist neutrosophic open sets T and N where $j_{m,n,p} \subseteq T$, $C \subseteq N$, and $\overline{T} \subseteq (\overline{N})^c$. Since $j_{m,n,p} \subseteq T \subseteq \overline{T}$, it follows that $j_{m,n,p} \subseteq T^\circ = T \subseteq (\overline{T})^\circ$. Define $T_1 = (\overline{T})^\circ$; consequently, T_1 is neutrosophic δ-open, and $j_{m,n,p} \subseteq T_1$. Since $C \subseteq N \subseteq \overline{N}$, we have $C \subseteq N^\circ = N \subseteq (\overline{N})^\circ$. Define $T_2 = (\overline{N})^\circ$; as a result, T_2 is neutrosophic δ-open, and $C \subseteq T_2$. Additionally, since $(\overline{T})^\circ \subseteq \overline{T} \subseteq (\overline{N})^c \subseteq ((\overline{N})^\circ)^c$, we conclude that $T_1 \subseteq (T_2)^c$. □

Definition 32. *A neutrosophic δ-normal space is one where, for any pair of neutrosophic δ-closed sets C and S in U with $C \subseteq S^c$, there exist neutrosophic δ-open sets T_1 and T_2 such that $C \subseteq T_1$, $S \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. A neutrosophic space U is called neutrosophic $\delta - T_4$ if it is both neutrosophic $\delta - T_1$ and neutrosophic δ-normal.*

Theorem 10. *Within a neutrosophic topological space (U, σ), the following statements are all true simultaneously:*

(1) U is neutrosophic δ-normal.
(2) Given a neutrosophic δ-closed set C and a neutrosophic δ-open set T containing C, there exists a neutrosophic δ-open set N with the properties $C \subseteq N \subseteq \overline{N} \subseteq T$.
(3) Given a neutrosophic δ-closed set C and a neutrosophic δ-open set T containing C, a neutrosophic open set N can be found with $C \subseteq N \subseteq \overline{N} \subseteq T$.
(4) Given a pair of neutrosophic δ-closed subsets C and S in U with $C \subseteq S^c$, neutrosophic open sets T_1 and T_2 exist, satisfying $C \subseteq T_1$, $S \subseteq T_2$, and $\overline{T_1} \subseteq (\overline{T_2})^c$.

Proof. (1) $\Rightarrow$ (2) Consider a neutrosophic δ-closed set C and a neutrosophic δ-open set T containing C. It follows that T^c is a neutrosophic δ-closed set, and $T^c \subseteq C^c$. As a result, there exist neutrosophic δ-open sets N_1 and N_2 where $C \subseteq N_1$, $T^c \subseteq N_2$, and $N_1 \subseteq (N_2)^c$. Consequently, we have $C \subseteq N_1 \subseteq (N_2)^c \subseteq T$ and $N_1 \subseteq \overline{N_1} \subseteq \overline{(N_2)^c} = (N_2)^c$. Hence, $C \subseteq N_1 \subseteq \overline{N_1} \subseteq \overline{N_{1\delta}} \subseteq T$.
(2) $\Rightarrow$ (3) It is apparent.
(3) $\Rightarrow$ (4) Given that C and S are neutrosophic δ-closed subsets of U with $C \subseteq S^c$, it follows that S^c is a neutrosophic δ-open set containing C. By (3), there exists a neutrosophic open set N where $C \subseteq N \subseteq \overline{N} \subseteq S^c$. Since $\overline{N}$ is neutrosophic δ-closed and S^c is a neutrosophic δ-open set containing $\overline{N}$, a neutrosophic open set T must exist such that $C \subseteq N \subseteq \overline{N} \subseteq T \subseteq \overline{T} \subseteq S^c$. Define $T_1 = N$ and $T_2 = (\overline{T})^c$, so T_1 and T_2 are neutrosophic open sets where $C \subseteq T_1$ and $S \subseteq T_2$. Additionally, $\overline{(T_2)} = \overline{((\overline{T})^c)} \subseteq \overline{(T^c)} = T^c$. Therefore, $\overline{(T_1)} = \overline{N} \subseteq T \subseteq (\overline{(T_2)})^c$.
(4) $\Rightarrow$ (1) Given neutrosophic δ-closed subsets C and S of U with $C \subseteq S^c$, according to (4), there exist neutrosophic open sets T and N satisfying $C \subseteq T$, $S \subseteq N$, and $\overline{T} \subseteq (\overline{N})^c$. Additionally, since $C \subseteq T \subseteq \overline{T}$, it follows that $C \subseteq T = T^\circ \subseteq (\overline{T})^\circ$. Define T_1 as $(\overline{T})^\circ$, so T_1 is neutrosophic δ-open and $C \subseteq T_1$. Likewise, define T_2 as $(\overline{N})^\circ$, which makes T_2 neutrosophic δ-open and $S \subseteq T_2$. Furthermore, since $(\overline{T})^\circ \subseteq \overline{T} \subseteq (\overline{N})^c \subseteq ((\overline{N})^\circ)^c$, we have $T_1 \subseteq (T_2)^c$. □

Example 2. *Let $U = [0,1]$ and, for each $\beta \in U$, $V_{T_\beta}(u) = \beta$, $Y_{T_\beta}(u) = \beta$, $W_{T_\beta}(u) = 1 - \beta$, for all $u \in U$. Meanwhile, let $\sigma = \{T_\beta : \beta \in U\}$. Then, σ is a neutrosophic topology and each T_β is neutrosophic δ-open. Therefore, σ is neutrosophic δ-normal and neutrosophic δ-regular. However, it is not neutrosophic $\delta - T_1$. So, it is neither neutrosophic $\delta - T_3$ nor neutrosophic $\delta - T_4$.*

Example 3. *Let $U = [0,1]$ and, for each $\beta \in U$,*

$$V_{T_\beta}(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq \beta \\ 0 & \text{if } \beta < u \leq 1 \end{cases}, \ Y_{T_\beta}(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq \beta \\ 0 & \text{if } \beta < u \leq 1 \end{cases}, \ W_{T_\beta}(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq \alpha \\ 1 & \text{if } \beta < u \leq 1 \end{cases}$$

$$V_{N_\beta}(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq \beta \\ 1 & \text{if } \beta < u \leq 1 \end{cases}, \ Y_{N_\beta}(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq \beta \\ 1 & \text{if } \beta < u \leq 1 \end{cases}, \ W_{N_\beta}(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq \beta \\ 0 & \text{if } \beta < u \leq 1 \end{cases}$$

Let σ be a neutrosophic topology on U generated by the subbase $\{T_\beta : \beta \in U\} \cup \{N_\beta : \beta \in U\}$. Then, σ is a neutrosophic topology and $(\overline{(T_\beta)})^\circ = T_\beta$ for all $\beta \in U$. So, every T_β is neutrosophic

δ-open. Similarly, every N_β is also neutrosophic δ-open. Therefore, (U, σ) is neutrosophic $\delta - T_4$ and also neutrosophic $\delta - T_3$.

5. Neutrosophic δ-Closure and δ-Interior in the Neutrosophic Subspace

Consider (U, σ) as a neutrosophic topological space and η as a neutrosophic subset of U. The neutrosophic subspace on η is denoted by (η, σ^η). If η is neutrosophic regular open (or regular closed) in X, then (η, σ^η) is referred to as a neutrosophic regular open (or regular closed) subspace, respectively.

Given any subset $Q \subseteq U$, suppose η is a neutrosophic subset defined as follows:

$$V_\eta(u) = \begin{cases} 1 & \text{if } u \in Q \\ 0 & \text{if } u \notin Q \end{cases}, \; Y_\eta(x) = \begin{cases} 1 & \text{if } u \in Q \\ 0 & \text{if } u \notin Q \end{cases}, \; W_\eta(x) = \begin{cases} 0 & \text{if } u \in Q \\ 1 & \text{if } u \notin Q \end{cases}$$

In that case, the neutrosophic subspace (η, σ^η) will be represented as χQ.

Definition 33. *Consider $E \in U^\eta$. We define E as neutrosophic regular open (or regular closed) in the subspace η, if $E = (\overline{E}^\eta)^{\eta\circ}$ (or $\overline{(E^{\eta\circ})}^\eta$.*

Definition 34. *Consider $E \in U^\eta$. A neutrosophic point $u_{m,n,p} \in \eta$ is defined as a neutrosophic δ-cluster point of E in η if and only if every neutrosophic regular open q-neighborhood T of $u_{m,n,p}$ in η is q-coincident with E in η. The set of all neutrosophic δ-cluster points of E in η is referred to as the neutrosophic δ-closure of E in η, denoted by $\overline{E}^\eta_\delta$.*

Theorem 11. *Consider $E \in U^\eta$ and $u_{m,n,p} \in \eta$. The element $u_{m,n,p}$ belongs to the set $\bigcap\{W \in U^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$ if and only if every neutrosophic regular open q-neighborhood T of $u_{m,n,p}$ in η is q-coincident with E in η.*

Proof. Consider H to be a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ such that $H\tilde{q}E$. Then, H is a neutrosophic open set in η where $u_{m,n,p}qH \subseteq H$ and $E\tilde{q}H$ are satisfied. Since H^c is neutrosophic regular closed and $E \subseteq H^c$, it follows that $\bigcap\{W \in U^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\} \subseteq H^c$. Furthermore, because $u_{m,n,p} \notin H^c$, we conclude that $u_{m,n,p} \notin \bigcap\{W \in X^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$.

On the other hand, suppose $u_{m,n,p} \notin \bigcap\{W \in U^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$. In this case, there exists a neutrosophic regular closed set W such that $u_{m,n,p} \notin W$ and $E \subseteq W$. As a result, W^c is a neutrosophic regular open set where $u_{m,n,p}qW^c$ and $E\tilde{q}W^c$ hold. Thus, $u_{m,n,p}$ cannot be a neutrosophic δ-cluster point of E in η.

According to the aforementioned theorem, in a neutrosophic subspace (η, σ^η), we have $\overline{E}^\eta_\delta = \bigcap\{W \in X^\eta : E \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}$ for any set $E \in U^\eta$. Next, we introduce the concept of the δ-interior in a subspace. □

Definition 35. *Assume $E \in U^\eta$. The δ-interior of E within η is defined in the following way:*

$$\begin{aligned}\overline{E}^\eta_\delta = \eta \cap (\overline{(\eta \cap E^c)}^\eta_\delta)^c &= \eta \cap (\bigcap\{W \in U^\eta : \eta \cap E^c \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\})^c \\ &= \eta \cap (\bigcup\{W^c : W \in U^\eta, \eta \cap E^c \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\}) \\ &= \bigcup\{\eta \cap W^c : W \in U^\eta, \eta \cap E^c \subseteq W, W = \overline{(W^{\eta\circ})}^\eta\} \\ &= \bigcup\{\eta \cap W^c : F \in U^\eta, \eta \cap W^c \subseteq E, \eta \cap W^c = \eta \cap (\overline{(W^{\eta\circ})}^\eta)^c\} \\ &= \bigcup\{T \in U^\eta, T \subseteq E, T = (\overline{T}^\eta)^{\eta\circ}\}\end{aligned} \quad (1)$$

We aim to demonstrate that for any neutrosophic set E in a neutrosophic subspace (η, σ^{η}), the following holds: $\overline{E}^{\eta}_{\delta} = \bigcap\{\overline{T}^{\eta} : E \subseteq \overline{T}^{\eta}, T \in \sigma^{\eta}\}$. To accomplish this, we will first prove two lemmas.

Lemma 1. *Consider (η, σ^{η}) as a neutrosophic subspace. If $N \in \sigma^{\eta}$, then $\overline{N}^{\eta}$ is a neutrosophic regular closed set in η.*

Proof. Since $N \subseteq \overline{N}^{\eta}$, $N = N^{\eta\circ} \subseteq (\overline{N}^{\eta})^{\eta\circ}$ and hence $\overline{N}^{\eta} \subseteq \overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta}$.

On the other hand, since $\overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta} \subseteq \overline{N}^{\eta}$, $\overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta} \subseteq \overline{(\overline{N}^{\eta})}^{\eta} = \overline{N}^{\eta}$. Hence, $\overline{N}^{\eta} = \overline{((\overline{N}^{\eta})^{\eta\circ})}^{\eta}$. □

Lemma 2. *Consider (η, σ^{η}) as a neutrosophic subspace. In that case, $\{\overline{T}^{\eta} : T \in \sigma^{\eta}\} = \{W \in U^{\eta} : W$ is neutrosophic regular closed in $\eta\}$.*

Proof. It is known that for every neutrosophic open set T in η, $\overline{T}^{\eta}$ is neutrosophic regular closed in η.

On the other hand, consider any neutrosophic regular closed set W in η. In that case, $W = \overline{(W^{\eta\circ})}^{\eta} = \overline{(\bigcup\{T : T \subseteq W, T \in \sigma^{\eta}\})}^{\eta} \in \{\overline{T}^{\eta} : T \in \sigma^{\eta}\}$.

There may be a challenge in determining the neutrosophic δ-closure of any neutrosophic set. However, based on the above lemmas, we have a hint on how to find it. □

Theorem 12. *Consider any neutrosophic set E in a neutrosophic subspace (η, σ^{η}), $\overline{E}^{\eta}_{\delta} = \bigcap\{\overline{T}^{\eta} : E \subseteq \overline{T}^{\eta}, T \in \sigma^{\eta}\}$.*

Proof. It is clear from Lemma 2. □

Furthermore, if η is neutrosophic open in U and if $(\eta, \sigma^{\eta}) = \chi Q$, $\overline{E}^{\eta}_{\delta} = \overline{E}_{\delta} \cap \eta$ for any neutrosophic subset E of η. At this point, we will demonstrate it.

Lemma 3. *Consider U as a neutrosophic topological space, $Q \subseteq U$, and η as a neutrosophic open subset of U, where $(\eta, \sigma^{\eta}) = \chi Q$. Assume $E \in U^{\eta}$. When E is neutrosophic regular open in U, it follows that E is also neutrosophic regular open in η.*

Proof. Consider any neutrosophic subset $E \in U^{\eta}$; the following holds: $(\overline{E}^{\eta})^{\eta\circ} = (\overline{E} \cap \eta)^{\eta\circ} = (\overline{E} \cap \eta)^{\circ} = (\overline{E})^{\circ} \cap \eta^{\circ} = (\overline{E})^{\circ} \cap \eta$. Hence, if $E = (\overline{E})^{\circ}$, then $(\overline{E}^{\eta})^{\eta\circ} = (\overline{E})^{\circ} \cap \eta = E \cap \eta = E$. □

Theorem 13. *Consider (U, σ) as a neutrosophic topological space and $Q \subseteq U$. Suppose that $(\eta, \sigma^{\eta}) = \chi Q$, and η is neutrosophic regular open in U. Under these conditions, for any neutrosophic subset $E \in U^{\eta}$, it follows that $\overline{E}^{\eta}_{\delta} = \overline{E}_{\delta} \cap \eta$.*

Proof. Suppose that $u_{m,n,p} \notin \overline{E}^{\eta}_{\delta}$. Then, there is a neutrosophic regular open q-neighborhood H of $u_{m,n,p}$ *in* η with $H\tilde{q}^{\eta}A$, i.e., $(\overline{H}^{\eta})^{\eta\circ} \subseteq E^{c}$. Since $u_{m,n,p}q^{\eta}H$, $u_{m,n,p}qH$ and N are neutrosophic open in U, $H = H^{\circ} \subseteq (\overline{H})^{\circ}$. Note that $(\overline{H})^{\circ}$ is a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ in U. Since $(\overline{H})^{\circ} = (\overline{(H \cap \eta)})^{\circ} = (\overline{H}^{\eta})^{\circ} \subseteq (\overline{H}^{\eta})^{\eta\circ}\tilde{q}^{\eta}A$, we have $(\overline{H})^{\circ}\tilde{q}E$. Thus, $u_{m,n,p} \notin \overline{E}_{\delta} \cap \eta$. Conversely, take $u_{m,n,p} \in \overline{E}^{\eta}_{\delta}$ and a neutrosophic regular open q-neighborhood H of $u_{m,n,p}$ in U. Then, $u_{m,n,p} \notin H^{c}$ and so $u_{m,n,p} \notin (H \cap \eta)$. Thus, $H \cap \eta$ is a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ in U. By the above lemma, $H \cap \eta$ is also a neutrosophic regular open q-neighborhood of $u_{m,n,p}$ in η. Since $u_{m,n,p} \in \overline{E}^{\eta}_{\delta}$, $(H \cap \eta)qE$. Hence, HqE. Therefore, $u_{m,n,p}$ is a neutrosophic δ-cluster point of E in U.

A neutrosophic regular open set in η does not automatically qualify as neutrosophic regular open in U. However, when η is a neutrosophic regular open set in U, where $(\eta, \sigma^{\eta}) = \chi Q$, it follows that any neutrosophic δ-open set in U, contained within U^{η}, will also be considered neutrosophic δ-open in η. This is illustrated by the subsequent theorem. □

Theorem 14. *Consider (U, σ) as a neutrosophic topological space and $Q \subseteq U$. Suppose that η is neutrosophic regular open in U and $E \in U^{\eta}$, where $(\eta, \sigma^{\eta}) = \chi Q$. It follows that $E_{\delta}^{\eta\circ} = E_{\delta}^{\circ}$.*

Proof. $E_{\delta}^{\eta\circ} = \eta \cap (\overline{(\eta \cap E^c)}_{\delta}^{\eta})^c = \eta \cap (\overline{(\eta \cap E^c)}_{\delta} \cap \eta)^c = \eta \cap ((\overline{(\eta \cap E^c)}_{\delta})^c \cup \eta^c) =$
$\eta \cap (\overline{(\eta \cap E^c)}_{\delta})^c = \eta \cap ((\eta \cap E^c)^c)_{\delta}^{\circ} = (\eta \cap (\eta \cap E^c)^c)_{\delta}^{\circ} = E_{\delta}^{\circ}$ □

6. Neutrosophic δ-Separation Axioms in the Neutrosophic Subspace η

We proceed by defining the neutrosophic δ-separation axioms within neutrosophic subspaces. It should be noted that for any set $E \in U^{\eta}$, the complement of A, denoted as E^c, in the neutrosophic subspace η is equivalent to $\eta \cap E^c$.

Definition 36. *A neutrosophic subspace (η, σ^{η}) is called neutrosophic $\delta - T_0$ whenever, given any pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in η, a neutrosophic δ-open set T in η exists, where $j_{m,n,p} \subseteq T \subseteq k_{r,s,t}^c$ or $k \subseteq T \subseteq j_{m,n,p}^c$.*

Definition 37. *Assume (U, σ) is a neutrosophic $\delta - T_0$ space, with $Q \subseteq U$ and η being a neutrosophic regular open set of U, where $(\eta, \sigma^{\eta}) = \chi Q$. It follows that χQ is neutrosophic $\delta - T_0$.*

Proof. Consider U as a neutrosophic $\delta - T_0$ space, and suppose η is a neutrosophic regular open subset of U, where $(\eta, \sigma^{\eta}) = \chi Q$. For neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with different supports in the subspace χQ, it follows that $j_{m,n,p}$ and $k_{r,s,t}$ also have different supports in the space U. Since U is neutrosophic $\delta - T_0$, a neutrosophic δ-open set $T \in U$ exists, satisfying $j_{m,n,p} \subseteq T \subseteq k_{r,s,t}^c$ or $k_{r,s,t} \subseteq T \subseteq j_{m,n,p}^c$. Furthermore, since $T \cap \eta$ is neutrosophic δ-open in U with $T \cap \eta \subseteq \eta$, $T \cap \eta$ is also neutrosophic δ-open in η. Additionally, either $j_{m,n,p} \subseteq T \cap \eta \subseteq (\eta \cap k_{r,s,t}^c)$ or $k_{r,s,t} \subseteq T \cap \eta \subseteq (\eta \cap j_{m,n,p}^c)$. Therefore, (η, σ^{η}) is neutrosophic $\delta - T_0$. □

Definition 38. *A neutrosophic subspace (η, σ^{η}) is termed neutrosophic $\delta - T_1$ when, given a pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in η, two neutrosophic δ-open sets T_1 and T_2 exist in η such that $j_{m,n,p} \subseteq T_1 \subseteq k_{r,s,t}^c$ and $k_{r,s,t} \subseteq T_2 \subseteq j_{m,n,p}^c$.*

Theorem 15. *A neutrosophic subspace (η, τ^{η}) is referred to as neutrosophic $\delta - T_1$ whenever a pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with different supports in η is neutrosophic δ-closed in η.*

Proof. Consider a crisp neutrosophic point $j_{m,n,p}$ in η. It will be shown that $\eta \cap j_{m,n,p}^c$ is neutrosophic δ-open in η. A neutrosophic point $k_{r,s,t} \in \eta \cap j_{m,n,p}^c$ with a different support from $j_{m,n,p}$ can be chosen. Since (η, σ^{η}) is neutrosophic $\delta - T_1$, there exists a neutrosophic δ-open set T in η such that $k_{r,s,t} \subseteq T \subseteq \eta \cap j_{m,n,p}^c$. Therefore, as $\eta \cap j_{m,n,p}^c$ can be expressed as $\bigcup_{k_{r,s,t} \in \eta \cap j_{m,n,p}^c} T : k_{r,s,t} \subseteq T \subseteq \eta \cap j_{m,n,p}^c$, U is neutrosophic δ-open in η. Since this union is neutrosophic δ-open in η, it follows that $\eta \cap j_{m,n,p}^c$ is neutrosophic δ-open in η. As a result, $j_{m,n,p}$ is neutrosophic δ-closed in η. □

Corollary 2. *Consider (η, σ^{η}) as a neutrosophic subspace. η is neutrosophic $\delta - T_1$ if and only if $u_{m,n,p} = \bigcap \overline{T}\delta : u_{m,n,p} \subseteq \overline{T}\delta$ holds when $u_{m,n,p} \in \eta$.*

Theorem 16. *Consider (U, σ) as a neutrosophic $\delta - T_1$ space, where $Q \subseteq U$ and η is neutrosophic regular open in U, with $(\eta, \sigma^\eta) = \chi Q$. It follows that χQ is neutrosophic $\delta - T_1$.*

Proof. This is obvious. □

Definition 39. *A neutrosophic subspace (η, σ^η) is termed neutrosophic δ-Hausdorff, or neutrosophic $\delta - T_2$, when, given any pair of neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with distinct supports in η, there exist neutrosophic δ-open sets T_1 and T_2 in η such that $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$, and $T_1 \subseteq (T_2)^c$.*

It is clear that every neutrosophic $\delta - T_2$ subspace is also a neutrosophic $\delta - T_1$ subspace. In addition, it is easily seen that since every open set is δ-open, every T_i space is also $\delta - T_i$ space for i = 0, 1, 2.

Theorem 17. *Given that U is a neutrosophic $\delta - T_2$ space and $Q \subseteq U$, with η being neutrosophic regular open in U, where $(\eta, \sigma^\eta) = \chi Q$, it follows that χQ is neutrosophic $\delta - T_2$.*

Proof. Consider neutrosophic points $j_{m,n,p}$ and $k_{r,s,t}$ with different supports in a subspace (η, σ^η). It follows that $j_{m,n,p}$ and $k_{r,s,t}$ are also neutrosophic points with different supports in the space U. Since U is neutrosophic $\delta - T_2$, neutrosophic δ-open sets T_1 and T_2 exist in U where $j_{m,n,p} \subseteq T_1 \subseteq k^c_{r,s,t}$, $k_{r,s,t} \subseteq T_2 \subseteq j^c_{m,n,p}$, and $T_1 \subseteq (T_2)^c$. Consequently, neutrosophic δ-open sets $T_1 \cap \eta$ and $T_2 \cap \eta$ exist in η, where $j_{m,n,p} \subseteq T_1 \cap \eta \subseteq k^c_{r,s,t} \cap \eta$, $j_{m,n,p} \subseteq T_2 \cap \eta \subseteq j^c_{m,n,p} \cap \eta$, and $T_1 \cap \eta \subseteq (T_2)^c \cap \eta$. Therefore, η is neutrosophic $\delta - T_2$. □

Definition 40. *A neutrosophic subspace (η, σ^η) is defined as neutrosophic δ-regular if, for every pair consisting of a neutrosophic point $j_{m,n,p}$ in η and a neutrosophic δ-closed set C in η such that $j_{m,n,p} \subseteq C^c$, there exist neutrosophic δ-open sets T_1 and T_2 in η with $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$, and $T_1 \subseteq (T_2)^c$. A neutrosophic subspace (η, σ^η) is referred to as neutrosophic $\delta - T_3$ if it is both neutrosophic δ-regular and neutrosophic $\delta - T_1$.*

It is straightforward to demonstrate that every neutrosophic $\delta - T_3$ subspace is also a neutrosophic $\delta - T_2$ subspace.

It is known that for any neutrosophic closed set C in η, $C^{\eta\circ}$ is neutrosophic regular open in η. As a result, it is also neutrosophic δ-open. This leads to the conclusion of the following theorem.

Theorem 18. *The following conditions are equivalent for a neutrosophic subspace (η, σ^η):*

(1) *(η, σ^η) is neutrosophic δ-regular.*

(2) *Given a neutrosophic point $j_{m,n,p}$ and a neutrosophic δ-open set N containing $j_{m,n,p}$ in (η, σ^η), there exists a neutrosophic δ-open set T in η such that $j_{m,n,p} \subseteq T \subseteq \overline{T}^\eta_\delta \subseteq N$.*

(3) *In (η, σ^η), consider a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ where $j_{m,n,p} \subseteq \eta \cap C^c$. There exist neutrosophic δ-open sets T_1 and T_2 in η, with $j_{m,n,p} \subseteq T_1$, $K \subseteq T_2$, and $\overline{T_1}^\eta \subseteq \eta \cap (\overline{T_2}^\eta)^c$.*

(4) *In (η, σ^η), given a neutrosophic δ-closed set C and a neutrosophic point $j_{m,n,p}$ with $j_{m,n,p} \subseteq \eta \cap C^c$, there exist neutrosophic open sets T_1 and T_2 in (η, σ^η), where $j_{m,n,p} \subseteq T_1$, $C \subseteq T_2$, and $\overline{T_1}^\eta \subseteq \eta \cap (\overline{T_2}^\eta)^c$.*

Proof. (1) $\Rightarrow$ (2) Let $j_{m,n,p}$ be a neutrosophic point set in η and N be a neutrosophic δ-open set in η containing j. Then, there exist neutrosophic δ-open sets T_1 and T_1 in η such that $j \subseteq T_1$, $\eta \cap N^c \subseteq T_2$ and $T_1 \subseteq \eta \cap (T_2)^c$. So $j_{m,n,p} \subseteq T_1 \subseteq \eta \cap (T_2)^c \subseteq N$. Thus $j_{m,n,p} \subseteq T_1 \subseteq \overline{T_1}^\eta_\delta \subseteq \overline{(\eta \cap (T_2)^c)}^\eta_\delta = \eta \cap (T_2)^c \subseteq N$.

(2) $\Rightarrow$ (3) Let C be a neutrosophic δ-closed subset in η and $j_{m,n,p}$ be a neutrosophic point in η such that $j_{m,n,p} \subseteq \eta \cap C^c$. Then, $\eta \cap C^c$ is a neutrosophic δ-open set in η with $j_{m,n,p} \subseteq \eta \cap C^c$. By (2), there is a neutrosophic δ-open set T in η such that $j_{m,n,p} \subseteq T \subseteq \overline{T}^\eta_\delta \subseteq \eta \cap C^c$. Since

U is a neutrosophic δ-open set in η containing $j_{m,n,p}$, there is a neutrosophic δ-open set N in η such that $j_{m,n,p} \subseteq N \subseteq \overline{N}^{\eta}_{\delta} \subseteq T \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap C^c$. Put $U_1 = N$ and $U_2 = \eta \cap (\overline{T}^{\eta}_{\delta})^c$. Then, T_1 and T_2 are neutrosophic δ-open sets in η with $j \subseteq T_1$,$K \subseteq T_2$. Furthermore, $\overline{T_2}^{\eta} \subseteq \overline{(\eta \cap (\eta \cap (\overline{T}^{\eta}_{\delta})^c)}^{\eta} \subseteq \overline{(\eta \cap T^c)}^{\eta} = \eta \cap T^c$. Since, $\overline{N}^{\eta} \subseteq \overline{N}^{\eta}_{\delta}$, $\overline{T_1}^{\eta} = \overline{N}^{\eta} \subseteq T \subseteq \eta \cap (\overline{T_2}^{\eta})^c$.
(3) $\Rightarrow$ (4) This is obvious.
(4) $\Rightarrow$ (1) Let C be a neutrosophic δ-closed subset in η and $j_{m,n,p}$ be a neutrosophic point in η such that $j_{m,n,p} \subseteq \eta \cap C^c$. By (4), there are neutrosophic open sets T and N in η such that $j_{m,n,p} \subseteq T$, $C \subseteq N$ and $\overline{T}^{\eta} \subseteq \eta \cap (\overline{N}^{\eta})^c$. Since $j_{m,n,p} \subseteq T \subseteq \overline{T}^{\eta}$, $j_{m,n,p} \subseteq T^{\eta\circ} = T \subseteq (\overline{U}^{\eta})^{\eta\circ}$. Put $T_1 = (\overline{T}^{\eta})^{\eta\circ}$, then T_1 is neutrosophic δ-open in η and $j \subseteq T_1$. Since $C \subseteq N \subseteq \overline{N}^{\eta}$, $C \subseteq N^{\eta\circ} = N \subseteq (\overline{N}^{\eta})^{\eta\circ}$. Put $T_2 = (\overline{N}^{\eta})^{\eta\circ}$, then T_2 is neutrosophic δ-open in η and $C \subseteq T_2$. Furthermore, since $(\overline{T}^{\eta})^{\eta\circ} \subseteq \overline{T}^{\eta} \subseteq \eta \cap (\overline{V}^{\eta})^c \subseteq \eta \cap ((\overline{V}^{\eta})^{\eta\circ})^c$, $U_1 \subseteq \eta \cap (U2)^c$. □

Definition 41. *A neutrosophic subspace (η, σ^{η}) is called neutrosophic δ-normal if for any pair of neutrosophic δ-closed subsets C, S in η with $C \subseteq S^c$, there are neutrosophic δ-open sets T_1, T_2 in η with $K \subseteq T_1$, $S \subseteq T_2$ and $T_1 \subseteq (T_2)^c$. A neutrosophic subspace (η, σ^{η}) is called neutrosophic $\delta - T_4$ if it is neutrosophic $\delta - T_1$ and neutrosophic δ-normal.*

Clearly every neutrosophic $\delta - T_4$ subspace is neutrosophic $\delta - T_3$.

Theorem 19. *For a neutrosophic subspace (η, σ^{η}), the following are equivalent:*

(1) (η, σ^{η}) is neutrosophic δ-normal.
(2) For any neutrosophic δ-closed set C and any neutrosophic δ-open set T containing C in (η, σ^{η}), there exists a neutrosophic δ-open set N in η such that $C \subseteq N \subseteq \overline{N}^{\eta}_{\delta} \subseteq T$.
(3) For any neutrosophic δ-closed set C and any neutrosophic δ-open set T containing C in (η, σ^{η}), there exists a neutrosophic δ-open set N in η such that $C \subseteq N \subseteq \overline{N}^{\eta} \subseteq T$.
(4) For any neutrosophic δ-closed set C and any neutrosophic δ-open set T containing C in (η, σ^{η}), there exists a neutrosophic open set N in η such that $C \subseteq N \subseteq \overline{N}^{\eta} \subseteq T$.
(5) For any pair of neutrosophic δ-closed subsets C and S with $C \subseteq \eta \cap S^c$ in (η, σ^{η}), there are neutrosophic open sets T_1 and T_2 in η with $C \subseteq T_1$, $S \subseteq T_2$ and $\overline{T_1}^{\eta} \subseteq \eta \cap (\overline{T_2}^{\eta})^c$.

Proof. (1) $\Rightarrow$ (2) Let C be a neutrosophic δ-closed set in η and T be a neutrosophic δ-open set in η containing C. Then, $\eta \cap T^c$ is a neutrosophic δ-closed set in η with $\eta \cap T^c \subseteq \eta \cap C^c$. Thus, there are neutrosophic δ-open sets N_1 and N_2 in η such that $C \subseteq N_1$, $\eta \cap T^c \subseteq N_2$ and $N_1 \subseteq \eta \cap (N_2)^c$. So, $K \subseteq N_1 \subseteq \eta \cap (N_2)^c \subseteq T$ and $N_1 \subseteq \overline{N_1}^{\eta}_{\delta} \subseteq \overline{(\eta \cap (N_2)^c)}^{\eta}_{\delta} = \eta \cap (N_2)^c$. Hence, $C \subseteq N_1 \subseteq \overline{(N_1)}^{\eta}_{\delta} \subseteq T$.
(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) This is obvious.
(4) $\Rightarrow$ (5) Let C and S be neutrosophic δ-closed subsets in η with $C \subseteq \eta \cap S^c$. Then, $\eta \cap S^c$ is a neutrosophic δ-open set in η containing C. By (3), there is a neutrosophic open set N in η such that $C \subseteq N \subseteq \overline{V}^{\eta}_{\delta} \subseteq \eta \cap S^c$. Since $\overline{N}^{\eta}_{\delta}$ is neutrosophic δ-closed in η and $\eta \cap S^c$ is a neutrosophic δ-open set in η containing $\overline{N}^{\eta}_{\delta}$, there is a neutrosophic open set T in η such that $K \subseteq N \subseteq \overline{N}^{\eta}_{\delta} \subseteq T \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap S^c$. Let $T_1 = N$ and $T_2 = \eta \cap (\overline{T}^{\eta}_{\delta})^c$, then T_1 and T_2 are neutrosophic open sets in η with $C \subseteq T_1$ and $S \subseteq T_2$. Also, $(\overline{T_2}^{\eta}) = \overline{(\eta \cap (\overline{T}^{\eta}_{\delta})^c)}^{\eta} \subseteq \overline{(\eta \cap T^c)}^{\eta} = \eta \cap T^c$. So, $(\overline{T_1}^{\eta}) = \overline{N}^{\eta} \subseteq T \subseteq \eta \cap ((\overline{T_2}^{\eta}))^c$.
(5) $\Rightarrow$ (1) Let C and S be neutrosophic δ-closed subsets in η with $C \subseteq \eta \cap S^c$. Then, by (4), there are neutrosophic open sets T and N in η such that $C \subseteq T$,$S \subseteq N$ and $\overline{T}^{\eta}_{\delta} \subseteq \eta \cap (\overline{N}^{\eta}_{\delta})^c$. Since $K \subseteq T \subseteq \overline{T}^{\eta}_{\delta}$, we have $C \subseteq T = T^{\eta\circ} \subseteq (\overline{T}^{\eta}_{\delta})^{\eta\circ}$. Put $T_1 = (\overline{T}^{\eta}_{\delta})^{\eta\circ}$, then T_1 is neutrosophic δ-open in η and $C \subseteq T_1$. Similarly put $T_2 = (\overline{N}^{\eta}_{\delta})^{\eta\circ}$, then T_2 is neutrosophic δ-open in η and $S \subseteq T_2$. Since $(\overline{T}^{\eta}_{\delta})^{\eta\circ} \subseteq \overline{T}^{\eta}_{\delta} \subseteq \eta \cap (\overline{N}^{\eta}_{\delta})^c \subseteq \eta \cap ((\overline{N}^{\eta}_{\delta})^{\eta\circ})^c$, we have $T_1 \subseteq \eta \cap (T_2)^c$. □

Theorem 20. *Let (U, σ) be a neutrosophic δ-regular space. Suppose that $Q \subseteq U$ and η is neutrosophic regular open in U, where $(\eta, \sigma^{\eta}) = \chi Q$. Then, χQ is neutrosophic δ-regular.*

Proof. Let T be a neutrosophic δ-open set in η and $j_{m,n,p}$ a neutrosophic point in η with $j_{m,n,p} \subseteq T$. Since η is neutrosophic regular open in U, T is also neutrosophic δ-open in U. Since X is neutrosophic δ-regular, there is a neutrosophic δ-open set G of U such that $j_{m,n,p} \subseteq G \subseteq \overline{G}_{\delta} \subseteq T$. Thus, $G \cap \eta$ is a neutrosophic δ-open set in η such that $j_{m,n,p} \subseteq G \cap \eta \subseteq \overline{G}_{\delta} \cap \eta = \overline{(G \cap \eta)}_{\delta}^{\eta} \subseteq T \cap \eta = T$. Hence, η is neutrosophic δ-regular. □

Lemma 4. *Let (U, σ) be a neutrosophic δ-normal space. Suppose that $Q \subseteq U$ and η is neutrosophic regular closed in U, where $(\eta, \sigma^{\eta}) = \chi Q$. If $C \in U^{\eta}$ is neutrosophic regular closed in η, then C is also neutrosophic regular closed in U.*

Proof. $C \subseteq \overline{C}_{\delta} = \overline{(C \cap \eta)}_{\delta} \subseteq \overline{C}_{\delta} \cap \overline{\eta}_{\delta} = \overline{C}_{\delta} \cap \eta = C$. □

Theorem 21. *Let (U, σ) be a neutrosophic δ-normal space. Suppose that $Q \subseteq U$ and η is neutrosophic regular open in U, where $(\eta, \sigma^{\eta}) = \chi Q$. Then, χQ is neutrosophic δ-normal.*

Proof. Let U be a neutrosophic δ-normal space and η be a neutrosophic δ-open subspace of U. Let C, S be neutrosophic δ-closed subsets in η with $C \subseteq \eta \cap S^c$. Since η is neutrosophic δ-closed in U, C and S are also neutrosophic δ-closed in U with $C \subseteq S^c$. Since U is neutrosophic δ-normal, there exist neutrosophic δ-open sets T_1 and T_2 in U with $C \subseteq T_1$, $S \subseteq T_2$ and $T_1 \subseteq (T_2)^c$. So, there exist neutrosophic δ-open sets $\eta \cap T_1$ and $\eta \cap T_2$ in η with $C \subseteq \eta \cap T_1$, $S \subseteq \eta \cap T_2$ and $\eta \cap T_2 \subseteq \eta \cap (\eta \cap T_2)^c$ in the subspace η. □

7. Conclusions

In this study, the concept of δ-separation axioms, which have been defined in different ways in general topological spaces and various types of topological spaces, was extended to neutrosophic topological spaces. The relationships between these newly introduced separation axioms, which are defined for the first time in this paper, were analyzed and clarified with the aid of a diagram. Additionally, the properties of these novel separation axioms in neutrosophic subspaces were investigated. This work is anticipated to lay the groundwork for further exploration in the field of mathematics and contribute to human life.

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