Triangulation of a Triangle with Triangles having Equal Inscribed Circles

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In this article, we solve the following problem:

Any triangle can be divided by a cevian in two triangles that have congruent inscribed circles.

Solution

We consider a given triangle ABC and we show that there is a point D on the side (BC) so that the inscribed circles in the triangles ABD, ACD are congruent. If ABC is an isosceles triangle $(AB = AC)$, where D is the middle of the base (BC), we assume that ABC is a non-isosceles triangle.

We note I_1, I_2 the centers of the inscribed congruent circles; obviously, I_1I_2 is parallel to the BC (1).

We observe that $m(\sphericalangle I_1AI_2) = \frac{1}{2}m(\hat{A})$ (2).

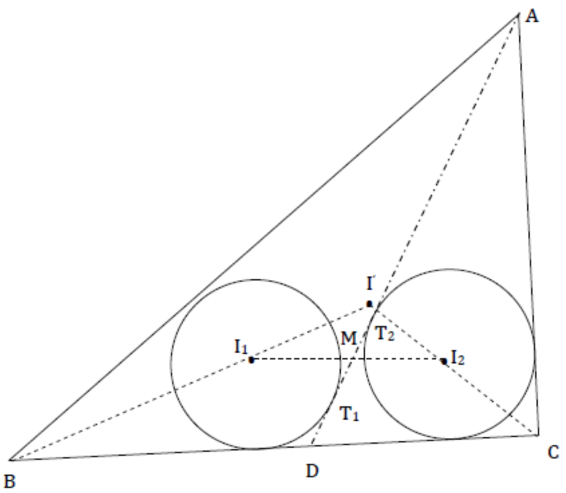

Fig. 1

If T_1, T_2 are contacts with the circles of the cevian AD, we have $\Delta\, I_1T_1M \equiv \Delta\, I_2T_2M$; let M be the intersection of I_1I_2 with AD, see *Fig. 1*.

From this congruence, it is obvious that $(I_1M) \equiv (I_2M)$ (3).

Let I be the center of the circle inscribed in the triangle ABC; we prove that: **AI is a simedian in the triangle I_1AI_2** (4).

Indeed, noting $\alpha = m\left(\widehat{BAI_1}\right)$, it follows that $m(\sphericalangle I_1AM) = \alpha$. From $\sphericalangle I_1AI_2 = \sphericalangle BAI$, it follows that $\sphericalangle BAI_1 \equiv \sphericalangle IAI_2$, therefore $\sphericalangle I_1AM \equiv \sphericalangle IAI_2$, indicating that AM and AI are isogonal cevians in the triangle I_1AI_2. Since in this triangle AM is a median, it follows that AI is a bimedian.

Now, we show how we build the point D, using the conditions (1) – (4), and then we prove that this construction satisfies the enunciation requirements.

Building the point *D*

1^0: We build the circumscribed circle of the given triangle ABC; we build the bisector of the angle BAC and denote by P its intersection with the circumscribed circle (see *Fig. 2*).

2^0: We build the perpendicular on C to CP and (BC) side mediator; we denote O_1 the intersection of these lines.

3^0: We build the circle $\mathrm{C}(O_1; O_1C)$ and denote A' the intersection of this circle with the bisector AI (A' is on the same side of the line BC as A).

4^0: We build through A the parallel to $A'O_1$ and we denote it IO_1.

5^0: We build the circle $\mathrm{C}(O_1'; O_1'A)$ and we denote I_1, I_2 its intersections with BI, and CI respectively.

6^0: We build the middle M of the segment $(I_1 I_2)$ and denote by D the intersection of the lines AM and BC.

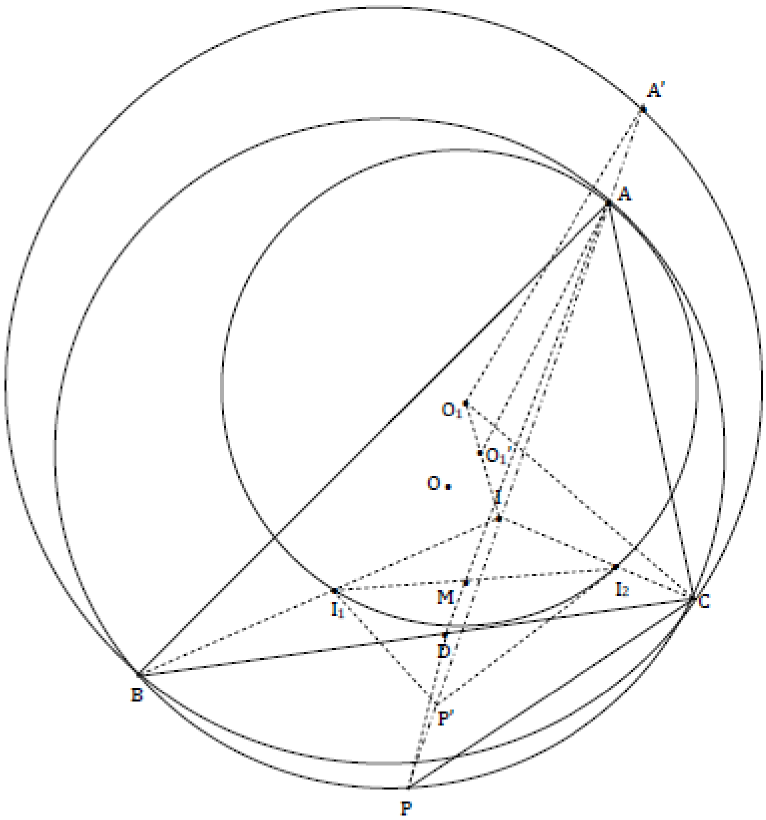

Fig. 2

Proof

The point P is the middle of the arc $\breve{BC}$, then $m(\widehat{PCB}) = \frac{1}{2} m(\hat{A})$.

The circle $\mathcal{C}(O_1;\ O_1C)$ contains the arc from which points the segment (BC) „is shown" under angle measurement $\frac{1}{2} m(\hat{A})$.

The circle $\mathcal{C}(O_1';\ O_1'A)$ is homothetical to the circle $\mathcal{C}(O_1;\ O_1C)$ by the homothety of center I and by the report $\frac{IA'}{IA}$; therefore, it follows that I_1I_2 will be parallel to the BC, and from the points of circle $\mathcal{C}(O_1';\ O_1'A)$ of the same side of BC as A, the segment (I_1I_2) „is shown" at an angle of measure $\frac{1}{2} m(\hat{A})$. Since the tangents taken in B and C to the circle $\mathcal{C}(O_1;\ O_1C)$

intersect in P, on the bisector AI, as from a property of simedians, we get that $A'I$ is a simedian in the triangle $A'BC$. Due to the homothetical properties, it follows also that the tangents in the points I_1, I_2 to the circle $\complement(O_1'; O_1'A)$ intersect in a point P' located on AI, *i.e.* AP' contains the simedian (AS) of the triangle I_1AI_2, noted $\{S\} = AP' \cap I_1I_2$. In the triangle I_1AI_2, AM is a median, and AS is simedian, therefore $\sphericalangle I_1AM \equiv I_2AS$; on the other hand, $\sphericalangle BAS \equiv \sphericalangle I_1AI_2$; it follows that $\sphericalangle BAI_1 \equiv I_2AS$, and more: $\sphericalangle I_1AM \equiv BAI_1$, which shows that AI_1 is a bisector in the triangle BAD; plus, I_1, being located on the bisector of the angle B, it follows that this point is the center of the circle inscribed in the triangle BAD. Analogous considerations lead to the conclusion that I_2 is the center of the circle inscribed in the triangle ACD. Because I_1I_2 is parallel to BC, it follows that the rays of the circles inscribed in the triangles ABD and ACD are equal.

Discussion

The circles $\complement(O_1; O_1A')$, $\complement(O_1'; O_1'A)$ are unique; also, the triangle I_1AI_2 is unique; therefore, determined as before, the point D is unique.

Remark

At the beginning of the *Proof*, we assumed that ABC is a non-isosceles triangle with the stated property. There exists such triangles; we can construct such a triangle starting "backwards". We consider two given congruent external circles and, by tangent constructions, we highlight the ABC triangle.

Open problem

Given a scalene triangle ABC, could it be triangulated by the cevians AD, AE, with D, E belonging to (BC), so that the inscribed circles in the triangles ABD, DAE and the EAC to be congruent?