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# Two applications of the concepts of pole and polar with respect to a circle

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## Abstract

In this article, we present in a unified form the theoretical results regarding the concepts of pole and polar with respect to a circle, and as applications of these, we offer a demonstration of Newton's theorem related to the circumscribed quadrilateral and a demonstration of a theorem usually derived as a particular case of Pascal's theorem related to the inscribed hexagon.

**Keywords:** Pole, Polar, Circle Geometry, Conjugate Points, Conjugate Diameters, Orthogonality, Duality, Projective Geometry, Inversion, Geometric Transformations.

## 1 | Introduction

In the realm of geometry, few concepts are as elegant and versatile as the notions of pole and polar with respect to a circle. These concepts, rooted in projective geometry, have found profound applications across various mathematical disciplines and beyond. In this paper, we explore two fascinating applications that highlight the beauty and utility of these concepts.

The concept of pole and polar is deeply intertwined with the idea of duality in projective geometry. Duality, a fundamental principle, establishes a correspondence between points and lines in a plane, wherein statements about points translate to statements about lines and vice versa. This duality underpins the power of the pole-polar relationship, allowing us to seamlessly transition between geometric entities [1].

Our exploration begins with a discussion on the basic definitions of pole and polar with respect to a circle. We then delve into our first application, which demonstrates how these concepts provide a geometric framework for understanding the orthogonality of circles. This application not only reveals the elegance of pole and polar but also showcases their utility in solving practical problems, such as determining the common tangents to two circles [2].

Moving forward, we shift our focus to the second application, which explores the use of poles and polars in constructing inversive transformations. Inversion, a powerful tool in geometry and complex analysis,

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can be elegantly described and analyzed using the concepts of pole and polar. Through this application, we highlight the deep connections between seemingly disparate areas of mathematics [3].

This paper aims to illuminate the beauty and versatility of the concepts of pole and polar with respect to a circle. By showcasing their applications in the context of orthogonality of circles and inversive transformations, we hope to inspire further exploration and appreciation of these fundamental ideas.

# 2 | Theoretical Concepts regarding the notions of pole and polar with respect to a circle

### Definition 1

Let $\Gamma(0,r)$ be a given circle, and $P$ an external point to the circle. If a secant through $P$ intersects the circle at points $M$ and $N$, then the point $Q$ that lies on the chord $MN$ and satisfies the condition $\frac{PM}{PN}=\frac{QM}{QN}$ (1) is said to be the harmonic conjugate of point $P$ with respect to points $M$ and $N$ or that it is the harmonic conjugate of point $P$ with respect to the circle. It can also be said that point $Q$ is the harmonic conjugate of $P$ with respect to the circle if condition (1) is met.

### Definition 2

If $P$ is a point in the plane of the circle $\Gamma(O,r)$, $P \neq O$ and $Q$ is its conjugate with respect to the circle, then the line perpendicular to $OP$ drawn from $Q$ is said to be the polar of point $P$ with respect to the circle [4-5].

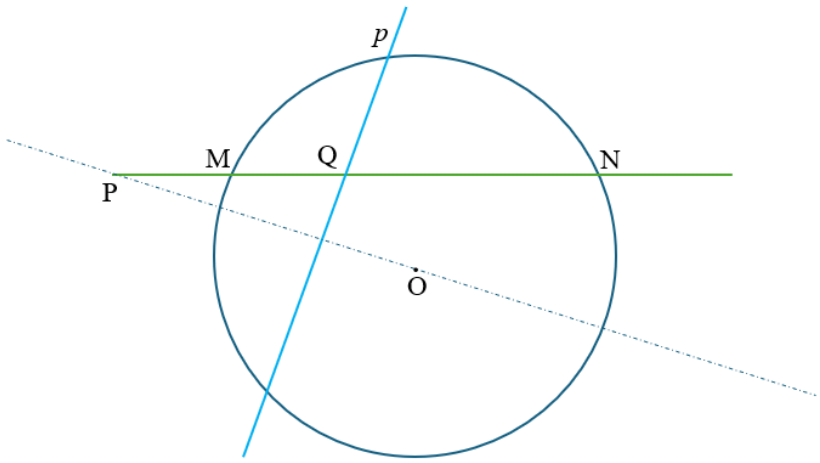


Figure 1. Polar of point P with respect to circle Γ(O,r).

### Observation 1

In Figure 1, the line p is the polar of P with respect to the circle.

### Remark 1

a) If $P \in \Gamma(0,r)$, the polar of $P$ is the tangent drawn at $P$ to the circle.

b) If $P$ is an interior point of the circle, its polar is a line outside the circle.

## Proposition 1

If $p$ is the polar of point $P$ outside the circle $\Gamma(O, r)$ with respect to it, and $P'$ is the intersection of the polar with the line $OP$, then: $OP \cdot OP' = r^2$.

## Proof

Let $Q$ be the harmonic conjugate of $P$ with respect to the circle, then: $QP' \perp OP$ and $\frac{PM}{PN} = \frac{QM}{QN}$ (1) – see Figure 2.

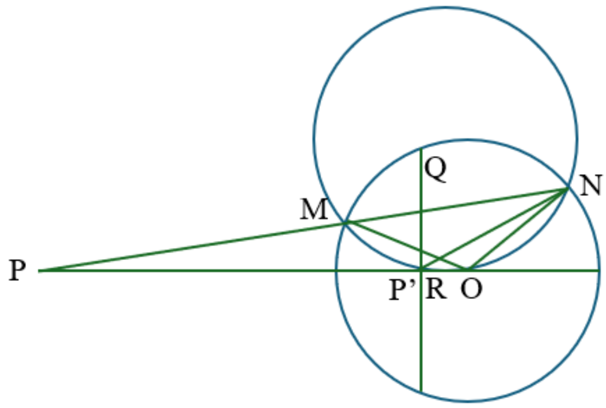


Figure 2: Illustration for Proposition 1. Point P' is the intersection of the polar p with line OP.

If $R$ is the point in which the circumscribed circle of the triangle $MON$ re-intersects the line $OP$, because $\sphericalangle OMP = \sphericalangle ORM$ (have the same supplement), $\sphericalangle OMN = \sphericalangle ONM$ and $\sphericalangle MOP = \sphericalangle ROM$, it follows that $\Delta OMP \sim \Delta ORM$.

From here, we note that $\frac{OM}{OR} = \frac{OP}{OM}$, meaning that $OP \cdot OR = r^2$ (2).

From the inscriptible quadrilateral $NORM$ we have that $\sphericalangle MRP = \sphericalangle ONM$ (3).

But $ON = OM$, therefore $\sphericalangle ONM = \sphericalangle OMN$ (4).

We also have $\sphericalangle NRO = \sphericalangle OMN$ (5).

The relations (3), (4) and (5) lead to $\sphericalangle MRP = \sphericalangle NRO$, which shows that $RP$ is the external angle bisector $\sphericalangle MRN$.

If we denote by $Q'$ the foot of the internal angle bisector $\sphericalangle MRN$, we obtain that $\frac{Q'M}{Q'N} = \frac{RM}{RN}$.

But $\frac{PM}{PN} = \frac{RM}{RN}$ (angle bisector theorem), we also have $\frac{PM}{PN} = \frac{Q'M}{Q'N}$ (6).

The relations (6) and (1) show that $Q' = Q$, therefore $QR \perp RP$.

But since $QP' \perp OP$, from the uniqueness of the perpendicular drawn from a point to a line, it follows that points $R$ and $P'$ coincide. Then relation (2) becomes $OP \cdot OP' = r^2$, which is what needed to be proven.

## Remark 2

a) From the proof of this property, it follows that the harmonic conjugate points of $P$ with respect to the circle $\Gamma(O,r)$ belong to the polar of point $P$.

b) If $PU$ and $PV$ are the tangents drawn from $P$ to the circle $\Gamma(O,r)$ and $UV \cap OP = \{P'\}$, then from the cathetus theorem applied in the right triangles $POU$ and $POV$, we have $OU^2 = OV^2 = r^2 = OP \cdot OP'$, so points $U$ and $V$ belong to the polar of $P$ with respect to the circle.

This remark justifies the construction of the polar $p$ of $P$ with respect to circle $\Gamma(O,r)$ as the line determined by the points of contact of the tangents drawn from $P$ to the circle [6].

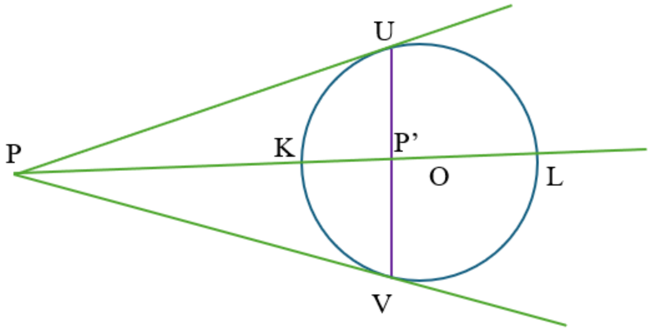


Figure 3. Illustration showing the point P' as the harmonic conjugate of P with respect to the circle Γ(O,r).

c) The point P′ in Figure 3 is the harmonic conjugate of P with respect to the circle.

Indeed, if we denote by $K$ and $L$ the intersection of the secant $PO$ with the circle $\Gamma$ and $OP = d$, $OP' = x$, therefore $PL = d + r$, $PK = d - r$, $P'K = r - x$.

We know that $OP \cdot OP' = r^2$, therefore $d \cdot x = r^2$.

$$\frac{PK}{PL} = \frac{d-r}{d+r}, \frac{P'K}{P'L} = \frac{r-x}{r+x}.$$

$$\frac{PK}{PL} = \frac{P'K}{P'L} \Leftrightarrow \frac{d-r}{d+r} = \frac{r-x}{r+x} \Leftrightarrow d \cdot x = r^2.$$

d) From the demonstrated results, it follows that the polar of a point contains the harmonic conjugates of that point with respect to the circle.

## Theorem 1 (Characterization of the Polar)

The point $M$ belongs to the polar of point $P$ with respect to circle $\Gamma(O,r)$ if and only if $MO^2 - MP^2 = 2r^2 - OP^2$ (1).

## Proof

We denote by $p$ the polar of $P$ with respect to $\Gamma$ (see *Figure 4*) and let $P'$ be the projection of $M$ pe $OP$. Then $MO^2 - MP^2 = P'O^2 - P'P^2 = OU^2 - UP^2 = r^2 - (OP^2 - r^2) = 2r^2 - OP^2$. $U$ is the point of contact between $\Gamma(O,r)$ and the tangent drawn from $P$.

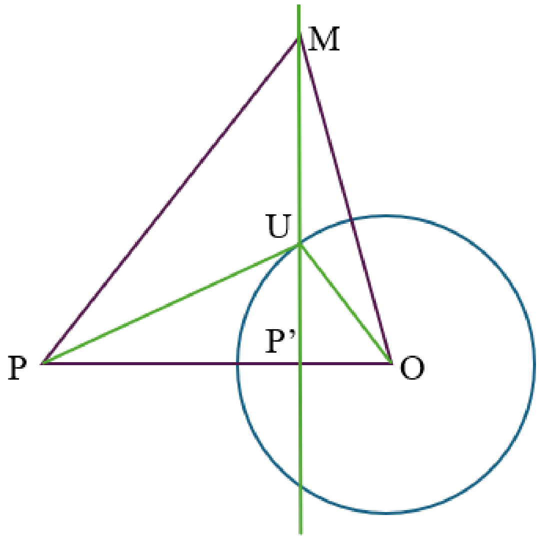


Figure 4: Illustration showing the polar p of point P with respect to circle Γ.

## Reciprocal

If $M$ is a point in the plane of the circle $\Gamma(O,r)$ and it satisfies the relation (1), denoting by $M' = pr_{OP}^{M}$, we have: $M'O^2 - M'P^2 = (MO^2 - M'M^2) - (MP^2 - M'M^2) = MO^2 - MP^2 = 2r^2 - OP^2$ (2).

On the other hand, $P'O^2 - P'P^2 = 2r^2 - OP^2$ (3).

The relations (2) and (3) show that $M' = P'$, therefore $M$ belongs to the polar $p$ of $P$ with respect to Γ.

## Theorem 2 (Philippe de La Hire)

If $P$, $Q$, $R$ are points in the plane of circle $\Gamma(O,r)$, and $p$, $q$, $r$ are their respective polars, then:

1. $P \in q \Leftrightarrow Q \in p$ (If a point belongs to the polar of another point, then this second point belongs to the polar of the first point).

2. If $R \in p \cap q \Leftrightarrow PQ = r$ (The pole of a line determined by two points is the intersection of the polars of these points with respect to a given circle).

## Proof

From *Theorem 1*, if $P \in q \Leftrightarrow PO^2 - PQ^2 = 2r^2 - OQ^2$. This relationship is equivalent to $QO^2 - PO^2 = 2r^2 - OP^2 \Leftrightarrow Q \in P$.

From $R \in p \cap q$ and (1), it follows that $P \in r$ and $Q \in r$, therefore $r = PQ$.

## Remark 3

From this theorem, we note:

a) The polar of a point, which is the intersection of two lines, is determined by the poles of these lines with respect to a given circle.

b) The poles of concurrent lines are collinear points, and conversely: the polars of collinear points are concurrent lines.

## Theorem 3

Let $ABCD$ be a convex quadrilateral inscribed in the circle $\Gamma$. Let $P$ be the intersection of the lines $AB$ and $AC$, $Q$ the intersection of the lines $BC$ and $AD$, and $\{R\} = AC \cap BD$. Let $U$ and $V$ be the points of contact between $\Gamma$ and the tangents drawn from $P$, and $K$ and $L$ the intersections of the tangents drawn at $A$ and $B$, respectively at $C$ and $D$, with $\Gamma$.

Then the points $Q$, $K$, $U$, $R$, $V$ and $L$ belong to the polar of point $P$ with respect to the circle.

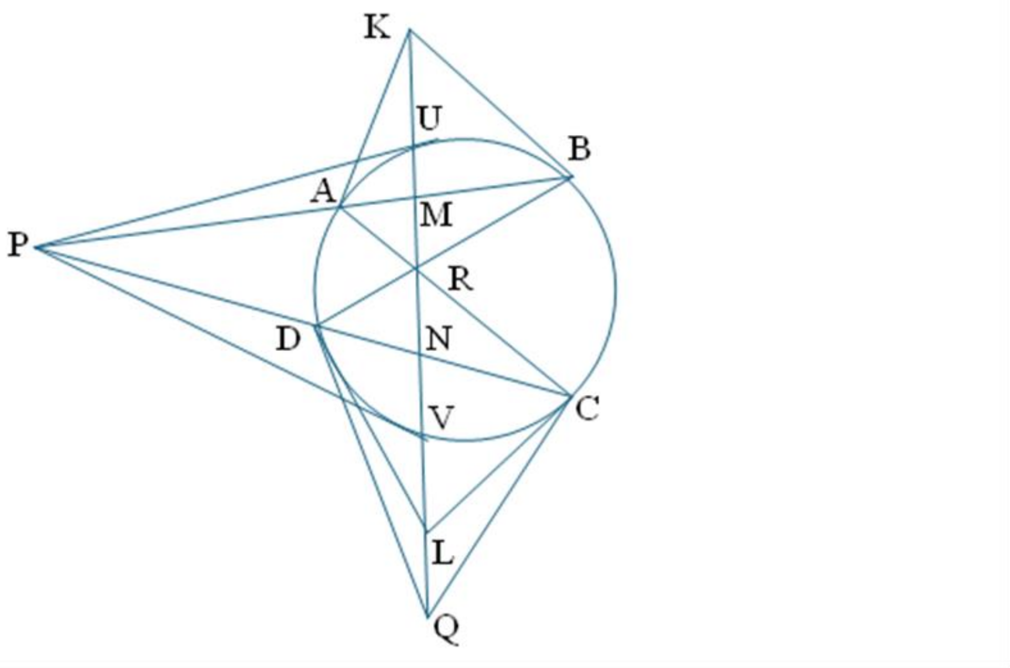


Figure 5: Illustration of the quadrilateral ABCD inscribed in circle Γ.

## Proof

We denote by $\{M\} = QR \cap AB$ and $\{N\} = QR \cap CD$ (see *Figure 5*). We prove that $M$ is the conjugate of $P$ with respect to $A$ and $B$ and that $N$ is the conjugate of $P$ with respect to $D$ and $C$.

It is sufficient to show that $\frac{PA}{PB} = \frac{MA}{MB}$ . We apply Ceva's theorem and Menelaus' theorem in triangle $QAB$ for the transversal $P - D - C$. We have:

$\frac{MA}{MB} \cdot \frac{CB}{QC} \cdot \frac{DQ}{DA} = 1$, (1)

$\frac{PA}{PB} \cdot \frac{BC}{QC} \cdot \frac{DQ}{DA} = 1$. (2)

From the relations (1) and (2) we obtain that $\frac{PA}{PB} = \frac{NA}{MB}$.

Similarly, it can be shown that $\frac{PD}{PC} = \frac{ND}{NC}$.

The points $M$ and $N$ belong to the polar of the point $P$, so this is the line $QR$.

From Theorem 2, we know that the polar of point $\{P\} = AB \cap CD$ is the line determined by the poles of lines AB and CD, namely K and L. Therefore, points K and L are also on the polar of P with respect to circle Γ. On the other hand, we have seen that the polar of P is determined by the points of contact of the tangents drawn from P to Γ, which are points U and V. Since the polar of a point with respect to a circle is a unique line, it follows that the aforementioned points are collinear.

### Observation 2

From *Theorem 2*, it follows that the polar of point $Q$ is line $PR$, and the polar of point $R$ is line $PQ$.

## 3 | Applications of the concepts of pole and polar with respect to a circle

### Theorem 4 (I. Newton)

If ABCD is a quadrilateral such that its sides AB, BC, CD and DA are tangent at M, N, P, and Q, respectively to a circle Γ, then AC, BD, MP and NQ are concurrent.

### Proof

Let $AB \cap CD = \{R\}$, $AD \cap BC = \{S\}$, $AC \cap BD = \{I\}$ – see *Figure 6*. Then the polar of $R$ is $SI$ and also the polar of $R$ is $PM$. From the uniqueness of the polar of a point with respect to a circle, we get that the points $S$, $P$, $I$, $M$ are collinear (1). Similarly, the polar of $S$ is $RI$ and the polar of $S$ is $QM$, resulting in the points $R$, $Q$, $I$, $N$ being collinear (2). From relations (1) and (2), we find that the lines $AC$, $BD$, $PM$ and $NQ$ are concurent.

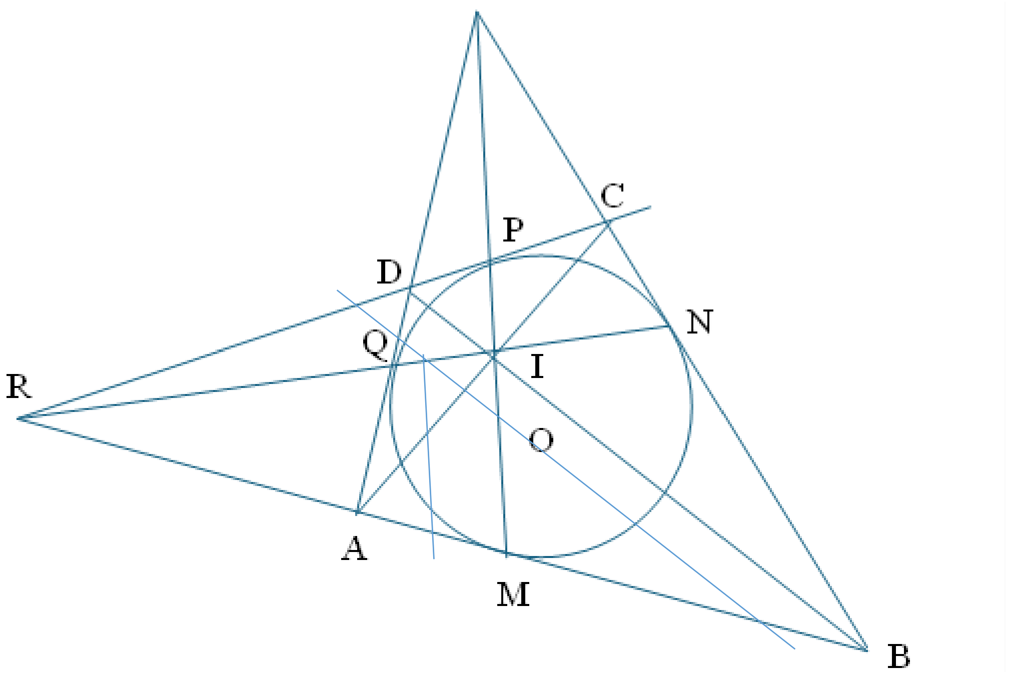


Figure 6. Illustration demonstrating Theorem 4 by I. Newton.

## Theorem 5

Let $ABCD$ be a quadrilateral inscribed in the circle $\Gamma(O, r)$, with $\{I\} = AC \cap BD$. Let $\{R\} = AB \cap CD$, $\{S\} = AD \cap BC$, $Q$ be the intersection of the tangents drawn at $A$ and $C$ to the circle $\Gamma$, and $P$ be the intersection of the tangents drawn at $B$ and $D$ to the circle $\Gamma$. Then the points $P$, $Q$, $R$, $S$ are collinear.

## Proof

The polar of point $P$ is $BD$. The polar of point $Q$ is $AC$. The polar of point $S$ is line $RI$, and the polar of point $R$ is $SI$. Since the polars $AC$, $BD$, $RI$, and $SI$ are concurrent lines at point $I$, then, according to *Theorem 2*, it follows that their poles, that is, the points $Q$, $P$, $S$ and $R$, are collinear points.

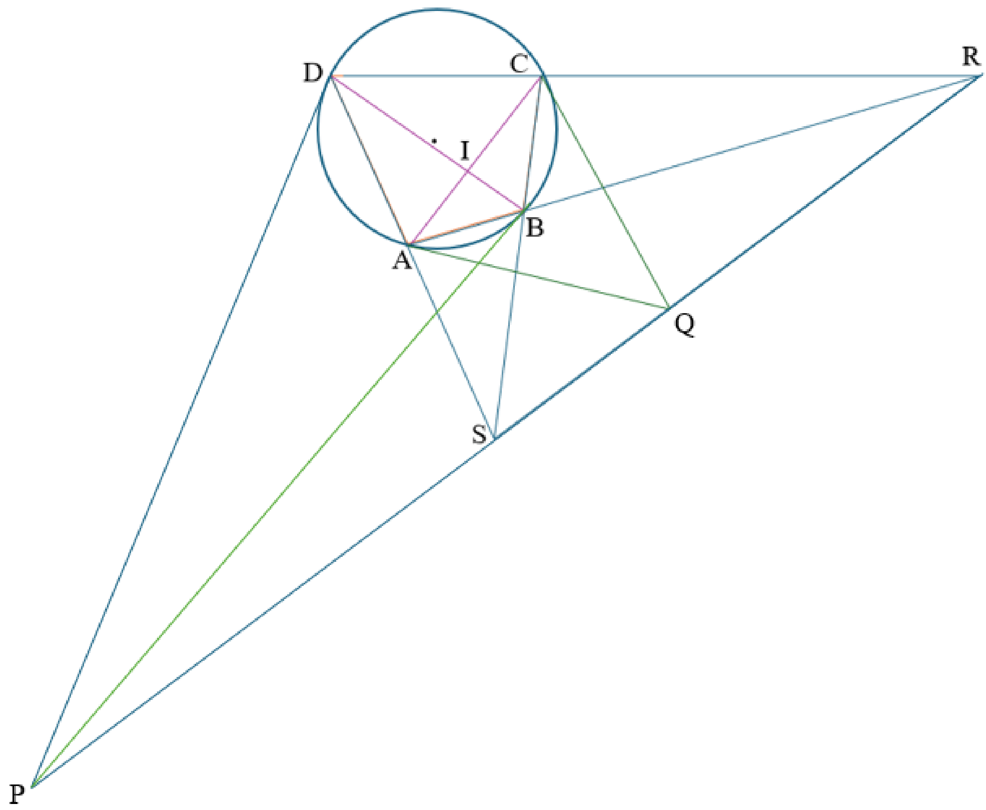


Figure 7. Illustration demonstrating the collinearity of points P, Q, R, and S in quadrilateral ABCD inscribed in circle Γ(O,r).

## 4 | Conclusion

The concepts of pole and polar with respect to a circle emerge as powerful tools in the study of geometry, offering elegant solutions to a diverse array of problems. Through our exploration of various applications, from determining the orthogonality of circles to establishing the concurrence of lines in inscribed quadrilaterals, we have witnessed the versatility and beauty of these concepts. Whether applied to classic theorems or contemporary problems, pole and polar relationships illuminate the inherent structure and symmetry within geometric configurations. Moreover, their connections to harmonic conjugates, inversive transformations, and projective geometry underscore their significance beyond the realm of pure mathematics, extending their utility to fields such as physics, engineering, and computer science. As we conclude this endeavor, we are reminded of the enduring relevance and timeless elegance of these fundamental concepts, inviting further inquiry and exploration into the depths of geometric theory and its myriad applications.

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### Author Contribution

All authors contributed equally to this work.

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## Conflicts of Interest

The authors declare that there is no conflict of interest in the research.

## Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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