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Unit disk graphs in Fuzzy and Neutrosophic Graphs

Takaaki Fujita[1*] and **Florentin Smarandache**[2]

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan; t171d603@gunma-u.ac.jp.,

[2] Department of Mathematics & Sciences, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu.

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Abstract

This paper investigates unit disk graphs within the frameworks of fuzzy and neutrosophic graphs, expanding upon the traditional study of intersection graphs. Unit disk graphs are well-known for their applications in wireless networks, where vertices represent equal-sized disks, and edges denote overlapping disks. Fuzzy and neutrosophic graphs, which incorporate uncertainty through degrees of membership, are explored in the context of unit disk graphs. We provide a detailed analysis of these intersections, aiming to advance research in fuzzy and neutrosophic graph theory.

Keywords: Neutrosophic graph, Unit disk graphs, Fuzzy graph, Intersection graphs

1 | Introduction

1.1 | Graph Theory and real-world elements

Graph theory is a fundamental branch of mathematics that explores networks consisting of nodes (vertices) and connections (edges). It is vital for analyzing the paths, structures, and properties of these networks [30]. A key strength of graph theory is its ability to visually and conceptually model relationships between real-world elements, making it an indispensable tool across numerous disciplines [49, 86, 37, 8].

1.2 | Intersection graph and Unit disk graph

One well-known example of these studies is the intersection graph. An intersection graph represents sets where vertices correspond to the sets, and edges exist between vertices if the corresponding sets intersect [75, 45, 94]. Examples of intersection graphs include interval graphs[40], proper interval graphs[52], weighted interval graphs[15], unit disk graphs[27], quasi-unit disk graphs[25, 54], Weak Unit Disk Graphs[5, 4], circular arc graphs[44, 109, 51], proper circular arc graphs[110, 105, 91], and polygon-circle graphs[67].

Corresponding Author: t171d603@gunma-u.ac.jp

In this paper, we focus on the study of unit disk graphs [27]. A unit disk graph is a type of graph where the vertices represent equal-sized disks in the plane, and edges exist between disks that overlap. One of the most well-known applications of unit disk graphs is in wireless networks, particularly in ad hoc wireless communication networks [68]. These graphs are also widely used in various wireless networking applications [25, 56, 76]. Furthermore, many computational problems, such as the maximum independent set problem, the graph coloring problem, and the minimum dominating set problem, can be efficiently solved using unit disk graphs [71, 72, 62].

1.3 | Fuzzy Graphs and Neutrosophic Graphs

To better represent uncertainties in real-world scenarios, various types of graphs such as fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been introduced. These graphs incorporate a relationship function that expresses the degree of uncertainty associated with each vertex and edge. This approach has led to extensive research into both the theoretical properties of these graphs and their practical applications.

A fuzzy graph assigns a membership value between 0 and 1 to each vertex and edge, representing the degree of uncertainty or imprecision. Essentially, a fuzzy graph is a graphical representation of a fuzzy set [115, 114]. Fuzzy graphs have been widely applied in various domains such as social networks, decision-making, and transportation systems, where relationships are not precisely defined or involve uncertainty [90, 77]. Due to their wide applicability, fuzzy graphs have attracted significant attention in research.

More recently, neutrosophic graphs [103, 101, 50, 20, 2] have emerged within the framework of neutrosophic set theory [6, 104]. Neutrosophic logic extends classical and fuzzy logic by incorporating three distinct degrees: truth, indeterminacy, and falsity, making it a more flexible tool for handling uncertainty.

Building upon these concepts, the Turiyam Neutrosophic graph was introduced as an extension of neutrosophic and fuzzy graphs. In a Turiyam Neutrosophic graph, each vertex and edge is assigned four attributes: truth, indeterminacy, falsity, and a liberal state, further expanding the framework established by neutrosophic and fuzzy graphs [41]. In addition, plithogenic graphs have emerged as a more generalized form, and are currently a subject of active research [99, 99].

Despite significant advancements in the study of fuzzy and neutrosophic graphs, as well as their intersection variants (such as fuzzy intersection graphs[93, 74, 89, 28] and neutrosophic intersection graphs[18]), there has been relatively little exploration of unit disk graphs in the context of fuzzy, neutrosophic, and plithogenic graphs.

1.4 | Our Contribution

While the study of fuzzy and neutrosophic graphs, along with various other graph structures, is highly important, these areas remain far from fully explored. In this paper, we focus on a comprehensive investigation of unit disk graphs within the frameworks of fuzzy and neutrosophic graphs. We hope that our research will contribute to the ongoing development of graph theory and open new pathways for further advancements in these areas.

2 | Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, neutrosophic graphs, and plithogenic graphs.

Additionally, please note that this paper may also incorporate concepts from set theory alongside graph theory. For a more comprehensive understanding of set theory, you may refer to the relevant surveys or notes [60].

2.1 | Basic Concepts

2.1.1 | Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [30].

Definition 1 (Graph). [30] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Degree). [30] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 3 (Subgraph). [30] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Definition 4 (Connected graph). A graph $G = (V, E)$ is said to be a **connected graph** if for any two distinct vertices $u, v \in V$, there exists a path in G that connects u and v. In other words, every pair of vertices in the graph is reachable from each other, meaning there is a sequence of edges that allows traversal between any two vertices.

Mathematically, for all $u, v \in V$, there exists a sequence of vertices $v_1 = u, v_2, \ldots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.

Definition 5 (Induced subgraph). Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. For a subset $V' \subseteq V$, the *induced subgraph* $G[V']$ is the graph whose vertex set is V' and whose edge set consists of all edges from E that have both endpoints in V'. Formally, the induced subgraph $G[V'] = (V', E')$ is defined as follows:

$$E' = \{(u, v) \in E \mid u \in V', v \in V'\}.$$

In other words, $G[V']$ is the subgraph of G that contains all vertices in V' and all edges from G whose endpoints are both in V'.

Example 6 (Induced Subgraph). Consider a graph $G = (V, E)$ where the vertex set is

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

and the edge set is

$$E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}.$$

Let us select a subset of vertices $V' = \{v_1, v_2, v_3\}$. The induced subgraph $G[V']$ consists of the vertices v_1, v_2, v_3 and all edges in G that connect these vertices. The edges in G that have both endpoints in V' are $(v_1, v_2), (v_1, v_3), (v_2, v_3)$.

Thus, the induced subgraph $G[V']$ is the graph:

- Vertex set: $V' = \{v_1, v_2, v_3\}$
- Edge set: $E' = \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}$

In this case, the induced subgraph $G[V']$ forms a triangle with vertices v_1, v_2, v_3 and the edges connecting each pair of these vertices.

Definition 7 (Complete Graph). A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.

- Each vertex has degree $\deg(v) = n - 1$ for all $v \in V$.

Example 8 (Examples of Complete Graphs)**.** The concept of a complete graph is best understood through specific examples:

- The complete graph K_1 consists of a single vertex with no edges.
- The complete graph K_2 has two vertices, v_1 and v_2, and a single edge connecting them, $\{v_1, v_2\}$.
- The complete graph K_3 consists of three vertices, v_1, v_2, and v_3, with edges $\{v_1, v_2\}, \{v_2, v_3\}$, and $\{v_1, v_3\}$. This forms a triangle.
- The complete graph K_4 has four vertices, with every possible pair of vertices connected by an edge. The edges are $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}$, and $\{v_3, v_4\}$. This graph forms a tetrahedron when represented in three dimensions.
- The complete graph K_5 includes five vertices, with each pair of vertices connected by a unique edge. The total number of edges is $\binom{5}{2} = 10$, making it a highly connected structure.

In general, for a complete graph K_n with n vertices, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, and each vertex has a degree of $n - 1$.

Definition 9 (Bipartite Graph)**.** A *bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- Every edge in E connects a vertex from V_1 to a vertex from V_2. In other words, there are no edges connecting two vertices within the same subset V_1 or V_2.

Formally, $G = (V, E)$ is bipartite if there exists a partition (V_1, V_2) such that for every edge $e = \{u, v\} \in E$, either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

A graph G is bipartite if and only if it contains no odd-length cycles.

Definition 10 (Complete Bipartite Graph)**.** (cf.[34]) A *complete bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- There is an edge between every vertex in V_1 and every vertex in V_2.
- There are no edges between vertices within the same subset V_1 or V_2.

The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. It has the following properties:

- The number of vertices is $|V| = m + n$.
- The number of edges is $|E| = m \times n$.
- Each vertex in V_1 has degree n, and each vertex in V_2 has degree m.

2.1.2 | Forbidden Graph

In the study of graph classes in graph theory, Forbidden Graphs have been extensively researched [24, 70, 80]. In simple terms, Forbidden Graphs are used to identify which subgraph structures must not be contained in the graphs belonging to a particular class. The concept helps in understanding and defining the structural constraints of a graph class. The formal definition is provided below.

Definition 11 (Forbidden Graph)**.** [24, 70, 80] A *forbidden graph* is a graph H such that a given class of graphs $\mathcal{C}$ does not contain any graph that has H as a subgraph. In other words, a graph G belongs to the class $\mathcal{C}$ if and only if it does not contain any graph from a predetermined set of forbidden graphs as a subgraph.

The forbidden substructures may be of different types, including:

- **Subgraph**: A smaller graph obtained by removing vertices or edges from the original graph.
- **Induced Subgraph**: A smaller graph obtained by choosing a subset of vertices and including all edges with both endpoints in that subset [92, 66, 22].
- **Homeomorphic Subgraph (Topological Minor)**: A smaller graph obtained by collapsing paths of degree-two vertices into single edges.
- **Graph Minor**: A smaller graph obtained by contracting edges or deleting edges and vertices[16, 57, 73].

In this paper, we use the concept of homomorphism in our discussion of the aforementioned forbidden graphs. The definition is provided below.

Definition 12 (homomorphic)**.** (cf.[17, 55, 31, 36, 112]) Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be *homomorphic* if there exists a mapping $\phi : V \to V'$ such that for every edge $(u, v) \in E$, the image $(\phi(u), \phi(v))$ is an edge in E'. In other words, there is a structure-preserving mapping from G to H that maintains the adjacency relationships between vertices.

2.1.3 | Intersection graph

In this paper, we focus on unit disk graphs, which are known as intersection graphs. Intersection graphs have been extensively studied[75, 45, 94]. Variants such as intersection digraphs[29, 116], random intersection graphs[46, 117, 81] , and geometric intersection graphs[35, 48] have also been studied. The definition is provided below[75, 45, 94].

Definition 13 (Intersection graph)**.** [75, 45, 94] A *intersection graph* is a graph that represents the intersection relationships between sets. Formally, let $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a collection of sets. The *intersection graph* $G = (V, E)$ associated with $\mathcal{S}$ is a graph where:

- The vertex set V corresponds to the sets in $\mathcal{S}$, i.e., $V = \{v_1, v_2, \dots, v_n\}$, where each vertex v_i represents the set $S_i \in \mathcal{S}$.
- There is an edge $(v_i, v_j) \in E$ if and only if the corresponding sets S_i and S_j have a non-empty intersection, i.e., $S_i \cap S_j \neq \emptyset$.

The following is well-known regarding the relationship between intersection graphs and general graphs.

Theorem 14. *Every graph can be represented as an intersection graph.*

2.1.4 | Euclidean space and Euclidean distance

In unit disk graphs, the concepts of Euclidean space and Euclidean distance are commonly employed. In geometric graph theory[82, 83], Euclidean space plays a central role. A Euclidean graph is a type of graph where the vertices correspond to points in the plane, and each edge is assigned a length equal to the Euclidean distance between its endpoints[64, 58]. The relevant definitions are provided below.

Definition 15 (n-dimensional Euclidean Space)**.** (cf.[87]) The *n-dimensional Euclidean space*, denoted as $\mathbb{R}^n$, is the set of all ordered n-tuples of real numbers. Formally, it is defined as:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for all } i = 1, 2, \dots, n\}.$$

Each element $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is called a point (or vector) in n-dimensional Euclidean space, where $x_1, x_2, \dots, x_n$ are the coordinates of the point, and $\mathbb{R}$ denotes the set of real numbers.

Definition 16 (Euclidean Distance)**.** Let $\mathbf{p} = (p_1, p_2, \dots, p_n)$ and $\mathbf{q} = (q_1, q_2, \dots, q_n)$ be two points in the n-dimensional Euclidean space $\mathbb{R}^n$. The *Euclidean distance* $d(\mathbf{p}, \mathbf{q})$ between $\mathbf{p}$ and $\mathbf{q}$ is defined as:

$$d(\mathbf{p}, \mathbf{q}) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$$

or equivalently,

$$d(\mathbf{p},\mathbf{q}) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}.$$

Example 17. Consider two points in the 2-dimensional Euclidean space, $\mathbb{R}^2$. Let $\mathbf{p} = (1,3)$ and $\mathbf{q} = (4,7)$. The Euclidean distance between these two points is calculated as follows:

$$d(\mathbf{p},\mathbf{q}) = \sqrt{(1-4)^2 + (3-7)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

Thus, the Euclidean distance between $\mathbf{p}$ and $\mathbf{q}$ is 5.

2.2 | Classic class about unit disk graph

We introduce several representative classical graph classes. The Unit Disk Graph is a well-known example of an intersection graph, which has been extensively studied in graph theory and related fields[27, 11, 78, 61, 111, 13]. These graph classes have also been explored in fuzzy, neutrosophic, and plithogenic contexts.

Definition 18. A *Unit Disk Graph (UDG)* is a graph $G = (V,E)$ where each vertex corresponds to a disk of equal radius (typically 1) in the plane, and there is an edge between two vertices if and only if their corresponding disks intersect. Formally, for each pair of vertices $u, v \in V$, there exists an edge $(u,v) \in E$ if and only if the Euclidean distance between the centers of the disks corresponding to u and v is at most 1.

Mathematically, if p_u and p_v are the centers of the disks corresponding to vertices u and v, respectively, then:

$$(u,v) \in E \iff \|p_u - p_v\| \leq 1.$$

Unit Disk Graphs are commonly used to model wireless networks and other spatial systems where connections depend on proximity.

The following is an example of Unit Disk Graphs.

Example 19. Consider three vertices $V = \{v_1, v_2, v_3\}$, where each vertex corresponds to a disk of radius 1 in the plane, centered at points $p_1 = (0,0)$, $p_2 = (1,0)$, and $p_3 = (2,0)$, respectively.

Check the distances between the centers.

- The distance between p_1 and p_2 is:
$$\|p_1 - p_2\| = \sqrt{(1-0)^2 + (0-0)^2} = 1.$$
- The distance between p_2 and p_3 is:
$$\|p_2 - p_3\| = \sqrt{(2-1)^2 + (0-0)^2} = 1.$$
- The distance between p_1 and p_3 is:
$$\|p_1 - p_3\| = \sqrt{(2-0)^2 + (0-0)^2} = 2.$$

Define the edges based on the distances.

- Since $\|p_1 - p_2\| = 1$, there is an edge between v_1 and v_2, i.e., $(v_1, v_2) \in E$.
- Since $\|p_2 - p_3\| = 1$, there is an edge between v_2 and v_3, i.e., $(v_2, v_3) \in E$.
- Since $\|p_1 - p_3\| = 2$, which is greater than 1, there is no edge between v_1 and v_3, i.e., $(v_1, v_3) \notin E$.

We consider about Graph description. The resulting graph $G = (V, E)$ has the vertex set $V = \{v_1, v_2, v_3\}$ and the edge set $E = \{(v_1, v_2), (v_2, v_3)\}$. This forms a path graph where v_1 is connected to v_2, and v_2 is connected to v_3, but v_1 and v_3 are not connected.

Thus, the Unit Disk Graph for this example is:

$$G = (\{v_1, v_2, v_3\}, \{(v_1, v_2), (v_2, v_3)\}).$$

The definition of a minimal non-unit disk graph is provided below [96, 11, 32]. This concept is related to the study of forbidden graphs.

Definition 20. [11] A graph $G = (V, E)$ is called a *minimal non-unit disk graph* (minimal-non-UDG) if:

- G is not a unit disk graph, i.e., it cannot be represented as a graph where vertices correspond to points in the plane such that two vertices are adjacent if and only if their corresponding points are at a Euclidean distance of at most 1.
- For every proper induced subgraph $G' = (V', E')$ of G, the subgraph G' is a unit disk graph.

Minimal non-unit disk graphs have been studied in various contexts. For example, the following theorem is well-known[11, 53, 59].

Theorem 21. [53, 59] *The complete bipartite graphs $K_{1,6}$ and $K_{2,3}$ are minimal non-unit disk graphs.*

2.3 | Unit disk graph in Fuzzy Graphs

We focus on the concept of Unit Disk Graphs within the context of Fuzzy Graphs. To begin, the definition of a fuzzy graph is provided below. A fuzzy graph extends traditional graph theory by incorporating the principles of fuzzy sets [114, 23, 113]. Extensive research has been conducted on fuzzy graphs [39, 84, 90].

Definition 22. [90] A *fuzzy graph* $\psi = (V, \sigma, \mu)$ is defined as follows:

- V is a set of vertices.
- $\sigma : V \to [0, 1]$ is a function that assigns a membership degree to each vertex $v \in V$, indicating the degree of membership of v in the fuzzy graph.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation that represents the strength of the connection between each pair of vertices $(u, v) \in V \times V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

In this definition, the following properties hold:

- The fuzzy function μ is symmetric, meaning $\mu(u, v) = \mu(v, u)$ for all $u, v \in V$.
- Additionally, $\mu(v, v) = 0$ for all $v \in V$, meaning that there is no self-loop in the fuzzy graph.

The fuzzy graph ψ allows for the representation of uncertainty in the presence or strength of connections between vertices, making it a valuable tool for modeling complex systems with ambiguous or imprecise relationships.

An example of a fuzzy graph is provided below.

Example 23. (cf.[21]) Consider a fuzzy graph $G = (\sigma, \mu)$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$.

The membership degrees of the vertices are as follows:

$$\sigma(v_1) = 0.1, \quad \sigma(v_2) = 0.3, \quad \sigma(v_3) = 0.2, \quad \sigma(v_4) = 0.4$$

The fuzzy relation on the edges is defined by the values of μ, where $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. The fuzzy membership degrees of the edges are as follows:

$$\mu(v_1, v_2) = 0.1, \quad \mu(v_2, v_3) = 0.1, \quad \mu(v_3, v_4) = 0.1$$

$$\mu(v_4, v_1) = 0.1, \quad \mu(v_2, v_4) = 0.3$$

In this example, the fuzzy graph G illustrates how vertices v_1, v_2, v_3, v_4 are connected by edges with varying membership degrees. The fuzzy relations ensure that the membership degree of an edge does not exceed the minimum membership degree of the vertices it connects. This fuzzy graph captures uncertainty in both vertex presence and edge connections.

Next, we define the fuzzy unit disk graph, which combines the concepts of a fuzzy graph and a unit disk graph, as follows.

Definition 24. A *fuzzy unit disk graph* $\psi = (V, \sigma, \mu)$ is a graph where:

- V is a set of vertices, each corresponding to a disk of equal radius (typically 1) centered in the plane.
- $\sigma : V \to [0, 1]$ is a membership function assigning a degree of membership to each vertex $v \in V$, representing the presence or relevance of the vertex.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the strength of the connection between each pair of vertices (u, v), defined by:

$$\mu(u, v) = \begin{cases} \min\{\sigma(u), \sigma(v)\}, & \text{if } \|p_u - p_v\| \leq 1, \\ 0, & \text{if } \|p_u - p_v\| > 1, \end{cases}$$

  where p_u and p_v are the centers of the disks corresponding to vertices u and v, respectively, and $\|p_u - p_v\|$ denotes the Euclidean distance between them.

In this definition:

- The fuzzy relation $\mu(u, v)$ satisfies $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$.
- The relation is symmetric: $\mu(u, v) = \mu(v, u)$.
- There are no self-loops: $\mu(u, u) = 0$ for all $u \in V$.

2.4 | Unit disk graph in Intuitionistic fuzzy Graphs

Next, we consider Unit Disk Graphs in Intuitionistic Fuzzy Graphs. Intuitionistic fuzzy graphs are an extended version of fuzzy graphs and have been the subject of extensive study for over 15 years [88, 85, 43]. Intuitionistic fuzzy graphs are related to the concept of intuitionistic fuzzy sets [108, 33, 10, 9]. The definitions of intuitionistic fuzzy graphs and intuitionistic fuzzy unit disk graphs are provided below.

Definition 25 (Intuitionistic Fuzzy Graph (IFG))**.** [85] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

(1) (μ_A, v_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $v_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + v_A(x) \leq 1$$

The value $1 - \mu_A(x) - v_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

(2) (μ_B, v_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $v_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + v_B(x, y) \leq 1$$

Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$
$$v_B(x, y) \leq v_A(x) \vee v_A(y)$$

In this definition:

- $\mu_A(x)$ and $v_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.

- $\mu_B(x, y)$ and $v_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $v_A(x) = 0$ and $v_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

The following is an example of an Intuitionistic Fuzzy Graph.

Example 26 (Intuitionistic Fuzzy Graph)**.** Consider the Intuitionistic Fuzzy Graph $G = (V, E, \mu_A, v_A, \mu_B, v_B)$, where $V = \{v_1, v_2, v_3, v_4\}$ and the edges $E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_2, v_4)\}$. The membership and non-membership degrees for the vertices are given as:

$$\mu_A(v_1) = 0.1, \quad v_A(v_1) = 0.4$$
$$\mu_A(v_2) = 0.3, \quad v_A(v_2) = 0.3$$
$$\mu_A(v_3) = 0.2, \quad v_A(v_3) = 0.4$$
$$\mu_A(v_4) = 0.4, \quad v_A(v_4) = 0.6$$

For the edges, the membership and non-membership degrees are given as follows:

- $\mu_B(v_1, v_2) = 0.1, \quad v_B(v_1, v_2) = 0.4$
- $\mu_B(v_1, v_4) = 0.1, \quad v_B(v_1, v_4) = 0.6$
- $\mu_B(v_2, v_3) = 0.1, \quad v_B(v_2, v_3) = 0.4$
- $\mu_B(v_3, v_4) = 0.1, \quad v_B(v_3, v_4) = 0.6$
- $\mu_B(v_2, v_4) = 0.3, \quad v_B(v_2, v_4) = 0.6$

This example illustrates how an Intuitionistic Fuzzy Graph can be structured, representing uncertainty in both vertex and edge membership using membership and non-membership degrees.

We define the Intuitionistic Fuzzy Unit Disk Graph as follows. The Intuitionistic Fuzzy Unit Disk Graph is a concept that combines the ideas of an Intuitionistic Fuzzy Graph and a Unit Disk Graph.

Definition 27. An *Intuitionistic Fuzzy Unit Disk Graph* $G_{IFUD} = (V, E, \mu_A, \nu_A, \mu_B, \nu_B)$ is defined as follows:

(1) **Vertices and Edges**: V is a set of vertices, each representing a disk of equal radius (typically 1) centered in the plane. $E \subseteq V \times V$ is the set of edges, where an edge $(u, v) \in E$ exists if $\|p_u - p_v\| \leq 1$, where p_u and p_v are the centers of the disks corresponding to vertices u and v, respectively.

(2) **Vertex Membership Functions**: $\mu_A : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$, and $\nu_A : V \to [0, 1]$ assigns a non-membership degree to each vertex, such that:

$$0 \leq \mu_A(v) + \nu_A(v) \leq 1.$$

The hesitation degree (indeterminacy) for each vertex v is defined as:

$$\pi_A(v) = 1 - \mu_A(v) - \nu_A(v).$$

(3) **Edge Membership Functions**: $\mu_B : E \to [0, 1]$ assigns a membership degree to each edge $(u, v) \in E$, and $\nu_B : E \to [0, 1]$ assigns a non-membership degree to each edge, such that:

$$0 \leq \mu_B(u, v) + \nu_B(u, v) \leq 1.$$

The hesitation degree for each edge (u, v) is:

$$\pi_B(u, v) = 1 - \mu_B(u, v) - \nu_B(u, v).$$

(4) **Constraints Between Vertices and Edges**: For all $u, v \in V$ with $(u, v) \in E$, the following constraints hold:

$$\mu_B(u, v) \leq \min\{\mu_A(u), \mu_A(v)\},$$
$$\nu_B(u, v) \geq \max\{\nu_A(u), \nu_A(v)\}.$$

2.5 | Unit disk graph in Neutrosophic Graphs

First, the definition of a neutrosophic graph is provided. As mentioned in the introduction, neutrosophic graphs are an extension of fuzzy graphs and Intuitionistic Fuzzy Graphs. Similar to fuzzy graphs, neutrosophic graphs have been the subject of extensive research [63, 3, 103, 101]. Neutrosophic graphs are related to the concept of Neutrosophic sets [7, 69, 19]. The definition is provided below[103, 38, 98].

Definition 28. [103] A neutrosophic graph $G = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a graph where:

- $\sigma : V \to [0,1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.
- $\mu : E \to [0,1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.
- For every edge $e = v_i v_j \in E$, the following condition holds:
$$\mu_T(e) \leq \min(\sigma_T(v_i), \sigma_T(v_j)).$$

(1) σ is called the *neutrosophic vertex set*.

(2) μ is called the *neutrosophic edge set*.

(3) The number of vertices $|V|$ is the *order* of G, denoted by $O(G)$.

(4) The sum of the truth values over all vertices, $\sum_{v \in V} \sigma_T(v)$, is the *neutrosophic order* of G, denoted by $On(G)$.

(5) The number of edges $|E|$ is the *size* of G, denoted by $S(G)$.

(6) The sum of the truth values over all edges, $\sum_{e \in E} \mu_T(e)$, is the *neutrosophic size* of G, denoted by $Sn(G)$.

The Examples of neutrosophic graph is following.

Example 29. (cf.[21]) Consider a neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ with four vertices $V = \{v_1, v_2, v_3, v_4\}$, as shown in the diagram.

The neutrosophic membership degrees of the vertices are as follows:

$$\sigma(v_1) = (0.5, 0.1, 0.4), \quad \sigma(v_2) = (0.6, 0.3, 0.2),$$
$$\sigma(v_3) = (0.2, 0.3, 0.4), \quad \sigma(v_4) = (0.4, 0.2, 0.5)$$

The neutrosophic membership degrees of the edges are as follows:

$$\mu(v_1 v_2) = (0.2, 0.3, 0.4), \quad \mu(v_2 v_3) = (0.3, 0.3, 0.4),$$
$$\mu(v_3 v_4) = (0.2, 0.3, 0.4), \quad \mu(v_4 v_1) = (0.1, 0.2, 0.5)$$

In this case, the neutrosophic graph NTG has the following properties:

- Vertices v_1, v_2, v_3, v_4 are connected by edges with varying neutrosophic membership degrees.
- The neutrosophic relations ensure that for every edge $v_i v_j \in E$, $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, where $\wedge$ denotes the minimum operation.

We define the Neutrosophic Unit Disk Graph as follows. The Neutrosophic Unit Disk Graph is a concept that combines the ideas of an Neutrosophic Graph and a Unit Disk Graph.

Definition 30. A *Neutrosophic Unit Disk Graph* $G_{NUDG} = (V, E, \sigma, \mu)$ is defined as follows:

(1) **Vertices and Edges**: V is a set of vertices, each corresponding to a disk of equal radius centered in the plane. $E \subseteq V \times V$ is the set of edges, where $(u, v) \in E$ if $\|p_u - p_v\| \leq 1$.

(2) **Vertex Neutrosophic Membership Functions**: $\sigma : V \to [0, 1]^3$, where for each vertex $v \in V$:

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v)),$$

representing the truth-membership, indeterminacy-membership, and falsity-membership, respectively, such that:

$$0 \leq \sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 1.$$

(3) **Edge Neutrosophic Membership Functions**: $\mu : E \to [0, 1]^3$, where for each edge $e = (u, v) \in E$:

$$\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e)),$$

such that:

$$0 \leq \mu_T(e) + \mu_I(e) + \mu_F(e) \leq 1.$$

(4) **Constraints Between Vertices and Edges**: For all $e = (u, v) \in E$:

$$\mu_T(e) \leq \min\{\sigma_T(u), \sigma_T(v)\},$$
$$\mu_I(e) \geq \max\{\sigma_I(u), \sigma_I(v)\},$$
$$\mu_F(e) \geq \max\{\sigma_F(u), \sigma_F(v)\}.$$

2.6 | Unit disk graph in Turiyam Neutrosophic Graph

Research on Turiyam Neutrosophic Graphs, which incorporate parameters into Neutrosophic Graphs, is currently being conducted [41]. These graphs are a graphical representation of the Turiyam Neutrosophic Set [95, 42]. Similar concepts include four-valued logic [26, 14].

The definition is provided below. Note that Turiyam Neutrosophic Set is actually a particular case of the Quadruple Neutrosophic Set, by replacing "Contradiction" with "Liberal" (cf.[97]).

Definition 31 (Turiyam Neutrosophic Graph)**.** [41, 42] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

(1) *Turiyam Neutrosophic Vertex Set*: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$

where:

- $t(v_i)$ is the truth value (tv) of the vertex v_i,
- $iv(v_i)$ is the indeterminacy value (iv) of v_i,
- $fv(v_i)$ is the falsity value (fv) of v_i,
- $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

for all $v_i \in V$, such that the following condition holds for each vertex:

$$0 \leq t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4.$$

(2) *Turiyam Neutrosophic Edge Set*: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

- $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),

- $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
- $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

We define the Turiyam Neutrosophic Unit Disk Graph as follows. The Turiyam Neutrosophic Unit Disk Graph is a concept that combines the ideas of an Turiyam Neutrosophic Graph and a Unit Disk Graph.

Definition 32. A *Turiyam Neutrosophic Unit Disk Graph* $G_{TUDG} = (V, E, t_V, iv_V, fv_V, lv_V, t_E, iv_E, fv_E, lv_E)$ is defined as follows:

(1) **Vertices and Edges**: V is a set of vertices corresponding to disks of equal radius in the plane. $E \subseteq V \times V$ is the set of edges, where $(u, v) \in E$ if $\|p_u - p_v\| \leq 1$.

(2) **Turiyam Neutrosophic Vertex Membership Functions**: For each vertex $v \in V$, the Turiyam Neutrosophic membership functions are:

$$t_V(v),\ iv_V(v),\ fv_V(v),\ lv_V(v) \in [0, 1],$$

representing the truth value, indeterminacy value, falsity value, and liberal value, respectively, such that:

$$0 \leq t_V(v) + iv_V(v) + fv_V(v) + lv_V(v) \leq 1.$$

(3) **Turiyam Neutrosophic Edge Membership Functions**: For each edge $e = (u, v) \in E$, the Turiyam Neutrosophic membership functions are:

$$t_E(e),\ iv_E(e),\ fv_E(e),\ lv_E(e) \in [0, 1],$$

such that:

$$0 \leq t_E(e) + iv_E(e) + fv_E(e) + lv_E(e) \leq 1.$$

2.7 | Unit disk graph in Plithogenic Graph

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [102, 1, 47, 106, 100]. Plithogenic Graphs have been developed and are currently being actively studied [107] The definition is provided below.

Definition 33. [107] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph PG* is defined as:

$$PG = (PM, PN)$$

where:

(1) *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the range of possible attribute values.
- $adf : M \times Ml \to [0, 1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
- $aCf : Ml \times Ml \to [0, 1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.

(2) *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.

- m is an attribute associated with the edges.
- Nm is the range of possible attribute values.
- $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
- $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

(1) *Edge Appurtenance Constraint*: For all $(x,a),(y,b) \in M \times Ml$:

$$bdf((xy),(a,b)) \leq \min\{adf(x,a), adf(y,b)\}$$

where $xy \in N$ is an edge between vertices x and y, and $(a,b) \in Nm \times Nm$ are the corresponding attribute values.

(2) *Contradiction Function Constraint*: For all $(a,b),(c,d) \in Nm \times Nm$:

$$bCf((a,b),(c,d)) \leq \min\{aCf(a,c), aCf(b,d)\}$$

(3) *Reflexivity and Symmetry of Contradiction Functions*:

$$\begin{aligned} aCf(a,a) &= 0, && \forall a \in Ml \\ aCf(a,b) &= aCf(b,a), && \forall a,b \in Ml \\ bCf(a,a) &= 0, && \forall a \in Nm \\ bCf(a,b) &= bCf(b,a), && \forall a,b \in Nm \end{aligned}$$

Example 34. (cf.[39]) The following examples are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Graph*.
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Graph*.
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Graph*.
- When $s = 7, t = 1$, PG is called a *Plithogenic heptapartitioned Neutrosophic Graph*.
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Graph*.
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Graph*.

We define the Plithogenic Unit Disk Graph as follows. The Plithogenic Unit Disk Graph is a concept that combines the ideas of an Plithogenic Graph and a Unit Disk Graph.

Definition 35. A *Plithogenic Unit Disk Graph* $PG_{UDG} = (PM, PN)$ is defined as follows:

(1) **Vertices and Edges**: V is a set of vertices corresponding to disks of equal radius in the plane. $E \subseteq V \times V$ is the set of edges, where $(u,v) \in E$ if $\|p_u - p_v\| \leq 1$.

(2) **Plithogenic Vertex Set** $PM = (M, l, Ml, adf, aCf)$:

- $M \subseteq V$ is the set of vertices.
- l is an attribute associated with the vertices.
- Ml is the set of possible values for attribute l.
- $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function* for vertices.
- $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function* for vertices.

(3) **Plithogenic Edge Set** $PN = (N, m, Nm, bdf, bCf)$:

- $N \subseteq E$ is the set of edges.
- m is an attribute associated with the edges.
- Nm is the set of possible values for attribute m.
- $bdf : N \times Nm \to [0, 1]^s$ is the Degree of Appurtenance Function for edges.
- $bCf : Nm \times Nm \to [0, 1]^t$ is the Degree of Contradiction Function for edges.

Example 36. The following is an example of a Plithogenic unit disk Graph.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy unit disk Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy unit disk Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic unit disk Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic unit disk Graph*.

3 | Result in this paper

In this section, we present the results of this paper.

3.1 | Property of Plithogenic Unit Disk Graph

We consider about Plithogenic Unit Disk Graph. These properties also hold similarly for fuzzy graphs, intuitionistic fuzzy graphs, neutrosophic graphs, and Turiyam Neutrosophic graphs.

Theorem 37. *Plithogenic Unit Disk Graph can be transformed into a fuzzy, neutrosophic, or Turiyam Neutrosophic Unit Disk Graph.*

Proof: This result holds because Plithogenic graphs generalize fuzzy, neutrosophic, and Turiyam Neutrosophic graphs, and their corresponding transformations can be made by adjusting the appurtenance, contradiction, and other related parameters to align with the specific framework (fuzzy, neutrosophic, or Turiyam) being considered. □

Theorem 38. *A Plithogenic Unit Disk Graph can be transformed into a classical Unit Disk Graph.*

Proof: Let $PG_{UDG} = (PM, PN)$ be a Plithogenic Unit Disk Graph, where PM is the vertex set and PN is the edge set, each with associated plithogenic attributes.

To transform this into a classical Unit Disk Graph $G = (V, E)$, we perform the following steps:

(1) Remove the plithogenic attributes (degree of appurtenance and degree of contradiction) from both vertices and edges.

(2) Retain the vertex set $V = M$ and edge set $E = N$, where M and N are the vertex and edge sets of PG_{UDG}, respectively.

(3) Verify that the resulting graph satisfies the condition for a classical Unit Disk Graph: there is an edge between two vertices if and only if the Euclidean distance between their corresponding disk centers is at most 1.

Since the original Plithogenic Unit Disk Graph satisfies this spatial condition, the transformed graph $G = (V, E)$ is a valid classical Unit Disk Graph. □

Theorem 39. *For any Plithogenic Unit Disk Graph PG_{UDG}, the degrees of the vertices are identical to those in its underlying classical Unit Disk Graph G_{UDG}.*

Proof: The degree of a vertex v in any graph is defined as the number of edges incident to it. In the case of a Plithogenic Unit Disk Graph PG_{UDG}, the vertices and edges are associated with additional attributes, such as the degree of appurtenance and contradiction. However, these attributes only modify the characteristics of the vertices and edges, not their existence or connectivity.

Let $G_{UDG} = (V, E)$ be the underlying classical Unit Disk Graph of PG_{UDG}. The edge set E in both PG_{UDG} and G_{UDG} is identical, meaning that the connections between vertices remain the same. Therefore, for any vertex $v \in V$, the number of edges incident to v in PG_{UDG} is the same as in G_{UDG}, because the set of edges and their adjacency relationships are preserved.

While the plithogenic attributes (such as degrees of appurtenance and contradiction) may modify the properties of the edges and vertices, they do not alter the fundamental structure of the graph. In particular, the count of edges incident to each vertex, which determines the vertex degree, remains unchanged between PG_{UDG} and G_{UDG}.

Thus, the degrees of the vertices in PG_{UDG} are identical to those in its underlying classical Unit Disk Graph G_{UDG}. □

Theorem 40. *For any threshold $\alpha \in [0, 1]$, the subgraph G_α of a Plithogenic Unit Disk Graph PG_{UDG} induced by vertices and edges with degrees of appurtenance greater than or equal to α is a subgraph of the underlying classical Unit Disk Graph G_{UDG}.*

Proof: Let V_α be the set of vertices in the Plithogenic Unit Disk Graph PG_{UDG} whose degree of appurtenance is greater than or equal to α. Specifically, we define:

$$V_\alpha = \{v \in V \mid adf(v, l_v) \geq \alpha\},$$

where $adf(v, l_v)$ is the Degree of Appurtenance Function (DAF) for vertex v with respect to its attribute l_v. Similarly, let E_α be the set of edges where the degree of appurtenance is greater than or equal to α. Formally, we define:

$$E_\alpha = \{e = (u, v) \in E \mid bdf(e, m_e) \geq \alpha\},$$

where $bdf(e, m_e)$ is the Degree of Appurtenance Function (DAF) for the edge e with respect to its attribute m_e.

By construction, the set V_α is a subset of V (the vertex set of the original Unit Disk Graph G_{UDG}), and E_α is a subset of E (the edge set of G_{UDG}). Therefore, the subgraph $G_\alpha = (V_\alpha, E_\alpha)$ is a subgraph of the classical Unit Disk Graph G_{UDG}, as both the vertices and edges in G_α are filtered based on their degrees of appurtenance but still belong to the original graph G_{UDG}.

Thus, G_α is a subgraph of G_{UDG}, as the appurtenance degrees act as a filtering mechanism to select specific vertices and edges from G_{UDG}. This completes the proof. □

Theorem 41. *A Plithogenic Unit Disk Graph PG_{UDG} is connected if and only if its underlying classical Unit Disk Graph G_{UDG} is connected.*

Proof: Assume that the Plithogenic Unit Disk Graph PG_{UDG} is connected. By definition, PG_{UDG} shares the same vertices and edges as the underlying classical Unit Disk Graph G_{UDG}, with the only difference being the additional plithogenic attributes such as degrees of appurtenance and contradiction. Since PG_{UDG} is connected, there exists a path between any pair of vertices in PG_{UDG}, and because the edges of G_{UDG} are present in PG_{UDG}, the same path must exist in G_{UDG}. Therefore, if PG_{UDG} is connected, G_{UDG} must also be connected.

Now assume that the classical Unit Disk Graph G_{UDG} is connected. In this case, there exists a path between any pair of vertices in G_{UDG}. The Plithogenic Unit Disk Graph PG_{UDG} inherits the same vertex and edge sets from G_{UDG}, and the plithogenic attributes do not alter the existence of edges but only add additional information. As a result, the connectivity of G_{UDG} implies that there is a path between any pair of vertices in PG_{UDG}, making PG_{UDG} connected as well.

Thus, the Plithogenic Unit Disk Graph PG_{UDG} is connected if and only if the classical Unit Disk Graph G_{UDG} is connected. □

Theorem 42. *If a graph is a Plithogenic unit disk graph, then it contains no subgraph homomorphic to* $K_{1,6}$ *or* $K_{2,3}$.

Proof: Let $PG = (V, E, PM, PN)$ be a Plithogenic unit disk graph. By definition, the vertices of PG correspond to disks of equal radius in the plane, and two vertices are adjacent if and only if the distance between their corresponding disk centers is at most 1.

Now, assume for the sake of contradiction that PG contains a subgraph homomorphic to $K_{1,6}$ or $K_{2,3}$. Such a configuration is impossible in a unit disk graph, as the spatial arrangement required for $K_{1,6}$ or $K_{2,3}$ would violate the unit distance condition between adjacent vertices. Therefore, PG cannot contain a subgraph homomorphic to $K_{1,6}$ or $K_{2,3}$. Refer to Theorem 21 for details. □

3.2 | Relation to Fuzzy Intersection Graph

One of the well-known graph classes of fuzzy graphs is the Fuzzy Intersection Graph. The following definition is commonly recognized.

Definition 43 (Fuzzy Intersection Graph)**.** A *Fuzzy Intersection Graph* is a graph $G = (V, E, \sigma, \mu)$ where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $\sigma : V \to [0, 1]$ is a membership function that assigns a degree of membership to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation representing the strength of the connection (degree of membership) between each pair of vertices $(u, v) \in V \times V$.

The edge set E of the fuzzy intersection graph is defined based on the membership functions of the vertices and the fuzzy relation. Specifically, for each pair $(u, v) \in V \times V$, the edge (u, v) exists in the fuzzy intersection graph with the membership degree:

$$\mu(u, v) = \min(\sigma(u), \sigma(v))$$

if the Euclidean distance between the corresponding points of u and v satisfies the condition for intersection, and $\mu(u, v) = 0$ otherwise.

In this way, the fuzzy intersection graph generalizes the concept of an intersection graph by incorporating fuzzy set theory, allowing for partial membership and gradual relationships between vertices and edges.

Theorem 44. *Any undirected fuzzy graph* $G = (V, \sigma, \mu)$ *can be represented as a fuzzy intersection graph.*

Proof: Let $G = (V, \sigma, \mu)$ be an undirected fuzzy graph, where:

- V is the set of vertices.
- $\sigma : V \to [0, 1]$ is the membership function for vertices.
- $\mu : V \times V \to [0, 1]$ is the fuzzy relation representing the strength of connections between vertices, subject to $\mu(u, v) \leq \min(\sigma(u), \sigma(v))$.

We need to show that G can be represented as a fuzzy intersection graph, where the edges between vertices are determined by their membership degrees.

We consider about Set Representation. For each vertex $v_i \in V$, define a set S_i that contains the edges (v_i, v_j) where v_i is adjacent to v_j. The set S_i represents the neighbors of v_i based on the fuzzy membership degrees assigned to each edge.

We consider about Fuzzy Relation Preservation. We construct a fuzzy intersection graph by defining the fuzzy membership of the edges between any two vertices v_i and v_j as follows:

$$\mu(v_i, v_j) = \min(\sigma(v_i), \sigma(v_j))$$

This definition is consistent with the properties of the fuzzy graph G, where the membership degree of an edge cannot exceed the minimum membership degree of the vertices it connects.

We consider about Symmetry and No Self-Loops. Since the fuzzy graph G is undirected, the fuzzy relation μ is symmetric. That is, $\mu(v_i, v_j) = \mu(v_j, v_i)$, which is preserved in the fuzzy intersection graph. Furthermore, the absence of self-loops (i.e., $\mu(v_i, v_i) = 0$) is also preserved in the fuzzy intersection graph.

We consider about Intersection Property. The sets S_i for each vertex define the neighbors based on the fuzzy relation. Two sets S_i and S_j intersect if and only if the corresponding vertices v_i and v_j share an edge in G. Since the fuzzy membership degree is defined as $\mu(v_i, v_j) = \min(\sigma(v_i), \sigma(v_j))$, the edge exists if and only if there is a non-zero intersection, which satisfies the conditions for a fuzzy intersection graph.

Thus, the fuzzy graph G can be represented as a fuzzy intersection graph, with edges determined by the intersection of the sets S_i and S_j, and fuzzy membership degrees assigned to the edges as $\mu(v_i, v_j) = \min(\sigma(v_i), \sigma(v_j))$. □

Theorem 45. *A Plithogenic Fuzzy Unit Disk Graph is a Fuzzy Intersection Graph.*

Proof: Let $PG_{UDG} = (PM, PN)$ be a *Plithogenic Fuzzy Unit Disk Graph*, where:

- V is the set of vertices corresponding to disks of equal radius in the plane.
- $E \subseteq V \times V$ is the set of edges, with an edge $(u, v) \in E$ if and only if the Euclidean distance between the centers of the disks corresponding to u and v is less than or equal to 1.
- The vertices have a *Degree of Appurtenance Function (adf)* $adf(v, l_v)$, representing the membership degree of each vertex v in its attribute class l_v, and the edges have a similar *Degree of Appurtenance Function* for edges.

In PG_{UDG}, the vertices have a *Degree of Appurtenance Function (adf)*, $adf(v, l_v)$, which assigns a membership degree to each vertex v. This is functionally equivalent to the membership function $\sigma(v)$ in the *Fuzzy Intersection Graph*. Therefore, we can set:

$$\sigma(v) = adf(v, l_v).$$

Thus, the vertices in PG_{UDG} behave as vertices in a *Fuzzy Intersection Graph* in terms of their membership degree.

In PG_{UDG}, the edges also have a *Degree of Appurtenance Function (bdf)* $bdf(e, m_e)$, where $e = (u, v)$, representing the degree of membership of the edge in the graph. For a *Fuzzy Intersection Graph*, the membership of the edge is determined by:

$$\mu(u, v) = \min(\sigma(u), \sigma(v)) = \min(adf(u, l_u), adf(v, l_v)),$$

which is exactly how the membership degree of the edges in PG_{UDG} can be interpreted. Therefore, the edges of PG_{UDG} follow the same rule as in the *Fuzzy Intersection Graph*. □

Corollary 46. *A Fuzzy Unit Disk Graph is a Fuzzy Intersection Graph.*

Proof: Obviously holds. □

3.3 | Plithogenic Intersection Graph and Unit disk graph

We propose the concept of a Plithogenic Intersection Graph, which extends the Fuzzy Intersection Graph within the framework of Plithogenic Graph theory.

Definition 47 (Plithogenic Intersection Graph)**.** A *Plithogenic Intersection Graph* $PG = (V, E, PM, PN)$ is defined as follows:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic vertex set, where:

  - $M \subseteq V$ is the set of vertices.
  - l is an attribute associated with the vertices.
  - Ml is the range of possible values for attribute l.
  - $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function* for vertices.
  - $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function* for vertices.
- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic edge set, where:
  - $N \subseteq E$ is the set of edges.
  - m is an attribute associated with the edges.
  - Nm is the range of possible values for attribute m.
  - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function* for edges.
  - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function* for edges.

The edge set E of the Plithogenic Intersection Graph is defined based on the degree of appurtenance of the vertices and the plithogenic relation. Specifically, for each pair $(u,v) \in V \times V$, the edge (u,v) exists in the Plithogenic Intersection Graph with the degree of appurtenance:

$$bdf(u,v) = \min(adf(u,l_u), adf(v,l_v))$$

if the Euclidean distance between the corresponding points of u and v satisfies the condition for intersection, and $bdf(u,v) = 0$ otherwise.

Example 48. (cf.[39]) The following is an example of a Plithogenic Intersection Graph.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Intersection Graph.*
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Intersection Graph.*
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Intersection Graph.*
- When $s = 4, t = 1$, PG is called a *Plithogenic quadripartitioned Neutrosophic Intersection Graph.*
- When $s = 5, t = 1$, PG is called a *Plithogenic pentapartitioned Neutrosophic Intersection Graph.*
- When $s = 6, t = 1$, PG is called a *Plithogenic hexapartitioned Neutrosophic Intersection Graph.*
- When $s = 7, t = 1$, PG is called a *Plithogenic Heptapartitioned Neutrosophic Intersection Graph.*
- When $s = 8, t = 1$, PG is called a *Plithogenic octapartitioned Neutrosophic Intersection Graph.*
- When $s = 9, t = 1$, PG is called a *Plithogenic nonapartitioned Neutrosophic Intersection Graph.*

Theorem 49. *A Plithogenic Unit Disk Graph is a Plithogenic Intersection Graph.*

Proof: Let $PG_{UDG} = (PM, PN)$ be a *Plithogenic Unit Disk Graph*, where:

- V is the set of vertices, each corresponding to a disk of equal radius in the plane.
- $E \subseteq V \times V$ is the set of edges, where $(u,v) \in E$ if the Euclidean distance between the centers of the disks corresponding to u and v is less than or equal to 1.
- The vertices have a *Degree of Appurtenance Function (adf)* $adf(v, l_v)$, representing the membership degree of each vertex v in its attribute class l_v.
- The edges have a *Degree of Appurtenance Function (bdf)* $bdf(e, m_e)$, representing the degree of membership of each edge e in its attribute class m_e.

We consider about Definition of a Plithogenic Intersection Graph. A *Plithogenic Intersection Graph* is defined as a graph where the vertices have degrees of appurtenance and contradiction, and the edge membership degree is determined by the minimum appurtenance degree of the vertices it connects. Specifically, the degree of appurtenance for the edge (u, v) is given by:

$$bdf(u, v) = \min(adf(u, l_u), adf(v, l_v))$$

if the condition for intersection is satisfied.

We consider about Establishing the Plithogenic Unit Disk Graph as an Intersection Graph. In the case of a *Plithogenic Unit Disk Graph*, the vertices represent disks in the plane, and edges exist if the Euclidean distance between the centers of two disks is at most 1. The degree of appurtenance for each edge is defined based on the degree of appurtenance of the vertices it connects, which aligns with the definition of a Plithogenic Intersection Graph. Specifically, for an edge (u, v), we have:

$$bdf(u, v) = \min(adf(u, l_u), adf(v, l_v))$$

This satisfies the requirement that the degree of appurtenance for the edge is determined by the vertices it connects, as in a Plithogenic Intersection Graph.

We consider about Edge Formation and Intersection Condition. In the *Plithogenic Unit Disk Graph*, an edge between two vertices exists if and only if the Euclidean distance between their corresponding disk centers is at most 1. This corresponds to the intersection condition in the Plithogenic Intersection Graph, where edges are formed based on the intersection of the vertex sets, represented by the disks in the unit disk graph. The edge membership degree is then determined by the appurtenance degrees of the intersecting vertices.

Therefore, a *Plithogenic Unit Disk Graph* is indeed a *Plithogenic Intersection Graph*. □

Theorem 50. *Any undirected Plithogenic graph can be represented as a Plithogenic intersection graph.*

Proof: Let $PG = (V, E, PM, PN)$ be an undirected Plithogenic graph, where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic edge set.

We need to show that the Plithogenic graph PG can be represented as a Plithogenic intersection graph, where the edges between vertices are determined by the degree of appurtenance.

We consider about Set Representation of Vertices. For each vertex $v_i \in V$, define a set S_i containing the edges (v_i, v_j), where v_i is adjacent to v_j according to the degree of appurtenance $adf(v_i, l_{v_i})$. The set S_i represents the neighbors of v_i based on the Plithogenic attributes assigned to the vertices and edges.

We consider about Defining Edge Membership in the Intersection Graph. For each pair of vertices v_i and v_j, define the degree of membership for the edge (v_i, v_j) in the Plithogenic intersection graph by:

$$bdf(v_i, v_j) = \min(adf(v_i, l_{v_i}), adf(v_j, l_{v_j})),$$

where $adf(v_i, l_{v_i})$ and $adf(v_j, l_{v_j})$ are the degrees of appurtenance of vertices v_i and v_j, respectively. This definition ensures that the edge membership degree is based on the minimum appurtenance of the two connected vertices, similar to the construction of fuzzy intersection graphs.

We consider about Plithogenic Relation Preservation. The plithogenic nature of the graph is preserved by considering the additional parameters of contradiction. The contradiction function bCf ensures that any contradictions between vertex or edge attributes are respected in the Plithogenic intersection graph, where:

$$bCf(m_i, m_j) \leq \min(aCf(l_{v_i}, l_{v_j}), aCf(l_{v_j}, l_{v_i})).$$

This ensures that the degree of contradiction between edges remains consistent in the Plithogenic intersection graph.

We consider about Symmetry and No Self-Loops. Since the original Plithogenic graph PG is undirected, the edge membership function bdf is symmetric, meaning $bdf(v_i, v_j) = bdf(v_j, v_i)$. Furthermore, no self-loops are allowed in the graph, so $bdf(v_i, v_i) = 0$.

We consider about Intersection Property. The sets S_i for each vertex define the neighbors based on the Plithogenic relation. Two sets S_i and S_j intersect if and only if the corresponding vertices v_i and v_j share an edge in PG. Since the degree of appurtenance of the edge is defined as:

$$bdf(v_i, v_j) = \min(adf(v_i, l_{v_i}), adf(v_j, l_{v_j})),$$

the edge exists if and only if there is a non-zero intersection, satisfying the conditions for a Plithogenic intersection graph.

Thus, any undirected Plithogenic graph can be represented as a Plithogenic intersection graph, with edges determined by the intersection of the sets S_i and S_j, and edge membership degrees assigned as $bdf(v_i, v_j) = \min(adf(v_i, l_{v_i}), adf(v_j, l_{v_j}))$. □

Theorem 51. *A Plithogenic Fuzzy Intersection Graph is a Fuzzy Intersection Graph.*

Proof: Let $PG = (V, E, PM, PN)$ be a *Plithogenic Fuzzy Intersection Graph*, where:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- $PM = (M, l, Ml, adf, aCf)$ is the Plithogenic vertex set, where $adf(v, l_v)$ is the *Degree of Appurtenance Function* for each vertex $v \in V$, assigning a membership degree to each vertex based on its attribute l_v.
- $PN = (N, m, Nm, bdf, bCf)$ is the Plithogenic edge set, where $bdf(u, v)$ is the *Degree of Appurtenance Function* for each edge (u, v), representing the membership degree of the edge in the graph.

In PG, each vertex has a degree of appurtenance function $adf(v, l_v)$, which assigns a membership degree to each vertex v. This function is conceptually similar to the membership function $\sigma(v)$ in a *Fuzzy Intersection Graph*. Therefore, we can map the membership function $\sigma(v)$ in the fuzzy graph as:

$$\sigma(v) = adf(v, l_v),$$

where $\sigma(v)$ represents the membership degree of vertex v in the fuzzy graph.

Next, for each edge $(u, v) \in E$, the degree of appurtenance in PG is represented by $bdf(u, v)$, which defines the membership degree of the edge based on the vertices u and v. In a *Fuzzy Intersection Graph*, the membership degree of an edge is defined as:

$$\mu(u, v) = \min(\sigma(u), \sigma(v)) = \min(adf(u, l_u), adf(v, l_v)).$$

This is exactly how the membership degree of the edges is interpreted in the Plithogenic Fuzzy Intersection Graph. The edge (u, v) exists in both graphs based on the minimum degree of appurtenance of the vertices it connects.

Additionally, in both the *Plithogenic Fuzzy Intersection Graph* and the *Fuzzy Intersection Graph*, the edges are defined by the intersection condition, i.e., an edge exists between two vertices if and only if their degree of membership (or appurtenance) satisfies the intersection condition.

Since the vertices and edges in PG behave identically to those in a *Fuzzy Intersection Graph* in terms of membership degrees, and the rules for edge existence are the same, we conclude that a *Plithogenic Fuzzy Intersection Graph* is a *Fuzzy Intersection Graph*. □

4 | Conclusion and Future works

In this section, we present the conclusion and outline potential future work based on the findings of this paper.

4.1 | Conclusion in this paper

In conclusion, this paper provides a comprehensive exploration of unit disk graphs within the frameworks of fuzzy and neutrosophic graphs. By extending unit disk graphs to these frameworks, our work offers new insights and potential applications in modeling real-world networks where uncertainty and imprecision are inherent. We believe that our research will contribute to the broader understanding of these graph structures and serve as a foundation for future studies in both theoretical and applied graph theory.

4.2 | Future works

In this paper, we focused on unit disk graphs. Moving forward, we aim to extend other types of intersection graphs within the frameworks of fuzzy and neutrosophic graphs.

Additionally, we plan to explore intersection hypergraphs and their extensions to fuzzy and neutrosophic settings. Intersection hypergraphs, the hypergraph counterpart of intersection graphs, have been the subject of some research, and we intend to build upon these studies [79, 65, 12].

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

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