SCIENTIA MAGNA

An international journal

Edited by School of Mathematics
Northwest University, P.R.China

Vol. 11, No. 2, 2016 **ISSN 1556-6706**

SCIENTIA MAGNA

An international journal

Edited by

School of Mathematics

Northwest University

Xi'an, Shaanxi, China

Information for Authors

Scientia Magna is a peer-reviewed, open access journal that publishes original research articles in all areas of mathematics and mathematical sciences. However, papers related to Smarandache's problems will be highly preferred.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. Submission of a manuscript implies that the work described has not been published before, that it is not under consideration for publication elsewhere, and that it will not be submitted elsewhere unless it has been rejected by the editors of Scientia Magna.

Manuscripts should be submitted electronically, preferably by sending a PDF file to ScientiaMagna@hotmail.com.

On acceptance of the paper, the authors will also be asked to transmit the TeX source file. PDF proofs will be e-mailed to the corresponding author.

Contents

Scientia Magna
Vol. 11 (2016), No. 2, 1-6

The mean value of $P^*(n)$ over square-full numbers

Hualin Si[1], Xuejiao Liu[2] and Dan Liu[3]

[1]Department of Mathematical and Statistics Sciences, Shandong Normal University
Jinan, Shandong, China
E-mail: sihualin123@163.com
[2]Department of Mathematical and Statistics Sciences, Shandong Normal University
Jinan, Shandong, China
E-mail: xiaoshitiao@163.com
[3]Department of Mathematical and Statistics Sciences, Shandong Normal University
Jinan, Shandong, China
E-mail: liudanprime@163.com

Abstract Let $n > 1$ be an integer, $P^*(n)$ be the unitary analogue of the gcd-sum function. In this paper, we consider the mean value of $P^*(n)$ over square-full numbers, that is

$$\sum_{\substack{n \leq x \\ n\ is\ square-full}} P^*(n) = \sum_{n \leq x} P^*(n) f_2(n),$$

where $f_2(n)$ is the characteristic function of square-full integers, i.e.

$$f_2(n) = \begin{cases} 1, & \text{n is square-full,} \\ 0, & \text{otherwise.} \end{cases}$$

.

Keywords divisor problem, Dirichlet convolution method, mean value.
2010 Mathematics Subject Classification 11N37.

§1. Introduction and preliminaries

An integer $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ is called $k - full$ number if all the exponents $a_1 \geq k$, $a_2 \geq k$, $\cdots$, $a_r \geq k$. When $k = 2$, n is called $square - full$ integer.

American-Romanian number theorist Florentin Smarandache introduced hundreds of interesting sequences and arithmetical functions. In 1991, he published a book named 'Only problems, Not solutions!' He presented 105 unsolved arithmetical problems and conjectures about these functions and sequences in it. In the unsolved problem 32 (see [3]), Smarandache introduced the irrational root sieve. We can get the irrational root sieve by taking off all k-powers, $k \geq 2$, of all square free numbers from the set of natural numbers (except 0 and 1). In fact, the complementary set of the irrational root sieve in the set of natural numbers (except 0 and 1) is the set of $square - full$ numbers. Let $f_2(n)$ be the characteristic function of *square* $-$ *full* integers, i.e.

$$f_2(n) = \begin{cases} 1, & \text{n is square-full}, \\ 0, & \text{otherwise}. \end{cases}$$

In 1982, M. V. Subbarao [4] gave the definition of the exponential divisor, i.e. $n > 1$ is an integer and $n = \prod_{i=1}^{r} p_i^{a_i}$, $d = \prod_{i=1}^{r} p_i^{c_i}$, if $c_i \mid a_i$, $i = 1, 2, \cdots, r$, then d is an exponential divisor of n. We denote $d \mid_e n$. Two integers n, $m > 1$ have common exponential divisors if they have the same prime factors. For $n = \prod_{i=1}^{r} p_i^{a_i}$, $m = \prod_{i=1}^{r} p_i^{b_i}$, a_i, $b_i \geq 1 (1 \leq i \leq r)$, the greatest common exponential divisor of n and m is $(n, m)_e = \prod\limits_{i=1}^{r} p_i^{(a_i, b_i)}$. Here $(1, 1)_e = 1$ by convention and $(1, m)_e$ does not exist for $m > 1$.

The integers n, $m > 1$ are called exponentially coprime, if they have the same prime factors and $(a_i, b_i) = 1$ for every $1 \leq i \leq r$, with the notation of above. In this case, one gets $(n, m)_e = S_r(n) = S_r(m)$. The function $S_r(n) = P_1 * \cdots * P_r$ can be found in the unsolved problem 63 (see [3]). 1 and $m > 1$ are not exponentially coprime. Let

$$P^*(n) = \sum_{k=1}^{n} (k, n)_*,$$

where $(k, n)_* := \max\{d \in \mathbb{N} : d|k, d\|n\}$, which was introduced by Tóth [5]. The function $P^*(n)$ is also multiplicative and $P^*(p^a) = 2p^a - 1$ for every prime power p^a $(a \geq 1)$.

Many authors have investigated the properties of the function $P^*(n)$, see [6] and [1]. Recently, L. Tóth [6] proved the following result:

$$\sum_{n \leq x} P^*(n) = \frac{\alpha}{2\zeta(2)} x^2 \log x + \beta x^2 + O(x^{3/2} \log x),$$

where $\alpha = \prod_p (1 - 1/(p+1)^2) \approx 0.775883$, α, β are constants.

The aim of this paper is to establish the following asymptotic formula for the mean value of the function $P^*(n)$ over square-full numbers.

Theorem 1.1. *We have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \text{ is } square-full}} P^*(n) = \frac{1}{3} x^{3/2} R_{1,1}(\log x) + \frac{1}{4} x^{4/3} R_{1,2}(\log x) + O(x^{5/4} \exp(-D(\log x)^{3/5} (\log\log x)^{-1/5})),$$

where $R_{1,k}(t)$, $k = 1, 2$ *are polynomials of degree* 1 *in* t, $D > 0$ *is an absolute constant.*

Notation. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Some lemmas

Lemmas 2.1. *Let*

$$d(2, 2, 3, 3; k) := \sum_{k = n^2 m^3} d(n) d(m),$$

$$D(2, 2, 3, 3; x) := \sum_{1 \leq k \leq x} d(2, 2, 3, 3; k),$$

such that

$$D(2,2,3,3;x) = x^{1/2}P_{1,1}(\log x) + x^{1/3}P_{1,2}(\log x) + O(x^{19/80+\epsilon}),$$

where $P_{1,1}(t)$, $P_{1,2}(t)$ *are polynomials of degree* 1 *in* t.

Proof. This is Lemma 6 of D. Zhang [7]. □

Lemmas 2.2. *Let* $f(m)$, $g(n)$ *are arithmetical functions such that*

$$\sum_{m\le x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^{\alpha}),$$

$$\sum_{n\le x} |g(n)| = O(x^{\beta}),$$

where $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_J > \alpha > \beta > 0$, $P_j(t)$ *are polynomials in* t. *If* $h(n) = \sum_{n=md} f(m)g(d)$ *then*

$$\sum_{n\le x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha}),$$

where $Q_j(t)$ *are polynomials in* t, $(j = 1, \cdots J)$.

Proof. This is Theorem 14.1 of Ivić [2]. □

Lemmas 2.3. *Let* $f(n)$ *be an arithmetical function for which*

$$\sum_{n\le x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^{a}),$$

$$\sum_{n\le x} |f(n)| = O(x^{a_1}(\log x)^r),$$

where $a_1 \ge a_2 \ge \cdots \ge a_l > 1/c > a \ge 0$, $r \ge 0$, $P_j(t)$ *are polynomials in* t *of degrees not exceeding* r, $(j = 1, \cdots J)$, *and* $c \ge 1$, $b \ge 1$ *are fixed integers. Suppose for* $\Re s > 1$ *that*

$$\sum_{n=1}^{\infty} \frac{\mu_d(n)}{n^s} = \frac{1}{\zeta^b(s)},$$

if

$$h(n) = \sum_{d^c|n} \mu_b(d) f(n/d^c),$$

then

$$\sum_{n\le x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x),$$

where $R_j(t)$ *are polynomials in* t *of degrees not exceeding* r, $(j = 1, \cdots l)$, *and for some* $D > 0$,

$$E_c(x) \ll x^{1/c} \exp(-D(\log x)^{3/5}(\log\log x)^{-1/5}).$$

Proof. See Theorem 14.2 of Ivić [2]. □

Lemmas 2.4. *Let* $P'(n) = \frac{P^*(n)}{n}$, $\Re s > 1$, *we have*

$$\sum_{\substack{n=1 \\ n \text{ is square-full}}}^{\infty} \frac{P'(n)}{n^s} = \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)} G(s),$$

where the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ *is absolutely convergent for* $\Re s > 1/5$.

Proof.

$$\sum_{\substack{n=1 \\ n \text{ is square-full}}}^{\infty} \frac{P'(n)}{n^s} = \sum_{n=1}^{\infty} \frac{P'(n) f_2(n)}{n^s}$$

$$= \prod_p \left(1 + \frac{P'(p^2) f_2(p^2)}{p^{2s}} + \frac{P'(p^3) f_2(p^3)}{p^{3s}} + \frac{P'(p^4) f_2(p^4)}{p^{4s}} + \cdots + \frac{P'(p^r) f_2(p^r)}{p^{rs}}\right)$$

$$= \prod_p \left(1 + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \frac{2}{p^{4s}} - \frac{1}{p^{2+2s}} - \frac{1}{p^{3+3s}} - \frac{1}{p^{4+4s}} + \cdots\right)$$

$$= \zeta(2s) \prod_p \left(1 + \frac{1}{p^{2s}} + \frac{2}{p^{3s}} - \frac{1}{p^{2+2s}} + \frac{1}{p^{2+4s}} + \cdots\right)$$

$$= \zeta^2(2s) \prod_p \left(1 + \frac{2}{p^{3s}} - \frac{1}{p^{4s}} - \frac{2}{p^{5s}} + \cdots\right)$$

$$= \zeta^2(2s)\zeta(3s) \prod_p \left(1 + \frac{1}{p^{3s}} - \frac{1}{p^{4s}} - \frac{2}{p^{5s}} - \frac{2}{p^{6s}} + \cdots\right)$$

$$= \zeta^2(2s)\zeta^2(3s) \prod_p \left(1 - \frac{1}{p^{4s}} - \frac{2}{p^{5s}} - \frac{3}{p^{6s}} + \cdots\right)$$

$$= \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)} \prod_p \left(1 - \frac{2}{p^{5s}} - \frac{3}{p^{6s}} + \cdots\right)$$

$$= \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)} G(s),$$

where $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \prod_p \left(1 - \frac{2}{p^{5s}} - \frac{3}{p^{6s}} + \cdots\right)$, which is absolutely convergent for $\Re s > 1/5$, and

$$\sum_{n \le x} |g(n)| \ll x^{1/5+\epsilon}.$$

□

§3. Proof of Theorem 1.1

Let

$$\zeta^2(2s)\zeta^2(3s)G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \ \Re s > 1,$$

$$\zeta^2(2s)\zeta^2(3s) = \sum_{n=1}^{\infty} \frac{d(2,2,3,3;n)}{n^s},$$

such that

$$f(n) = \sum_{n=md} d(2,2,3,3;m)g(d). \tag{1}$$

From Lemma 2.1 and the definition of $d(2,2,3,3;m)$ we get

$$\sum_{m\le x} d(2,2,3,3;m) = x^{1/2}P_{1,1}(\log x) + x^{1/3}P_{1,2}(\log x) + O(x^{19/80+\epsilon}), \tag{2}$$

where $P_{1,k}(t)$ are polynomials of degree 1 in t, $k = 1, 2$.

In addition we have

$$\sum_{n\le x} |g(n)| = O(x^{1/5+\epsilon}). \tag{3}$$

Combining (1), (2) and (3), and applying Lemma 2.2, we have

$$\sum_{n\le x} f(n) = x^{1/2}Q_{1,1}(\log x) + x^{1/3}Q_{1,2}(\log x) + O(x^{19/80+\epsilon}), \tag{4}$$

where $Q_{1,1}(t)$, $Q_{1,2}(t)$ are polynomials of degrees 1 in t, then we can get

$$\sum_{n\le x} |f(n)| \ll x^{1/2}\log x. \tag{5}$$

Since $\frac{1}{\zeta(4s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{4s}}$, $\Re s > 1/4$, from Lemma 2.4 and (1) we have the relation

$$P'(n)f_2(n) = \sum_{n=md^4} f(m)\mu(d). \tag{6}$$

From (4), (5) and (6), in view of Lemma 2.3, we can get

$$\sum_{\substack{n\le x \\ n \ is \ square-full}} P'(n) = x^{1/2}R_{1,1}(\log x) + x^{1/3}R_{1,2}(\log x) + O(x^{1/4}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5}). \tag{7}$$

From the definition of $P'(n)$ and Abel's summation formula, we can easily get

$$\begin{aligned}
\sum_{\substack{n\le x \\ n \ is \ square-full}} P^*(n) &= \sum_{\substack{n\le x \\ n \ is \ square-full}} P'(n)n \\
&= \int_1^x td\Big(\sum_{\substack{n\le t \\ n \ is \ square-full}} P'(n)\Big) \\
&= \frac{1}{3}x^{3/2}R_{1,1}(\log x) + \frac{1}{4}x^{4/3}R_{1,2}(\log x) + O(x^{5/4}\exp(-D(\log x)^{3/5}(\log\log x)^{-1/5})),
\end{aligned}$$

where $R_{1,k}(t)$, $k = 1, 2$ are polynomials of degree 1 in t, $D > 0$ is an absolute constant.

Then, we complete the proof of Theorem 1.1.

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions, which highly improve the quality of this paper. This work is supported by Natural Science Foundation of Shandong Province(No. ZR2015AM010).

References

[1] S. Chen and W. Zhai. Reciprocals of the Gcd-Sum Functions. J. Integer Seq., 2011, 14 (8): Article 11.8.3, 13 pp.

[2] A. Ivić. The Riemann zeta-function: theory and applications. New York: John Wiley and Sons, 1985, 387–393.

[3] F. Smarandache. Only problems, Not solutions! Chicago: Xiquan Publishing House, 1993.

[4] M. V. Subbarao. On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol. 251, Springer, 1972, 247–271.

[5] L. Tóth. The unitary analogue of Pillai's arithmetical function. Collect. Math., 1989, 40 (1): 19–30.

[6] L. Tóth. The unitary analogue of Pillai's arithmetical function II. Notes Number Theory Discrete Math., 1996, 2 (2): 40–46.

[7] D. Zhang and W. Zhai. On certain divisor sum over square-full integers. J. Shandong Univ. Nat. Sci., 2006, 41 (2): 66–68.

Scientia Magna
Vol. 11 (2016), No. 2, 7-13

On the mean value of exponential divisor function

Kai Li[1] and Yankun Sui[2]

[1]Department of Mathematics and Statistics, Shandong Normal University
Jinan, Shandong, China
E-mail:likaisdnu@outlook.com
[2]Department of Mathematics and Statistics, Shandong Normal University
Jinan, Shandong, China
E-mail: 18353114095@163.com

Abstract Let $n > 1$ be an integer. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i | a_i$ for every $i \in \{1, 2, \ldots, s\}$. Let $\phi^{(e)}(n)$ denote the number of divisors d of n such that d and n have no common exponential divisors. In this paper, we study the sum $D(1, \underbrace{3, \ldots, 3}_{k}; x) = \sum_{n \leq x} d(1, \underbrace{3, \ldots, 3}_{k}; n)$ and get the asymptotic formula for it, where $d(1, \underbrace{3, \ldots, 3}_{k}; n) = \sum_{n = ab_1^3 \cdots b_k^3} 1$. We get the mean value for the exponential divisor function, which improves the previous result.

Keywords Dirichlet convolution; asymptotic formula; exponential divisor function.
2010 Mathematics Subject Classification 11N37.

§1. Introduction and preliminaries

Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems hasn't been solved. For example, F. Smarandache gave some unsolved problems in his book *ONLY PROBLEMS, NOT SOLUTIONS!*, and one problem is that, a number n is called simple number if the product of its proper divisors is less than or equal to n. Generally speaking, $n = p$, or $n = p^2$, or $n = p^3$, or pq, where p and q are distinct primes. The properties of this simple number sequence hasn't been studied yet. And other problems are introduced in this book, such as proper divisor products sequence and the largest exponent (of power p) which divides n, where $p \geq 2$ is an integer.

In this paper, we study the exponential divisor function, which is a class of the divisor problem. In 1982, Subbarao [3] firstly gave the definition of exponential divisor: suppose $n > 1$ is an integer, and $n = \prod_i^t p_i^{a_i}$. If $d = \prod_i^t p_i^{b_i}$ satisfies $b_i | a_i, i = 1, 2, \cdots, t$, then d is called an exponential divisor of n, notation $d|_e n$. By convention $1|_e 1$.

For $n = \prod_i^t p_i^{a_i} > 1$, $a_i \geq 1 (1 \leq i \leq r)$, $\phi^{(e)}(n)$ denotes the number of integers $\prod_i^t p_i^{c_i}$ such that $1 \leq c_i \leq a_i$, and $(c_i, a_i) = 1$ for $1 \leq i \leq r$, and let $\phi^{(e)}(1) = 1$. Thus $\phi^{(e)}(n)$ counts the number of divisors d of n such that d and n are exponentially coprime.

It is easy to see that $\phi^{(e)}$ is a prime independent multiplicative function and for $n > 1$,

$$\phi^{(e)}(n) = \prod_{i=1}^{r} \phi(a_i),$$

where ϕ is the Euler-function. Exponentially coprime integers and function $\phi^{(e)}$ were introduced by J.S*á*ndor [2]. He showed that

$$\lim_{n\to\infty} \sup \frac{\log \phi^{(e)}(n) \log\log n}{\log n} = \frac{\log 4}{5}. \tag{1}$$

In 2007, T*ó*th [5] obtained the asymptotic formula for the $r-$th power of the function $\phi^{(e)}(n)$, where for every integer $r \geq 1$

$$\sum_{n\leq x} (\phi^{(e)}(n))^r = B_r x + x^{1/3} R_{2^r-2}(\log x) + O(x^{t_r+\varepsilon}), \tag{2}$$

for every $\varepsilon > 0$, where $t_r := \frac{2^{r+1}-1}{3\cdot 2^r+1}$, $R_{2^r-2}(x)$ is a polynomial of degree $2^r - 2$ and

$$B_r := \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{\phi^r(a) - \phi^r(a-1)}{p^a}\right). \tag{3}$$

In the case $r = 1$, formula (1.2) was proved in [4] with a better error term, that is

$$\sum_{n\leq x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + O(x^{1/5+\varepsilon}), \tag{4}$$

for every $\varepsilon > 0$, where C_1, C_2 are constants given by

$$C_1 = \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{\phi(a) - \phi(a-1)}{p^a}\right),$$

$$C_2 = \zeta(1/3) \prod_p \left(1 + \sum_{a=5}^{\infty} \frac{\phi(a) - \phi(a-1) - \phi(a-3) + \phi(a-4)}{p^{a/3}}\right).$$

In this paper, we will study the asymptotic formula for the mean value of the r-th power of the function $\phi^{(e)}(n)$, where $r > 1$ is an integer, which improves Tóth's result.

Theorem 1.1. *For every integer $r > 1$, then we have*

$$\sum_{n\leq x} (\phi^{(e)}(n))^r = B_r x + x^{1/3} R_{2^r-2}(\log x) + O(x^{b(r)+\varepsilon}),$$

for every $\varepsilon > 0$, where $b(r) := \frac{1}{4-\alpha_{2^r-1}}$, α_k is as defined in Lemma 2.2, the $O-$term is related to r, $R_{2^r-2}(x)$ is a polynomial of degree $2^r - 2$ and

$$B_r := \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{\phi^r(a) - \phi^r(a-1)}{p^a}\right).$$

Remark 1. *Throughout this paper, the letter ε denotes a sufficiently small positive constant but may not be the same at each occurrences. Divisor functions $d(n) = \sum_{n=ab} 1$, $d_k(n) = \sum_{n=m_1\cdots m_k} 1$ and $d(1, \underbrace{3, \ldots, 3}_{k}; n) = \sum_{n=ab_1^3\cdots b_k^3} 1$. $f(x) \ll g(x)$ or $f(x) = O(g(x))$ denotes that $|f(x)| \leq Cg(x)$, where C is a positive constant.*

§2. Some lemmas

In this section, we give some lemmas which will be used in the proof of our theorem. Lemma 2.2 and 2.3 can be found in [1] and [6].

Lemma 2.1. *For $r \geq 1$, then we have*

$$\sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^r}{n^s} = \zeta(s)\zeta^{2^r-1}(3s)V(s),$$

where the infinite series $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$.

Proof. Note that the function $\phi^{(e)}(n)$ is multiplicative and for every prime power $p^a (a \geq 1)$, we have $\phi^{(e)}(p^a) = \phi(a)$, where ϕ is the Euler-function. By Euler's product formula, we can get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^r}{n^s} &= \prod_p \left(1 + \frac{\phi^r(1)}{p^s} + \frac{\phi^r(2)}{p^{2s}} + \frac{\phi^r(3)}{p^{3s}} + \frac{\phi^r(4)}{p^{4s}} + \frac{\phi^r(5)}{p^{5s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{2^r}{p^{3s}} + \frac{2^r}{p^{4s}} + \frac{4^r}{p^{5s}} + \cdots\right) \\
&= \zeta(s) \prod_p \left(1 + \frac{2^r - 1}{p^{3s}} + \frac{4^r - 2^r}{p^{5s}} + \cdots\right) \\
&= \zeta(s)\zeta^{2^r-1}(3s) \prod_p \left(1 + \frac{4^r - 2^r}{p^{5s}} + \cdots\right) \\
&= \zeta(s)\zeta^{2^r-1}(3s)V(s), \qquad (5)
\end{aligned}$$

where the infinite series $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$. □

Lemma 2.2. *Suppose $k \geq 2$ is an integer. Then*

$$D_k(x) = \sum_{n \leq x} d_k(n) = x \sum_{j=0}^{k-1} c_j (\log x)^j + O(x^{\alpha_k + \varepsilon}),$$

where c_j is a calculable constant, ε is a sufficiently small positive constant, α_k is the infimum of numbers a_k, such that

$$\Delta_k(x) = \sum_{n \leq x} d_k(n) - xP_{k-1}(\log x) \ll x^{a_k + \varepsilon}, \qquad (6)$$

and

$$\begin{aligned}
&\alpha_2 \leq \frac{131}{416}, \quad \alpha_3 \leq \frac{43}{94}, \\
&\alpha_k \leq \frac{3k-4}{4k}, \quad 4 \leq k \leq 8, \\
&\alpha_9 \leq \frac{35}{54}, \quad \alpha_{10} \leq \frac{41}{61} \quad \alpha_{11} \leq \frac{7}{10}, \\
&\alpha_k \leq \frac{k-2}{k+2}, \quad 12 \leq k \leq 25, \\
&\alpha_k \leq \frac{k-1}{k+4}, \quad 26 \leq k \leq 50,
\end{aligned}$$

$$\alpha_k \le \frac{31k-98}{32k}, \quad 51 \le k \le 57,$$
$$\alpha_k \le \frac{7k-34}{7k}, \quad k \ge 58.$$

Lemma 2.3. *Suppose $f(m)$, $g(n)$ are arithmetical functions such that*

$$\sum_{m\le x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^{\alpha}), \quad \sum_{n\le x} |g(n)| = O(x^{\beta}),$$

where $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_J > \alpha > \beta > 0$, $P_j(t)$ is a polynomial in t. If $h(n) = \sum_{n=md} f(m)g(d)$, then

$$\sum_{n\le x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha}),$$

where $Q_j(t)\{j = 1,\ldots,J\}$ is a polynomial in t.

§3. Estimate of $D(1,\underbrace{3,\ldots,3}_{k};x)$

Theorem 3.1. *Suppose $k \ge 2$ is an integer, then*

$$D(1,\underbrace{3,\ldots,3}_{k};x) = \sum_{n\le x} d(1,\underbrace{3,\ldots,3}_{k};n) = \zeta^k(3)x + x^{\frac{1}{3}} Q_{k-1}(\log x) + O(x^{\frac{1}{4-\alpha_k}+\varepsilon}),$$

where Q_{k-1} is a polynomial of degree $k-1$ in $\log x$, α_k is defined in Lemma 2.2.

Proof. Recall that $d(1,\underbrace{3,\ldots,3}_{k},n) = \sum_{n=ab_1^3\cdots b_k^3} 1$, by hyperbolic summation formula, we have

$$\begin{aligned} D(1,\underbrace{3,\ldots,3}_{k},x) &= \sum_{n\le x} d(1,\underbrace{3,\ldots,3}_{k};n) = \sum_{m^3l\le x} d_k(m) \\ &= \sum_{m\le y} d_k(m) \sum_{m^3l\le x} 1 + \sum_{l\le z} \sum_{m^3l\le x} d_k(m) - \sum_{m\le y} d_k(m) \sum_{l\le z} 1 \\ &:= S_1 + S_2 - S_3, \end{aligned} \tag{7}$$

where y,z are parameters that will be determined later, and satisfy that $y^3z = x, 1 \le y \le x$. Now, we deal with S_1, S_2 and S_3, separately.

$$\begin{aligned} S_1 &= \sum_{m\le y} d_k(m) \sum_{m^3l\le x} 1 = \sum_{m\le y} d_k(m) \left[\frac{x}{m^3}\right] \\ &= x \sum_{m\le y} \frac{d_k(m)}{m^3} + O\left(\sum_{m\le y} d_k(m)\right) \\ &= \zeta^k(3)x - x \sum_{m>y} \frac{d_k(m)}{m^3} + O(y^{1+\varepsilon}). \end{aligned} \tag{8}$$

Using Lemma 2.2 and partial summation formula, we have

$$
\begin{aligned}
\sum_{m>y}\frac{d_k(m)}{m^3} &= \int_{y^+}^{\infty}\frac{1}{t^3}d\left(\sum_{m\le t}d_k(m)\right)=\int_{y^+}^{\infty}\frac{1}{t^3}d\left(t\sum_{j=0}^{k-1}c_j(\log t)^j+O(t^{\alpha_k+\varepsilon})\right)\\
&= \sum_{j=0}^{k-1}c_j\int_{y^+}^{\infty}\frac{1}{t^3}d\left(t(\log t)^j\right)+O(y^{-3+\alpha_k+\varepsilon})\\
&= \sum_{j=0}^{k-1}c_jy^{-2}\left[\frac{1}{2}(\log y)^j+\frac{3}{4}j(\log y)^{j-1}+\frac{3}{8}j(j-1)(\log y)^{j-2}+\cdots+\frac{3}{2^{j+1}}j(j-1)\cdots 1\right]\\
&+ O(y^{-3+\alpha_k+\varepsilon}).
\end{aligned}
$$

Since $y=\sqrt[3]{\frac{x}{z}}$, we have $\log y=\frac{1}{3}(\log x-\log z)$, inserting this into (8), we can get

$$S_1=\zeta^k(3)x-S_{11}-S_{12}+O(y^{1+\varepsilon}+xy^{-3+\alpha_k+\varepsilon}), \tag{9}$$

where

$$
\begin{aligned}
S_{11} &= \frac{1}{2}x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=1}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\log z)^i,\\
S_{12} &= \frac{3}{2}x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=1}^{k-1}c_j\sum_{i=0}^{j-1}\frac{j!}{i!2^{j-i}3^i}\sum_{s=0}^{i}C_i^s(\log x)^{i-s}(-1)^s(\log z)^s.
\end{aligned}
$$

By Lemma 2.2, we get

$$
\begin{aligned}
S_2 &= \sum_{l\le z}\sum_{m\le\sqrt[3]{\frac{x}{l}}}d_k(m)=\sum_{l\le z}\left(\sqrt[3]{\frac{x}{l}}\sum_{j=0}^{k-1}c_j\left(\log\sqrt[3]{\frac{x}{l}}\right)^j+O\left(\left(\sqrt[3]{\frac{x}{l}}\right)^{\alpha_k+\varepsilon}\right)\right)\\
&= x^{\frac{1}{3}}\sum_{j=0}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i\sum_{l\le z}l^{-\frac{1}{3}}(\log l)^i+O(xy^{-3+\alpha_k+\varepsilon}), \qquad (10)
\end{aligned}
$$

where

$$\sum_{l\le z}l^{-\frac{1}{3}}(\log l)^i=\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^id[t]=\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^idt+\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^id\Delta(t). \tag{11}$$

We can easily get that $\Delta(t)=O(1)$. Using partial integral formula, we have

$$\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^id\Delta(t)=w_i+O(z^{-\frac{1}{3}+\varepsilon}), \tag{12}$$

where w_i is a constant. We can also obtain that

$$\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^idt=\frac{3}{2}z^{\frac{2}{3}}(\log z)^i-\left(\frac{3}{2}\right)^2iz^{\frac{2}{3}}(\log z)^{i-1}+\cdots+(-1)^{i+1}\left(\frac{3}{2}\right)^{i+1}i!. \tag{13}$$

Combing (10)-(13), we have

$$S_2=x^{\frac{1}{3}}\tilde{Q}_{k-1}(\log x)+S_{21}+S_{22}+O(xy^{-3+\alpha_k+\varepsilon}), \tag{14}$$

where

$$\tilde{Q}_{k-1}(\log x) = \sum_{j=0}^{k-1} \frac{c_j}{3^j} \sum_{i=0}^{j} C_j^i (\log x)^{j-i} (-1)^i \left(w_i - (-1)^i \left(\frac{3}{2}\right)^{i+1} i! \right),$$

$$S_{21} = \frac{3}{2} x^{\frac{1}{3}} z^{\frac{2}{3}} \sum_{j=0}^{k-1} \frac{c_j}{3^j} \sum_{i=0}^{j} C_j^i (\log x)^{j-i} (-1)^i (\log z)^i,$$

$$S_{22} = \frac{3}{2} x^{\frac{1}{3}} z^{\frac{2}{3}} \sum_{j=0}^{k-1} \frac{c_j}{3^j} \sum_{i=0}^{j} C_j^i (\log x)^{j-i} (-1)^i \sum_{s=0}^{i-1} (-1)^{s-i} \left(\frac{3}{2}\right)^{i-s} \frac{i!}{s!} (\log z)^s.$$

For S_3, we have

$$\begin{aligned} S_3 &= \sum_{m \le y} d_k(m) \sum_{l \le z} 1 = zy \sum_{j=0}^{k-1} c_j (\log y)^j + O(y^{\alpha_k + \varepsilon} z) + O(y^{1+\varepsilon}) \\ &= yz \sum_{j=0}^{k-1} c_j (\log y)^j + O(y^{\alpha_k+\varepsilon} z + y^{1+\varepsilon}). \end{aligned} \tag{15}$$

Inserting $y = \sqrt[3]{\frac{x}{z}}$, and $\log y = \frac{1}{3}(\log x - \log z)$ into (15), then

$$S_3 = S_{31} + O(y^{\alpha_k+\varepsilon} z + y^{1+\varepsilon}), \tag{16}$$

where

$$S_{31} = x^{\frac{1}{3}} z^{\frac{2}{3}} \sum_{j=0}^{k-1} \frac{c_j}{3^j} \sum_{i=0}^{j} C_j^i (\log x)^{j-i} (-1)^i (\log z)^i.$$

Note that $C_j^i = \frac{i!}{j!(i-j)!}$. After some simplification we can easily get that $S_{11} + S_{31} = S_{21}$, $S_{12} = S_{22}$. Taking $y = x^{\frac{1}{4-\alpha_k}}$, $z = x^{\frac{1-\alpha_k}{4-\alpha_k}}$, then Theorem 3.1 is proved. □

§4. Proof of Theorem 1.1

For $r \ge 1$, from Lemma 2.1, we have $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$, and then

$$\sum_{n \le x} |v(n)| \ll x^{\frac{1}{5}+\varepsilon}. \tag{17}$$

Let $F(s) = \zeta(s)\zeta^{2^r-1}(3s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, where $f(n) = d(1, \underbrace{3, \ldots, 3}_{2^r-1}, n)$.

From Theorem 3.1, we have

$$\sum_{n \le x} f(n) = \sum_{n \le x} d(1, \underbrace{3, \ldots, 3}_{2^r-1}, n) = \zeta^{2^r-1}(3)x + x^{\frac{1}{3}} \tilde{Q}_{2^r-2}(\log x) + O(x^{\frac{1}{4-\alpha_{2^r-1}}+\varepsilon}), \tag{18}$$

where $\tilde{Q}_{2^r-2}(\log x)$ is a polynomial in $\log x$ of degree 2^r-2, α_k is as defined in Lemma 2.2. From Lemma 2.1, we have

$$(\phi^{(e)}(n))^r = \sum_{n=kl} v(k) f(l), \tag{19}$$

then, by Lemma 2.3 we can get the Theorem 1.1.

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions, which highly improve the quality of this paper. This work is supported by Natural Science Foundation of Shandong Province(No. ZR2015AM010).

References

[1] A. Ivić. The Riemann zeta-function: theory and applications. Oversea Publishing House, 2003.

[2] J. Sándor. On an exponential totient function. Sudia Univ. Babes-Bolyai, Math., 1996, 41 (3): 91–94.

[3] M. V. Subbarao. On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol. 251, Springer, 1972, 247–271.

[4] L. Tóth. On certain arithmetic functions involving exponential divisors. Ann. Univ. Sci. Budapest. Sect. Comput., 2004, 24: 285–294.

[5] L. Tóth. An order result for the exponential divisor function. Publ. Math. Debrecen, 2007, 71(1-2): 165–171.

[6] L. Zhang, M. Lü and W. Zhai. On the Smarandache ceil function and the Dirichlet divisor function. Sci. Magna, 2008, 4(4): 55–57.

Scientia Magna
Vol. 11 (2016), No. 2, 14-20

Ideal on generalized topological spaces

Shyamapada Modak

Department of Mathematics, University of Gour Banga
P.O. Mokdumpur, Malda-732103, India
e-mail: spmodak2000@yahoo.co.in

Abstract The aim of this paper is to introduce ideal generalized topological spaces and to investigate the relationships between generalized topological spaces and ideal generalized topological spaces. For establishment of their relationships, we define some closed sets in these spaces. Basic properties and characterization related to these sets are also discussed.

Keywords topological ideal, generalized topological space, ideal generalized topological space, g_μ-closed set, μ^*-closed set, μ-I_g-closed set.
2000 Mathematics Subject Classification: 54A05, 54C10.

§1. Introduction and preliminaries

The study of ideal topological space [8] has been started from 1933 and till, it is developing by several mathematicians. Generalized closed sets [9] in topological space as well as in ideal topological space [5,11] has been discussed at various research papers. We have introduced the generalized closed sets in ideal generalized topological space (generalized topological space (GTS) [2,3] with ideal), and characterized the same at different aspect. We also obtain the relations with earlier generalized closed sets in topological space, generalized topological space and ideal generalized topological space etc.

Definition 1.1.[8] *An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions:*

(i) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$;

(ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X, if $\wp(X)$ is the set of all subsets of X, a set operator $()^* : \wp(X) \to \wp(X)$, is called a local function with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $(A)^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$ [8].

A Kuratowski closure operator cl^* for a topology $\tau^*(\mathcal{I}, \tau)$, called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [16]. We will simply write A^* for $A^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space.

Definition 1.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is τ^*-closed [7] (resp. *-dense in itself [6], *-perfect [6]), if $A^* \subseteq A$(resp. $A \subseteq A^*$, $A = A^*$). Through the paper, we will use *-closed instead of τ^*-closed.*

Definition 1.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is I_g-closed* [5] *if $A^* \subseteq U$ whenever U is open and $A \subseteq U$.*

Definition 1.4. *Let (X, τ) be a topological space. A subset A of a space (X, τ) is said to be g-closed set* [9] *if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.*

Remark 1.1.[5] *Every g-closed set is an I_g-closed but not vice versa.*

Remark 1.2.[16] *Every closed set is g-closed.*

Very interesting notion in literature has been introduced by Császár [1] in 1997. Using this notion, topology has been constructed. The concept is:

A map $\gamma : \exp\,(X) \to \exp\,(X)$ possessing the property monotony (i.e. such that $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$). We denote by $\Gamma(X)$ the collections of all mapping having this property.

One of the consequence of the above concept is generalized topological space (GTS) [2,3], its formal definition is:

Definition 1.5. *Let X be a non-empty set, and $\mu \subseteq exp(X)$. μ is called a generalized topology (GT) on X if $\emptyset \in \mu$ and the union of elements of μ belongs to μ.*

The member of μ is called μ-open set and the complement of μ-open set is called μ-closed set. Again c_μ is the notation of μ-closure [2,3,14,10].

Definition 1.6.[14] *Let (X, μ) be a generalized topological space. Then the generalized kernel of $A \subseteq X$ is denoted by g-ker(A) and defined as g-ker(A) $= \cap\{G \in \mu :\ A \subseteq G\}$.*

Lemma 1.1.[14] *Let (X, μ) be a generalized topological space and $A \subseteq X$. Then g-ker(A) $= \{x \in X :\ c_\mu(\{x\}) \cap A \neq \emptyset\}$.*

If $\mathcal{I}$ be an ideal on X, then $(X, \mu, \mathcal{I})$ is called an ideal generalized topological space.

§2. Ideal generalized topological space

Definition 2.1. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. A mapping $()^{*\mu}$: $exp\ X \to exp\ X$ is defined as follows:*

$(A)^{*\mu} = (A)^{*\mu}(\mathcal{I}) = \{x \in X :\ A \cap U \notin \mathcal{I}\}$, where $U \in \psi(x)$ [2].

The mapping is called the Local function associated with the ideal $\mathcal{I}$ and generalized topology μ.

Properties:

Theorem 2.1. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Then*

(1) $(\emptyset)^{*\mu} = \emptyset$.

(2) for $A, B \subseteq X$ and $A \subseteq B$, $(A)^{*\mu} \subseteq (B)^{*\mu}$.

(3) $(A)^{*\mu} \subseteq c_\mu(A)$.

(4) $((A)^{*\mu})^{*\mu} \subseteq c_\mu(A)$.

(5) $(A)^{*\mu}$ is a μ-closed set.

(6) $((A)^{*\mu})^{*\mu} \subseteq (A)^{*\mu}$.

(7) for $\mathcal{I} \subseteq \mathcal{I}_1$ implies $(A)^{*\mu}(\mathcal{I}_1) \subseteq (A)^{*\mu}(\mathcal{I})$.

(8) for $U \in \mu$, $U \cap (U \cap A)^{*\mu} \subseteq U \cap (A)^{*\mu}$.

(9) for $I \in \mathcal{I}$, $(A \setminus I)^{*\mu} = (A)^{*\mu} = (A \cup I)^{*\mu}$.

Proof. (1) It is obvious from definition.

(2) It is done by the fact of $A \cap V \notin \mathcal{I}$ implies $B \cap V \notin \mathcal{I}$.

(3) Obvious from [13].

(4) $((A)^{*\mu})^{*\mu} \subseteq c_\mu(c_\mu(A)) = c_\mu(A)$.[3]

(5) From [2], for $G \in \mu$ and $x \in G$, there exists $V \in \psi(x)$ such that $V \subseteq G$. Now if $A \cap G \in \mathcal{I}$ then for $A \cap V \subseteq A \cap G$, $A \cap V \in \mathcal{I}$. It follows that $X \setminus (A)^{*\mu}$ is the union of μ-open sets. We know that the arbitrary union of μ-open sets is an μ-open set. So $X \setminus (A)^{*\mu}$ is an μ-open set and hence $(A)^{*\mu}$ is a μ-closed set.

(6) From above, $((A)^{*\mu})^{*\mu} \subseteq c_\mu((A)^{*\mu}) = (A)^{*\mu}$, since $(A)^{*\mu}$ is a μ-closed set.

(7) Obvious from the fact that $A \cap V \notin \mathcal{I}_1$ implies $A \cap V \notin \mathcal{I}$.

(8) Since $U \cap A \subseteq A$, then $(U \cap A)^{*\mu} \subseteq (A)^{*\mu}$. So $U \cap (U \cap A)^{*\mu} \subseteq U \cap (A)^{*\mu}$.

(9) Let $x \in (A)^{*\mu}$. If possible suppose that $x \notin (A \setminus I)^{*\mu}$. Then there is an $V \in \psi(x)$, $V \cap (A \setminus I) \in \mathcal{I}$. Therefore $(V \cap (A \setminus I)) \cup I \in \mathcal{I}$, i.e., $I \cup (A \cap V) \in \mathcal{I}$. Then $V \cap A \in \mathcal{I}$, a contradiction to the fact that $x \in (A)^{*\mu}$. Hence $(A \setminus I)^{*\mu} = (A)^{*\mu}$.

Proof of 2nd part is similar. □

It is obvious from (2), $()^{*\mu} \in \Gamma(X)$ [1].

Definition 2.2. *Let (X, μ) be a generalized topological space with an ideal $\mathcal{I}$ on X.*

The set operator $c^{*\mu}$ is called a generalized *-closure and is defined as $c^{*\mu}(A) = A \cup (A)^{*\mu}$, for $A \subseteq X$. We will denote by $\mu^*(\mu;\mathcal{I})$ the generalized structure, generated by $c^{*\mu}$, that is, $\mu^*(\mu;\mathcal{I})= \{U \subseteq X : c^{*\mu}(X \setminus U) = (X \setminus U)\}$. $\mu^*(\mu;\mathcal{I})$ is called *-generalized structure which is finer than μ.

The element of $\mu^*(\mu;\ \mathcal{I})$ are called μ^*-open and the complement of an μ^*-open is called μ^*-closed.

Theorem 2.2. *The set operator $c^{*\mu}$ satisfy following conditions:*

(a) $A \subseteq c^{*\mu}(A)$, for $A \subseteq X$.

(b) $c^{*\mu}(\emptyset) = \emptyset$ and $c^{*\mu}(X) = X$.

(c) $c^{*\mu}(A) \subseteq c^{*\mu}(B)$ if $A \subseteq B \subseteq X$.

(d) $c^{*\mu}(A) \cup c^{*\mu}(B)) \subseteq c^{*\mu}(A \cup B)$.

(e) $c^{*\mu} \in \Gamma(X)$.

Proof. Proof is obvious from Theorem 2.1. □

Although some results of the Theorem 2.1 and the Theorem 2.2 have been proved by Á. Császár [4] in his paper "Modification of generalized topologies via hereditary classes" published in Acta Math. Hungar. in 2007 using Hereditary class.

Definition 2.3. *Let (X, μ) be a generalized topological space. A subset A of X is said to be g_μ-closed* [10] *if $c_\mu(A) \subseteq M$ whenever $A \subseteq M$ and $M \in \mu$.*

Definition 2.4. *A subset A of an ideal generalized topological space $(X, \mu, \mathcal{I})$ is μ^*-dense in itself (resp. μ^*-perfect) if $A \subseteq (A)^{*\mu}$ (resp. $(A)^{*\mu} = A$).*

Definition 2.5. *A subset A of an ideal generalized topological space $(X, \mu, \mathcal{I})$ is called μ -I-generalized closed (briefly, μ-I_g-closed) if $(A)^{*\mu} \subseteq U$ whenever U is μ-open and $A \subseteq U$. A subset A of an ideal generalized topological space $(X, \mu, \mathcal{I})$ is called μ -I-generalized open(briefly, μ-I_g-open) if $X \setminus A$ is μ-I_g-closed.*

Theorem 2.3. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Every g_μ-closed set is μ-I_g-closed.*

Proof. Let U any μ-open set containing A. Since A is g_μ-closed, then $c_\mu(A) \subseteq U$. By Theorem 2.1 (3), we have $(A)^{*\mu} \subseteq U$. □

Remark 2.1. *Let (X, τ) be a topological space. If we take $\mu = \tau$, then g_μ-closed sets coincide with g-closed sets.*

Proposition 2.1. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Then*

(a) Every μ^*-prefect set is μ^*-dense in itself.

(b) Every μ^*-perfect set is μ^*-closed.

Proof. The proof can be easily done. □

Remark 2.2. *Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X. If we take $\mu = \tau$, then μ-I_g-closed (resp. μ^*-closed, μ^*-dense in itself) sets coincide with I_g-closed [5] (resp. *-closed [7], *-dense in itself [7]).*

Theorem 2.4. *If $(X, \mu, \mathcal{I})$ is an ideal generalized topological space and $A \subseteq X$, then A is μ-I_g-closed if and only if $c^{*\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is μ-open in X.*

Proof. Since A is μ-I_g-closed, we have $(A)^{*\mu} \subseteq U$ whenever $A \subseteq U$ and U is μ-open in X. $c^{*\mu}(A) = A \cup (A)^{*\mu} \subseteq U$ whenever $A \subseteq U$ and U is μ-open in X.

Converse part: Let $A \subseteq U$ and U be μ-open in X. By hypothesis $c^{*\mu}(A) \subseteq U$. Since $c^{*\mu}(A) = A \cup (A)^{*\mu}$, we have $(A)^{*\mu} \subseteq U$. □

Theorem 2.5. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $A \subseteq X$. Then the following are equivalent:*

(a) A is μ-I_g-closed.

(b) $c^{*\mu}(A) \subseteq U$ whenever $A \subseteq U$ and U is μ-open in X.

(c) $c^{*\mu}(A) \subseteq g$-ker(A).

(d) $c^{*\mu}(A) \setminus A$ contains no nonempty μ-closed set.

(e) $(A)^{*\mu} \setminus A$ contains no nonempty μ-closed set.

Proof. (a) $\Leftrightarrow$ (b) It follows from Theorem 2.4.

(b) $\Rightarrow$ (c) Suppose $x \in c^{*\mu}(A)$ and $x \notin g$-ker(A). Then $c_\mu(\{x\}) \cap A = \emptyset$. Implies that $A \subseteq X \setminus (c_\mu(\{x\}))$. Now from (b), $c^{*\mu}(A) \subseteq X \setminus c_\mu(\{x\})$. This implies $c^{*\mu}(A) \cap \{x\} = \emptyset$, a contradiction. Hence the result.

(c) $\Rightarrow$ (d) Suppose $F \subseteq (c^{*\mu}(A)) \setminus A$, F is μ-closed and $x \in F$. Since $F \subseteq (c^{*\mu}(A)) \setminus A$, $F \cap A = \emptyset$. We have $c_\mu(\{x\}) \cap A = \emptyset$ because F is μ-closed and $x \in F$. From (c), this is a contradiction.

(d) $\Rightarrow$ (e) This is obvious from the definition of $c^{*\mu}(A)$.

(e) $\Rightarrow$ (a) Let U be an μ-open subset containing A. Since $(A)^{*\mu}$ is μ-closed by means of Theorem 2.1 (5). Now $(A)^{*\mu} \cap (X \setminus U) \subseteq (A)^{*\mu} \setminus A$. Since intersection of two μ-closed sets is a μ-closed set, then $(A)^{*\mu} \cap (X \setminus U)$ is an μ-closed set contained in $(A)^{*\mu} \setminus A$. By assumption, $(A)^{*\mu} \cap (X \setminus U) = \emptyset$. Hence, we have $(A)^{*\mu} \subseteq U$. □

Remark 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal generalized topological space. If $\mu = \tau$ then the above theorem coincides with Theorem 2.1 in [12].*

Proposition 2.2. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Every μ^*-closed set is μ-I_g-closed.*

Proof. Let A be a subset of X and A be μ^*-closed. Assume that $A \subseteq U$ and U is μ-open. Since A is μ^*-closed, we have $(A)^{*\mu} \subseteq A$ and so A is μ-I_g-closed. □

For the relationship related to several sets defined in the paper, we have the following diagram:

$$\mu^*\text{-dense in itself} \Longleftarrow \mu^*\text{-perfect} \Longrightarrow \mu^*\text{-closed} \Longrightarrow \mu\text{-}I_g\text{-closed} \Longleftarrow g_\mu\text{-closed} \Longleftarrow \mu\text{-closed}.$$

The following examples show that the converse implications of the diagram are not satisfied.

Example 2.1. *(i) Let $X = \{a, b, c, d\}$, $\mu = \{X, \emptyset, \{a\}, \{a, b\}, \{b, c, d\}\}$, $\mathcal{I} = \{\emptyset, \{c\}, \{b\}, \{b, c\}\}$ and $A = \{a, b\}$. It is obvious that the μ-open sets containing A are X and $\{a, b\}$. $(A)^{*\mu} = \{a\}$ is also contained in X and $\{a, b\}$. Thus, A is μ-I_g-closed. But A is not g_μ-closed, since $c_\mu(A) = X$ is not a subset of $\{a, b\}$.*

(ii) In (i), let $B = \{a, c\}$. Note that the only μ-open set containing A is X. $c_\mu(A) = X$ is also contained in X. Therefore A is g_μ-closed but not μ-closed.

(iii) In (i), B is μ^*-closed but not μ^*-perfect.

(iv) Let $X = \{a, b, c\}$, $\mu = \{X, \emptyset, \{a\}, \{a, b\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$ and $A = \{a, c\}$. Notice that only μ-open set containing A is X. $(A)^{*\mu} = X$ also contained in X. Hence, A is μ-I_g-closed but not μ^*-closed.

(v) In (iv), A is μ^*-dense in itself but not μ^*-perfect.

Definition 2.6.[15] *A space (X, μ) is called μ-T_1 if any pair of distinct points x and y of X, there exists a μ-open set U of X containing x but not y and a μ-open set V of X containing y but not x.*

It is obvious from definition that every singleton set is μ-closed if and only if the space is μ-T_1.

Remark 2.4. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $A \subseteq X$. If (X, μ) is a μ-T_1 space, then A is μ^*-closed if and only if A is μ-I_g-closed.*

Theorem 2.6. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $A \subseteq X$. If A is an μ-I_g-closed set, then the following are equivalent:*

(a) A is a μ^*-closed set.

(b) $c^{*\mu}(A) \setminus A$ is a μ-closed set.

(c) $(A)^{*\mu} \setminus A$ is a μ-closed set.

Proof. (a)⇒(b) If A is μ^*-closed, then $c^{*\mu}(A) \setminus A = \emptyset$. $c^{*\mu}(A) \setminus A$ is μ-closed.

(b)⇒(c) Since $c^{*\mu}(A) \setminus A = (A)^{*\mu} \setminus A$, it is clear.

(c)⇒(a) If $(A)^{*\mu} \setminus A$ is μ-closed and A is μ-I_g-closed, from Theorem 2.5 (e), $(A)^{*\mu} \setminus A = \emptyset$ and so A is μ^*-closed. □

Lemma 2.1. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $A \subseteq X$. If A is μ^*-dense in it self, then $(A)^{*\mu} = c_\mu((A)^{*\mu}) = c_\mu(A) = c^{*\mu}(A)$.*

Proof. Let A be μ^*-dense in itself. Then we have $A \subseteq (A)^{*\mu}$ and hence $c_\mu(A) \subseteq c_\mu((A)^{*\mu})$. We know that $(A)^{*\mu} = c_\mu((A)^{*\mu}) \subseteq c_\mu(A)$ from Theorem 2.1 (3). In this case $c_\mu(A) = c_\mu((A)^{*\mu}) = (A)^{*\mu}$. Since $(A)^{*\mu} = c_\mu(A)$, we have $c^{*\mu}(A) = c_\mu(A)$. □

We obtained that every g_μ-closed set is μ-I_g-closed in Theorem 2.3 but not vice versa. For μ^*-dense in itself sets, g_μ-closedness and μ-I_g-closedness are equivalent.

Theorem 2.7. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $A \subseteq X$. If A is μ^*-dense in itself and μ-I_g-closed, then A is g_μ-closed.*

Proof. Assume A is μ^*-dense in itself and μ-I_g-closed in X. If U is an μ-open set containing A, then we have $(A)^{*\mu} \subseteq U$. Since A is μ^*-dense it self, Lemma 2.1 implies $c_\mu(A) \subseteq U$ and so A is g_μ-closed. □

Theorem 2.8. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $A \subseteq X$. If A is μ-I_g-closed and μ-open then A is μ^*-closed.*

Proof. Let A be an μ-open. Since A is μ-I_g-closed, we have $(A)^{*\mu} \subseteq A$. Hence A is μ^*-closed. □

References

[1] Á. Császár. Generalized open sets. Acta Math. Hungar., 1997, 75 (1-2): 65–87.

[2] Á. Császár. Generalized topology, generalized continuity. Acta Math. Hungar., 2002, 96 (4): 351–357.

[3] Á. Császár. Generalized open sets in generalized topologies. Acta Math. Hungar., 2005, 106 (1-2): 57–66.

[4] Á. Császár. Modification of generalized topologies via hereditary classes. Acta Math. Hungar., 2007, 115 (1-2): 29–36.

[5] J. Dontchev, M. Ganster and T. Noiri. Unified operation approach of generalized closed sets via topological ideals. Math. Japon., 1999, 49 (3): 395–401.

[6] E. Hayashi. Topologies defined by local properties. Math. Ann., 1964, 156: 205–215.

[7] D. Jankovic and T. R. Hamlett. New topologies from old via ideals. Amer. Math. Monthly, 1990, 97 (4): 295–310.

[8] K. Kuratowski. Topology, Vol. *I*. New York: Academic Press, 2014.

[9] N. Levine. Generalized closed sets in topology. Rend. Circ. Mat. Palermo, 1970, 19 (2): 89–96.

[10] S. Maragathavalli, M. Sheik John and D. Sivaraj. On g-closed sets in genaralized topological spaces. J. Adv. Res. Pure maths., 2010, 2 (1): 57–64.

[11] M. Navaneethakrishnan and J. Paulraj Joseph. g-closed sets in ideal topological spaces. Acta Math. Hungar., 2008, 119: 365–371.

[12] T. Noiri and V. Popa. Between closed sets and g-closed sets. Rend. Circ. Mat. Palermo, 2006, 55 (2): 175–184.

[13] T. Noiri and B. Roy. Unification of generalized open sets on topological spaces. Acta Math. Hungar., 2011, 130 (4): 349–357.

[14] B. Roy. On generalized R_0 and R_1 spaces. Acta Math. Hungar., 2010, 127 (3): 291–300.

[15] M. S. Sarsak. Weak separation axioms in generalized topological spaces. Acta Math. Hungar., 2011, 131 (1-2): 110–121.

[16] R. Vaidyanathaswamy. The localization theory in set topology. Proc. Indian Acad. Sci. Sect A, 1944, 20: 51–61.

Scientia Magna
Vol. 11 (2016), No. 2, 21-32

Edge-distance pattern distinguishing graph

Kishori P. Narayankar[1], S. B. Lokesh[2] and H. S. Ramane[3]

[1]Department of Mathematics, Mangalore University
Mangalagangothri, Mangalore-574199, India.
E-mail: kishori_pn@yahoo.co.in
[2]Department of Mathematics, Mangalore University
Mangalagangothri, Mangalore-574199, India.
E-mail: sbloki83@gmail.com
[3]Department of Mathematics, Karnatak University
Dharwad-580003, India.
E-mail: hsramane@yahoo.com

Abstract Let $G = (V, E)$ be a given non trivial and connected simple (p, q)-graph, and M be an arbitrary nonempty subset of an edge set $E(G)$ of G. For each $e \in E(G)$, define $N_j^M[e] = \{f \in M : d_2(e, f) = j\}$, where $d_2(e, f)$ denotes the distances of f from the edge e. B.D. Acharya, defined the M-eccentricity of f as the largest j for which $N_j^M[f] \neq \emptyset$, $d_2(G)$ as the largest M-eccentricity of edges in G and the nonnegative integer $q \times (d_2(G)$-matrix $D_2^M(G) = (|N_j^M[e_i]|)$ as the 'Edge-M-distance neighborhood pattern' (or, Edge-$M - dnp$) matrix of G. The associated $(0, 1)$-matrix $D_2^{*M}(G)$ is obtained from $D_2^M(G)$ by replacing each nonzero entry in it by 1. Let $f_M(e) = \{j : N_j^M[e] \neq \emptyset\}$ for each $e \in E(G)$. If $f_M : e \longmapsto f_M(e)$ is an injective function, then the set M is a 'Edge-M-distance-pattern distinguishing set' (or, a "Edge-DPD-set" in short) of G and G is a 'Edge-DPD-graph'. If $f_M(e) \setminus \{0\}$ is independent of the choice of e in G then M is said to be a 'Edge-open distance-pattern uniform' (or, 'Edge-ODPU') set of G. A study of these sets is useful in a number of areas of application such as facility location and design of indices of "quantitative structure activity relationships" (QSAR) in chemistry. This paper is a study of Edge- M-dnp matrices of a Edge-dpd-graph for a class of graphs.

Keywords distance(in graph), edge-to-edge-set distance-pattern distinguishing sets, edge-distance neighborhood pattern matrix, edge-to-edge-set distance-pattern distinguishing graph.
2010 Mathematics Subject Classification 05C12, 05C50.

§1. Introduction

For all terminology which are not defined in this paper, we refer the reader to F.Harary [5]. Unless mentioned otherwise, all the graphs considered in this paper are finite, connected, simple non trivial. Distance between two elements(vertex to vertex, vertex to edge, edge to vertex, and edge to edge) in graphs is already defined in the literature (refer [9]), but here we are using Edge to edge-distance , and call it as Edge-distance. A formal definition is given bellow.

Definition 1.1. [9] *For any connected graph G, the Edge-to-edge-distance $d_2(e,f)$ (in short* Edge-distance*) between two edge e and f is the number of edges between $(e-f)$ path. For any edge e in a connected graph G, the* Edge-eccentricity $e_2(e)$ *of e is $e_2(e) = max\,\{d(e,f) : f \in E(G)\}$. Any edge e for which $e_2(e)$ is minimum is called an* Edge-central *edge of G and the set of all Edge-central edges of G is the* Edge-center C_{2G} *of G.* Edge-diameter $d_{2G} = max\,\{e_2(e)\}$ *and* Edge-radius $r_{2G} = min\,\{e_2(e)\}$. *Any edge f for which $e_2(e) = d_2(e,f) = d_{2G}$ is called an Edge-eccentric edge of e.*

The Edge-to-edge-eccentricities (or Edge-eccentricity) of the Figure 1 is shown in the Table 1.

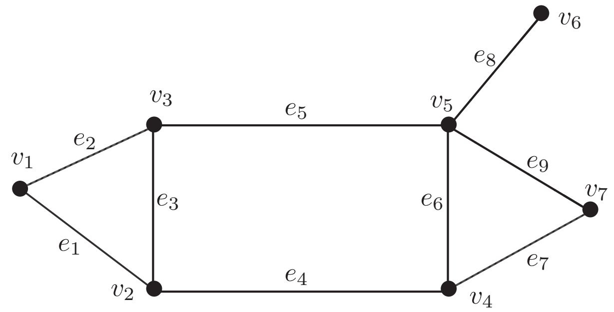

Figure 1: A Graph of Edge-diameter $d_{2G} = 2$

e	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
$e_2(e)$	2	2	1	1	1	1	2	2	2

Table 1: Showing an Edge-eccentricity of all the edges of Figure 1.

For an arbitrarily fixed edge e in G and for any nonnegative integer j, we let $N_j[e] = \{f \in E(G) : d_2(e,f) = j\}$, and $N_j = E(G) - E(\xi_e)$ whenever j exceeds the eccentricity $\epsilon(e)$ of e in the component ξ_e to which e belongs. Thus, if G is connected then, $N_j[e] = \phi$ if and only if $j > \epsilon(e)$. If G is a connected graph then the vectors $\bar{e} = (|N_0[e]|, |N_1[e]|, |N_2[e]|, ..., |N_{\epsilon(e)}[e]|)$ associated $e \in E(G)$ can be arranged as a $q \times (d_{2G}+1)$ nonnegative integer matrix D_{2G} given by

$$\begin{bmatrix} |N_0[e_1]| & |N_1[e_1]| & |N_2[e_1]| & ... & |N_{\epsilon_1(e_1)}[e_1]| & 0 & 0 & 0 \\ |N_0[e_2]| & |N_1[e_2]| & |N_2[e_2]| & ... & ... & |N_{\epsilon_1(e_2)}[e_2]| & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ |N_0[e_q]| & |N_1[e_q]| & |N_2[e_q]| & ... & ... & ... & ... & |N_{\epsilon_1(e_q)}[e_q]| \end{bmatrix}$$

where d_{2G} denotes the diameter of G; we call D_{2G} *edge-to-edge distance neighborhood pattern (or, Edge–dnp-matrix)* of G.

Example:

If we consider the above Figure 1 then the below matrix gives Edge–dnp-matrix

$$D_{2G} = \begin{bmatrix} |N_0[e_1]| = 4 & |N_1[e_1]| = 3 & |N_2[e_1]| = 2 \\ |N_0[e_2]| = 4 & |N_1[e_2]| = 4 & |N_2[e_2]| = 1 \\ |N_0[e_3]| = 5 & |N_1[e_3]| = 4 & |N_2[e_3]| = 0 \\ |N_0[e_4]| = 5 & |N_1[e_4]| = 4 & |N_2[e_4]| = 0 \\ |N_0[e_5]| = 6 & |N_1[e_5]| = 3 & |N_2[e_5]| = 0 \\ |N_0[e_6]| = 6 & |N_1[e_6]| = 3 & |N_2[e_6]| = 0 \\ |N_0[e_7]| = 4 & |N_1[e_7]| = 4 & |N_2[e_7]| = 1 \\ |N_0[e_8]| = 4 & |N_1[e_8]| = 3 & |N_2[e_8]| = 2 \\ |N_0[e_9]| = 5 & |N_1[e_9]| = 3 & |N_2[e_9]| = 1 \end{bmatrix}$$

For a Edge–dnp-matrix the following observations are immediate.

Observation 1.2. *Entries in the first column of D_{2G} corresponds to the nonzero entries.*

Observation 1.3. *In each row of D_{2G}, entry zero will be after the nonzero entries.*

Proposition 1.4. *For each $e \in E(G)$ of a non-trivial connected graph G, $\{N_j[e] : N_j[e] \neq \phi, 0 \leq j \leq d_{2G}\}$ gives a partition of $E(G)$.*

Proof. If possible, let $N_j[e] \cap N_k[e] = f$, for some $e, f \in E(G)$ which implies $d_2(e, f) = j$ and $d_2(e, f) = k$, and hence $j = k$. Therefore,$N_j[e] \cap N_k[e] = \phi$ for any (j, k) with $j \neq k$. Now, clearly, $\bigcup_{j=o}^{d_{2G}} N_j[e] \subseteq E(G)$. Also, for any $f \in E(G)$, since G is connected, $d_2(e, f) = k$, for some $k \in \{0, 1, 2, ..., d_{2G}\}$. That is, $f \in N_k[e]$ for some $k \in \{0, 1, 2, ..., d_{2G}\}$ which implies $E(G) \subseteq \bigcup_{j=o}^{d_{2G}} N_j[e]$. Hence $\bigcup_{j=o}^{d_{2G}} N_j[e] = E(G)$. □

Corollary 1.5. *Each row of the Edge–dnp-matrix D_{2G} of a graph G is the partition of $E(G)$. Hence, sum of the entries in each row of the Edge–dnp-matrix D_{2G} of a graph G is equal to the number of edges of G.*

§2. M-distance Neighborhood Pattern Matrix of a Graph

Given an arbitrary nonempty subset $M \subseteq E(G)$ of G and for each $e, f \in E(G)$, define $N_j^M[e] = \{f \in M : d_(e, f) = j\}$; clearly then $N_j^{E(G)}[e] = N_j[e]$. One can define the M-eccentricity of e as the largest integer for which $N_j^M[e] \neq \phi$ and the $q \times (d_{2G} + 1)$ nonnegative integer matrix $D_{2G}^M = (|N_j^M[e]|)$ is called the M-distance neighborhood pattern (or, M-Edge–dnp) matrix D_{2G}^{*M} is obtained from D_{2G}^M by replacing each nonzero entry by 1. B. D. Acarya [1] defined Edge–dnp-matrix of any graph and in particular, M-Edge–dnp matrix of dpd-graph as follows:

Definition 2.1. [4] *Let $G = (V, E)$ be a given non-trivial connected simple (p,q)-graph, $\phi \neq M \subseteq E(G)$ and $e \in E(G)$. Then the M-Edge–distance-pattern of e is the set $f_M(e) = \{d_2(e, f) : f \in M\}$. Clearly, $f_M(e) = \{j : N_j^M[e] \neq \phi\}$. Hence, in particular, if $f_M : e \longmapsto f_M(e)$ is an injective function, then the set M is a Edge–distance-pattern distinguishing set (or, a "Edge–dpd-set" is short) of G and if $f_M(e) - \{0\}$ is independent of the choice of e in G then M is an Edge–open distance-pattern uniform (or, Edge–odpu) set of G. A graph G with a dpd-set(Edge–odpu-set) is called a Edge–dpd-(Edge–odpu)-graph.*

Following are some interesting results on M-Edge–dnp matrix of connected non-trivial graph G.

Observation 2.2. *Both D_{2G}^{M} and D_{2G}^{*M} do not admit null rows.*

Proposition 2.3. *For each $e_i \in E(G)$, $N_0^M[e_i] = \begin{cases} N[e_i] & if e_i \in M \\ \emptyset & if e_i \notin M \end{cases}$*

Therefore, the entries in the first column of D_{2G}^{*M} will either be 0 or 1.

Corollary 2.4. *If $G \cong K_n, P_2, K_{m,n}$ then $N_0^M[e_i] = \begin{cases} e_i & if e_i \in M \\ \emptyset & if e_i \notin M \end{cases}$*

i.e For all graph of diameter $d_{2G} = 1$.

Remark 2.5. *It should note that Observation is not true in the case of D_{2G}^{*M}.*

Lemma 2.6 is similar to Proposition 1.4.

Lemma 2.6. *For each $e \in E(G)$ of a non-trivial connected graph G, $\{N_j[e] : N_j[e] \neq \phi, 0 \leq j \leq d_{2G}\}$ gives a partition of $E(G)$.*

Proof. If possible, let $N_j[e] \cap N_k[e] = f$, for some $e, f \in E(G)$ which implies $d_2(e, f) = j$ and $d_2(e, f) = k$, and hence $j = k$. Therefore,$N_j[e] \cap N_k[e] = \phi$ for any (j, k) with $j \neq k$. Now, clearly, $\bigcup_{j=o}^{d_{2G}} N_j[e] \subseteq E(G)$. Also, for any $f \in E(G)$, since G is connected, $d_2(e, f) = k$, for some $k \in \{0, 1, 2, ..., d_{2G}\}$. That is, $f \in N_k[e]$ for some $k \in \{0, 1, 2, ..., d_{2G}\}$ which implies $E(G) \subseteq \bigcup_{j=o}^{d_{2G}} N_j[e]$. Hence $\bigcup_{j=o}^{d_{2G}} N_j[e] = E(G)$. $\square$

Corollary 2.6. *Each row of D_{2G}^{M} is a partition of $|M|$.*

Corollary 2.7. *Sum of the entries in each row of D_{2G}^{M} gives $|M|$ and sum of the entries in each row of D_{2G}^{*M} is less than or equal to $|M|$.*

§3. M-Edge–distance Neighborhood Pattern Matrix of a distance Neighborhood Pattern Graph.

In this section we find out some results of D_{2G}^{*M} of a Edge–dpd-graph. From the definition of D_{2G}^{*M}, we have the following observations.

Observation 3.1. *In any graph G, a nonempty $M \subseteq E(G)$ is a Edge–dpd-set if and only if no two rows of D_{2G}^{*M} are identical.*

Observation 3.2. *If any graph of $d_{2G} < 1$ then, D_{2G}, D_{2G}^{M}, and D_{2G}^{*M} are all constant matrix. For Example $G \cong K_{n \leq 3}$ or $K_{1,n-1}$.*

Theorem 3.3. *A Graph $G \cong P_m$ of size $m \geq 2$ admits a Edge–dpd-set if and only if $m \geq 5$.*

Proof. Case:1, Let $G \cong P_m$ and $m \geq 5$.
Let $P_n = (v_1, e_1, v_2, e_2, v_3, e_3, ..., e_m, v_n,)$ be a path on m edges.
Let $M = \{e_1, e_2, e_5\}$. Then

$$D_{2G}^{*M} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now, we can partition D_{2G}^{*M} in to two sub matrices say, A and B where A is a $5 \times (d_{2G}+1)$ submatrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 5×4 sub-matrix A_1 of A which is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

The remaining entries of $5 \times (d_{2G} - 3)$ submatrix A_2 of A has all the entries zero.

And the sub matrix B of order $(m-5) \times (d_{2G}+1)$ has entries 1 only in the$(m)^{th}$, $(m-1)^{th}$, and $(m-4)^{th}$ columns. Clearly we can observe that the rows of A and B of D_{2G}^{*M} are not identical, and hence $\{e_1, e_2, e_5\}$ form a Edge–dpd-set.

Therefore, for any graph $G \cong P_m$ of size $m \geq 5$ admits a Edge–dpd-set.

Now to complete the proof we need to show that the P_m is not a Edge–dpd-graph for $m \leq 4$.

Case: 2, Let $G \cong P_m$ and $m \leq 4$.

Proof follows directly from Lemma 3.8. □

Theorem 3.4. *A cycle $G \cong C_n$ of order n admits a Edge–dpd-set if and only if $n \geq 10$*

Proof. Let $C_n = (v_1, e_1, v_2, e_2, v_3, e_3, ..., e_m, v_1,)$ be a cycle on n vertices.
Case 1: n, is an even integer and ≥ 8
Let $M = \{e_1, e_2, e_5\}$. Then

$$D_{2G}^{*M} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we can partition D_{2G}^{*M} in to four sub matrices say, A,B,C and D where A is a $5 \times (d_{2G} + 1)$ sub-matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 5×4 sub-matrix A_1 in A which is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Here the remaining entries of $4 \times (d_{2G} - 3)$ sub-matrix A_2 of A has all the entries zero The sub matrix B of order $[\frac{(n-8)}{2}] \times (d_{2G} + 1)$ of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In this matrix B the entry 1 appears only in $(m)^{th}$, $(m-1)^{th}$, $(m-4)^{th}$ columns.

And we choose sub matrix C of order $(n - 5 - \frac{(n-8)}{2} - \left[\frac{n-8}{2}\right]) \times (d_{2G} + 1)$ of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Finally we can choose a submatrix D as $(\left[\frac{n-8}{2}\right]) \times (d_{2G} + 1)$ of the form and its exactly reverse matrix of B

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly we can observe that the rows of A,B,C and D of D_{2G}^{*M} are not identical.

Therefore, for any graph $G \cong C_n$ of order $n \geq 10$ admits a Edge–dpd-set.

Case 2: n, an odd integer and ≥ 11

Let $M = \{e_1, e_2, e_5\}$. Then

$$D_{2G}^{*M}=\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we can partition D_{2G}^{*M} in to four sub matrices say, A,B,C and D where A is a $5 \times (d_{2G}+1)$ sub-matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again we can find the 5×4 sub-matrix A_1 in A which is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Here the remaining entries of $5 \times (d_{2G}-3)$ sub-matrix A_2 of A has all the entries zero The sub matrix B of order $[\frac{(n-9)}{2}] \times (d_{2G}+1)$ of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In this matrix B the entry 1 appears only in $(m)^{th}$, $(m-1)^{th}$, $(m-4)^{th}$ columns.

And we choose sub matrix C of order $(n - 5 - \frac{(n-9)}{2} - [\frac{n-9}{2}]) \times (d_{2G} + 1)$ of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Finally we can choose a submatrix D as $([\frac{n-9}{2}]) \times (d_{2G} + 1)$ of the form and its exactly reverse matrix of B

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly we can observe that the rows of A,B,C and D of D^{*M}_{2G} are not identical.

Therefore, for any graph $G \cong C_n$ of order $n \geq 11$ admits a Edge–dpd-set. □

Theorem 3.6. *For any graph $G = (V, G)$ there exists no Edge–dpd-set M of cardinality 2.*

Proof. Suppose there exists a Edge–dpd-graph with $|M| = 2$, say e and f.

If these e and f are adjacent then $d_2(e, f) = 0 = d_2(f, e)$, then D^{*M}_{2G} contains a sub matrix $[2 \times (d_{2G} + 1)]$ so that the rows of submatrix represents the M-Edge–dnp of the edges e and f in D^{*M}_{2G} that is entry 1 is at the first column of submatrix and the rows are as shown in below

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If these two edges are independent edges then the rows of the submatrix D^{*M}_{2G} is as shown below and here the entry 1 appears only at the first and $(d_2(e, f) + 1)^{th}$ columns, and the rows will be of the following form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, D^{*M}_{2G} contains identical rows and so M is not a Edge–dpd-set. □

Lemma 3.6. *If $G \cong P_m$ of size $m \geq 2$ admits a Edge–dpd set then $3 \leq |M| \leq m-2$.*

Proof. we need to prove that the result is not true for $|M| = 2$ and $|M| \geq m-1$

Case;-1. If $|M| = 2$ the proof fallows from theorem

Case;-2.1, If $|M| = m$, consider path on size 3 and let $M = \{e_1, e_2, e_3\} = m$, then

$$D^{*M}_{2G} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is clear that two rows are identical.

Case;-2.2, If $|M| = m-1$ consider path on size 4 and for any choice of $|M| = 3 = m-1$, let $M_1 = \{e_1, e_2, e_3\}$, $M_2 = \{e_1, e_2, e_4\}$, $M_3 = \{e_2, e_3, e_4\}$, $M_4 = \{e_1, e_3, e_4\}$ Edge–dnp-matrix D^{*M}_{2G} shown bellow respectively,

$$D^{*M_1}_{2G} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \ D^{*M_2}_{2G} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$D^{*M_3}_{2G} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \ D^{*M_4}_{2G} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is observe that two of its rows are identical for any choice of $|M| = m-1$. □

Theorem 3.7. *If G is a Edge–dpd-graph with $|M| = 3$ then the edges should be at distinct distances from each other.*

Proof. Let G be a Edge–dpd-graph with Edge–dpd-set $|M| = \{e_1, e_2, e_3\}$. Consider $d_2(e_1, e_2) = k_1, d_2(e_2, e_3) = k_2, d_2(e_1, e_3) = k_3$.

Case:-1-

If $d_2(e_1, e_2) = d_2(e_2, e_3) = d_2(e_1, e_3) = k$.

The sub matrices $3 \times (d_{2G} + 1)$ represented by edges e_1, e_2 and e_3 respectively of D^{*M}_{2G} will have the entry 1 at first and $(k+1)$th column,

i.e

$$D^{*M}_{2G} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

It is observe that D_{2G}^{*M} contains identical rows and hence M is not Edge–dpd-set

Case:-2- If $k_1 \neq k_2 = k_3$ Here also the submatrix $(2 \times d_{2G} + 1)$ represented by e_1 and e_2 respectively have the entry 1 at the first, $(k_1 + 1)$th and $(k_3 + 1)$th column, then

$$D_{2G}^{*M} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Case:-3- If $k_1 \neq k_2 \neq k_3$

The sub matrices $3 \times (d_{2G} + 1)$ represented by edges e_1, e_2 and e_3 respectively in D_{2G}^{*M} have the entry 1 at first, and $(k_1 + 1)$th, $(k_2 + 1)$th, and $(k_3 + 1)$th columns,

$$D_{2G}^{*M} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0\cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 0\cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0\cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

It is possible to form a Edge–dpd-set M with $|M| = 3$. $\square$

Here these is not a sufficient condition for M to be a Edge–dpd-set.
For Example Consider path on size 5 i.e $\{v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6\}$ and $M = \{e_1, e_2, e_4\}$.

Lemma 3.8. *If G of size $m \geq 2$ admits a Edge–dpd set then* $3 \leq |M| \leq m - 2$.

Proof. we need to prove that the result is not true for $|M| = 2$ and $|M| \geq m - 1$
Case;-1. If $|M| = 2$.
The proof fallows from Theorem 3.5.
Case;-2. If $|M| \geq m - 1$.
We know that for any graph G of size $m \geq 2$ has at least two diametral edge, then for any choice of $|M| \geq m - 1$ in D_{2G}^{*M} the sub matrix of these diametral edges have the same entry in each column. Because $N_0(e) \geq 2$. $\square$

Theorem *For any graph $E(G)$ is a Edge–dpd set if and only if $G \cong K_2$.*

Proof. If $G \cong K_2$, then Edge–dpd-set of $k_2 = e_1$ and $e(k_2) = \{e\}$.

For Converse. If $M = E(G)$ in D_{2G}^{*M} is a square matrix its row and column have same element and G has exactly one row and column hence $G \cong K_2$. $\square$

Corollary *The complete graph K_n posses a Edge–dpd set if and only if $n = 2$.*

Corollary *Complete bipartite graph $K_{m,n}$ posses a Edge–dpd-set if and only if $m = n = 1$.*

Acknowledgements

Authors thank B. D. Acharya for his valuable suggestion during group discussion on 17^{th} June 2011. This research work is supported by DST(SERB). Govt. of India, for supporting through MRP. No–SB/EMEQ-119/2013.

References

[1] B. D. Acharya. Group discussion held in Mangalore University. India, *on* 17^{th} *June* 2011.

[2] S. C. Basak, D. Mills and B.D. Gute. Predicting bioactivity and toxicity of chemicals from mathematical descriptors: A chemical-cum-biochemical approach. In D.J. Klein and D.Brandas, editors, Advances in Quantum Chemistry:Chemical Graph Theory: *Where from, where for and where to*, Elsevier-Academic Press, 1-91, 2007.

[3] F. Buckley and F. Harary., Distance in graphs. Redwood City: Addison Wesley Publishing Company, 1990.

[4] K. A. Germina, A. Joseph and S. Jose. Distance Neighborhood Pattern Matrices. Eur. J. Pure Appl. Math., 2010, 3 (4): 748–764.

[5] F. Harary. Graph Theory. Massachusetts: Addison Wesley Publ. Comp., 1969.

[6] F. Harary and R. A. Melter. On the metric dimension of a graph. Ars Combin., 1976, 2: 191–195.

[7] D. H. Rouvrey. Predicting chemistry from topology. Sci. Amer., 1986, 255(3): 40–47.

[8] Kishori P. Narayankar, S. B. Lokesh and V. Mathad. Vertex-Edge-set Distance Neighborhood Pattern Matrices. International J. Math. Combin., 2015, 3: 105–115.

[9] A. P. Santhakumaran. Center of a graph with respect to edge. Sci. Ser. A Math. Sci. (N.S.), 2010, 19: 13–23.

Scientia Magna
Vol. 11 (2016), No. 2, 33-40

Certain results on a class of entire Dirichlet series in two variables

Niraj Kumar[1], Lakshika Chutani[2] and Garima Manocha[3]

[1]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail: nirajkumar2001@hotmail.com
[2]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail: lakshika91.chutani@gmail.com
[3]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail: garima89.manocha@gmail.com

Abstract The present paper deals with the class K of entire functions represented by Dirichlet series in two variables s_1, s_2 for which

$$(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\ \ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\ \|a_{m,n}\|$$

is bounded. Various results on Division Algebra, Topological Zero Divisor and Continuous linear Functional are then established for the set K.

Keywords Dirichlet series, Banach algebra, topological zero divisor, division algebra, continuous linear functional.
2010 Mathematics Subject Classification 30B50, 46J15, 47A10, 46A11, 54D65.

§1. Introduction and preliminaries

Let

$$f(s_1,s_2)=\sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1+\mu_n s_2)}, \qquad (s_j=\sigma_j+it_j,\ j=1,2) \tag{1}$$

be a Dirichlet series of two complex variables s_1 and s_2. Let E be a commutative Banach Algebra such that $a_{m,n}{}'s \in E$. Also $\lambda_m{}'s,\ \mu_n{}'s \in \mathbb{R}$ satisfying

$$0<\lambda_1<\lambda_2<\ldots<\lambda_m\to\infty \text{ as } m\to\infty$$
$$\text{and}\quad 0<\mu_1<\mu_2<\ldots<\mu_n\to\infty \text{ as } n\to\infty.$$

If

$$\limsup_{m+n\to\infty}\frac{\log(m+n)}{\lambda_m+\mu_n}=L<\infty \tag{2}$$

and

$$\limsup_{m+n\to\infty}\frac{\log\|a_{m,n}\|}{\lambda_m+\mu_n}=-\infty \tag{3}$$

Then from [2], the series (1) represents an entire function. Let K be a class of entire functions represented by series (1) for which

$$(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \; \|a_{m,n}\|$$

is bounded where $c_1, c_2 \geq 0$ and c_1, c_2 are simultaneously not zero. It is also clear that K defines a linear space over $\mathbb{C}^2$. Let

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

$$\text{and} \qquad g(s_1, s_2) = \sum_{m,n=1}^{\infty} b_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

then the binary operations in K are defined as follows

$$f(s_1, s_2) + g(s_1, s_2) = \sum_{m,n=1}^{\infty} (a_{m,n} + b_{m,n})\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

$$\xi.f(s_1, s_2) = \sum_{m,n=1}^{\infty} (\xi.a_{m,n})\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

$$f(s_1, s_2).g(s_1, s_2) = \sum_{m,n=1}^{\infty} \{(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \; a_{m,n}\, b_{m,n}\}\, e^{(\lambda_m s_1 + \mu_n s_2)}.$$

The norm in K is defined as

$$\|f\| = \sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \; \|a_{m,n}\| \tag{4}$$

During the last two decades a lot of research has been done in the field of Dirichlet series and many important results have been proved wherein a result showed that every entire function can be represented in the form of Dirichlet series but this representation is not unique. Daoud in his papers [2]- [3] considered a function of two variables represented by Dirichlet series and proved results which could be easily extended to finite number of variables. Kamthan in [5] considered different classes of entire functions represented by Dirichlet series in several variables and gave different characterizations of continuous linear functionals.

Hussein and Srivastava in their paper [4] discussed bornological properties of the space of entire functions of several complex variables. Behnam and Srivastava in [1] equipped the space of several complex variables with natural locally convex topology and proved it to be Frechet space. Also they gave different representations of continuous linear functionals.

So far many authors considered set of entire functions with weighted norms and studied results on it. Kumar and Manocha in [6] generalized the condition of weighted norm for a Dirichlet series of one variable and thus established some results. Present work is an extension of [6] to a Dirichlet series of two complex variables defined by (1). The purpose of this paper is to give a broader view to the study of Dirichlet series in two variables.

§2. Main Results

In this section main results are proved. For the definitions of terms used refer [7]- [8].

Theorem 2.1. *K is a commutative Banach algebra with identity.*

Proof. In order to prove this theorem we need to show that K is complete under the norm defined in (4). Let $\{f_{r_1}\}$ be any cauchy sequence in K where

$$f_{r_1}(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}^{(r_1)}\, e^{(\lambda_m s_1 + \mu_n s_2)}$$

Then for a given $\epsilon > 0$ we can find a constant $r \geq 1$ such that

$$\|f_{r_1} - f_{r_2}\| < \epsilon \ \ \forall \ \ r_1, r_2 \geq r$$

that is

$$\sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \ \ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \ \|a_{m,n}^{(r_1)} - a_{m,n}^{(r_2)}\| < \epsilon \ \ \forall \ \ r_1, r_2 \geq r.$$

This shows that $\{a_{m,n}^{(r_1)}\}$ forms a cauchy sequence in a Banach space E for all values of $m, n \geq 1$. Hence

$$\lim_{r_1 \to \infty} a_{m,n}^{(r_1)} = a_{m,n} \ \ \forall \ \ m, n \geq 1.$$

Letting $r_2 \to \infty$,

$$\sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \ \ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \ \|a_{m,n}^{(r_1)} - a_{m,n}\| < \epsilon \ \ \forall \ \ r_1 \geq r.$$

Thus $f_{r_1} \to f$ as $r_1 \to \infty$. Also

$$\sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \|a_{m,n}\| \leq$$

$$\sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \|a_{m,n}^{(r_1)} - a_{m,n}\| +$$

$$\sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \ \|a_{m,n}^{(r_1)}\| \ < \ \infty.$$

The identity element in K is

$$e(s_1, s_2) = \sum_{m,n=1}^{\infty} e_{m,n}(\lambda_m + \mu_n)^{-c_1(\lambda_m+\mu_n)} \ \ e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)} \ e^{(\lambda_m s_1 + \mu_n s_2)}.$$

Now if $f, g \in K$ then

$$\|f.g\| = \sum_{m,n=1}^{\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}.$$

$$\|(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} a_{m,n}\, b_{m,n}\| \leq \|f\|.\|g\|$$

This proves the theorem. □

Theorem 2.2. *The function* $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}$ *is invertible in* K *if and only if*

$$\{\|d_{m,n}\ (\lambda_m + \mu_n)^{-c_1(\lambda_m+\mu_n)}\ e^{\{c_1 - c_2(m+n)\}(\lambda_m+\mu_n)}\|\}$$

is a bounded sequence where $d_{m,n}$ *is the inverse of* $a_{m,n}$.

Proof. Let $f(s_1, s_2) \in K$ be invertible and $g(s_1, s_2) = \sum_{m,n=1}^{\infty} b_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}$ be its inverse. Then $f(s_1, s_2).g(s_1, s_2) = e(s_1, s_2)$. Therefore

$$(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\ a_{m,n}\, b_{m,n} = e_{m,n}(\lambda_m+\mu_n)^{-c_1(\lambda_m+\mu_n)} e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)}$$

which implies

$$(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} b_{m,n} = e_{m,n}\{(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} a_{m,n}\}^{-1}.$$

This further implies

$$(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \|b_{m,n}\| =$$

$$\|e_{m,n}\{(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} a_{m,n}\}^{-1}\|$$

which is equivalent to

$$\|d_{m,n}(\lambda_m + \mu_n)^{-c_1(\lambda_m+\mu_n)} e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)}\|$$

and is a bounded sequence since $g(s_1, s_2) \in K$.

Conversely suppose $\{\|d_{m,n}\ (\lambda_m + \mu_n)^{-c_1(\lambda_m+\mu_n)}\ e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)}\|\}$ be a bounded sequence. Define $g(s_1, s_2)$ such that

$$g(s_1, s_2) = \sum_{m,n=1}^{\infty} e_{m,n}(\lambda_m + \mu_n)^{-2c_1(\lambda_m+\mu_n)} e^{\{2c_1-2c_2(m+n)\}(\lambda_m+\mu_n)} a_{m,n}^{-1}\, e^{(\lambda_m s_1+\mu_n s_2)}$$

Clearly $g(s_1, s_2) \in K$. Further

$$f(s_1, s_2).g(s_1, s_2) = \sum_{m,n=1}^{\infty} \{(a_{m,n} e_{m,n}(\lambda_m + \mu_n)^{-2c_1(\lambda_m+\mu_n)} e^{\{2c_1-2c_2(m+n)\}(\lambda_m+\mu_n)} a_{m,n}^{-1})$$

$$(\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\} e^{(\lambda_m s_1+\mu_n s_2)} = e(s_1, s_2)$$

Hence the theorem. □

Theorem 2.3. *A necessary and a sufficient condition that an element* $f(s_1, s_2) \in K$ *be a topological zero divisor is*

$$\lim_{m,n\to\infty} (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)}\ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\ \|a_{m,n}\| = 0.$$

Proof. Let the given condition holds. Construct a sequence $\{g_{m,n}\}$ such that

$$g_{m,n}(s_1,s_2)=\sum_{m,n=1}^{\infty}(\lambda_m+\mu_n)^{-c_1(\lambda_m+\mu_n)}\ e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)}e^{(\lambda_m s_1+\mu_n s_2)}$$

Thus for all $m,n\geq 1$, $g_{m,n}\in K$ and $\|g_{m,n}\|=1$. Now

$$\begin{aligned} f(s_1,s_2).g_{m,n}(s_1,s_2) &= g_{m,n}(s_1,s_2).f(s_1,s_2) \\ &= \sum_{m,n=1}^{\infty} a_{m,n}e^{(\lambda_m s_1+\mu_n s_2)} \end{aligned}$$

Therefore

$$\|f.g_{m,n}\|=\|g_{m,n}.f\|=\sum_{m,n=1}^{\infty}(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\ \|a_{m,n}\|$$

As $m,n\to\infty$,

$$\|f.g_{m,n}\|=\|g_{m,n}.f\|\to 0$$

Thus $f(s_1,s_2)$ is a topological zero divisor.

Conversely suppose the given condition is not true that is

$$\lim_{m,n\to\infty}(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\ \|a_{m,n}\|=\beta>0$$

Then given γ with $0<\gamma<\beta$ we can find integers $n_0\geq 1$, $m_0\geq 1$ such that for all $n\geq n_0$, $m\geq m_0$

$$(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\ \|a_{m,n}\|\geq\beta-\gamma$$

hold true. Also since $f(s_1,s_2)$ is a topological zero divisor therefore there exists an arbitrary sequence $\{g_{h_1,h_2}\}$ of elements in K with unit norm such that for all
$h_1,h_2\geq 1$ one has

$$g_{h_1,h_2}(s_1,s_2)=\sum_{h_1,h_2=1}^{\infty}b_{h_1,h_2}\ e^{(\lambda_{h_1}s_1+\mu_{h_2}s_2)}$$

which implies

$$\sum_{h_1,h_2=1}^{\infty}(\lambda_{h_1}+\mu_{h_2})^{c_1(\lambda_{h_1}+\mu_{h_2})}\ e^{\{c_2(h_1+h_2)-c_1\}(\lambda_{h_1}+\mu_{h_2})}\ \|b_{h_1,h_2}\|=1.$$

Next, for ϵ such that $0<\epsilon<1$ there exists integers N_{h_1,h_2}, M_{h_1,h_2} and subsequences $\{n_i\}$ of sequence of indices $\{n\}$ and $\{m_i\}$ of sequence of indices $\{m\}$ such that

$$(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\ e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\|b_{m_{h_1},n_{h_2}}\|>1-\epsilon$$

$$\text{for all } n=n_i\geq N_{h_1,h_2},\ m=m_i\geq M_{h_1,h_2}.$$

This implies

$$(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\{(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}\}.$$

$$\|a_{m,n}.b_{m_{h_1},n_{h_2}}\| > c > 0 \text{ for all } n_i \geq N_{h_1,h_2},\; m_i \geq M_{h_1,h_2}.$$

Therefore

$$\|f(s_1,s_2).g_{h_1,h_2}(s_1,s_2)\| \;\nrightarrow\; 0$$

which is a contradiction. Hence the theorem. □

Theorem 2.4. *K is not a Division Algebra.*

Proof. Let

$$h(s_1,s_2) = \sum_{m,n=1}^{\infty} \{(m+n)^{-1}\,(\lambda_m+\mu_n)^{-c_1(\lambda_m+\mu_n)}\;e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)}\;e^{(\lambda_m s_1+\mu_n s_2)}$$

Clearly $h(s_1,s_2) \in K$ and does not possess inverse in K. Let if possible

$$z'(s_1,s_2) = \sum_{m,n=1}^{\infty} z_{m,n}\,e^{(\lambda_m s_1+\mu_n s_2)}$$

be its inverse. Hence $h(s_1,s_2).z'(s_1,s_2) = e(s_1,s_2)$. This implies

$$z_{m,n} = e_{m,n}\,(m+n)\,(\lambda_m+\mu_n)^{-c_1(\lambda_m+\mu_n)}\;e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)} \text{ does not belong to } K.$$

This completes the proof of the theorem. □

Theorem 2.5. *Every continuous linear functional $\theta : K \to E$ is of the form*

$$\theta(f) \;=\; \sum_{m,n=1}^{\infty} a_{m,n}\;l_{m,n}\,(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\;\;e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}$$

where

$$f(s_1,s_2) = \sum_{m,n=1}^{\infty} a_{m,n}\,e^{(\lambda_m s_1+\mu_n s_2)}$$

and $\{l_{m,n}\}$ is a bounded sequence in E.

Proof. Let us first assume that $\theta : K \to E$ be a continuous linear functional. Since θ is continuous,

$$\theta(f) \;=\; \theta(\lim_{M,N\to\infty} f^{(M,N)})$$

where

$$f^{(M,N)}(s_1,s_2) \;=\; \sum_{m,n=1}^{M,N} a_{m,n}\,e^{(\lambda_m s_1+\mu_n s_2)}.$$

Let us define a sequence $\{f_{m,n}\} \subseteq K$ as

$$f_{m,n}(s_1,s_2) \;=(\lambda_m+\mu_n)^{-c_1(\lambda_m+\mu_n)}\;\;e^{\{c_1-c_2(m+n)\}(\lambda_m+\mu_n)}e^{(\lambda_m s_1+\mu_n s_2)}$$

Therefore

$$\theta(f) \;\;=\;\; \theta(\lim_{M,N\to\infty} \sum_{m,n=1}^{M,N} a_{m,n}\,(\lambda_m+\mu_n)^{c_1(\lambda_m+\mu_n)}\;e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}f_{m,n})$$

$$= \lim_{M,N\to\infty} \sum_{m,n=1}^{M,N} a_{m,n} \, (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \;\; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \theta(f_{m,n}).$$

Since θ is a linear functional therefore

$$\theta(f_{m,n}) = l_{m,n}.$$

This implies

$$\theta(f) = \lim_{M,N\to\infty} \sum_{m,n=1}^{M,N} a_{m,n} \, l_{m,n} \, (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \;\; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}.$$

We now show that $\{l_{m,n}\}$ is a bounded sequence in E.

$$\|l_{m,n}\| = \|\theta(f_{m,n})\| \;\leq\; \tau\|f_{m,n}\|$$

and $\|f_{m,n}\| = 1$ which further implies

$$\|l_{m,n}\| \;\leq \tau.$$

Thus $\{l_{m,n}\}$ is a bounded sequence in E.

Conversely let $\{l_{m,n}\}$ be a bounded sequence in E satisfying

$$\theta(f) = \sum_{m,n=1}^{\infty} a_{m,n} \, l_{m,n} \, (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \;\; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)}.$$

Then, θ is well defined and linear. Now

$$\begin{aligned}
\|\theta(f)\| &= \sum_{m,n=1}^{\infty} \|a_{m,n} \, l_{m,n}\| \, (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \;\; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \\
&\leq \sum_{m,n=1}^{\infty} \|a_{m,n}\| \, \|l_{m,n}\| \, (\lambda_m + \mu_n)^{c_1(\lambda_m+\mu_n)} \;\; e^{\{c_2(m+n)-c_1\}(\lambda_m+\mu_n)} \\
&\leq \tau \, \|f\|.
\end{aligned}$$

Thus θ is a continuous linear functional which proves the theorem. □

References

[1] H. S. Behnam and G. S. Srivastava. Spaces of analytic functions represented by Dirichlet Series of two complex variables. Approx. Theory Appl. (N.S.), 2002, 18(3): 1–14.

[2] S. Daoud. On a class of entire Dirichlet functions of several complex variables having finite order point. Port. Math., 1986, 43: 417–427.

[3] S. Daoud. On the class of entire functions defined by Dirichlet series of several complex variables. J. Math. Anal. Appl., 1991, 162: 294–299.

[4] M. S. A. Hussein and G. S. Srivastava. A study on Bornological properties of the space of entire functions of several complex variables. Tamkang J. Math., 2002, 33(4): 289–301.

[5] P. K. Kamthan. A study on the space of entire functions of several complex variables. Yokohama Math. J., 1973, 21: 11–20.

[6] N. Kumar and G. Manocha. On a class of entire functions represented by Dirichlet series. J. Egyptian Math. Soc., 2013, 21: 21–24.

[7] R. Larsen. Banach Algebras - An Introduction. New York: Marcel Dekker Inc., 1973.

[8] R. Larsen. Functional Analysis - An Introduction. New York: Marcel Dekker Inc., 1973.

Scientia Magna
Vol. 11 (2016), No. 2, 41-48

Exponential sums over primes formed with coefficients of primitive Maass forms

Cui Yao[1] and Huixue Lao[2]

[1,2]School of Mathematics and Statistics, Shandong Normal University
Ji'nan 250014, P. R. China
E-mail: [1]ycsdnu314@sina.com and [2]lhxsdnu@163.com

Abstract Let $f(z) = 2\sqrt{y}\sum\limits_{n\neq 0} a_f(n)K_{ir}(2\pi|n|y)e(nx)$ be a Maass cusp form for $SL(2,\mathbb{Z})$ with Laplace eigenvalue $\frac{1}{4}+r^2$, where K_{ir} denotes the K-Bessel function. In this paper, we establish the mean value estimate for the coefficients of Maass cusp forms in exponential sums over primes.

Keywords Fourier coefficients, Maass cusp form, exponential sums, zero-density.
2010 Mathematics Subject Classification 11F30, 11L07, 11N37.

§1. Introduction

Let $f(z)$ be a primitive holomorphic cusp form of even integral weight $k \geq 2$ for the full modular group $SL(2,\mathbb{Z})$. The Fourier series expansion of $f(z)$ at infinity is

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz)$$

for $\Re s > 0$. Moreover, we assume that $f(z)$ is a normalized Hecke eigenform such that $a_f(1) = 1$. It is known that $a_f(n)$ satisfies the Ramanujan-Petersson conjecture, proved by Deligne [2]:

$$|a_f(n)| \leq d(n),$$

where $d(n)$ is the divisor function. If $f(z)$ is a Maass cusp form for $SL(2,\mathbb{Z})$ with Laplace eigenvalue $\frac{1}{4}+r^2$, then its Fourier expansion at infinity is

$$f(z) = 2\sqrt{y}\sum_{n\neq 0} a_f(n)K_{ir}(2\pi|n|y)e(nx),$$

where K_{ir} denotes the K-Bessel function and $a_f(1) = 1$. In contrast with holomorphic cusp form, the Ramanujan-Petersson conjecture for Maass cusp form, has not been proved yet. The best record till now is $a_f(n) \ll n^{\frac{7}{64}+\varepsilon}$, which is due to Kim and Sarnak [7].

For $\sigma = \Re s > 1$, let $L(f,s)$ be the corresponding Hecke L-function associated to $f(z)$, then

$$L(f,s) = \sum_{n=1}^{\infty} a_f(n)n^{-s} = \prod_p (1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1}, \tag{1.1}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are local roots at p, and

$$\alpha_f(p)+\beta_f(p)=a_f(p), \qquad \alpha_f(p)\beta_f(p)=1.$$

Taking logarithmic differentiation in (1.1), we have

$$-\frac{L'}{L}(f,s)=\sum_{n=1}^{\infty}\Lambda(n,f)n^{-s},$$

where

$$\Lambda(n,f)=\begin{cases}(\alpha_f(p^k)+\beta_f(p^k))\log p, & n=p^k;\\ 0, & \text{otherwise.}\end{cases}$$

With an additive character $e(\alpha\sqrt{n}),\ \alpha>0$, we have

$$S_f(x)=\sum_{x<n\le 2x}\Lambda(n,f)e(\alpha\sqrt{n}),$$

where $x\ge 2$. Note that

$$S_f(x)=\sum_{x<p\le 2x}a_f(p)\log p\ e(\alpha\sqrt{p})+O(x^{\frac{1}{2}}\log x).$$

It should be mentioned that Lao [8] has studied the exponential sums over primes connected with the coefficients of holomorphic cusp forms. She showed that

$$S_f(x)=\sum_{x<n\le 2x}\Lambda(n,f)e(\alpha\sqrt{n})\ll x^{\frac{5}{6}+\varepsilon}.$$

In this paper we want to study the mean value estimate for the coefficients of Maass cusp forms in exponential sums over primes.

Another reason we study the problem is from Vinogradov's exponential sums over primes. Vinogradov [13] is the first person to study the following sum

$$S(x)=\sum_{x<n\le 2x}\Lambda(n)e(\alpha\sqrt{n}),$$

where $\Lambda(n)$ refers to the Mangoldt function. And it was shown that

$$S(x)\ll x^{\frac{7}{8}+\varepsilon}.$$

Later, Iwaniec and Kowalski [6] obtained a better result

$$S(x)\ll x^{\frac{5}{6}+\varepsilon}.$$

Ren [9] made a further study with a new method and found that

$$S(x)\ll x^{\frac{4}{5}+\varepsilon}.$$

The main aim of this paper is to prove

Theorem 1.1. *Let $f(z)$ be a Maass cusp form for the group $SL(2,\mathbb{Z})$, and assume that it satisfies the Ramanujan-Petersson conjecture. For any $\alpha > 0$ and any sufficiently small $\varepsilon > 0$, we have*

$$S_f(x) = \sum_{x<n\le 2x} \Lambda(n,f)e(\alpha\sqrt{n}) \ll x^{\frac{5}{6}+\varepsilon},$$

where the implied constant depends on α and $f(z)$.

§2. Preliminaries

First we recall some basic notations and knowledge. We use $L(f,s)$ to denote any normalized L-function. It is well-known that when $\sigma = \Re s > 1$, all its nontrivial zeros are in the critical strip $0 \le \sigma = \Re s \le 1$. However the Grand Riemann Hypothesis asserts that they all lie on the critical line $\Re s = \frac{1}{2}$.

In the absence of a proof of the Grand Riemann Hypothesis, it is natural to ask how many zeros of a given L-function can lie off the critical line $\sigma = \frac{1}{2}$. Therefore we define

$$N_L(T) := \#\{\rho = \beta + i\gamma : L(\rho, f) = 0, |\gamma| \le T\} \tag{2.1}$$

$$N_L(\sigma, T) := \#\{\rho = \beta + i\gamma : L(\rho, f) = 0, \sigma \le \beta \le 1, |\gamma| \le T\} \tag{2.2}$$

where $\frac{1}{2} \le \sigma \le 1$ and $T \ge 3$. As we all know, zero-density theorems for L-functions to the right of the critical line are objects of intensive studies in analytic number theory. These results have been established by many mathematicians for various L-functions.

For the Riemann zeta-function $\zeta(s)$, Ingham [5] showed that

$$N_\zeta(\sigma, T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}}(\log T)^5,$$

this result was further refined as

$$N_\zeta(\sigma, T) \ll T^{\frac{12(1-\sigma)}{5}}(\log T)^{100}.$$

See [3] for details.

For the Dirichlet L-function, Bombieri [1] stated that when $T \le Q$,

$$\sum_{q\le Q} \sideset{}{^*}\sum_{\chi} N_\chi(\sigma, T) \ll TQ^{\frac{8(1-\sigma)}{3-2\sigma}}(\log Q)^{10},$$

where $\sideset{}{^*}\sum_{\chi}$ means that the sum is over primitive characters.

If $f(z)$ is a holomorphic cusp form, we quote Ivic's result [4], which stated that

$$N_L(\sigma, T) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon}, \qquad for \quad \frac{1}{2} \le \sigma \le \frac{3}{4};$$

$$N_L(\sigma, T) \ll T^{\frac{2-2\sigma}{\sigma}+\varepsilon}, \qquad for \quad \frac{3}{4} \le \sigma \le 1.$$

If $f(z)$ is a primitive Maass cusp form for $SL(2,\mathbb{Z})$ with Laplace eigenvalue $\frac{1}{4}+r^2$, we have known that

$$N_L(\sigma, T) \ll T\log T, \qquad for \quad \frac{1}{2} \le \sigma \le \frac{1}{2} + \frac{1}{\log T}. \tag{2.3}$$

Sankaranarayanan and Sengupta [10] obtained a result for Maass cusp form for $SL(2,\mathbb{Z})$, which showed that

$$N_L(\sigma,T) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon}, \qquad for \qquad \frac{1}{2}+\frac{1}{\log T} \le \sigma \le 1. \tag{2.4}$$

Later, Xu [14] improved the previous result (2.4) when $\sigma \in [\frac{3}{4},1)$,

$$N_L(\sigma,T) \ll T^{\frac{(1-\sigma)(8\sigma-5)}{-2\sigma^2+6\sigma-3}+\varepsilon}, \qquad for \qquad \frac{3}{4} \le \sigma \le 1. \tag{2.5}$$

On the basis of Xu, Tang [12] obtained a better estimate for $N_L(\sigma,T)$,

$$N_L(\sigma,T) \ll T^{\frac{2-2\sigma}{\sigma}+\varepsilon}, \qquad for \qquad \frac{3}{4} \le \sigma \le 1-\varepsilon_0, \tag{2.6}$$

for arbitrarily small $\varepsilon_0 > 0$.

In order to get the results we want, we assume $f(z)$ satisfies the Ramanujan-Petersson conjecture. Derive (using Perron's formula) the following approximate expansion

$$\Psi_f(x) = \sum_{n\le x} \Lambda(n,f) = -\sum_{|\gamma|\le T} \frac{x^\rho}{\rho} + R(x,T), \tag{2.7}$$

where $R(x,T) = O(\frac{x}{T}\log^2 x)$ and $\rho = \beta + i\gamma$ runs over the zeros of $L(f,s)$ in the critical strip of height up to T, with $1 \le T \le x$, and the implied constant is absolute. See [6] for details.

We also need the following result.

Lemma 2.1. *Let $F(x)$ and $G(x)$ be real functions in $[a,b]$ with $G(x)$ and $1/F'(x)$ monotonic. Suppose that $|G(x)| \le M$.*

(i)*If $F'(x) \ge u > 0$ or $F'(x) \le -u < 0$, then*

$$\int_a^b G(x)e(F(x))dx \ll \frac{M}{u}. \tag{2.8}$$

(ii)*If $F''(x) \ge v > 0$ or $F''(x) \le -v < 0$, then*

$$\int_a^b G(x)e(F(x))dx \ll \frac{M}{\sqrt{v}}. \tag{2.9}$$

See [11] for details.

§3. Proof of Theorem 1.1.

Integrating by parts, we have

$$S_f(x) = \sum_{x<n\le 2x} \Lambda(n,f)e(\alpha\sqrt{n}) = \int_x^{2x} e(\alpha\sqrt{u})d\sum_{n\le u}\Lambda(n,f),$$

then applying the explicit formula in (2.7), we get

$$S_f(x) = -\sum_{|\gamma|\le T}\int_x^{2x} u^{\rho-1}e(\alpha\sqrt{u})du + \int_x^{2x} e(\alpha\sqrt{u})dR(u,T). \tag{3.1}$$

The error term above is bounded by

$$\int_x^{2x} e(\alpha\sqrt{u})dR(u,T) \ll (1+\pi|\alpha|x^{\frac{1}{2}})\frac{x}{T}\log^2 x.$$

On taking

$$T = (1+\pi|\alpha|x^{\frac{1}{2}})x^{\frac{1}{4}}, \tag{3.2}$$

we find that the error term in (3.1) is $O(x^{\frac{3}{4}}\log^2 x)$, which is obviously acceptable.

To prove the Theorem, it suffices to show that

$$\sum_{|\gamma|\le T}\int_x^{2x} u^{\rho-1}e(\alpha\sqrt{u})du \ll x^{\frac{5}{6}+\varepsilon}. \tag{3.3}$$

Making change of variable $\sqrt{u}=v$ in (3.3), we have

$$\int_x^{2x} u^{\beta+i\gamma-1}e(\alpha\sqrt{u})du = 2\int_{x^{\frac{1}{2}}}^{(2x)^{\frac{1}{2}}} v^{2\beta-1}e(\alpha v+\frac{\gamma\log v}{\pi})dv.$$

By Lemma 2.1, the last integral satisfies

$$\begin{aligned}
&\ll x^{\beta}\min\{1, \frac{1}{\min\limits_{x^{\frac{1}{2}}<v\le(2x)^{\frac{1}{2}}}|\gamma+\pi\alpha v|}, \frac{1}{\sqrt{|\gamma|}}\}\\
&\ll x^{\beta}\begin{cases} \frac{1}{(1+|\alpha|x^{\frac{1}{2}})^{\frac{1}{2}}}, & |\gamma|\le 2\pi|\alpha|(2x)^{\frac{1}{2}};\\ \frac{1}{1+|\gamma|}, & 2\pi|\alpha|(2x)^{\frac{1}{2}}<|\gamma|\le T.\end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{|\gamma|\le T}\int_x^{2x} u^{\rho-1}e(\alpha\sqrt{u})du \qquad (3.4)\\
&\ll \frac{1}{(1+|\alpha|x^{\frac{1}{2}})^{\frac{1}{2}}}\sum_{|\gamma|\le 2\pi|\alpha|(2x)^{\frac{1}{2}}} x^{\beta} + \sum_{2\pi|\alpha|(2x)^{\frac{1}{2}}<|\gamma|\le T}\frac{x^{\beta}}{1+|\gamma|}\\
&= S_1+S_2.
\end{aligned}$$

Now, we define a new function

$$F(u,\beta) = \begin{cases} 1, & 0\le u\le\beta;\\ 0, & \beta\le u\le 1.\end{cases}$$

By (2.1), we have

$$\begin{aligned}
\sum_{|\gamma|\le t} x^{\beta} &= \sum_{|\gamma|\le t}(\log x\int_0^{\beta} x^u du+1)\\
&= N_L(t)+\log x\sum_{|\gamma|\le t}\int_0^1 x^u F(u,\beta)du.
\end{aligned}$$

From the definitions of $F(u,\beta)$ and $N_L(u,t)$, we have

$$\sum_{|\gamma|\le t} F(u,\beta) = N_L(u,t).$$

Therefore we get

$$\begin{aligned}\sum_{|\gamma|\le t} x^{\beta} &= N_L(t) + \log x \int_0^1 x^u N_L(u,t)du \\ &= N_L(t) + \log x \int_0^{\frac{1}{2}} x^u N_L(u,t)du + \log x \int_{\frac{1}{2}}^1 x^u N_L(u,t)du \\ &\ll x^{\frac{1}{2}} t \log t + \log x \int_{\frac{1}{2}}^1 x^u N_L(u,t)du,\end{aligned}$$

where we have used the fact that, for $0 \le u \le \frac{1}{2}$,

$$N_L(u,t) \ll N_L(t) \ll t \log t.$$

From (2.5) and (2.6), we can get

$$N_L(u,t) \ll t^{\frac{8}{3}(1-u)+\varepsilon}, \qquad for \qquad \frac{3}{4} \le u \le 1-\varepsilon_0; \tag{3.5}$$

$$N_L(u,t) \ll t^{3(1-u)+\varepsilon}, \qquad for \qquad 1-\varepsilon_0 \le u \le 1, \tag{3.6}$$

for arbitrarily small $\varepsilon_0 > 0$.

Using (2.3), (2.4), (3.5) and (3.6), we find that

$$\begin{aligned}\sum_{|\gamma|\le t} x^{\beta} &\ll x^{\frac{1}{2}} t \log t + \log x \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{\log t}} x^u t \log t \; du + \log x \int_{\frac{1}{2}+\frac{1}{\log t}}^{\frac{3}{4}} x^u t^{\frac{4(1-u)}{3-2u}+\varepsilon} du \\ &+ \log x \int_{\frac{3}{4}}^{1-\varepsilon_0} x^u t^{\frac{8}{3}(1-u)+\varepsilon} du + \log x \int_{1-\varepsilon_0}^1 x^u t^{3(1-u)+\varepsilon} du \\ &\ll x^{\frac{1}{2}+\frac{1}{\log t}} t \log t + \log x \int_{\frac{1}{2}+\frac{1}{\log t}}^{\frac{3}{4}} x^u t^{\frac{4(1-u)}{3-2u}+\varepsilon} du + \log x \int_{\frac{3}{4}}^{1-\varepsilon_0} x^u t^{\frac{8}{3}(1-u)+\varepsilon} du \\ &+ \log x \int_{1-\varepsilon_0}^1 x^u t^{3(1-u)+\varepsilon} du \\ &\ll x^{\frac{1}{2}+\frac{1}{\log t}} t \log t + \log x \max_{\frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4}} x^u t^{\frac{4(1-u)}{3-2u}+\varepsilon} + \log x \max_{\frac{3}{4}\le u\le 1-\varepsilon_0} x^u t^{\frac{8}{3}(1-u)+\varepsilon} \\ &+ \log x \max_{1-\varepsilon_0\le u<1} x^u t^{3(1-u)+\varepsilon}.\end{aligned} \tag{3.7}$$

Now we estimate S_1. Taking $t = 2\pi|\alpha|(2x)^{\frac{1}{2}}$ in (3.7), we have

$$S_1 = \frac{1}{(1+|\alpha|x^{\frac{1}{2}})^{\frac{1}{2}}} \sum_{|\gamma|\le 2\pi|\alpha|(2x)^{\frac{1}{2}}} x^{\beta}$$

$$\begin{aligned}&\ll x^{\frac{3}{4}+\varepsilon} + \log x \max_{\frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4}} x^{u+\frac{2(1-u)}{3-2u}-\frac{1}{4}+\varepsilon} + \log x \max_{\frac{3}{4}\le u\le 1-\varepsilon_0} x^{u+\frac{4}{3}(1-u)-\frac{1}{4}+\varepsilon} \\ &+ \log x \max_{1-\varepsilon_0\le u<1} x^{u+\frac{3}{2}(1-u)-\frac{1}{4}+\varepsilon}.\end{aligned}$$

Note that

$$\max_{\frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4}} (u + \frac{2(1-u)}{3-2u} - \frac{1}{4}) = \frac{5}{6},$$

$$\max_{\frac{3}{4}\le u\le 1-\varepsilon_0}(u+\frac{4}{3}(1-u)-\frac{1}{4})=\frac{5}{6},$$

$$\max_{1-\varepsilon_0\le u\le 1}(u+\frac{3}{2}(1-u)-\frac{1}{4})=\frac{3}{4}+\frac{\varepsilon_0}{2}.$$

Taking $\varepsilon_0=\frac{1}{6}$, which is obviously acceptable, then we obtain

$$S_1\ll x^{\frac{5}{6}+\varepsilon}. \tag{3.8}$$

Now we estimate S_2, we have

$$\sum_{2\pi|\alpha|(2x)^{\frac{1}{2}}<|\gamma|\le T}\frac{x^{\beta}}{1+|\gamma|}\ll\log x\max_{2\pi|\alpha|(2x)^{\frac{1}{2}}<t\le T}t^{-1}\sum_{|\gamma|\sim t}x^{\beta}.$$

Using the same method, we obtain

$$\begin{aligned}S_2&=\sum_{2\pi|\alpha|(2x)^{\frac{1}{2}}<|\gamma|\le T}\frac{x^{\beta}}{1+|\gamma|}\\&\ll x^{\frac{1}{2}+\varepsilon}+\log x\max_{2\pi|\alpha|(2x)^{\frac{1}{2}}<t\le T}\max_{\frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4}}x^{u}t^{\frac{4(1-u)}{3-2u}-1+\varepsilon}\\&+\log x\max_{2\pi|\alpha|(2x)^{\frac{1}{2}}<t\le T}\max_{\frac{3}{4}\le u\le 1-\varepsilon_0}x^{u}t^{\frac{8(1-u)}{3}-1+\varepsilon}\\&+\log x\max_{2\pi|\alpha|(2x)^{\frac{1}{2}}<t\le T}\max_{1-\varepsilon_0\le u<1}x^{u}t^{3(1-u)-1+\varepsilon}.\end{aligned}$$

According to

$$\begin{cases}\frac{4(1-u)}{3-2u}-1\le 0, & \frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4};\\ \frac{8(1-u)}{3}-1<0, & \frac{3}{4}\le u\le 1-\varepsilon_0;\\ 3(1-u)-1<0, & 1-\varepsilon_0\le u\le 1,\end{cases}$$

we have

$$\begin{aligned}S_2&=\sum_{2\pi|\alpha|(2x)^{\frac{1}{2}}<|\gamma|\le T}\frac{x^{\beta}}{1+|\gamma|}\\&\ll x^{\frac{1}{2}+\varepsilon}+\log x\max_{\frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4}}x^{u+\frac{2(1-u)}{3-2u}-\frac{1}{2}+\varepsilon}+\log x\max_{\frac{3}{4}\le u\le 1-\varepsilon_0}x^{u+\frac{4(1-u)}{3}-\frac{1}{2}+\varepsilon}\\&+\log x\max_{1-\varepsilon_0\le u<1}x^{u+\frac{3}{2}(1-u)-\frac{1}{2}+\varepsilon}.\end{aligned}$$

Note that

$$\max_{\frac{1}{2}+\frac{1}{\log t}\le u\le\frac{3}{4}}(u+\frac{2(1-u)}{3-2u}-\frac{1}{2})=\frac{7}{12},$$

$$\max_{\frac{3}{4}\le u\le 1-\varepsilon_0}(u+\frac{4}{3}(1-u)-\frac{1}{2})=\frac{7}{12},$$

$$\max_{1-\varepsilon_0\le u\le 1}(u+\frac{3}{2}(1-u)-\frac{1}{2})=\frac{1}{2}+\frac{\varepsilon_0}{2}.$$

Taking $\varepsilon_0=\frac{1}{6}$, then we obtain

$$S_2\ll x^{\frac{7}{12}+\varepsilon}. \tag{3.9}$$

From (3.4), (3.8) and (3.9), we complete the proof of Theorem 1.1.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No.11101249). The authors wish to thank the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] E. Bombieri. On the large sieve. Mathematatika, 1965, 12: 201–225.

[2] P. Deligne. La conjecture de Weil. I. Inst. Hautes Études Sci. Publ. Math., 1974, 43: 273–307.

[3] M. N. Huxley. On the difference between consecutive primes. Invent. Math., 1972, 15: 164–170.

[4] A. Ivić. On zeta-functions associated with Fourier coefficients of cusp forms. In: Proceedings of the Amalfi Conference on Analytic Number Theory, Università di Salerno, 1992, 231–246.

[5] A. E. Ingham. On the estimation of $N(\sigma, T)$. Quart. I. Math., Oxford Ser., 1940, 11: 291–292.

[6] H. Iwaniec and E. Kowalski. Analytic number theory. Amer. Math. Soc., 2004.

[7] H. H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, J. Amer. Math. Soc., 2003, 16: 139–183.

[8] H. X. Lao. Exponential sums over primes formed with coefficients of primitive cusp forms. Acta Math. Sin. (Engl. Ser.), 2009, 25(4): 687–692.

[9] X. M. Ren. Vingradov's exponential sum over primes. Acta Arith., 2006, 124: 269–285.

[10] A. Sankaranarayanan and J. Sengupta. Zero-density estimate of L-functions attached to Maass forms. Acta Arith., 2007, 127(3): 273–284.

[11] E. C. Titchmarsh. The theory of the Riemann zeta-function. 2nd ed. Oxford University Press, Oxford, 1986.

[12] H. C. Tang. Zero density of L-functions related to Maass forms. Front. Math. China, 2013, 8(4): 923–932.

[13] I. M. Vinogradov. Special variant of the method of trigonometric sums. Selected works, Springer, Berlin, 1985.

[14] Z. Xu. A new zero-density result of L-functions attached to Maass forms. Acta Math. Sin. (Engl. Ser.), 2011, 27(6): 1149–1162.

Scientia Magna
Vol. 11 (2016), No. 2, 49-59

Certain indefinite integrals involving Lucas polynomials

Salahuddin[1] and R. K. Khola[2]

[1]Mewar University,Gangrar, Chittorgarh (Rajasthan) , India
E-mail: vsludn@gmail.com
[2]Mewar University,Gangrar, Chittorgarh (Rajasthan) , India
E-mail: rkmkhola176@gmail.com

Abstract In this paper we have established certain indefinite integrals involving Polylogarithm and Lucas Polynomials. The results represent here are assume to be new.

Keywords polylogarithm, Lucas polynomials, Gaussian hypergeometric function.
2010 Mathematics Subject Classification 33C05, 33C45, 33C15, 33D50, 33D60.

§1. Introduction and preliminaries

Lucas polynomials

The sequence of Lucas polynomials is a sequence of polynomials defined by the recurrence relation

$$L_n(x) = \begin{cases} 2x^0 = 2 & , \quad \text{if } \ n = 0 \\ 1x^1 = x & , \quad \text{if } \ n = 1 \\ x^1 L_{n-1}(x) + x^0 L_{n-2}(x) & , \quad \text{if } \ n \geq 2 \end{cases} \tag{1.1}$$

The first few Lucas polynomials are:

$$L_0(x) = 2$$

$$L_1(x) = x$$

$$L_2(x) = x^2 + 2$$

$$L_3(x) = x^3 + 3x$$

$$L_4(x) = x^4 + 4x^2 + 2$$

The ordinary generating function of the Lucas polynomials is

$$G_{\{L_n(x)\}}(t) = \sum_{n=0}^{\infty} L_n(x) t^n = \frac{2 - xt}{1 - t(x + t)}. \tag{1.2}$$

Polylogarithm

The polylogarithm is a special function $Li_s(z)$ that is defined by the infinite sum, or power series:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \tag{1.3}$$

It is in general not an elementary function, unlike the related logarithm function. The above definition is valid for all complex values of the order s and the argument z where$\mid z \mid < 1$. The polylogarithm is defined over a larger range of z than the above definition allows by the process of analytic continuation.

The special case s = 1 involves the ordinary natural logarithm ($Li_1(z) = -ln(1-z)$) while the special cases s = 2 and s = 3 are called the dilogarithm (also referred to as Spence's function) and trilogarithm respectively. The name of the function comes from the fact that it may alternatively be defined as the repeated integral of itself, namely that

$$Li_{s+1}(z) = \int_0^z \frac{Li_s(t)}{t} dt \tag{1.4}$$

Thus the dilogarithm is an integral of the logarithm, and so on. For nonpositive integer orders s, the polylogarithm is a rational function.

The polylogarithm also arises in the closed form of the integral of the Fermi-Derac distribution and the Bose-Einstein distribution and is sometimes known as the Fermi-Derac integral or Bose-Einstein integral. Polylogarithms should not be confused with polylogarithmic functions nor with the offset logarithmic integral which has a similar notation.

Generalized Hypergeometric Functtions

A generalized hypergeometric function ${}_pF_q(a_1, ...a_p; b_1, ...b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\ z. \tag{1.5}$$

Where $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{lc} a_1, a_2, \cdots, a_p & ; \\ & z \\ b_1, b_2, \cdots, b_q & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{1.6}$$

or

$${}_pF_q\left[\begin{array}{lc} (a_p) & ; \\ & z \\ (b_q) & ; \end{array}\right] \equiv {}_pF_q\left[\begin{array}{lc} (a_j)_{j=1}^p & ; \\ & z \\ (b_j)_{j=1}^q & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_p))_k z^k}{((b_q))_k k!} \tag{1.7}$$

where the parameters $b_1, b_2, \cdots, b_q$ are neither zero nor negative integers and $p,\ q$ are non-negative integers.
The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $\mid z \mid < 1$ if $p \neq q+1$, diverges for all z , $z \neq 0$ if $p > q+1$.

The ${}_pF_q$ series absolutely converges for $\mid z \mid = 1$ if $R(\zeta) < 0$, conditionally converges for $\mid z \mid = 1, z \neq 0$ if $0 \leq R(\zeta) < 1$, diverges for $\mid z \mid = 1$, if $1 \leq R(\zeta)$, $\zeta = \sum\limits_{i=1}^{p} a_i - \sum\limits_{i=0}^{q} b_i$.
The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.
The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0 \tag{1.8}$$

The solution of this equation is

$$y = A_0\Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] \tag{1.9}$$

This is the so-called regular solution,denoted

$${}_2F_1(a, b; c; z) = \Big[1 + \frac{ab}{1!\ c}z + \frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2 + \cdots\cdots\Big] = \sum_{k=0}^{\infty} \frac{(a)_k\ (b)_k z^k}{(c)_k k!} \tag{1.10}$$

which converges if c is not a negative integer for all of $\mid z \mid < 1$ and on the unit circle $\mid z \mid = 1$ if $R(c - a - b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

Many of the common mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Some typical examples are

$$(1-z)^{-a} = z\ {}_2F_1(1, 1; 2; -z) \tag{1.11}$$

$$\sin^{-1} z = z\ {}_2F_1(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2) \tag{1.12}$$

The special case of (1.3.4) when $a = c$ and $b = 1$, or $a = 1$ and $b = c$, yields the elementary geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots \tag{1.13}$$

Hence the term "Hypergeometric" is given. The term hypergeometric was first used by Wallis in his work "Arithmetrica Infinitorum". Hypergeometric series or more precisely Gauss series is due to Carl Friedrich Gauss(1777-1855) who in year 1812 introduced and studied this series in his thesis presented at Gottingen and gave the F-notation for it. Here z is a real or complex variable. If c is zero or negative integer, the series (1.10) does not exist and hence the function ${}_2F_1(a,b;\,c;\,z)$ is not defined unless one of the parameters a or b is also a negative integer such that $-c < -a$. If either of the parameters a or b is a negative integer, say $-m$ then in this case (1.10) reduce to the hypergeometric polynomial defined as

$$ {}_2F_1(-m,b;\,c;\,z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n\, z^n}{(c)_n\, n!} \tag{1.14} $$

Hypergeometric Function of Second Kind

$$ G(a,b;\,c;\,z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)\,\Gamma(b-c+1)} \times {}_2F_1\left[\begin{array}{ll} a,\, b & ; \\ & z \\ c & ; \end{array}\right] + $$

$$ + \frac{\Gamma(c-1)\, z^{(1-c)}}{\Gamma(a)\,\Gamma(b)} \times {}_2F_1\left[\begin{array}{ll} 1+a-c,\, 1+b-c & ; \\ & z \\ 2-c & ; \end{array}\right] \tag{1.15} $$

where $c \neq 0, \pm 1, \pm 2, \ldots$

$$ G(a,b;\,c;\,z) = z^{(1-c)}\, G(1+a-c, 1+b-c;\, 2-c;\, z) \tag{1.16} $$

Each of the following functions is a solution of differential equation (1.8). A system of two linearly independent solutions of differential equation (1.8) in

the vicinity of the singular point $z = 0, 1$ and ∞ are given by

$$ w_1^{(0)}(z) = {}_2F_1\left[\begin{array}{ll} a,\, b & ; \\ & z \\ c & ; \end{array}\right] $$

$$ w_1^{(1)}(z) = {}_2F_1\left[\begin{array}{ll} a,\, b & ; \\ & 1-z \\ 1+a+b-c & ; \end{array}\right] \tag{1.17} $$

$$ w_2^{(0)}(z) = z^{(1-c)}\, {}_2F_1\left[\begin{array}{ll} 1+a-c,\, 1+b-c & ; \\ & z \\ 2-c & ; \end{array}\right] $$

$$ w_2^{(1)}(z) = (1-z)^{(c-a-b)}\, {}_2F_1\left[\begin{array}{ll} c-a,\, c-b & ; \\ & 1-z \\ 1+c-a-b & ; \end{array}\right] \tag{1.18} $$

$$w_1^{(\infty)}(z) = (-z)^{-a}\, {}_2F_1 \left[\begin{array}{lll} a,\, 1+a-c & ; & \\ & & \frac{1}{z} \\ 1+a-b & ; & \end{array} \right]$$

$$w_2^{(\infty)}(z) = (-z)^{-b}\, {}_2F_1 \left[\begin{array}{lll} 1+b-c,\, b & ; & \\ & & \frac{1}{z} \\ 1+b-a & ; & \end{array} \right] \tag{1.19}$$

where $c \neq 0, \pm 1, \pm 2, \ldots$; $(c-a-b)$ and $(a-b)$ are not integers.

The equation (1.8) is also denoted by

$${}_2F_1 \left[\begin{array}{lll} a,\, b & ; & \\ & & z \\ c & ; & \end{array} \right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m\, z^m}{(c)_m\, m!}$$

$$= 1 + \frac{a\, b\, z}{c} + \frac{a\,(a+1)\, b\,(b+1)\, z^2}{c\,(c+1)\, 2!} +$$

$$+ \frac{a\,(a+1)\,(a+2)\, b\,(b+1)\,(b+2)\, z^3}{c\,(c+1)\,(c+2)\, 3!} + \cdots + \text{ad inf.} \tag{1.20}$$

It is convergent for $|z| < 1$.

Note:

$${}_2F_1 \left[\begin{array}{lll} a,\, b & ; & \\ & & 0 \\ c & ; & \end{array} \right] = {}_2F_1 \left[\begin{array}{lll} 0,\, b & ; & \\ & & z \\ c & ; & \end{array} \right] = 1 \tag{1.21}$$

$$(1-z)^{-a} = \sum_{r=0}^{\infty} \frac{(a)_r\, z^r}{r!} = {}_1F_0 \left[\begin{array}{lll} a & ; & \\ & & z \\ \text{---} & ; & \end{array} \right];\ |z| < 1 \tag{1.22}$$

Generalized Ordinary Hypergeometric Function of One Variable

The generalized Gaussian hypergeometric function of one variable is defined as follows

$${}_AF_B \left[\begin{array}{lll} a_1,\, a_2,\, a_3, \ldots, a_A & ; & \\ & & z \\ b_1,\, b_2,\, b_3, \ldots, b_B & ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n\, (a_2)_n\, (a_3)_n \cdots (a_A)_n\, z^n}{(b_1)_n\, (b_2)_n\, (b_3)_n \cdots (b_B)_n\, n!} \tag{1.23}$$

or,

$${}_AF_B \left[\begin{array}{lll} (a_A) & ; & \\ & & z \\ (b_B) & ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{[(a_A)]_n\, z^n}{[(b_B)]_n\, n!} \tag{1.24}$$

where for the sake of convenience (in the contracted notation), (a_A) denotes thearray of "A" number of parameters given by $a_1, a_2, a_3, \ldots, a_A$. The denominatorparameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. A and B are positive integers or zero. Empty sum is to be interpreted as zero and empty product as unity.

$$\sum_{n=a}^{b} \text{ and } \prod_{n=a}^{b} \text{ are empty if } b < a.$$

$$[(a_A)]_{-n} = \frac{(-1)^{nA}}{[1-(a_A)]_n} \tag{1.25}$$

$$[(a_A)]_n = (a_1)_n(a_2)_n(a_3)_n \cdots (a_A)_n = \prod_{m=1}^{A}(a_m)_n = \prod_{m=1}^{A}\frac{\Gamma(a_m+n)}{\Gamma(a_m)} \tag{1.26}$$

where $a_1, a_2, a_3, \ldots, a_A$; $b_1, b_2, b_3, \ldots, b_B$ and z may be real and complex numbers.

$$ {}_3F_2\left[\begin{array}{lll} a,\, b,\, 1 & ; & \\ & & z \\ c,\, 2 & ; & \end{array}\right] = \frac{(c-1)}{(a-1)(b-1)\,z} \times$$

$$\times\left\{ {}_2F_1\left[\begin{array}{lll} a-1,\, b-1 & ; & \\ & & z \\ c-1 & ; & \end{array}\right] - 1\right\} \tag{1.27}$$

The convergence conditions of ${}_AF_B$ are given below

Suppose that numerator parameters are neither zero nor negative integers (otherwise question of convergence will not arise).

(i) If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex) i.e., $|z| < \infty$.

(ii) If $A = B+1$ and $|z| < 1$, then series ${}_AF_B$ is convergent.

(iii) If $A = B+1$ and $|z| > 1$, then series ${}_AF_B$ is divergent.

(iv) If $A = B+1$ and $|z| = 1$, then series ${}_AF_B$ is absolutely convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0$$

(v) If $A = B+1$ and $z = 1$, then series ${}_AF_B$ is convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > 0$$

(vi) If $A = B + 1$ and $z = 1$, then series ${}_AF_B$ is divergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0$$

(vii) If $A = B + 1$ and $z = -1$, then series ${}_AF_B$ is convergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} > -1$$

(viii) If $A = B + 1$ and $|z| = 1$, but $z \neq 1$, then series ${}_AF_B$ is conditionally convergent, when

$$-1 < \mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq 0$$

(ix) If $A > B + 1$, then series ${}_AF_B$ is convergent, when $z = 0$.

(x) If $A = B + 1$ and $|z| \geq 1$, then it is defined as an analytic continuation of this series.

(xi) If $A = B + 1$ and $|z| = 1$, then series ${}_AF_B$ is divergent, when

$$\mathrm{Re}\left\{\sum_{m=1}^{B} b_m - \sum_{n=1}^{A} a_n\right\} \leq -1$$

(xii) If $A > B + 1$, then a meaningful independent attempts were made to define MacRobert's E-function, Meijer's G-function, Fox's H-function and its related functions.

(xiii) If one or more of the numerator parameters are zero or negative integers, then series ${}_AF_B$ terminates for all finite values of z i.e., ${}_AF_B$ will be a hypergeometric polynomial and the question of convergence does not enter the discussion.

§2. Main Indefinite Integrals

$$\int \frac{\cosh x\ L_1(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} =$$

$$= -\frac{1}{\sqrt{1-\cos x}}\Big(\frac{8}{25}-\frac{6\iota}{25}\Big)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\Big[2e^{2x}{}_3F_2\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\Big)+$$
$$+2e^{\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\Big)-(2-\iota)xe^{2x}{}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)+$$
$$+(2-\iota)xe^{\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)+(2-\iota)xe^{2x}-2e^{2x}\Big]+Constant \tag{2.1}$$

$$\int \frac{\sinh x\ L_1(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} =$$

$$= -\frac{1}{\sqrt{1-\cos x}}\Big(\frac{8}{25}-\frac{6\iota}{25}\Big)e^{(-1-\frac{\iota}{2})x}\sin\frac{x}{2}\Big[2e^{2x}{}_3F_2\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\Big)-$$
$$-2e^{\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\Big)-(2-\iota)xe^{2x}{}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)-$$
$$-(2-\iota)xe^{\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)+(2-\iota)xe^{2x}-2e^{2x}\Big]+Constant \tag{2.2}$$

$$\int \frac{\cos x\ L_2(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} = \frac{2}{\sqrt{1-\cos x}}\sin\frac{x}{2}\Big[4\iota x Li_2(-e^{\frac{\iota x}{2}})-4\iota x Li_2(e^{\frac{\iota x}{2}})-8Li_3(-e^{\frac{\iota x}{2}})+8Li_3(e^{\frac{\iota x}{2}})+$$
$$+x^2\log(1-e^{\frac{\iota x}{2}})-x^2\log(1+e^{\frac{\iota x}{2}})+2x^2\cos\frac{x}{2}-8x\sin\frac{x}{2}-12\cos\frac{x}{2}+2\log(\tan\frac{x}{4})\Big]+Constant \tag{2.3}$$

$$\int \frac{\cos x\ L_1(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} = \frac{2}{\sqrt{1-\cos x}}\sin\frac{x}{2}\Big[2\iota Li_2(-e^{\frac{\iota x}{2}})-2\iota Li_2(e^{\frac{\iota x}{2}})+x\log(1-e^{\frac{\iota x}{2}})-x\log(1+e^{\frac{\iota x}{2}})-$$
$$-4\sin\frac{x}{2}+2x\cos\frac{x}{2}\Big]+Constant \tag{2.4}$$

$$\int \frac{\cos x\ L_1(x)}{\sqrt{1-\cosh x}}\ \mathrm{dx} = -\frac{1}{25\sqrt{1-\cosh x}}e^{-\iota x}(e^x-1)\Big[(6-8\iota){}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)+$$
$$+(6+8\iota)e^{2\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;\cosh x+\sinh x\Big)+5x\Big\{(1+2\iota){}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big)+$$
$$+(1-2\iota)e^{2\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;\cosh x+\sinh x\Big)\Big\}\Big]+Constant \tag{2.5}$$

$$\int \frac{\sin x\ L_1(x)}{\sqrt{1-\cosh x}}\ \mathrm{dx} = \frac{1}{25\sqrt{1-\cosh x}}e^{-\iota x}(e^x-1)\Big[-(8+6\iota){}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)-$$
$$-(8-6\iota)e^{2\iota x}{}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;\cosh x+\sinh x\Big)+5x\Big\{(2-\iota){}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big)+$$
$$+(2+\iota)e^{2\iota x}{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;\cosh x+\sinh x\Big)\Big\}\Big]+Constant \tag{2.6}$$

$$\int \frac{\cos x\ L_2(x)}{\sqrt{1-\cosh x}}\ \mathrm{dx} =$$

$$= \frac{2}{125\sqrt{1-\cosh x}} e^{(-\iota+\frac{1}{2})x} \sinh\frac{x}{2}\Big[(2+11\iota)e^{2\iota x}\Big\{-(8-4\iota)x\ {}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^x\Big)-$$

$$-8\iota\ {}_4F_3\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^x\Big)+(4+3\iota)(x^2+2)\ {}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^x\Big)\Big\}-$$

$$-(11+2\iota)\Big\{(4-8\iota)x\ {}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)-$$

$$-8\ {}_4F_3\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)+(3+4\iota)(x^2+2)\,{}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big)\Big\}\Big]+Constant \tag{2.7}$$

$$\int\frac{\sin x\ L_3(x)}{\sqrt{1-\sin x}}\ \mathrm{dx} = \frac{1}{\sqrt{1-\sin x}}(1+\iota)\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\Big[(-1)^{\frac{3}{4}}\Big\{-6\iota(x^2+1)Li_2\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+$$

$$+6\iota(x^2+1)Li_2\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+24xLi_3\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-24xLi_3\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+48\iota Li_4\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-$$

$$-48\iota Li_4\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+x^3\Big(-\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)+x^3\Big(\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)-$$

$$-3x\Big(\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)+3x\Big(\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)\Big\}-(1-\iota)(x^3-6x^2-21x+42)\sin\frac{x}{2}+$$

$$+(-1+\iota)(x^3+6x^2-21x-42)\cos\frac{x}{2}\Big]+Constant \tag{2.8}$$

$$\int\frac{\sin x\ L_2(x)}{\sqrt{1-\cosh x}}\ \mathrm{dx} =$$

$$= \frac{2}{125\sqrt{1-\cosh x}} e^{(-\iota+\frac{1}{2})x}\sinh\frac{x}{2}\Big[(2-11\iota)\Big\{(4-8\iota)x\ {}_3F_2\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}-\iota;e^x\Big)-$$

$$-8\ {}_4F_3\Big(\frac{1}{2}-\iota,\frac{1}{2}-\iota,\frac{1}{2}-\iota,1;\frac{3}{2}-\iota,\frac{3}{2}+\iota,\frac{3}{2}-\iota;e^x\Big)+(3+4\iota)(x^2+2)\ {}_2F_1\Big(\frac{1}{2}-\iota,1;\frac{3}{2}-\iota;e^x\Big)\Big\}+$$

$$+(2+11\iota)e^{2\iota x}\Big\{(4+8\iota)x\ {}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^x\Big)-$$

$$-8\ {}_4F_3\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^x\Big)+(3-4\iota)(x^2+2)\,{}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^x\Big)\Big\}\Big]+Constant \tag{2.9}$$

$$\int\frac{\cosh x\ L_2(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} =$$

$$= \frac{1}{\sqrt{1-\cos x}}\Big(\frac{4}{125}+\frac{22\iota}{125}\Big)e^{(-\iota-\frac{1}{2})x}\sin\frac{x}{2}\Big[(4+8\iota)xe^{2x}\ {}_3F_2\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\Big)+$$

$$+(4+8\iota)xe^{\iota x}\ {}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\Big)-$$

$$-8\iota e^{2x}\ {}_4F_3\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota,\frac{1}{2}-\iota,;e^{\iota x}\Big)+$$

$$+8\iota e^{\iota x}\ {}_4F_3\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota,\frac{3}{2}+\iota,;e^{\iota x}\Big)-$$

$$-(4+3\iota)x^2e^{2x}\ {}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)+(4+3\iota)x^2e^{\iota x}\ {}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)-$$

$$-(8+6\iota)e^{2x}\ {}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)+(8+6\iota)e^{\iota x}\ {}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)+$$

$$+(4+3\iota)x^2e^{2x}-(4+8\iota)xe^{2x}+(8+14\iota)e^{2x}\Big]+Constant \tag{2.10}$$

$$\int \frac{\sinh x\ L_2(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} =$$

$$= -\frac{1}{\sqrt{1-\cos x}}\Big(\frac{4}{125}+\frac{22\iota}{125})e^{(-\iota-\frac{1}{2})x}\sin\frac{x}{2}\Big[-(4+8\iota)xe^{2x}\ {}_3F_2\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota;e^{\iota x}\Big)+$$

$$+(4+8\iota)xe^{\iota x}\ {}_3F_2\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota;e^{\iota x}\Big)+$$

$$+8\iota e^{2x}\ {}_4F_3\Big(-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota,\frac{1}{2}-\iota,\frac{1}{2}-\iota,;e^{\iota x}\Big)+$$

$$+8\iota e^{\iota x}\ {}_4F_3\Big(\frac{1}{2}+\iota,\frac{1}{2}+\iota,\frac{1}{2}+\iota,1;\frac{3}{2}+\iota,\frac{3}{2}+\iota,\frac{3}{2}+\iota,;e^{\iota x}\Big)+$$

$$+(4+3\iota)x^2e^{2x}\ {}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)+(4+3\iota)x^2e^{\iota x}\ {}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)-$$

$$+(8+6\iota)e^{2x}\ {}_2F_1\Big(-\frac{1}{2}-\iota,1;\frac{1}{2}-\iota;e^{\iota x}\Big)+(8+6\iota)e^{\iota x}\ {}_2F_1\Big(\frac{1}{2}+\iota,1;\frac{3}{2}+\iota;e^{\iota x}\Big)+$$

$$+(-4-3\iota)x^2e^{2x}+(4+8\iota)xe^{2x}-(8+14\iota)e^{2x}\Big]+Constant \tag{2.11}$$

$$\int \frac{\sin x\ L_4(x)}{\sqrt{1-\sin x}}\ \mathrm{dx} =$$

$$= \frac{1}{\sqrt{1-\sin x}}(1+\iota)\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)\Big[(-1)^{\frac{3}{4}}\Big\{48x^2Li_3\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-48x^2Li_3\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-$$

$$-8\iota(x^2+2)xLi_2\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+8\iota(x^2+2)xLi_2\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+192\iota xLi_4\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-$$

$$-192\iota xLi_4\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+32Li_3\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-32Li_3\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-384Li_5\Big(-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+$$

$$+384Li_5\Big((-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+x^4\Big(-\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)\Big)+x^4\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)-4x^2\log\Big(1-(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+$$

$$+4x^2\log\Big(1+(-1)^{\frac{3}{4}}e^{\frac{\iota x}{2}}\Big)+4\iota\tan^{-1}\Big((-1)^{\frac{1}{4}}e^{\frac{\iota x}{2}}\Big)\Big\}-(1-\iota)(x^4-8x^3-44x^2+176x+354)\sin\frac{x}{2}+$$

$$+(-1+\iota)(x^4+8x^3-44x^2-176x+354)\cos\frac{x}{2}\Big]+Constant \tag{2.12}$$

$$\int \frac{\cos x\ L_5(x)}{\sqrt{1-\cos x}}\ \mathrm{dx} =$$

$$= \frac{1}{12\sqrt{1-\cos x}}\sin\frac{x}{2}\Big[1920x^3Li_3\Big(e^{-\frac{\iota x}{2}}\Big)-1920x^3Li_3\Big(-e^{\frac{\iota x}{2}}\Big)+240\iota(x^2+3)x^2Li_2\Big(e^{-\frac{\iota x}{2}}\Big)-$$

$$-11520\iota x^2Li_4\Big(e^{-\frac{\iota x}{2}}\Big)-11520\iota x^2Li_4\Big(-e^{\frac{\iota x}{2}}\Big)+240\iota(x^4+3x^2+1)Li_2\Big(-e^{\frac{\iota x}{2}}\Big)+2880xLi_3\Big(e^{-\frac{\iota x}{2}}\Big)-$$

$$-2880xLi_3\Big(-e^{\frac{\iota x}{2}}\Big)-46080xLi_5\Big(e^{-\frac{\iota x}{2}}\Big)+46080xLi_5\Big(-e^{\frac{\iota x}{2}}\Big)-240\iota Li_2\Big(e^{\frac{\iota x}{2}}\Big)-$$

$$-5760\iota Li_4\Big(e^{-\frac{\iota x}{2}}\Big)-5760\iota Li_4\Big(-e^{\frac{\iota x}{2}}\Big)+92160\iota Li_6\Big(e^{-\frac{\iota x}{2}}\Big)+92160\iota Li_6\Big(-e^{\frac{\iota x}{2}}\Big)+$$

$$+2\iota x^6+24x^5\log\Big(1-e^{-\frac{\iota x}{2}}\Big)-24x^5\log\Big(1+e^{\frac{\iota x}{2}}\Big)+48x^5\cos\frac{x}{2}+15\iota x^4-480x^4\sin\frac{x}{2}+$$

$$+120x^3\log\Big(1-e^{-\frac{\iota x}{2}}\Big)-120x^3\log\Big(1+e^{\frac{\iota x}{2}}\Big)-3600x^3\cos\frac{x}{2}+21600x^2\sin\frac{x}{2}+120x\log\Big(1-e^{\frac{\iota x}{2}}\Big)-$$

$$-120x\log\Big(1+e^{\frac{\iota x}{2}}\Big)-173280\sin\frac{x}{2}+86640x\cos\frac{x}{2}-64\iota\pi^6-120\iota\pi^4\Big]+Constant \tag{2.13}$$

§3.Derivation of the Integrals

Applying the method which is used in ref[11] , one can derive the integrals.

Conclusion

In our work we have established certain indefinite integrals involving Lucas Polynomials and Hypergeometric function. However, one can establish such type of integrals which are very useful for different field of engineering and sciences by involving these integrals. Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions.

References

[1] E. Jahnke and F. Emde. Tables of functions with formulae and curves (4th ed.). New York: Dover Publications, 1945.

[2] E. Stein and G. Weiss. Introduction to Fourier analysis on Euclidean spaces. Princeton: Princeton University Press, 1971.

[3] E. T. Whittaker and G. N. Watson. A course of modern analysis. London: Cambridge University Press, 1962.

[4] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clarke. NIST handbook of mathematical functions. Cambridge University Press, 2010.

[5] G. Arfkeni. Mathematical methods for physicists. Academic Press, 1985.

[6] J. Guillera., J. Sondow. Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent. Ramanujan J., 2008, 16 (3): 247–270.

[7] M. Abramowitz and I. Stegun. Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards, 1970.

[8] N. Temme. Special functions: an Introduction to the classical functions of mathematical physics. New York: Wiley, 1996.

[9] N. Wiener. The Fourier integral and certain of its applications. New York: Dover Publications,1958.

[10] P. E. Ricci. Generalized Lucas polynomials and Fibonacci polynomials. Riv. Mat. Univ. Parma, 1995, 4: 137–146.

[11] Salahuddin. Hypergeometric form of certain indefinite integrals. Global Journal of Science Frontier Research(F). 2012, 12: 37–41.

[12] Z. W. Sun and H. Pan. Identities concerning Bernoulli and Euler polynomials. Acta Arith., 2006, 125 (1): 21–39.

Scientia Magna
Vol. 11 (2016), No. 2, 60-66

Some indefinite integrals

Salahuddin

Mewar University, Gangrar, Chittorgarh (Rajasthan) , India
E-mail: vsludn@gmail.com

Abstract In this paper we have developed some indefinite integrals in the form of Hypergeometric function. The results represent here are assume to be new.

Keywords hypergeometric function, elliptic integral.
2010 Mathematics Subject Classification 33C75, 33E05.

§1. Introduction and preliminaries

Elliptical Integral

IIn integral calculus, elliptic integrals originally arose in connection with the problem of giving the arc length of an ellipse. They were first studied by Giulio Fagnano and Leonhard Euler. Modern mathematics defines an "elliptic integral" as any function f which can be expressed in the form

$$f(x) = \int_c^x R\Big(t, \sqrt{P(t)}\Big)dt \tag{1.1}$$

where R is a rational function of its two arguments, P is a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

In general, elliptic integrals cannot be expressed in terms of elementary functions. Exceptions to this general rule are when P has repeated roots, or when $R(x, y)$ contains no odd powers of y. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e. the elliptic integrals of the first, second and third kind).

Besides the Legendre form , the elliptic integrals may also be expressed in Carlson symmetric form. Additional insight into the theory of the elliptic integral may be gained through the study of the Schwarz-Christoffel mapping. Historically, elliptic functions were discovered as inverse functions of elliptic integrals.

Incomplete elliptic integrals are functions of two arguments; complete elliptic integrals are functions of a single argument.

The incomplete elliptic integral of the first kind F is defined as

$$F(\psi, k) = F(\psi \mid k^2) = F(\sin\psi; k) = \int_0^{\psi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \tag{1.2}$$

This is the trigonometric form of the integral; substituting $t = \sin\theta, x = \sin\psi$,one obtains Jacobi's form:

$$F(x; k) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \tag{1.3}$$

Equivalently, in terms of the amplitude and modular angle one has:

$$F(\psi\backslash\alpha) = F(\psi, \sin\alpha) = \int_0^{\psi} \frac{d\theta}{\sqrt{1-(\sin\theta\sin\alpha)^2}} \tag{1.4}$$

In this notation, the use of a vertical bar as delimiter indicates that the argument following it is the "parameter" (as defined above), while the backslash indicates that it is the modular angle. The use of a semicolon implies that the argument preceding it is the sine of the amplitude:

$$F(\psi, \sin\alpha) = F(\psi \mid \sin^2\alpha) = F(\psi\backslash\alpha) = F(\sin\psi; \sin\alpha) \tag{1.5}$$

Incomplete elliptic integral of the second kind E is defined as

$$E(\psi, k) = E(\psi \mid k^2) = E(\sin\psi; k) = \sqrt{1-k^2\sin^2\theta}\; d\theta \tag{1.6}$$

Substituting $t = \sin\theta$ and $x = \sin\psi$, one obtains Jacobi's form:

$$E(x; k) = \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\; dt \tag{1.7}$$

Equivalently, in terms of the amplitude and modular angle:

$$E(\psi\backslash\alpha) = E(\psi, \sin\alpha) = \int_0^{\psi} \sqrt{1-(\sin\theta\sin\alpha)^2}\; d\theta \tag{1.8}$$

Incomplete elliptic integral of the third kind Π is defined as

$$\Pi(n; \psi\backslash\alpha) = \int_0^{\psi} \frac{1}{1-n\sin^2\theta}\, \frac{d\theta}{1-(\sin\theta\sin\alpha)^2} \tag{1.9}$$

or

$$\Pi(n; \psi \mid m) = \int_0^{\sin\psi} \frac{1}{1-nt^2}\, \frac{dt}{(1-mt^2)(1-t^2)} \tag{1.10}$$

The number n is called the characteristic and can take on any value, independently of the other arguments.

Complete elliptic integral of the first kind is defined as

Elliptic Integrals are said to be 'complete' when the amplitude $\psi = \frac{\pi}{2}$ and therefore x=1. The complete elliptic integral of the first kind K may thus be defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{1.11}$$

or more compactly in terms of the incomplete integral of the first kind as

$$K(k) = F\left(\frac{\pi}{2}, k\right) = F(1; k) \tag{1.12}$$

It can be expressed as a power series

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\sum_{n=0}^{\infty}\Big[P_{2n}(0)\Big]^2 k^{2n} \tag{1.13}$$

where P_n is the Legendre polynomial, which is equivalent to

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 k^4 + \,.........\, + \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 k^{2n} + \,......\right] \tag{1.14}$$

where $n!!$ denotes the double factorial. In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$K(k) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) \tag{1.15}$$

The complete elliptic integral of the first kind is sometimes called the quarter period. It can most efficiently be computed in terms of the arithmetic-geometric mean:

$$K(k) = \frac{\frac{\pi}{2}}{agm(1-k, 1+k)} \tag{1.16}$$

Complete elliptic integral of the second kind is defined as

The complete elliptic integral of the second kind E is proportional to the circumference of the ellipse C:

$$C = 4aE(e)$$

where a is the semi-major axis, and e is the eccentricity.

E may be defined as

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\ d\theta = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\ dt \tag{1.17}$$

or more compactly in terms of the incomplete integral of the second kind as

$$E(k) = E\left(\frac{\pi}{2}, k\right) = E(1; k) \tag{1.18}$$

It can be expressed as a power series

$$E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{1-2n} \tag{1.19}$$

which is equivalent to

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 \frac{k^4}{3} - \,.........\, - \left\{\frac{(2n-1)!!}{(2n)!!}\right\}^2 \frac{k^{2n}}{2n-1} - \,......\right] \tag{1.20}$$

In terms of the Gauss hypergeometric function, the complete elliptic integral of the second kind can be expressed as

$$E(k) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k^2\right) \tag{1.21}$$

Complete elliptic integral of the third kind is defined as

The complete elliptic integral of the third kind Π can be defined as

$$\Pi(n, k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} \tag{1.22}$$

Generalized Hypergeometric Functions

A generalized hypergeometric function ${}_pF_q(a_1,...a_p;b_1,...b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\ z. \tag{1.23}$$

Where $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{lc} a_1,a_2,\cdots,a_p & ; \\ & & z \\ b_1,b_2,\cdots,b_q & ; \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!} \tag{1.24}$$

or

$${}_pF_q\left[\begin{array}{lc} (a_p) & ; \\ & & z \\ (b_q) & ; \end{array}\right]\equiv {}_pF_q\left[\begin{array}{lc} (a_j)_{j=1}^{p} & ; \\ & & z \\ (b_j)_{j=1}^{q} & ; \end{array}\right]=\sum_{k=0}^{\infty}\frac{((a_p))_k z^k}{((b_q))_k k!} \tag{1.25}$$

where the parameters $b_1,b_2,\cdots,b_q$ are neither zero nor negative integers and $p,\ q$ are non-negative integers.

The ${}_pF_q$ series converges for all finite z if $p\leq q$, converges for $\mid z\mid<1$ if $p\neq q+1$, diverges for all $z,\ z\neq 0$ if $p>q+1$.

The ${}_pF_q$ series absolutely converges for $\mid z\mid=1$ if $R(\zeta)<0$, conditionally converges for $\mid z\mid=1, z\neq 0$ if $0\leq R(\zeta)<1$, diverges for $\mid z\mid=1$, if $1\leq R(\zeta)$, $\zeta=\sum\limits_{i=1}^{p}a_i-\sum\limits_{i=0}^{q}b_i$.

The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2, q=1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (Gauss 1812, Barnes 1908). To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y''+[c-(a+b+1)z]y'-aby=0 \tag{1.26}$$

The solution of this equation is

$$y=A_0\Big[1+\frac{ab}{1!\ c}z+\frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2+\cdots\cdots\Big] \tag{1.27}$$

This is the so-called regular solution,denoted

$${}_2F_1(a,b;c;z)=\Big[1+\frac{ab}{1!\ c}z+\frac{a(a+1)b(b+1)}{2!\ c(c+1)}z^2+\cdots\cdots\Big]=\sum_{k=0}^{\infty}\frac{(a)_k\ (b)_k z^k}{(c)_k k!} \tag{1.28}$$

which converges if c is not a negative integer for all of $| z |< 1$ and on the unit circle $| z |= 1$ if $R(c-a-b) > 0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

§2. Main Integrals

$$\int \sqrt{1+x^{-n}}\ dx = \frac{nx\ {}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right)-2x\ \sqrt{x^{-n}+1}}{n-2}+Constant \tag{2.1}$$

If $n = 38$ then the integral(2.1) becomes

$$\int \sqrt{1+x^{-38}}\ dx = \frac{1}{360}\sqrt{1+x^{-38}}\ x\left[\frac{19x^{38}{}_2F_1\left(\frac{1}{2},\frac{10}{19};\frac{29}{19};-x^{38}\right)}{\sqrt{1+x^{38}}}-20\right]+Constant \tag{2.2}$$

If $n = 20$ then the integral(2.1) becomes

$$\int \sqrt{1+x^{-20}}\ dx = \frac{1}{99}\sqrt{1+x^{-20}}\ x\left[\frac{10x^{20}{}_2F_1\left(\frac{1}{2},\frac{11}{20};\frac{31}{20};-x^{20}\right)}{\sqrt{1+x^{20}}}-11\right]+Constant \tag{2.3}$$

If $n = 10$ then the integral(2.1) becomes

$$\int \sqrt{1+x^{-10}}\ dx = \frac{1}{24}\sqrt{1+x^{-10}}\ x\left[\frac{5x^{10}{}_2F_1\left(\frac{1}{2},\frac{3}{5};\frac{8}{5};-x^{10}\right)}{\sqrt{1+x^{10}}}-6\right]+Constant \tag{2.4}$$

$$\int \sqrt{1+x^{n}}\ dx = \frac{nx\ {}_2F_1\left(\frac{1}{2},\frac{1}{n};\frac{n+1}{n};-x^{n}\right)+2x\ \sqrt{x^{n}+1}}{n+2}+Constant \tag{2.5}$$

If $n = 3$ then the integral(2.5) becomes

$$\int \sqrt{1+x^{3}}\ dx = \frac{1}{5}\ x\left[3\ {}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{4}{3};-x^{3}\right)+2\sqrt{1+x^{3}}\right]+Constant \tag{2.6}$$

If $n = 5$ then the integral(2.5) becomes

$$\int \sqrt{1+x^{5}}\ dx = \frac{1}{7}\ x\left[5\ {}_2F_1\left(\frac{1}{5},\frac{1}{2};\frac{6}{5};-x^{5}\right)+2\sqrt{1+x^{5}}\right]+Constant \tag{2.7}$$

If $n = 6$ then the integral(2.5) becomes

$$\int \sqrt{1+x^{6}}\ dx =$$

$$= \frac{2(x^6+1)x^2+\dfrac{3^{\frac{3}{4}}(x^4-x^2+1)\sqrt{\frac{x^4+x^2}{[(1+\sqrt{3})x^2+1]^2}}F\left(\cos^{-1}\left(\frac{1-(-1+\sqrt{3})x^2}{(1+\sqrt{3})x^2+1}\right)\middle|\frac{1}{4}(2+\sqrt{3})\right)}{\sqrt{\frac{x^4-x^2+1}{[(1+\sqrt{3})x^2+1]^2}}}}{8x\sqrt{x^6+1}}+Constant \tag{2.8}$$

If $n = 7$ then the integral(2.5) becomes

$$\int \sqrt{1+x^7}\ dx = \frac{1}{9}\ x\left[7\ {}_2F_1\left(\frac{1}{7},\frac{1}{2};\frac{8}{7};-x^7\right) + 2\sqrt{1+x^7}\right] + Constant \tag{2.8}$$

If $n = 17$ then the integral(2.5) becomes

$$\int \sqrt{1+x^{17}}\ dx = \frac{1}{19}\ x\left[17\ {}_2F_1\left(\frac{1}{17},\frac{1}{2};\frac{18}{17};-x^{17}\right) + 2\sqrt{1+x^{17}}\right] + Constant \tag{2.9}$$

$$\int \frac{1}{\sqrt{1+x^n}}\ dx = x\ {}_2F_1\left(\frac{1}{2},\frac{1}{n};\frac{n+1}{n};-x^n\right) + Constant \tag{2.10}$$

If $n = 3$ then the integral(2.10) becomes

$$\int \frac{1}{\sqrt{1+x^3}}\ dx =$$

$$= \frac{1}{\sqrt[4]{3}\ \sqrt{1+x^3}}\ 2\ \sqrt[6]{-1}\sqrt{-\sqrt[6]{-1}\ \left(x+(-1)^{\frac{2}{3}}\right)}\sqrt{(-1)^{\frac{2}{3}}x^2+\sqrt[3]{-1}\ x+1}\ \times$$

$$\times\ F\left(\sin^{-1}\left(\frac{\sqrt{-(-1)^{\frac{5}{6}}(x+1)}}{\sqrt[4]{3}}\right)\middle|\sqrt[3]{-1}\right) + Constant \tag{2.11}$$

If $n = 11$ then the integral(2.10) becomes

$$\int \frac{1}{\sqrt{1+x^{11}}}\ dx = x\ {}_2F_1\left(\frac{1}{11},\frac{1}{2};\frac{12}{11};-x^{11}\right) + Constant \tag{2.12}$$

If $n = 14$ then the integral(2.10) becomes

$$\int \frac{1}{\sqrt{1+x^{14}}}\ dx = x\ {}_2F_1\left(\frac{1}{14},\frac{1}{2};\frac{15}{14};-x^{14}\right) + Constant \tag{2.13}$$

$$\int \frac{1}{\sqrt{1+x^{-n}}}\ dx = x\ {}_2F_1\left(\frac{1}{2},-\frac{1}{n};\frac{n-1}{n};-x^{-n}\right) + Constant \tag{2.14}$$

If $n = 8$ then the integral(2.14) becomes

$$\int \frac{1}{\sqrt{1+x^{-8}}}\ dx = \frac{x\ \sqrt{1+x^8}\ {}_2F_1\left(\frac{1}{2},\frac{5}{8};\frac{13}{8};-x^8\right)}{5\ \sqrt{1+x^{-8}}} + Constant \tag{2.15}$$

If $n = 16$ then the integral(2.14) becomes

$$\int \frac{1}{\sqrt{1+x^{-16}}}\ dx = \frac{x\ \sqrt{1+x^{16}}\ {}_2F_1\left(\frac{1}{2},\frac{9}{16};\frac{25}{16};-x^{16}\right)}{9\ \sqrt{1+x^{-16}}} + Constant \tag{2.16}$$

References

[1] A. G. Greenhill. The applications of elliptic functions. New York: Macmillan, 1892.

[2] L. C. Andrews. Special function of mathematics for engineers. Second Edition. New York: McGraw-Hill Co Inc., 1992.

[3] R. Bells and R. Wong. Special functions. Cambridge Studies in Advanced Mathematics, 2010.

[4] E. Borowski and J. Borwein. Mathematics, collins dictionary (2nd ed.). Glasgow: HarperCollins, 2002.

[5] M. Abramowitz and I. Stegun. Handbook of mathematical functions with formulas, graphs, and mathematical tables. National Bureau of Standards, 1970.

[6] R. Crandall and C. Pomeramce. Prime numbers: a computational perspective. New York: Springer, 2001.

[7] L. Ahlfors. Complex analysis: an introduction to the theory of analytic functions of one complex variable (3rd ed.). New York: McGraw-Hill, 1979.

[8] J. Derbyshire. Prime obsession: Bernhard Riemann and the greatest unsolved problem in mathematics. New York: Penguin, 2004.

[9] L. R. Graham, D. E. Knuth and O. Patashnik. Concrete mathematics. Reading: Addison- Wesley, 1994.

[10] G. H. Hardy and E. M. Wright. An introduction to the theory of numbers (Fifth edition). Oxford: Oxford University Press, 1980.

[11] H. Hancock. Lectures on the theory of elliptic functions. New York: J. Wiley and sons, 1910.

[12] J. W. S. Cassels. An introduction to Diophantine approximation. Cambridge University Press, 1957.

[13] S. G. Krantz. Handbook of complex variables. Boston: Birkhauser, 1999.

[14] F. Lemmermeyer. Reciprocity laws: from Euler to Eisenstein. Berlin: Springer, 2000.

[15] L. V. King. On the direct numerical calculation of elliptical functions and integrals. Cambridge University Press, 1924.

[16] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery. Numerical recipes: the art of scientific computing (3rd ed.). New York: Cambridge University Press, 2007.

[17] S. Ramanujan. Collected papers. Providence RI: AMS / Chelsea, 2000.

[18] P. Ribenboim. The new book of prime number records. New York: Springer, 1996.

[19] Salahuddin. Hypergeometric form of certain indefinite integrals. Global Journal of Science Frontier Research(F), 2012, 12: 33–37.

[20] R. A. Silverman. Introductory complex analysis. New York: Dover, 1984.

[21] E. C. Titchmarsh. The theory of the Riemann zeta function (2nd ed.). Oxford: Oxford University Press, 1986.

Scientia Magna
Vol. 11 (2016), No. 2, 67-82

Subclass of analytic functions involving generalized Ruscheweyh derivative operator

J. J. Bhamare[1] and S. M. Khairnar[2]

[1]Department of Applied Sciences,
S.S.V.P.S.s B. S. Deore College of Engineering
Deopur, Dhule, India
E-mail: jjbhamre2002@yahoo.co.in
[2]Department of Engineering Sciences
MIT Academy of Engineering
Alandi, Pune-412105, M. S., India
E-mail: smkhairnar2007@gmail.com

Abstract In this paper we introduce a subclass of analytic and univalent functions defined by the operator D_λ^n, which is a generalized Ruscheweyh derivatives operator. We derive some results which are sharp on coefficient inequalities, growth and distortion theorems, extreme points, convolution. We also investigate some inclusion theorem, radius of convexity and starlikeness, integral mean, inequalities for fractional derivatives of functions belonging to the class $S_{m,n,\lambda,\gamma}(\alpha)$.

Keywords Ruscheweyh derivatives operator, growth and distortion theorems, convolution, radius of starlikeness and convexity.
2010 Mathematics Subject Classification 30C45.

§1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathcal{C}|z| < 1\}$. D_λ^n, the operator introduced by authors [3] and is given by

$$D_\lambda^0 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z),\ \lambda \geq 0$$

$$D_\lambda^1 f(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))',$$

$$D_\lambda^n f(z) = D_\lambda \left(\frac{z(z^{n-1}f(z))^n}{n!}\right),\ (n \in \mathbb{N}_0 = \mathbb{N}U\{0\}).$$

If the function f is given by (1), then we write

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty}[1+\lambda(k-1)]\delta(n,k)a_k z^k$$

where

$$\delta(n,k) = \begin{pmatrix} k+n-1 \\ n \end{pmatrix} = \frac{\prod_{j=2}^{k-2}(j+n)}{(k-1)!},\ k \geq 2.$$

The hadamard product(or convolution) of two functions $f(z)$ given by (1) and

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

The class

$$S_{\gamma}(z) = \frac{z}{(1-z)^{2(1-\gamma)}},\quad (z \in \mathcal{U};\ 0 \leq \gamma < 1),$$

is the well-known extremal function for $\mathcal{S}^*(\gamma)$. Setting

$$c_k(\gamma) = \frac{\prod_{n=2}^{k}(n-2\gamma)}{(k-1)!},\ (k \in \mathbb{N} \setminus 1;\ \mathbb{N} := 1, 2, 3, ...),$$

$\mathcal{S}_{\gamma}(z)$ can be written in the form:

$$S_{\gamma}(z) = z + \sum_{k=2}^{\infty} c_k(\gamma) z^k.$$

Then we can see that $c_k(\gamma)$ is an decreasing function in γ $(0 \leq \gamma < 1)$ and that

$$\lim_{k\to\infty} c_k(\gamma) = \begin{cases} \infty, & \left(\gamma < \frac{1}{2}\right), \\ 1, & \left(\gamma = \frac{1}{2}\right), \\ 0, & \left(\gamma > \frac{1}{2}\right). \end{cases}$$

Let $\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$ the subclass of $\mathcal{A}$ consisting of function f which satisfy the inequality

$$\mathcal{R}e\left(\frac{D_{\lambda}^{m}(f * S_{\gamma}(z))}{D_{\lambda}^{n}(f * S_{\gamma}(z))}\right) > \alpha,\ (z \in \mathcal{U})$$

for some $0 \leq \alpha < 1,\ m \in \mathbb{N},\ n \in \mathbb{N}_0$.

§2. Coefficient Estimate

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty} \psi(m,n,k,\lambda,\alpha) c_k(\gamma) |a_k| \leq 2(1-\alpha). \tag{2}$$

where

$$\begin{aligned}\psi(m,n,k,\lambda,\alpha)c_k(\gamma) &= [1+\lambda(k-1)]c_k(\gamma)\\ &\quad\times\{|\delta(m,k)-(1+\alpha)\delta(n,k)|+[\delta(m,k)+(1-\alpha)\delta(n,k)]\}\end{aligned} \tag{3}$$

for some α $(0\le\alpha<1)$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, *then* $f(z)\in\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$.

Proof. Suppose (2) is true for α $(0\le\alpha<1)$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, and $\lambda\ge 0$. For $f(z)\in\mathcal{A}$ defined the function $F(z)$ by

$$F(z)=\left(\frac{D_\lambda^m(f*S_\gamma(z))}{D_\lambda^n(f*S_\gamma(z))}\right)-\alpha$$

It is sufficient to show that

$$\left|\frac{F(z)-1}{F(z)+1}\right|<1,\ (z\in\mathcal{U}). \tag{4}$$

Note that

$$\begin{aligned}\left|\frac{F(z)-1}{F(z)+1}\right| &= \left|\frac{\dfrac{D_\lambda^m(f*S_\gamma(z))}{D_\lambda^n(f*S_\gamma(z))}-\alpha-1}{\dfrac{D_\lambda^m(f*S_\gamma(z))}{D_\lambda^n(f*S_\gamma(z))}-\alpha+1}\right|\\ &= \left|\frac{D_\lambda^m(f*S_\gamma(z))-(1+\alpha)D_\lambda^n(f*S_\gamma(z))}{D_\lambda^m(f*S_\gamma(z))-(1-\alpha)D_\lambda^n(f*S_\gamma(z))}\right|.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\frac{F(z)-1}{F(z)+1}\right|\\ &=\left|\frac{\alpha+\sum_{k=2}^{\infty}[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)-(1+\alpha)\delta(n,k)]a_kz^{k-1}}{(2-\alpha)-\sum_{k=2}^{\infty}[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)+(1-\alpha)\delta(n,k)]a_kz^{k-1}}\right|\\ &=\frac{\alpha+\sum_{k=2}^{\infty}|[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)-(1+\alpha)\delta(n,k)]||a_k||z^{k-1}|}{(2-\alpha)-|\sum_{k=2}^{\infty}[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)+(1-\alpha)\delta(n,k)]||a_k||z^{k-1}|}\\ &=\frac{\alpha+\sum_{k=2}^{\infty}|[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)-(1+\alpha)\delta(n,k)]||a_k|}{(2-\alpha)-|\sum_{k=2}^{\infty}[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)+(1-\alpha)\delta(n,k)]||a_k|}.\end{aligned}$$

This expression is bounded above by 1, using (4)

$$\begin{aligned}&\alpha+\sum_{k=2}^{\infty}|[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)-(1+\alpha)\delta(n,k)]||a_k|\\ &\le(2-\alpha)-|\sum_{k=2}^{\infty}[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)+(1-\alpha)\delta(n,k)]||a_k|.\end{aligned}$$

which is equivalent to condition (2).
This completes the proof of Theorem 2.1. □

Now derive the coefficient of inequalities for $f(z)$ belonging to the class $\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$.

Theorem 2.2. *If $f(z) \in \mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$, then for $k \geq 2$,*

$$|a_j| \leq \frac{2(1-\alpha)\sum_{k=1}^{j-1}[1+\lambda(j-k-1)]c_{j-k}(\gamma)\delta(n,j-k)\,|a_{j-k}|}{[1+\lambda(j-1)]c_j(\gamma)|\delta(m,j)-\delta(n,j)|}$$

Proof. Define the function $\phi(z)$ by

$$\phi(z) = \frac{1}{1-\alpha}\left(\frac{D_\lambda^m(f * S_\gamma(z))}{D_\lambda^n(f * S_\gamma(z))}\right) = 1 + \sum_{k=1}^{\infty} C_k z^k$$

Since $\phi(z)$ is Caratheodory function,

$$|C_k| \leq 2, \ (k = 1, 2, 3, ...)$$

The definition of $\phi(z)$ implies that

$$\frac{1}{1-\alpha}\left(D_\lambda^m(f * S_\gamma(z)) - D_\lambda^n(f * S_\gamma(z))\right) = D_\lambda^n f(z)\left(1 + \sum_{k=1}^{\infty} C_k z^k\right)$$

We have

$$\begin{aligned}
&\frac{1}{1-\alpha}\left(D_\lambda^m(f * S_\gamma(z)) - \alpha D_\lambda^n(f * S_\gamma(z))\right)\\
&= z + (1+\lambda)\left[c_2(\gamma)\frac{\delta(m,2)-\alpha\delta(n,2)}{1-\alpha}\right]a_2 z^2\\
&+ (1+2\lambda)\left[c_3(\gamma)\frac{\delta(m,3)-\alpha\delta(n,3)}{1-\alpha}\right]a_3 z^3\\
&+ ...\\
&+ [1+\lambda(j-1)]\left[c_j(\gamma)\frac{\delta(m,j)-\alpha\delta(n,j)}{1-\alpha}\right]a_j z^j + ...
\end{aligned} \tag{5}$$

. Also

$$\begin{aligned}
&D_\lambda^n(f * S_\gamma(z))\left(1 + \sum_{k=1}^{\infty} C_k z^k\right)\\
&= \left(z + \sum_{k=1}^{\infty}[1+\lambda(k-1)]c_k(\gamma)a_k z^k\right)(1 + C_1 z + C_2 z^2 + ... + C_j z^j + ...)
\end{aligned} \tag{6}$$

From (5) and (6)

$$\begin{aligned}
&z + (1+\lambda)\left[\frac{\delta(m,2)-\alpha\delta(n,2)}{1-\alpha}\right]c_2(\gamma)a_2z^2 \\
&+ (1+2\lambda)\left[\frac{\delta(m,3)-\alpha\delta(n,3)}{1-\alpha}\right]c_3(\gamma)a_3z^3 + ... \\
&+ [1+\lambda(j-1)]\left[\frac{\delta(m,j)-\alpha\delta(n,j)}{1-\alpha}\right]c_j(\gamma)a_jz^j + ... \\
&= \left(z + \sum_{k=1}^{\infty}[1+\lambda(k-1)]c_k(\gamma)\delta(m,k)a_kz^k\right) \\
&\qquad\qquad \times (1 + C_1z + C_2z^2 + ... + C_jz^j + ...)
\end{aligned}$$

Consider coefficient of z^j of both sides in the above equality, then

$$\begin{aligned}
[1+\lambda(j-1)]c_j(\gamma)\left[\frac{\delta(m,j)-\alpha\delta(n,j)}{1-\alpha}\right]a_j &= [1+\lambda(j-1)]c_j(\gamma)\delta(n,j)a_j \\
&+ \sum_{k=1}^{j-1}[1+\lambda(j-k-1)]c_{j-k}(\gamma)\delta(n,j-k)a_{j-k}C_k
\end{aligned}$$

That is

$$\begin{aligned}
[1+\lambda(j-1)]c_j(\gamma)\left[\frac{\delta(m,j)-\alpha\delta(n,j)}{1-\alpha} - \delta(n,j)\right]a_j \\
= \sum_{k=1}^{j-1}[1+\lambda(j-k-1)]c_{j-k}(\gamma)\delta(n,j-k)a_{j-k}C_k
\end{aligned} \tag{7}$$

Therefore

$$\begin{aligned}
|a_j| &= \frac{1-\alpha}{[1+\lambda(j-1)]c_j(\gamma)|[\delta(m,j)-\delta(n,j)]|} \times \\
&\left|\sum_{k=1}^{j-1}[1+\lambda(j-k-1)]c_{j-k}(\gamma)\delta(n,j-k)a_{j-k}C_k\right| \\
&\leq \frac{(1-\alpha)\sum_{k=1}^{j-1}[1+\lambda(j-k-1)]c_{j-k}(\gamma)\delta(n,j-k)|a_{j-k}||C_k|}{[1+\lambda(j-1)]c_j(\gamma)|\delta(m,j)-\delta(n,j)|}
\end{aligned} \tag{8}$$

i.e.

$$|a_j| \leq \frac{2(1-\alpha)\sum_{k=1}^{j-1}[1+\lambda(j-k-1)]c_{j-k}(\gamma)\delta(n,j-k)\,|a_{j-k}|}{[1+\lambda(j-1)]c_j(\gamma)|\delta(m,j)-\delta(n,j)|}$$

□

Corollary 2.1. *If the function $f(z)$ is in the class $\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$ then*

$$a_k < \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}. \tag{9}$$

The result (9) is sharp for the function $f(z)$ of the form

$$f(z) = z + \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} z^k. \tag{10}$$

where $\psi(m,n,k,\lambda,\alpha)c_k(\gamma)$ given in equation (3).

§3. Extreme Point

In view of Theorem 2.1, we now introduce the subclass $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha) \subset \mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$ which consist of function

$$f(z) = z + \sum_{k=1}^{\infty} a_k z^k, \ (a_k \geq 0)$$

whose Taylor-Maclaurin coefficients satisfy inequality (2). Determining extreme points of the class $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$.

Theorem 4.1. *Let $f_1(z) = z$ and*

$$f_k(z) = z + \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} z^k, \quad for\ k = 2, 3 \ldots,$$

where $\psi(m,n,k,\lambda,\alpha)c_k(\gamma)$ is given by (3). Then $f(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$, if and only if $f(z)$ can be expressed in the form,

$$f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z),$$

where

$$\eta_k \geq 0 \quad for\ n \in \mathbb{N} = 1, 2, 3, ...\ and\ \sum_{k=1}^{\infty} \eta_k = 1.$$

Proof. Suppose that,

$$f(z) = \sum_{k=1}^{\infty} \eta_k f_k(z) = z + \sum_{k=2}^{\infty} \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} \eta_k z^k. \tag{11}$$

Then,

$$\begin{aligned} \sum_{k=2}^{\infty} \psi(m,n,k,\lambda,\alpha)c_k(\gamma) \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} \eta_k &= 2(1-\alpha) \sum_{k=2}^{\infty} \eta_k \\ &= 2(1-\alpha)(1-\eta_1) \\ &\leq 2(1-\alpha). \end{aligned}$$

which shows that f satisfies condition (2) and therefore, $f(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$.

Conversely, suppose that $f(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$. Thus

$$a_k \leq \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}.$$

We may set

$$\eta_k = \frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)}a_k$$

and

$$\eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$$

.

Then we obtain

$$f(z) = \sum_{k=2}^{\infty} \eta_k f_k(z),$$

which completes the proof of Theorem 4.1 □

§4. Closure Theorem

Theorem 4.1. *Let $f_j(z)$ be defined as,*

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k, \ a_{k,j} \geq 0, \ j = 1, 2, 3 \ldots m$$

belong to the class $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$. Then the function,

$$h(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z) = z + \frac{1}{m}\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} a_{k,j}\right) z^k$$

is also belongs to the class $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$.

Proof. Since $f_j(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$, in view of Theorem 2.1, we have,

$$\sum_{k=2}^{\infty} \frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)a_{k,j}}{2(1-\alpha)} \leq 1, j = 1, 2, 3 \ldots m. \tag{12}$$

Now

$$\frac{1}{m}\sum_{j=1}^{m} f_j(z) = z - \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=2}^{\infty} a_k\right) z^k = z - \sum_{n=2}^{\infty} e_k z^k,$$

where

$$e^k = \frac{1}{m}\sum_{j=1}^{m} a_{k,j} \leq 1.$$

Notice that,

$$\sum_{k=2}^{\infty} \frac{[\psi(m,n,k,\lambda,\alpha)c_k(\gamma)]}{2(1-\alpha)} \frac{1}{m}\sum_{j=1}^{m} a_{k,j} \leq 1, \text{ using(12)}.$$

Thus $h(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$. □

§5. Growth and Distortion Theorem

Theorem 5.1. *If the function $f(z)$ defined by (1) is in the class $\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$, $0 \leq \gamma < 1$, $0 \leq \alpha < 1$ and either $0 \leq \gamma \leq \frac{5}{6}$ or $|z| \leq \frac{3}{4}$ then,*

$$|f(z)| \geq max\left\{0, |z| - \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)}\,|z|^2\right\}$$

and

$$|f(z)| \leq |z| + \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)}\,|z|^2 .$$

The bounds are sharp.

Proof. By virtue of the theorem , we note that

$$|f(z)| \geq max\left\{0, |z| - \max_{n\in\mathbb{N}-\{1\}} \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}\,|z|^k\right\}$$

and

$$|f(z)| \leq |z| + \max_{n\in\mathbb{N}-\{1\}} \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}\,|z|^k, \text{ for } z \in \mathcal{U}.$$

Hence it suffices to deduce that

$$\mathcal{G}(m,n,k,\lambda,\alpha,|z|) = \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}\,|z|^k$$

is a decreasing function of k, $(k \geq 2)$. Since

$$c_{k+1}(\gamma) = \frac{k+1-2\gamma}{k}c_k(\gamma).$$

We can see that, for $|z| \neq 0$,

$$\mathcal{G}(m,n,k,\lambda,\alpha,|z|) \geq \mathcal{G}(m,n,k+1,\lambda,\alpha,|z|)$$

if and only if

$$\mathcal{H}(\gamma,k,|z|) = (k+1)(k+1-2\gamma) - k^2\,|z| \geq 0.$$

It is easy to see that $\mathcal{H}(\gamma,k,|z|)$ is a decreasing function of γ for fixed $|z|$. Consequently it follows that

$$\mathcal{H}(\gamma,k,|z|) \geq \mathcal{H}(\frac{5}{6},k,|z|) = k^2(1-|z|) + \frac{1}{3}(k-2) \geq 0,$$

for $0 \leq \gamma \leq \frac{5}{6}$, $z \in \mathcal{U}$ and $k \geq 2$.
Further, since $\mathcal{H}(\gamma,k,|z|)$ is decreasing in $|z|$ and increasing in k, we obtain that

$$\mathcal{H}(\gamma,k,|z|) > \mathcal{H}(1,k,|z|) \geq \mathcal{H}(1,2,\frac{3}{4}),$$

for $0 \leq \gamma \leq 1$, $|z| \leq \frac{3}{4}$ and $k \geq 2$. Thus

$$\max_{n\in\mathbb{N}-\{1\}} \mathcal{G}(m,n,k,\lambda,\alpha,|z|)$$

is attained at $k = 2$, and the proof is complete.
Finally, since the function $f_k(z)$, $(k \geq 0)$ defined in theorem are the extreme points of the

class $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$, we can see that the bound of the theorem are attained by the function $f_2(z)$ is

$$f_2(z) = z + \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)} z^2. \tag{13}$$

$\square$

Corollary 5.1 *Let the function $f(z)$ defined by (1) be in the class $\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$, with $0 \leq \gamma \leq \frac{5}{6}$ and $0 \leq \beta < 1$. Then $f(z)$ is included in a disk with its center at the origin and radius r given by*

$$r = 1 + \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)}$$

Remark 5.1 *The extremal function $f(z)$ given by (13) is equal to zero when*

$$z = -\frac{(1-\gamma)\psi(m,n,2,\lambda,\alpha)}{1-\alpha)}.$$

Letting $z \to 1^-$, it follows that

$$\alpha \to \frac{1-\alpha+\psi(m,n,2,\lambda,\alpha)}{\psi(m,n,2,\lambda,\alpha)}.$$

We thus have

$$|f(z)| \geq |z| - \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)} |z|^2 .$$

for all $z \in \mathcal{U}$ if and only if

$$0 \leq \alpha \leq \frac{1-\alpha+\psi(m,n,2,\lambda,\alpha)}{\psi(m,n,2,\lambda,\alpha)}.$$

Theorem 5.2. *If the function $f(z)$ defined by (1) is in the class $\mathcal{S}_{m,n,\lambda,\gamma}(\alpha)$, $0 \leq \gamma < 1$, $0 \leq \alpha < 1$ and either $0 \leq \gamma \leq \frac{1}{2}$ or $|z| \leq \frac{1}{2}$ then,*

$$1 - \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)} |z| \leq |f'(z)| \leq 1 + \frac{(1-\alpha)}{\psi(m,n,2,\lambda,\alpha)(1-\gamma)} |z|$$

The bounds are sharp.

Proof. By virtue of the theorem , we note that

$$|f'(z)| \geq 1 - \max_{n \in \mathbb{N}-\{1\}} \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} |z|^{k-1}$$

and

$$|f'(z)| \leq 1 + \max_{n \in \mathbb{N}-\{1\}} \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} |z|^{k-1} .$$

Hence it suffices to deduce that

$$\mathcal{G}_1(m,n,k,\lambda,\alpha,|z|) = \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} |z|^{k-1}$$

is a decreasing function of k, $(k \geq 2)$. Since

$$c_{k+1}(\gamma) = \frac{k+1-2\gamma}{k} c_k(\gamma).$$

We can see that, for $|z| \neq 0$,

$$\mathcal{G}_1(m, n, k, \lambda, \alpha, |z|) \geq \mathcal{G}_1(m, n, k+1, \lambda, \alpha, |z|)$$

if and only if

$$\mathcal{H}_1(\gamma, k, |z|) = k + 1 - 2\gamma - k\,|z| \geq 0.$$

Since $\mathcal{H}_1(\gamma, k, |z|)$ is a decreasing function in $|z|$. It follows that

$$\mathcal{H}_1(\gamma, k, |z|) \geq \mathcal{H}_1(\gamma, n, 1) = 1 - 2\alpha \geq 0, \text{ for } 0 \leq \gamma \leq \frac{1}{2}.$$

Further, since $\mathcal{H}_1(\gamma, k, |z|)$ is decreasing in α, we have

$$\mathcal{H}_1(\gamma, k, |z|) \geq \mathcal{H}_1(1, k, |z|) = k - 1 - k\,|z| \geq \mathcal{H}_1(1, k, \frac{1}{2}) \geq \mathcal{H}_1(1, 2, \frac{1}{2}) = 0,$$

for $|z| \leq \frac{1}{2}$. Finally, the bound of the theorem are attained by the function $f_2(z)$ given by (13). $\square$

§6. Convolution Theorem

Theorem 6.1. *Let the function $f(z)$ and $g(z)$ defined by,*

$$f(z) = z + \sum_{k=2}^{\infty} a_k z_k \tag{14}$$

and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z_k \tag{15}$$

belong to the class $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$ with $0 \leq \lambda < 1$, $-1 < \alpha \leq 1$. Then $(f * g)(z) \in \tilde{\mathcal{S}}_{m,n,\lambda}(\xi)$ where,

$$\xi \leq 1 - \frac{4(1-\alpha)^2[1+\lambda(k-1)][\delta(m,k) - \delta(n,k)]}{\psi^2(m,n,k,\lambda,\alpha)c_k(\gamma)},$$

and the result is sharp for,

$$f(z) = z - \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} z^k$$

Proof. $f(z)$ and $g(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$ and so we have,

$$\sum_{k=2}^{\infty} \frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)} a_k \leq 1. \tag{16}$$

$$\sum_{k=2}^{\infty} \frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)} b_k \leq 1. \tag{17}$$

By applying Cauchy-Schwarz inequity to (16) and (17), we have

$$\sum_{k=2}^{\infty} \frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)} \sqrt{a_k b_k} \leq 1. \tag{18}$$

We need to find smallest number ξ such that

$$\sum_{k=2}^{\infty} \frac{\psi(m,n,k,\lambda,\xi)c_k(\gamma)}{2(1-\xi)} a_k b_k \leq 1. \tag{19}$$

Thus from (18) and (19)

$$\frac{\psi(m,n,k,\lambda,\xi)c_k(\gamma)}{2(1-\xi)} a_k b_k \leq \frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)} \sqrt{a_k b_k} \tag{20}$$

That is

$$\sqrt{a_k b_k} \leq \frac{(1-\xi)\psi(m,n,k,\lambda,\alpha)}{(1-\alpha)\psi(m,n,k,\lambda,\xi)} \tag{21}$$

From (18)

$$\sqrt{a_k b_k} \leq \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}. \tag{22}$$

Therefore in view of (21) and (22)

$$\frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)} \leq \frac{(1-\xi)\psi(m,n,k,\lambda,\alpha)}{(1-\alpha)\psi(m,n,k,\lambda,\xi)}$$

which simplifies to

$$\xi \leq 1 - \frac{2(1-\alpha)^2[1+\lambda(k-1)]c_k(\gamma)\delta(m,k)+\delta(m,k)}{\psi^2(m,n,k,\lambda,\alpha)c_k(\gamma)+2(1-\alpha)^2[1+\lambda(k-1)]c_k(\gamma)\delta(n,k)+\delta(n,k)}.$$

Since

$$A(k) = 1 - \frac{2(1-\alpha)^2[1+\lambda(k-1)]c_k(\gamma)\delta(m,k)+\delta(m,k)}{\psi^2(m,n,k,\lambda,\alpha)c_k(\gamma)+2(1-\alpha)^2[1+\lambda(k-1)]c_k(\gamma)\delta(n,k)+\delta(n,k)}. \tag{23}$$

is an increasing function of n $(n \geq 1)$ for $0 \leq \gamma \leq \frac{1}{2}$, $0 \leq \alpha < 1$, letting $k = 2$ in (23), we obtain

$$A(2) = 1 - \frac{4(1-\alpha)^2[1+\lambda)](1-\gamma)\delta(m,2)+2\delta(m,2)}{\psi^2(m,n,2,\lambda,\alpha)(1-\gamma)+4(1-\alpha)^2[1+\lambda](1-\gamma)\delta(n,2)+\delta(n,2)}. \tag{24}$$

This completes the proof. □

§7. Inclusion Properties

Theorem 7.1. *Let the function $f(z)$ and $g(z)$ defined by (14) and (15) be in the class $\tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$. Then the function $h(z)$ defined by,*

$$h(z) = z + \sum_{k=2}^{\infty}(a_k^2 + b_k^2)z^k \text{ is the class } \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$$

where,

$$\rho \leq 1 - \frac{(1-\alpha)[1+\lambda(k-1)]c_k(\gamma)\delta(m,k)+\delta(m,k)}{2\psi(m,n,k,\lambda,\alpha)c_k(\gamma)-(1-\alpha)[1+\lambda(k-1)]c_k(\gamma)\delta(n,k)-\delta(n,k)}. \tag{25}$$

Proof. Now, $f(z)$ and $g(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$ and thus we have

$$\sum_{k=2}^{\infty}\left[\frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)}\right]^2 a_k^2 \leq \left[\sum_{k=2}^{\infty}\frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)}a_k\right]^2 \leq 1 \tag{26}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)}\right]^2 b_k^2 \leq \left[\sum_{k=2}^{\infty}\frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)}b_k\right]^2 \leq 1. \tag{27}$$

Adding (26) and (27), we get,

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)}{2(1-\alpha)}\right]^2 (a_k^2+b_k^2) \leq 1. \tag{28}$$

We must show that $h \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$, that is,

$$\sum_{k=2}^{\infty}\frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)}{2(1-\rho)}(a_k^2+b_k^2) \leq 1. \tag{29}$$

In view of (28) and (29),

$$\frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)}{1-\rho} \leq \left\{\frac{1}{2}\frac{[\psi(m,n,k,\lambda,\alpha)c_k(\gamma)]}{(1-\alpha)}\right\}$$

Simplifying, we get

$$\rho \leq 1 - \frac{2(1-\alpha)\{\delta(m,k)+[1+\lambda(k-1)]c_k(\gamma)[\delta(m,k)-2\delta(n,k)]\}}{\psi(m,n,k,\lambda,\alpha)c_k(\gamma)+2(1-\alpha)[\delta(n,k)-[1+\lambda(k-1)]c_k(\gamma)\delta(n,k)]}.$$

□

§8. Integral Means Inequalities for Fractional Derivative

We will make use of the following definitions of fractional derivatives by Owa [8] and Srivastava and Owa [13, 14].

Definition 8.1 *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \quad (0 \leq \lambda < 1) \tag{30}$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 8.2 *Under the hypothesis of Definition (8.1), the fractional derivative of order $p+\lambda$ is defined, for a function $f(z)$, by*

$$D_z^{p+\lambda} f(z) = \frac{d^p}{dz^p} D_z^{\lambda} f(z) \ (0 \leq \lambda < 1,\ p \in \mathbb{N}_0 = \mathbb{N}U0). \tag{31}$$

It readily follows from (30) in Definition that

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \ (0 \leq \lambda < 1). \tag{32}$$

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [5] in our investigation.

Definition 8.3 *For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, denoted by $f \prec g$, if their exist a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < |z| < 1 (z \in \mathcal{U})$, such that $f(z) = g(w(z))(z \in \mathcal{U})$. In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0), f(U) = g(U)$.* The Littlewood's subordination theorem which we will use in our investigation to obtain the integral mean inequality.

Lemma 8.1 *If the functions $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$, with $f(z) \prec g(z)$ or $f(z) \prec g(z)$, then*

$$\int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |g(re^{i\mu})|^{\eta} d\mu \tag{33}$$

where $\mu > 0, z = re^{i\mu}$ and $0 < r < 1$. Strict inequality holds for $0 < r < 1$ unless f is constant or $w(z) = \alpha z, |\alpha| = 1$

Theorem 8.1. *Let $f(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$ and suppose that*

$$\sum_{k=2}^{\infty} (k-p)_{p+1} a_k \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\delta-p)}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)(k+1-\delta-p)\Gamma(2-p)} \tag{34}$$

for some $k \geq p,\ 0 \leq \delta < 1$ and $(k-p)_{p-1}$ denote the Pochhammer symbol defined by

$$(k-p)_{p+1} = (k-p)k-p-1...k.$$

Also let the function

$$f_k(z) = z + \frac{2(1-\alpha)}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)} z^k,\ k \geq 2. \tag{35}$$

If their exist an analytic function $w(z)$ given by

$$(w(z))^{k-1} = \frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{k=2}^{\infty} (k-p)_{p+1} \frac{\Gamma(k-p)a_k z^{k-1}}{\Gamma(j+1-\delta-p)},$$

$(k \geq p)$. Then for $z = re^{i\theta}$ and $0 < r < 1$

$$\int_0^{2\pi} |D_z^{p+\delta} f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |D_z^{p+\delta} f_k(z)|^{\mu} d\theta,\ (0 \leq \delta < 1,\ \mu > 0). \tag{36}$$

Proof. By virtue of the fractional derivative formula (32) and definition , we find from (1) that

$$\begin{aligned} D_z^{p+\lambda} f(z) &= \frac{z^{1-\delta-p}}{\Gamma(2+\delta-p)}\left\{1+\sum_{k=2}^{\infty}\frac{\Gamma(2-\delta-p)\Gamma(k+1)}{\Gamma(k+1-\delta-p)}\right\} \\ &= \frac{z^{1-\delta-p}}{\Gamma(2-\delta-p)}\left\{1+\sum_{k=2}^{\infty}\Gamma(2-\delta-p)(k+1)_{p+1}\phi(k)a_k z^{k-1}\right\} \end{aligned}$$

where

$$\phi(k)=\frac{\Gamma(k-p)}{\Gamma(k+1-\delta-p)}\ (0\le\delta<1,\ k\ge p).$$

Since $\phi(k)$ is a decreasing function of j, we have

$$0<\phi(k)\le\phi(2)=\frac{\Gamma(2-p)}{\Gamma(3-\delta-p)}.$$

Similarly, from (31) and (34), we get

$$D_z^{p+\lambda} f(z)=\frac{z^{1-\delta-p}}{\Gamma(2-\delta-p)}\left\{1+\frac{2(1-\alpha)\Gamma(2-\delta-p)\Gamma(k+1)z^{k-1}}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}\right\}.$$

For $z=re^{i\theta},\ 0<r<1$, we must show that

$$\begin{aligned} &\int_0^{2\pi}\left|1+\sum_{k=2}^{\infty}\Gamma(2-\delta-p)(k-p)_{p+1}\phi(k)a_j z^{k-1}\right|^{\mu}d\theta \\ &\le\int_0^{2\pi}\left|1+\frac{2(1-\alpha)\Gamma(2-\delta-p)\Gamma(k+1)z^{k-1}}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}\right|^{\mu}d\theta,\ (\mu>0). \end{aligned} \tag{37}$$

Thus by applying Littlewood's subordination theorem, it would be suffice to show that

$$1+\sum_{k=2}^{\infty}\Gamma(2-\delta-p)(k-p)_{p+1}\phi(k)a_j z^{k-1}\prec 1+\frac{2(1-\alpha)\Gamma(2-\delta-p)\Gamma(k+1)z^{k-1}}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}.$$

By setting

$$\begin{aligned} &1+\sum_{k=2}^{\infty}\Gamma(2-\delta-p)(k-p)_{p+1}\phi(k)a_j z^{k-1} \\ &=1+\frac{2(1-\alpha)\Gamma(2-\delta-p)\Gamma(k+1)z^{k-1}}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}(w(z))^{k-1}. \end{aligned} \tag{38}$$

$$(w(z))^{k-1}=\frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{k=2}^{\infty}(k-p)_{p+1}\phi(k)a_k z^{k-1}.$$

which readily yields $w(0)=0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)|<1, z\in\mathcal{U}$. We know that

$$\begin{aligned} |w(z)|^{k-1} &\le \left|\frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{k=2}^{\infty}(k-p)_{p+1}\phi(k)a_k z^{k-1}\right| \\ &\le \frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}{2(1-\alpha)\Gamma(k+1)}\sum_{k=2}^{\infty}(k-p)_{p+1}\phi(k)a_k|z^{k-1}| \end{aligned}$$

$$\begin{aligned}
&\leq \quad |z| \frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta-p)}{2(1-\alpha)\Gamma(k+1)} \sum_{k=2}^{\infty}(k-p)_{p+1}\phi(k)a_k \\
&\leq \quad |z| < 1
\end{aligned}$$

By means of the hypothesis (2) of Theorem.
As special case $p = 0$, Theorem 8.1 readily yields. □

Corollary 8.1 *Let $f(z) \in \tilde{\mathcal{S}}_{m,n,\lambda,\gamma}(\alpha)$ and suppose that*

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2(1-\alpha)\Gamma(k+1)\Gamma(3-\delta)}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)(k+1-\delta)}.$$

For some $j \geq 0,\ 0 \leq \delta < 1$. Also let the function

$$f_k(z) = z + \frac{2(k-\alpha)}{\psi(m,n,k,\lambda,\rho)c_k(\gamma)} z^k,\ \ k \geq 2.$$

If their exist an analytic function $w(z)$ given by

$$(w(z))^{k-1} = \frac{\psi(m,n,k,\lambda,\rho)c_k(\gamma)\Gamma(k+1-\delta)}{2(1-\alpha)\Gamma(k+1)} \sum_{k=2}^{\infty} \frac{\Gamma(k+p)a_k z^{k-1}}{\Gamma(k+1-\delta)},$$

Then for $z = re^{i\theta}$ and $0 < r < 1$

$$\int_0^{2\pi} |D_z^{\delta} f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |D_z^{\delta} f_k(z)|^{\mu} d\theta,\ \ (0 \leq \delta < 1,\ \ \mu > 0).$$

References

[1] A. R. S. Juma and S. R. Kulkarni. On univalent functions with negative coefficients by using generalized Sălăgean operator. Filomate, 2007, 21(2)): 173–184.

[2] F. M. Al-Oboudi. On univalent functions defined by generalized Sălăgean operator. JMMS, 2004, 27: 1429–1436.

[3] K. Al-Shaqsi and M. Darus. Subclass of close-to-convex functions. Int. J. Contemp. Math. Sci., 2007, 2(15): 745–757.

[4] P. L. Duren. Univalent functions. New York: Springer, 1983.

[5] J. E. Littlewood. On inequalities in the theory of functions. Proc. London Math. Soc., 1925, 23(1): 481–519.

[6] S. M. Khairnar and M. More. On certain subclass of analytic functions involving Al-Oboudi differential operator. J. Inequal. Pure and Appl. Math., 2009, 10(2): Art 57, 11pp.

[7] S. M. Khairnar and M. More. Properties of a class of analytic and univalent functions using Ruscheweyh derivatives. Int. J. Contemp. Math. Sci., 2008, 3(20): 967–976.

[8] S. Owa. On the distortion theorems I. Kyungpook Math. J., 1978, 18(1): 53–59.

[9] S. Ruscheweyh. New criteria for univalent functions. Proc. Amer. Math. Soc., 1975, 49: 109–115.

[10] G. S. Sălăgean. Subclasses of univalent functions. In Complex Analysis - 5th Romanian - Finnish Seminar Part 1 (Bucharest, 1981), Vol. 1013 of Lecture Notes in Matehmatics, Springer, Berlin, Germany, 1983, 362–372.

[11] T. Sekine, K. Tsuraumi and H. M. Srivastava. Integralmeans inequalities for fractional derivatives of some general subclasses of analytic functions. J. of Ineq. in Pure and Appl. Math., 2001, 2(2): Art. 23.

[12] S. Sümer Eker and H. Özlem Güney. A new subclass of analytic functions involving Al-Oboudi differential operator. J. Inequal. Appl., 2008: Article ID 452057,10 pages.

[13] H. M. Srivastava and S. Owa. Univalent functions, fractional calculus and their applications. Ellis Horwood Series: Mathematical Its Appl., Ellis Horwood, Chichester, 1989.

[14] H. M. Srivastava and S. Owa. Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 1987, 39: 1057–1077.

Scientia Magna
Vol. 11 (2016), No. 2, 83-96

Generalization of some new continuous functions in topological spaces

P. G. Patil, S. S. Benchalli and Pallavi S. Mirajakar

Department of Mathematics, Karnatak University, Dharwad-580 003.
Karnataka, India.
pgpatil01@gmail.com, benchalliss@gmail.com and psmirajakar@gmail.com

Abstract: The main aim of this paper is to introduce a new class of continuous functions called generalized star $\omega\alpha$-continuous functions in topological spaces. It is shown that generalized star $\omega\alpha$-continuous functions are lies between continuous and g$\omega\alpha$-continuous functions. Further, we study the characterizations of generalized star $\omega\alpha$-continuous functions in topological spaces.

Keywords: $\omega\alpha$-closed sets, $\omega\alpha$-continuous functions, g$^*\omega\alpha$-continuous functions, g$^*\omega\alpha$-irresolute maps, g$^*\omega\alpha$-closed maps, g$^*\omega\alpha$-closed graphs.
2010 Mathematics Subject Classification: 54C08, 54C10.

§1. Introduction and preliminaries

Continuous functions stands among the most fundamental point in the whole of the Mathematical Science. Many different forms of stronger and weaker forms of functions have been introduced over the years. As a generalization of closed sets, Levine [9] introduced the concept of generalized closed (briefly g-closed) sets which are weaker than closed sets in topological spaces. Balachandran et. al. [1] introduced the concept of generalized continuous maps and generalized irresolute maps in topological spaces and Benchalli et. al. [2], [3], [5] introduced and studied the concepts of $\omega\alpha$-closed sets, $\omega\alpha$-continuous maps and g$\omega\alpha$-continuous maps in topological spaces. Recently Patil et. al. [15], [16] introduced the concept of generalized star $\omega\alpha$-closed (briefly g$^*\omega\alpha$-closed) sets and generalized star $\omega\alpha$-spaces (briefly g$^*\omega\alpha$-spaces) in topological spaces.

In this paper, we introduce the concepts of generalized star $\omega\alpha$-continuous (briefly g$^*\omega\alpha$-continuous) functions and generalized star $\omega\alpha$-irresolute (briefly g$^*\omega\alpha$-irresolute) maps in topological spaces. Further, we also introduce g$^*\omega\alpha$-closed maps, g$^*\omega\alpha$-open maps and g$^*\omega\alpha$-closed graphs in topological spaces.

Throughout this paper spaces (X, τ) and (Y, σ) (or simply X and Y) always denote topological spaces on which no separation axioms are assumed unless explicitly stated.

Definition 1.1. *A subset A of a topological space X is called a*
(i) semi-open [8] if $A \subseteq cl(int(A))$ and semi-closed if $int(cl(A)) \subseteq A$.
(iii) α-open [14] if $A \subseteq int(cl(int(A)))$ and α-closed if $cl(int(cl(A))) \subseteq A$.

Definition 1.2. *A subset A of a topological space X is called a*
(i) $T_{g^*\omega\alpha}$-space [16] if every $g^*\omega\alpha$-closed set is closed.
(ii) ${}_{g^*\omega\alpha}T$-space [16] if every $g^*\omega\alpha$-closed set is ω-closed.
(iii) ${}_{g\omega\alpha}T_{g^*\omega\alpha}$-space [16] if every $g\omega\alpha$-closed set is $g^*\omega\alpha$-closed.
(iv) T_ω-space [17] if every ω-closed set is closed.

Definition 1.3. *A subset A of X is said to be a*
(i) g-closed [9] (respectively αg-closed [6]) if cl(A) $\subseteq$ U (respectively $\alpha cl(A) \subseteq$ U) whenever A $\subseteq$ U and U is open in X.
(iii) ω-closed [17] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X.
(iv) $\omega\alpha$-closed [2] (resp. $g\omega\alpha$-closed [4]) if $\alpha cl(A) \subseteq$ U whenever A $\subseteq$ U and U is ω-open (resp. $\omega\alpha$-open) in X.
(v) $g^*\omega\alpha$-closed [15] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $\omega\alpha$-open in X.

Definition 1.4. *A function f : X $\rightarrow$ Y is called a*
(i) g-continuous [1] (resp. α-continuous [13], ω-continuous [17], αg-continuous [6], gp-continuous [11]) if $f^{-1}(G)$ is g-closed (resp. α-closed, ω-closed, αg-closed, gp-closed) set in X for every closed set G of Y.
(ii) g-closed [12] (resp. ω-closed [17], αg-closed [7]) if f(G) is g-closed (resp. ω-closed, αg-closed) in Y for every closed set G in X.
(iii) $\omega\alpha$-closed [3] (resp. $g\omega\alpha$-closed [5]) if f(G) is $\omega\alpha$-closed (resp. $g\omega\alpha$-closed) for every closed set G in X.
(iv) $g\omega\alpha$-continuous [5] if $f^{-1}(G)$ is $g\omega\alpha$-closed in X for every closed set G of Y.
(v) ω-irresolute [17] (resp. $\omega\alpha$-irresolute [3]) if $f^{-1}(G)$ is ω-closed (resp. $\omega\alpha$-closed) in X for each ω-closed (resp. $\omega\alpha$-closed) set G of Y.

Definition 1.5. [16] *The intersection of all $g^*\omega\alpha$-closed sets containing a subset A of X is called $g^*\omega\alpha$-closure of A and is denoted by $g^*\omega\alpha$-cl(A).*
If A is $g^*\omega\alpha$-closed then $g^*\omega\alpha$-cl(A)= A.

Definition 1.6. [16] *The union of all $g^*\omega\alpha$-open sets contained in a subset A of X is called $g^*\omega\alpha$-interior of A and is denoted by $g^*\omega\alpha$-int(A).*
If A is $g^*\omega\alpha$-open then $g^*\omega\alpha$-int(A) = A.

Definition 1.7. [10] *Let f : X $\rightarrow$ Y be a function. Then*
(i) the subset { (x, f(x)) : x $\in$ X } of the product space X $\times$ Y is called the graph of f and is denoted by G(f).
(ii) a closed graph, if its graph G(f) is closed set in the product space X $\times$ Y. A.

Definition 1.8. [10] *A function f : X $\rightarrow$ Y has a closed graph if for each (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist U $\in$ O(X, x) and V $\in$ O(Y, y) such that f(U) $\cap$ V = ϕ.*

Definition 1.9. *Let x $\in$ X and V $\subset$ X, then V is called $g^*\omega\alpha$-neighborhood of x in X if there exists $g^*\omega\alpha$-open set U of X such that x $\in$ U $\subseteq$ V.*

Theorem 1.10 *Let A be a subset of X. Then x $\in$ $g^*\omega\alpha$-cl(A) if and only if for any $g^*\omega\alpha$-nbd N_x of x in X such that $N_x \cap$ A $\neq \phi$.*

Proof. Let us assume that there is a $g^*\omega\alpha$-nbd N of x in X such that N $\cap$ A = ϕ. There exists a $g^*\omega\alpha$-open set G of X such that x $\in$ G $\subseteq$ N. Therefore we have G $\cap$ A = ϕ and so x $\in$ X-G.

Then g$^*\omega\alpha$-cl(A) $\in$ X-G and therefore x $\notin$ g$^*\omega\alpha$-cl(A), which is contradiction to the hypothesis x $\in$ g$^*\omega\alpha$-cl(A). Therefore N $\cap$ A $\neq \phi$.

Conversely, suppose x $\in$ g$^*\omega\alpha$-cl(A). Then there exist a g$^*\omega\alpha$-closed set G of X such that A $\subseteq$ G and x $\neq$ G. □

§2. $g^*\omega\alpha$-continuous functions in topological spaces

In this section, we introduce the concept of generalized star $\omega\alpha$-continuous (briefly g$^*\omega\alpha$-continuous) functions in topological spaces and study their properties.

Definition 2.1. *A function $f : X \to Y$ is called $g^*\omega\alpha$-continuous if the inverse image of every closed set in Y is $g^*\omega\alpha$-closed in X.*

Theorem 2.1. *Every continuous function is $g^*\omega\alpha$-continuous function.*
However the converse of the above Theorem need not be true as seen from the following example.

Example 2.1. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ a, c \}\}$ and $\sigma = \{ Y, \phi, \{ a, b \}\}$. Let $f : X \to Y$ be the identity function. Then f is $g^*\omega\alpha$-continuous but not continuous, since for the closed set $A = \{ c \}$ in Y, $f^{-1}(c) = \{ c \}$ is not closed in X.*

Reamrk 2.1. *The converse of the Theorem 2.1 holds if X is $T_{g^*\omega\alpha}$ space.*

Theorem 2.2. *Every $g^*\omega\alpha$-continuous function is $g\omega\alpha$-continuous, αg-continuous and gp-continuous.*

Proof. Let f : X $\to$ Y be a function. Let V be an open set in Y. Since f is g$^*\omega\alpha$-continuous, f^{-1}(V) is g$^*\omega\alpha$-open in X. Then by Theorem 3.2 [15], f^{-1}(V) is g$\omega\alpha$-open in X and from [4] every g$\omega\alpha$-closed set is αg-closed and gp-closed. Therefore f is g$\omega\alpha$-continuous, αg-continuous and gp-continuous.

The converse of the above theorem need not be true as seen from the following example. □

Example 2.2. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}\}$ and $\sigma = \{ Y, \phi, \{ a \}, \{ a, b \}\}$. The identity function $f : X \to Y$ is $g\omega\alpha$-continuous, αg-continuous and gp-continuous but not $g^*\omega\alpha$-continuous, since for the closed set $A = \{ c \}$ in Y, $f^{-1}(\{ c \}) = \{ c \}$ is not $g^*\omega\alpha$-closed in X.*

Remark 2.2. *The concept of $g^*\omega\alpha$-continuous function is independent with $\omega\alpha$-continuous.*

Example 2.3. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}\}$ and $\sigma = \{ Y, \phi, \{ a \}, \{ b \}, \{ a, b \}\}$. Define a function $f : X \to Y$ by f(a)=b, f(b)=a and f(c)=c. Then f is $\omega\alpha$-continuous but not $g^*\omega\alpha$-continuous, since for the closed set $A = \{ b, c \}$ in Y, $f^{-1}(\{ b, c \}) = \{ a, c \}$ is not $g^*\omega\alpha$-closed in X.*

Example 2.4. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ b, c \}\}$ and $\sigma = \{ Y, \phi, \{ a \}\}$. Define a function $f : X \to Y$ by f(a)=b, f(b)=a and f(c)=c. Then f is $g^*\omega\alpha$-continuous but not $\omega\alpha$-continuous, since for the closed set $A = \{ b, c \}$ in Y, $f^{-1}(\{ b, c \}) = \{ a, c \}$ is not $\omega\alpha$-closed in X.*

Remark 2.3. *The concept of $g^*\omega\alpha$-continuous function is independent with α-continuous.*

Example 2.5. *Let $X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ a, b \}\}$ and $\sigma = \{ Y, \phi, \{ a \}\}$. Let $f : X \to Y$ be the identity function. Then f is α-continuous but not $g^*\omega\alpha$-continuous, since for the closed set $A = \{ c \}$ in Y, $f^{-1}(\{ c \}) = \{ c \}$ is not $g^*\omega\alpha$-closed in X.*

Example 2.6. *Let $X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ b, c \}\}$ and $\sigma = \{ Y, \phi, \{ a \}, \{ b \}, \{ a, b \}\}$. The identity function $f : X \to Y$ is $g^*\omega\alpha$-continuous but not α-continuous, since for the closed set $A = \{ a, c \}$ in Y, $f^{-1}(\{ a, c \}) = \{ a, b \}$ is not α-closed in X.*

Theorem 2.3. *A function $f : X \to Y$ is $g^*\omega\alpha$-continuous if and only if $f^{-1}(V)$ is $g^*\omega\alpha$-open set in X for every open set V in Y.*

Proof. The proof is obvious. □

Remark 2.4. *The composition of $g^*\omega\alpha$-continuous functions need not be $g^*\omega\alpha$-continuous as seen from the following example.*

Example 2.7. *$X = Y = Z = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ a, b \}\}$, $\sigma = \{ Y, \phi, \{ a, b \}\}$ and $\eta = \{ Z, \phi, \{ a \}\}$. Let $f : X \to Y$ be the identity function and the function $g : Y \to Z$ is defined by g(a)=b, g(b)=a and g(c)=c. Then f and g are $g^*\omega\alpha$-continuous functions but $gof : X \to Z$ is not $g^*\omega\alpha$-continuous, since for the closed set $\{ b, c \}$ in Z, $(gof)^{-1}(\{ b, c \}) = f^{-1}(g^{-1}(\{ b, c \})) = f^{-1}(\{ a, c \}) = \{ a, c \}$ is not $g^*\omega\alpha$-closed set in X.*

Theorem 2.4. *Let $f : X \to Y$ and $g : Y \to Z$ are any two functions then $gof : X \to Z$ is $g^*\omega\alpha$- continuous if g is continuous and f is $g^*\omega\alpha$-continuous.*

Proof. Let f : X → Y is $g^*\omega\alpha$-continuous and g : Y → Z is continuous. Let F be any closed set in Z. Since g is continuous, $g^{-1}(F)$ is closed in Y. Since f is $g^*\omega\alpha$-continuous $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$ is $g^*\omega\alpha$-closed in X. Hence $(gof)^{-1}$ is $g^*\omega\alpha$-closed in X. Thus gof is $g^*\omega\alpha$-continuous. □

The characterization of $g^*\omega\alpha$-continuous functions.

Theorem 2.5. *Following statements are equivalent for the function $f : X \to Y$:*
(i) f is $g^*\omega\alpha$-continuous.
(ii) the inverse image of each open set in Y is $g^*\omega\alpha$-open in X.
(iii) the inverse image of each closed set in Y is $g^*\omega\alpha$-closed in X.
(iv) for each x in X, the inverse image of every neighborhood of f(x) is a $g^*\omega\alpha$-neighborhood of x.
(v) for each x in X and each neighborhood N of f(x) there is a $g^*\omega\alpha$-neighborhood W of x such that $f(W) \subseteq N$.
(vi) for each subset A of X, $f(g^*\omega\alpha cl(A)) \subseteq cl(f(A))$.
(vii) for each subset B of Y, $g^*\omega\alpha cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Proof. (i) → (ii) Follows from the Theorem 2.3.
(ii) → (iii) Follows from the Definition 2.1.
(ii) → (iv) Let $x \in X$ and let N be a neighborhood of f(x). Then there exists an open set V in Y such that $f(x) \in V \subseteq N$. Consequently $f^{-1}(V)$ is $g^*\omega\alpha$-open in X and $x \in f^{-1}(V) \subseteq f^{-1}(N)$. Thus $f^{-1}(N)$ is $g^*\omega\alpha$ neighborhood of f(x).
(iv) → (v) Let $x \in X$ and let N be a neighborhood of f(x). Then by assumption $W = f^{-1}(N)$ is a $g^*\omega\alpha$ neighborhood of x and $f(W) = f(f^{-1}(N)) \subseteq N$.

(v) → (vi) Let y ∈ f(g$^*\omega\alpha$-cl(A)) and let N be any neighborhood of y. Then there exists x ∈ X and a g$^*\omega\alpha$ neighborhood W of x such that f(x) = y, x ∈ W. Hence x ∈ g$^*\omega\alpha$-cl(A) and f(W) ⊆ N. By Theorem 1.10, W ∩ A ≠ ϕ and hence f(A) ∩ N ≠ ϕ. Hence y ∈ f(x) ∈ cl(f(A)). Therefore f(g$^*\omega\alpha$-cl(A)) ⊆ cl(f(A)).
(vi) → (vii) Let B be any subset of Y. Then replacing A by f^{-1}(B) in (vi), we obtain f(g$^*\omega\alpha$-cl(f^{-1}(B)) ⊆ cl(f(f^{-1}(B))) ⊆ cl(B). That is g$^*\omega\alpha$-cl(f^{-1}(B)) ⊆ f^{-1}(cl(B)).
(vii) → (i) Let G be an open set in Y, then Y-G is closed in Y. Therefore, f^{-1}(Y-G) = f^{-1}(cl(Y-G)) ⊆ g$^*\omega\alpha$-cl(f^{-1}(Y-G)) = X - (g$^*\omega\alpha$-int(f^{-1}(G)). This implies that g$^*\omega\alpha$-int(f^{-1}(G)) ⊆ X - f^{-1}(Y-G)) = f^{-1}(G). Thus, g$^*\omega\alpha$-int(f^{-1}(G)) ⊆ f^{-1}(G). But f^{-1}(G) ⊆ g$^*\omega\alpha$-int(f^{-1}(G)) is always true. Therefore f^{-1}(G) = g$^*\omega\alpha$-int(f^{-1}(G)). This implies f^{-1}(G) is g$^*\omega\alpha$-open set. Therefore f is g$^*\omega\alpha$-continuous. □

§3. $g^*\omega\alpha$-irresolute maps in topological spaces

This section gives the concept of generalized star $\omega\alpha$-irresolute (briefly g$^*\omega\alpha$-irresolute) maps and their properties in topological spaces.

Definition 3.1. *A map $f : X \to Y$ is called $g^*\omega\alpha$-irresolute if $f^{-1}(V)$ is $g^*\omega\alpha$-closed in X for every $g^*\omega\alpha$-closed set V in Y.*

Theorem 3.1. *A map $f : X \to Y$ is $g^*\omega\alpha$-irresolute if and only if for every $g^*\omega\alpha$-open set A in Y, $f^{-1}(A)$ is $g^*\omega\alpha$-open in X.*

Proof. The proof is obvious. □

Theorem 3.2. *If $f : X \to Y$ is $g^*\omega\alpha$-irresolute then for every subset A of X, f($g^*\omega\alpha$-cl(A)) ⊆ cl(f(A)).*

Proof. If A ⊆ X, then cl(f(A)) which is also g$^*\omega\alpha$-closed in Y. As f is g$^*\omega\alpha$-irresolute, f^{-1}(cl(f(A))) is g$^*\omega\alpha$-closed in X. Furthermore, A ⊆ f^{-1}(f(A)) ⊆ f^{-1}(cl(f(A))). Therefore by g$^*\omega\alpha$-closure, g$^*\omega\alpha$-cl(A) ⊆ f^{-1}(cl(f(A))). Consequently, f(g$^*\omega\alpha$-cl(A)) ⊆ f(f^{-1}(cl(f(A)))) ⊆ cl(f(A)). □

Theorem 3.3. *Every $g^*\omega\alpha$-irresolute map is $g^*\omega\alpha$-continuous.*

Proof. Let f : X → Y be a g$^*\omega\alpha$-irresolute map and V be a closed set in Y. Then from [15], V is g$^*\omega\alpha$-closed in Y. Since f is g$^*\omega\alpha$-irresolute map, f^{-1}(V) is g$^*\omega\alpha$-closed in Y. Therefore f is g$^*\omega\alpha$-continuous.

The converse of the above theorem need not be true as seen from the following example. □

Example 3.1. *X = Y = { a, b, c }, τ = { X, ϕ, { a }, { a, c }} and σ = { Y, ϕ, { a, b }}. Define a function $f : X \to Y$ by f(a)=a, f(b)=c and f(c)=b. Then f is $g^*\omega\alpha$-continuous but not $g^*\omega\alpha$-irresolute, since for the $g^*\omega\alpha$-closed set A = { a, c } in Y, f^{-1}({ a, c }) = { a, b } is not $g^*\omega\alpha$-closed in X.*

Theorem 3.4. *Let $f : X \to Y$ be a closed surjective and $g^*\omega\alpha$-irresolute map. If X is $T_{g^*\omega\alpha}$-space then Y is also $T_{g^*\omega\alpha}$-space.*

Proof. Let A be g$^*\omega\alpha$-closed in Y. Then f^{-1}(A) is g$^*\omega\alpha$-closed in X as f is g$^*\omega\alpha$-irresolute. Since X is $T_{g^*\omega\alpha}$-space, then f^{-1}(A) is closed in X. Since f is closed and surjective then A = f(f^{-1}(A)) is closed in Y. Hence Y is also $T_{g^*\omega\alpha}$-space. □

Theorem 3.5. *If $f : X \to Y$ is bijective closed and $\omega\alpha$-irresolute then the inverse map $f^{-1} : Y \to X$ is $g^*\omega\alpha$-irresolute.*

Proof. Let G be a g$^*\omega\alpha$-closed set in X. Let (f^{-1})$^{-1}$(G) = f(G) $\subseteq$ U where U is $\omega\alpha$-open in Y.Then G $\subseteq$ f^{-1}(U) holds. Since f^{-1}(U) is $\omega\alpha$-open in X and G is g$^*\omega\alpha$-closed in X, cl(G) $\subseteq$ f^{-1}(U) and hence f(cl(G)) $\subseteq$ U. Since f is closed and cl(G) is closed in X, f(cl(G)) is closed in Y. So f(cl(G)) is g$^*\omega\alpha$-closed in Y. Therefore cl(f(cl(G)))) $\subseteq$ U, so that cl(f(G)) $\subseteq$ U. Thus f(G) is g$^*\omega\alpha$-closed in Y. Hence f^{-1} is g$^*\omega\alpha$-irresolute. □

Theorem 3.6. *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If f is $g^*\omega\alpha$-continuous and g is $g^*\omega\alpha$-irresolute and Y is $T_{g^*\omega\alpha}$-space then $gof : X \to Z$ is $g^*\omega\alpha$-irresolute.*

§4. $g^*\omega\alpha$-closed maps in topological spaces

The concept of $g^*\omega\alpha$-closed maps are introduced and their properties are discussed in this section.

Definition 4.1. *A map $f : X \to Y$ is called generalized star $\omega\alpha$-closed (briefly $g^*\omega\alpha$-closed) map if for each closed set F of X, f(F) is $g^*\omega\alpha$-closed in Y.*

Remark 4.1. *From the Definition 4.1, every closed map is a $g^*\omega\alpha$-closed map but not conversely.*

Example 4.1. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}\}$ and $\sigma = \{ Y, \phi, \{ a \}, \{ b, c \}\}$. Let $f : X \to Y$ be a map defined as f(a)=b, f(b)=c and f(c)=a. Then f is $g^*\omega\alpha$-closed map but not closed, since the set $A = \{ b, c \}$ is $g^*\omega\alpha$-closed in X but f($\{ b, c \}$) = $\{ a, c \}$ is not closed in Y.*

Remark 4.2. *The converse of the Remark 4.1 is true if Y is $T_{g^*\omega\alpha}$ space.*

Theorem 4.1: *A map $f : X \to Y$ is $g^*\omega\alpha$-closed if and only if for any subset S of Y and for an open set U containing f^{-1}(S) there exists $g^*\omega\alpha$-open set K of Y containing S such that f^{-1}(K) $\subseteq$ S.*

Proof. Suppose f : X $\to$ Y is g$^*\omega\alpha$-closed. Let S be a subset of Y and U be an open set of X containing f^{-1}(S). Then K = Y - f(X - U) is g$^*\omega\alpha$-open set containing S such that f^{-1}(K) $\subseteq$ F.

Conversely, suppose F is closed in X. Then f^{-1}(Y- f(F)) $\subseteq$ X - f^{-1}(f(F)) $\subseteq$ X-F and X - F is open. Then by hypothesis, there exists g$^*\omega\alpha$-open set K of Y such that Y - f(F) $\subseteq$ K and f^{-1}(K) $\subseteq$ X - F. Therefore F $\subseteq$ X - f^{-1}(K). Hence Y - K $\subseteq$ f(F) $\subseteq$ f(X - f^{-1}(K)) $\subseteq$ Y - K, which implies f(F) $\subseteq$ Y - K. Since Y - K is g$^*\omega\alpha$-closed, f(F) is g$^*\omega\alpha$-closed and thus f is g$^*\omega\alpha$-closed map. □

Theorem 4.2. *If $f : X \to Y$ is $g^*\omega\alpha$-closed and A is a closed subset of X then $f|A : A \to Y$ is $g^*\omega\alpha$-closed.*

Proof. Let B $\subset$ A be a closed set in X. Then f(B) is g$^*\omega\alpha$-closed in Y as f is g$^*\omega\alpha$-closed in Y. But f(B) = (f | A)(B), so (f | A)(B) is g$^*\omega\alpha$-closed in Y. Therefore f | A is g$^*\omega\alpha$-closed. □

Theorem 4.3. *Let $f : X \to Y$ and $g : Y \to Z$ are any two maps such that $gof : X \to Z$ is $g^*\omega\alpha$-closed map:*
(i) if f is $g^*\omega\alpha$-continuous and surjective then g is $g^*\omega\alpha$-closed map.
(ii) if g is $g^*\omega\alpha$-irresolute and injective then f is $g^*\omega\alpha$-closed map.

Proof. (i) Let F be closed set of Y. Then f^{-1}(F) is closed set of X as f is continuous. Since gof is g$^*\omega\alpha$-closed map, (gof)(f^{-1}(F)) = g(F) is g$^*\omega\alpha$-closed in Z. Hence g : Y $\to$ Z is g$^*\omega\alpha$-closed map.
(ii) Let F be closed set in X. Then (gof)(F) is g$^*\omega\alpha$-closed in Z and so g^{-1}(gof)(F) = f(F) is g$^*\omega\alpha$-closed in Y, since g is g$^*\omega\alpha$-irresolute and injective. Hence f is g$^*\omega\alpha$-closed map. □

Theorem 4.4. *If A is $g^*\omega\alpha$-closed in X and $f : X \to Y$ is bijective $\omega\alpha$-irresolute and $g^*\omega\alpha$-closed then f(A) is $g^*\omega\alpha$-closed in Y.*

Proof. Let cl(A) $\subseteq$ G where G is $\omega\alpha$-open in Y. Since f is $\omega\alpha$-irresolute, f^{-1}(G) is $\omega\alpha$-open set containing A. Hence cl(A) $\subseteq$ f^{-1}(G) as A is g$^*\omega\alpha$-closed. Again, since f is g$^*\omega\alpha$-closed, f(cl(A)) is g$^*\omega\alpha$-closed contained in the set G, which implies cl(f(cl(A))) $\subseteq$ G and hence cl(f(A)) $\subseteq$ G. So f(A) is g$^*\omega\alpha$-closed in Y. □

Remark 4.3. *Composition of $g^*\omega\alpha$-closed maps need not be a $g^*\omega\alpha$-closed map.*

Example 4.2. *$X = Y = Z = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}\}$, $\sigma = \{ Y, \phi, \{ b \}\}$ and $\eta = \{ Z, \phi, \{ a \}, \{ a, b \}, \{ a, c \}\}$. Let $f : X \to Y$ be the identity map and define a function $g : Y \to Z$ as g(a)=b, g(b)=a and g(c)=c. Then f and g are $g^*\omega\alpha$-closed maps but their composition gof is not $g^*\omega\alpha$-closed map, since for the set $A = \{ b, c \}$ of Z, $(gof)(\{ b, c \}) = g(f(\{ b, c \})) = g(\{ b, c \}) = \{ a, c \}$ is not a $g^*\omega\alpha$-closed in Y.*

Theorem 4.5. *If $f : X \to Y$ and $g : Y \to Z$ are closed and $g^*\omega\alpha$-closed maps respectively then their composition $gof : X \to Z$ is $g^*\omega\alpha$-closed.*

Theorem 4.6. *If $f : X \to Y$ is ω-closed and Y is T_ω-space [17] then $f : X \to Y$ is $g^*\omega\alpha$-closed map.*

Proof. Let F be closed set in X. Then f(F) is ω-closed in Y as f is ω-closed. Since Y is T$_\omega$-space, we have f(F) is closed in Y and hence g$^*\omega\alpha$-closed in Y. Thus f is g$^*\omega\alpha$-closed map. □

Theorem 4.7. *If $f : X \to Y$ is $g^*\omega\alpha$-closed map and $g : Y \to Z$ is $\omega\alpha$-irresolute and closed then gof is $g^*\omega\alpha$-closed map.*

Proof. Let A be a closed set in X. Then f(A) is g$^*\omega\alpha$-closed set in Y as f is g$^*\omega\alpha$-closed map. Since g : Y $\to$ Z is $\omega\alpha$-irresolute and closed map, by Theorem 4.5, we have g(f(A)) = (gof)(A) is g$^*\omega\alpha$-closed in Z. Thus gof is g$^*\omega\alpha$-closed map. □

Theorem 4.8. *If $f : X \to Y$ is $g^*\omega\alpha$-closed map then $g^*\omega\alpha\text{-}cl(f(A)) \subset f(cl(A))$ for every subset A of X.*

Proof. Suppose f is $g^*\omega\alpha$-closed and $A \subset X$. Then cl(A) is closed in X and f(cl(A)) is $g^*\omega\alpha$-closed in Y. We have $f(A) \subset f(cl(A))$. But $g^*\omega\alpha\text{-}cl(f(A)) \subset g^*\omega\alpha\text{-}cl(f(cl(A)))$. Since f(cl(A)) is $g^*\omega\alpha$-closed in Y, $g^*\omega\alpha\text{-}cl(f(cl(A)))) = f(cl(A))$. Hence $g^*\omega\alpha\text{-}cl(f(A)) \subset f(cl(A))$ for every subset A of X. □

Remark 4.4. *The converse of the above theorem need not be true in general as seen from the following example.*

Example 4.3. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ a, b \}, \{ a, c \}\}$ and $\sigma = \{ Y, \phi, \{ a \}, \{ a, b \}\}$. Let $f : X \to Y$ be the identity map. Then $g^*\omega\alpha\text{-}cl(f(A)) \subseteq f(cl(A))$ for subset A of X but f is not $g^*\omega\alpha$-closed, since $f(\{ b \}) = \{ b \}$ is not $g^*\omega\alpha$-closed in Y.*

Theorem 4.9. *Let $f : X \to Y$ be an open continuous $g^*\omega\alpha$-closed and surjective and X is regular then Y is also regular.*

Proof. Let U be an open set in Y and $p \in U$. Since f is surjective there exist a point $x \in X$ such that $f(x) = p$. Since X is regular and f is continuous, there is an open set V in X such that $x \in V \subset cl(V) \subseteq f^{-1}(U)$. Hence $p \in f(V) \subset f(cl(V)) \subseteq U$. Since f is $g^*\omega\alpha$-closed, f(cl(V)) is $g^*\omega\alpha$-closed set contained in the open set U. By hypothesis $cl(f(cl(V))) = f(cl(V))$ and $cl(f(V)) = cl(f(cl(V)))$. Therefore, $p \in f(V) \subset cl(f(V)) \subseteq U$ and f(V) is open as f is open. Hence Y is regular. □

Theorem 4.10. *If A is $g^*\omega\alpha$-closed set of X and $f : X \to Y$ is $g^*\omega\alpha$-closed and $\omega\alpha$-irresolute then f(A) is $g^*\omega\alpha$-closed in Y.*

Proof. Let A be a $g^*\omega\alpha$-closed in X and G be an $\omega\alpha$-open in Y such that $f(A) \subseteq G$. Then $f^{-1}(G)$ is $\omega\alpha$-open in X such that $A \subseteq f^{-1}(G)$. Hence $cl(A) \subseteq f^{-1}(G)$, since A is $g^*\omega\alpha$-closed and $f^{-1}(G)$ is $\omega\alpha$-open. Again since f is $g^*\omega\alpha$-closed, f(cl(A)) is $g^*\omega\alpha$-closed set contained in the $\omega\alpha$-open set G. Therefore $cl(f(cl(A))) = f(cl(A)) \subseteq G$. This implies $cl(f(A)) \subseteq G$. Hence f(A) is $g^*\omega\alpha$-closed in Y. □

Theorem 4.11. *If A is $g^*\omega\alpha$-closed subset of Y and $f : X \to Y$ is bijective $g^*\omega\alpha$-continuous and $\omega\alpha$-open then $f^{-1}(A)$ is $g^*\omega\alpha$-closed in X.*

Proof. Let U be an $\omega\alpha$-open set in X such that $f^{-1}(A) \subseteq U$. Then $A \subseteq f(U)$. Since A is $g^*\omega\alpha$-closed in Y, $cl(A) \subseteq f(U)$. Since f is bijective and $g^*\omega\alpha$-continuous, $f^{-1}(cl(A)) \subseteq f^{-1}(f(U)) = U$. Therefore $f^{-1}(cl(A)) \subseteq U$. Now $cl(f^{-1}(A)) \subseteq cl(f^{-1}(cl(A))) = f^{-1}(cl(A)) \subseteq U$. This implies $cl(f^{-1}(A)) \subseteq U$. Hence $f^{-1}(A)$ is $g^*\omega\alpha$-closed in X. □

Theorem 4.12. *If $f : X \to Y$ is continuous $g^*\omega\alpha$-closed map from a normal space X on to a space Y then Y is also normal.*

Proof. Let A and B are disjoint closed sets of Y then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in X. Then there exist disjoint open sets U and V of X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is $g^*\omega\alpha$-closed, then by Theorem 4.1, there exist disjoint $g^*\omega\alpha$-open sets G and H in Y such that $A \subseteq G$, $B \subseteq H$ and $f^{-1}(G) \subseteq U$, $f^{-1}(H) \subseteq V$. That is $f^{-1}(G) \cap f^{-1}(H) = \phi$ and hence $G \cap H = \phi$. Since A is closed and G is $\omega\alpha$-open, $A \subseteq G$ and by Theorem 4.2 [15], $A \subseteq int(G)$ and $B \subseteq int(H)$. Therefore $int(G) \cap int(H) = \phi$. Hence Y is normal. □

Definition 4.2. A map f : X → Y is called g$^*\omega\alpha$-open map if for each open set U of X, f(U) is g$^*\omega\alpha$-open set in Y.

Theorem 4.13. *If a map $f : X \to Y$ is $g^*\omega\alpha$-open then $f^{-1}(g^*\omega\alpha\text{-}cl(A)) \subseteq cl(f^{-1}(A))$ for each subset A of Y.*

Proof. Suppose f is g$^*\omega\alpha$-open then for any A ⊆ Y, $f^{-1}(A) \subseteq cl(f^{-1}(A))$. By Theorem 4.1 there exist g$^*\omega\alpha$-closed set K of Y such that A ⊆ K and $f^{-1}(K) \subseteq cl(f^{-1}(A))$. Since K is g$^*\omega\alpha$-closed set, $f^{-1}(g^*\omega\alpha\text{-cl}(A)) \subseteq f^{-1}(K) \subseteq cl(f^{-1}(A))$. Hence $f^{-1}(g^*\omega\alpha\text{-cl}(A)) \subseteq cl(f^{-1}(A))$.
Following example shows that the converse of the above theorem need not be true in general. □

Example 4.4. *$X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a \}, \{ b \}, \{ a, b \}\}$ and $\sigma = \{ Y, \phi, \{ a \}\}$. Let f : X → Y be the identity function. Then for each subset A of Y, conclusion of the above theorem holds but f is not $g^*\omega\alpha$-open map, since for the open set A = { a, b } of X, f({ a, b }) = { a, b } is not $g^*\omega\alpha$-open in X.*

Theorem 4.14. *If a map f : X → Y is $g^*\omega\alpha$-open, then for each neighborhood U of x in X there exists a $g^*\omega\alpha$-neighborhood W of f(x) in Y such that W ⊂ f(U).*

Proof. Let f : X → Y be g$^*\omega\alpha$-open map. Let x ∈ X and U be an arbitrary neighborhood of x in X. Then there exists an open set G in X such that x ∈ G ⊆ U. Now f(x) ∈ f(G) ⊆ f(U) and f(G) is g$^*\omega\alpha$-open set in Y, as f is g$^*\omega\alpha$-open map. Then f(G) is g$^*\omega\alpha$-nbd of each of its points. Taking f(G) = W, W is g$^*\omega\alpha$-nbd of f(x) in Y such that W ⊆ f(U). □

Theorem 4.15. *For any function f : X → Y the following statements are equivalent:*
(i) f is $g^*\omega\alpha$-open map
(ii) f(int(A)) ⊆ $g^*\omega\alpha$-int(f(A)) for any subset A in X
(iii) for every x ∈ X and for every open set U in X containing x, there exists a $g^*\omega\alpha$-open set W in Y containing f(x) such that W ⊆ f(U).

Proof. (i) → (ii) Let A be any subset of X. Then g$^*\omega\alpha$-int(A) is open in X and g$^*\omega\alpha$-int(A) ⊆ A. By hypothesis, f(g$^*\omega\alpha$-int(A)) ⊆ f(A). Then g$^*\omega\alpha$-int(f(A)) is the largest g$^*\omega\alpha$-open set contained in f(A). Therefore f(g$^*\omega\alpha$-int(A)) ⊆ g$^*\omega\alpha$-int(f(A)).
(ii) → (iii) Let x ∈ X and U be an g$^*\omega\alpha$-open set in X containing x. Then there exists g$^*\omega\alpha$-open set V in X such that x ∈ V ⊆ U. By hypothesis, f(V) = f(g$^*\omega\alpha$-int(V)) ⊆ g$^*\omega\alpha$-int(f(V)). Then f(V) is g$^*\omega\alpha$-open in Y containing f(x) such that f(V) ⊆ f(U). Take W = f(V) then W satisfies our requirement.
(iii) → (i) Let U be an g$^*\omega\alpha$-open set in X and y be any point in f(U). By hypothesis there exists g$^*\omega\alpha$-open set W_y in Y containing y such that W_y ⊆ f(U). Therefore f(U) = ∪{ W_y : y ∈ f(U) }. Therefore f(U) is g$^*\omega\alpha$-open set in Y. □

Theorem 4.16. *A surjective map f : X → Y is $g^*\omega\alpha$-open if and only if f^{-1} : Y → X is $g^*\omega\alpha$-continuous.*

Proof. Necessity: Let U be an open set in X then by hypothesis $(f^{-1})^{-1}(U)$ = f(U) is g$^*\omega\alpha$-open in Y. Hence f^{-1} : Y → X is g$^*\omega\alpha$-continuous.
Sufficiency: Let U be an open set in X. Then by hypothesis f(U) = $(f^{-1})^{-1}(U)$ is g$^*\omega\alpha$-open in Y. Hence f : X → Y is g$^*\omega\alpha$-open. □

Proposition 4.17. *For any bijective function $f : X \to Y$ the following statements are equivalent:*
(i) $f^{-1} : Y \to X$ is $g^*\omega\alpha$-continuous
(ii) f is $g^*\omega\alpha$-open map
(iii) f is $g^*\omega\alpha$-closed map.

§5. $g^*\omega\alpha$-homeomorphism in topological spaces

In this section the concept and characterizations of $g^*\omega\alpha$-homeomorphism in topological spaces are introduced and discussed.

Definition 5.1. *A function f: $X \to Y$ is called $g^*\omega\alpha$-homeomorphism if f and f^{-1} are $g^*\omega\alpha$-continuous.*

Remark 5.1. *From the Definition 5.1 it is clear that every homeomorphism is $g^*\omega\alpha$-homeomorphism but not conversely.*

Example 5.1. *Let $X = Y = \{ a, b, c \}$, $\tau = \{ X, \phi, \{ a, b \}\}$ and $\sigma = \{ Y, \phi, \{ a \}, \{ b \}, \{ a, b \}\}$. Let $f : X \to Y$ be an identity function. Then f is $g^*\omega\alpha$-homeomorphism but not homeomorphism, as f is not continuous, since for the open set $A = \{ a \}$ in Y, $f^{-1}(\{ a \}) = \{ a \}$ is not open in X.*

Theorem 5.1. *Let $f : X \to Y$ be a bijective function. Then the following statements are equivalent:*
(i) f is $g^*\omega\alpha$-homeomorphism.
(ii) f is $g^*\omega\alpha$-continuous and $g^*\omega\alpha$-open map.
(iii) f is $g^*\omega\alpha$-continuous and $g^*\omega\alpha$-closed map.

Proof. Follows from the definitions. □

Theorem 5.2. *If $f : X \to Y$ and $g : Y \to Z$ are $g^*\omega\alpha$-homeomorphism and Y is $T_{g^*\omega\alpha}$-space then $gof : X \to Z$ is $g^*\omega\alpha$-homeomorphism.*

Proof. Let A be an open set in Z. Since g is g$^*\omega\alpha$-continuous, g^{-1}(A) is g$^*\omega\alpha$-open in Y. Then g^{-1}(A) is open in Y as Y is $T_{g^*\omega\alpha}$-space. Also, since f is g$^*\omega\alpha$-continuous, f^{-1}(g^{-1}(A)) = (gof)$^{-1}$(A) is g$^*\omega\alpha$-open in X. Therefore gof is g$^*\omega\alpha$-continuous.

Again, let A be an open set in X. Since f^{-1} is g$^*\omega\alpha$-continuous, (f^{-1})$^{-1}$ = f(A) is g$^*\omega\alpha$-open Y and so f(A) is open in Y as Y is $T_{g^*\omega\alpha}$-space. Also, g^{-1} is g$^*\omega\alpha$-continuous then, (g^{-1})$^{-1}$f(A) = g(f(A)) = (gof)(A) is g$^*\omega\alpha$-open in Z. Therefore ((gof)$^{-1}$)$^{-1}$(A) = (gof)(A) is g$^*\omega\alpha$-open set in Z. Hence (gof)$^{-1}$ is g$^*\omega\alpha$-continuous. Thus gof is g$^*\omega\alpha$-homeomorphism. □

Definition 5.2. *A bijective function $f : X \to Y$ is said to be strongly $g^*\omega\alpha$-homeomorphism if both f and f^{-1} are $g^*\omega\alpha$-irresolute.*
We say that spaces X and Y are strongly $g^*\omega\alpha$-homeomorphic if there exists a $g^*\omega\alpha$-homeomorphism from X on to Y.
We denote the family of all strongly $g^*\omega\alpha$-homeomorphism of a topological space X on to itself by strongly $g^*\omega\alpha$-hX.

Theorem 5.3. *Every strongly $g^*\omega\alpha$-homeomorphism is $g^*\omega\alpha$-homeomorphism.*

Proof. Follows from Theorem 3.3. □

Remark 5.2. *Composition of two strongly $g^*\omega\alpha$-homeomorphism is a strongly $g^*\omega\alpha$-homeomorphism.*

Theorem 5.4. *The set strongly $g^*\omega\alpha$-hX is group under the composition of maps.*

Proof. Define a binary operation * : strongly $g^*\omega\alpha$-hX $\rightarrow$ strongly $g^*\omega\alpha$-hX by f*g = gof for all f, g $\in$ strongly $g^*\omega\alpha$-hX and so o is the usual operation of composition of maps. Then by Remark 5.2, gof $\in$ strongly $g^*\omega\alpha$-hX. We know that the composition of maps is associative and identity map I : X $\rightarrow$ X belonging to strongly $g^*\omega\alpha$-hX serves as the identity element. If f $\in$ strongly $g^*\omega\alpha$-hX, then f^{-1} $\in$ strongly $g^*\omega\alpha$-hX such that fof^{-1} = $f^{-1}$of = I and so inverse exists for each element of strongly $g^*\omega\alpha$-hX. Therefore (strongly $g^*\omega\alpha$-hX, o) is a group under the operation of composition of maps. □

Theorem 5.5. *Let $f : X \rightarrow Y$ be strongly $g^*\omega\alpha$-homeomorphism. Then f induces an isomorphism from the group strongly $g^*\omega\alpha$-hX onto the group strongly $g^*\omega\alpha$-hY.*

Proof. Using the map f, we define a map η_f : strongly $g^*\omega\alpha$-hX $\rightarrow$ strongly $g^*\omega\alpha$-hY by η_f(h) = fohof^{-1} for every h $\in$ strongly $g^*\omega\alpha$-hX. Then η_f is a bijection. Further for all h_1 and h_2 $\in$ strongly $g^*\omega\alpha$-hX, η_f (h_1oh_2) = fo(h_1oh_2)of^{-1} = (fo$h_1$of^{-1}) o (fo$h_2$of^{-1}) = η_f(h_1) o η_f(h_2). Therefore η_f is homeomorphism and so it is an isomorphism induced by f. □

Theorem 5.6. *Strongly $g^*\omega\alpha$-homeomorphism is an equivalence relation in the collection of all topological spaces.*

Proof. Reflexivity and Symmetry are immediate and Transitivity follows from the Remark 5.2. □

Corollary 5.1. *If $A \subset B$ then $g^*\omega\alpha$-cl(A) $\subset$ $g^*\omega\alpha$-cl(B).*

Theorem 5.7. *If $f : X \rightarrow Y$ is strongly $g^*\omega\alpha$-homeomorphism then $g^*\omega\alpha$-cl(f^{-1}(B)) = f^{-1}($g^*\omega\alpha$-cl(B)) for every $B \subseteq Y$.*

Proof. Since f is strongly $g^*\omega\alpha$-homeomorphism, f is $g^*\omega\alpha$-irresolute. Since $g^*\omega\alpha$-cl(f(B)) is $g^*\omega\alpha$-closed set in Y, f^{-1}($g^*\omega\alpha$-cl(f(B))) is $g^*\omega\alpha$-closed in X. Now f^{-1}(B) $\subset$ f^{-1}($g^*\omega\alpha$-cl(B))) and so by Corollary 5.1, $g^*\omega\alpha$-cl(f^{-1}(B)) $\subseteq$ f^{-1}($g^*\omega\alpha$-cl(B))).
Again, since f is strongly $g^*\omega\alpha$-homeomorphism, f^{-1} is $g^*\omega\alpha$-irresolute. Since $g^*\omega\alpha$-cl(f^{-1}(B)) is $g^*\omega\alpha$-closed in X, $(f^{-1})^{-1}$($g^*\omega\alpha$-cl(f^{-1}(B))) = f($g^*\omega\alpha$-cl(f^{-1}(B))) is $g^*\omega\alpha$-closed in Y. Now, B $\subset$ $(f^{-1})^{-1}$(f^{-1}(B))) $\subseteq$ $(f^{-1})^{-1}$($g^*\omega\alpha$-cl(f^{-1}(B))) = f($g^*\omega\alpha$-cl(f^{-1}(B))) and so $g^*\omega\alpha$-cl(B) $\subseteq$ f($g^*\omega\alpha$-cl(f^{-1}(B))). Therefore f^{-1}($g^*\omega\alpha$-cl(B)) $\subseteq$ f^{-1}(f($g^*\omega\alpha$-cl(f^{-1}(B)))) $\subseteq$ $g^*\omega\alpha$-cl(f^{-1}(B)) and hence the equality holds. □

Corollary 5.2. *If $f : X \rightarrow Y$ is strongly $g^*\omega\alpha$-homeomorphism then $g^*\omega\alpha$-cl(f(B)) = f($g^*\omega\alpha$-cl(B)) for all subset B of X.*

Proof. Since f : X $\rightarrow$ Y is strongly $g^*\omega\alpha$-homeomorphism, f^{-1} : Y $\rightarrow$ X is also strongly $g^*\omega\alpha$-homeomorphism. Therefore by the Theorem 5.7, $g^*\omega\alpha$-cl($(f^{-1})^{-1}$(B)) = $(f^{-1})^{-1}$($g^*\omega\alpha$-cl(B)) for all B $\subset$ X, that is $g^*\omega\alpha$-cl(f(B)) = f($g^*\omega\alpha$-cl(B)). □

Corollary 5.3. *If $f : X \to Y$ is strongly $g^*\omega\alpha$-homeomorphism then f($g^*\omega\alpha$-int(B)) = $g^*\omega\alpha$-int(f(B)) for all $B \subseteq X$.*

Proof. For any subset $B \subseteq X$, $g^*\omega\alpha$-int(B) = $g^*\omega\alpha$-cl(B^c))^c. Thus by using Corollary 5.2, we obtain f($g^*\omega\alpha$-int(B)) = f(($g^*\omega\alpha$-cl(B^c))^c = (f($g^*\omega\alpha$-cl(B^c)))^c = ($g^*\omega\alpha$-cl(f(B^c)))^c = ($g^*\omega\alpha$-cl((f(B))^c))^c = $g^*\omega\alpha$-int(f(B)). □

§6. $g^*\omega\alpha$-closed graphs in topological spaces

In this section we discussed the properties of $g^*\omega\alpha$-closed graphs.

Definition 6.1. *A topological space X is said to be a*

(i) $g^*\omega\alpha$-T_1-space if for each pair of distinct points x and y of X there exist disjoint $g^*\omega\alpha$-open sets U containing x but not y and V containing y but not x.

(ii) $g^*\omega\alpha$-T_2-space if for each pair of distinct points x and y of X there exist disjoint $g^*\omega\alpha$-open sets U and V such that $x \in U$ and $y \in V$.

Definition 6.2. *A function $f : X \to Y$ has $g^*\omega\alpha$-closed graph if for each (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist $U \in G^*\omega\alpha O(X, x)$ and $V \in O(Y, y)$ such that (U $\times$ cl(V)) $\cap$ G(f) = ϕ.*

Theorem 6.1. *Let $f : X \to Y$ be a function. Then the following properties are equivalent:*

(i) f is $g^*\omega\alpha$-closed graph.

(ii) for each (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist $U \in G^*\omega\alpha O(X, x)$ and $V \in O(Y, y)$ such that f(U) $\cap$ cl(V) = ϕ.

(iii) for each (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist $U \in G^*\omega\alpha O(X, x)$ and $V \in G^*\omega\alpha O(Y, y)$ such that (U $\times$ $g^*\omega\alpha$-cl(V)) $\cap$ G(f) = ϕ.

(iv) for each (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist $U \in G^*\omega\alpha O(X, x)$ and $V \in G^*\omega\alpha O(Y, y)$ such that f(U) $\cap$ $g^*\omega\alpha$-cl(V) = ϕ.

Proof. (i) → (ii): Suppose (i) holds. Then (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist U $\in$ G$^*\omega\alpha$O(X, x) and V $\in$ O(Y, y) such that (U $\times$ cl(V)) $\cap$ G(f) = ϕ. Thus, for each x $\in$ X, U is $g^*\omega\alpha$-open set in X containing x, implies f(x) $\neq$ y. Therefore f(U) $\cap$ cl(V) = ϕ. Thus (b) holds.
(ii) → (i): By (ii) there exist U $\in$ G$^*\omega\alpha$O(X, x) and V $\in$ O(Y, y) such that f(U) $\cap$ cl(V) = ϕ. That is U is a $g^*\omega\alpha$-open set in X containing x and f(x) $\neq$ y. Thus (U $\times$ cl(V)) $\setminus$ G(f) = ϕ.
(i) → (iii) From (iii) there exist U $\in$ G$^*\omega\alpha$O(X, x) and V $\in$ O(Y, y) such that (U $\times$ cl(V)) $\cap$ G(f) = ϕ. Therefore (U $\times$ $g^*\omega\alpha$-cl(V)) $\cap$ G(f) = ϕ. Thus (iii) holds.
(ii) → (iv): Suppose (ii) holds, that is (x, y) $\in$ (X $\times$ Y) $\setminus$ G(f) there exist U $\in$ G$^*\omega\alpha$O(X, x) and V $\in$ O(Y, y) such that f(U) $\cap$ cl(V) = ϕ. Since every open set is $g^*\omega\alpha$-open [15], $g^*\omega\alpha$-cl(V) $\subseteq$ cl(V), implies f(U) $\cap$ $g^*\omega\alpha$-cl(V) = ϕ. Thus (iv) holds.
(i) → (iv): It follows from (ii). □

Theorem 6.2. *If $f : X \to Y$ is surjective $g^*\omega\alpha$-closed graph then Y is a T_1-space.*

Proof. Let y_1, y_2 $\in$ Y with $y_1 \neq y_2$. Let $x_0 \in$ X. Since f is surjective f(x_0) = y_2. Therefore (x_0, y_1) $\in$ (X $\times$ Y) $\setminus$ G(f). Since f is $g^*\omega\alpha$-closed graph there exist $U_1 \in$ G$^*\omega\alpha$O(X, x_0) and $V_1 \in$ O(Y, y_1) such that f(U_1) $\cap$ cl(V_1) = ϕ. Since $x_0 \in U_1$ and f(x_0) = $y_1 \in$ f(U_1) and f(U_1) $\cap$ cl(V_1) = ϕ, implies $y_2 \notin V_1$.

Let $x_1 \in$ X. Since f is surjective f(x_1) = y_1. Therefore (x_1, y_2) $\in$ (X $\times$ Y) $\setminus$ G(f). Since f is g$^*\omega\alpha$-closed graph there exist $U_2 \in$ G$^*\omega\alpha$O(X, x_1) and $V_2 \in$ O(Y, y_2) such that f(U_2) $\cap$ cl(V_2) = ϕ. Since $x_1 \in U_2$ and f(x_1) = $y_2 \in$ f(U_2) and f(U_2) $\cap$ cl(V_2) = ϕ, implies $y_1 \notin V_2$. Therefore, for each y_1, $y_2 \in$ Y there exist an open sets V_1 and V_2 such that $y_1 \in V_1$, $y_2 \notin V_1$ and $y_1 \notin V_2$, $y_2 \in V_2$. Hence Y is T_1-space. □

Corollary 6.1. *If $f: X \to Y$ is surjective $g^*\omega\alpha$-closed graph then Y is $g^*\omega\alpha$-T_1-space.*

Theorem 6.3. *If $f: X \to Y$ is injective $g^*\omega\alpha$-closed graph then X is $g^*\omega\alpha$-T_1-space.*

Proof. Let x_1, $x_2 \in$ X with $x_1 \neq x_2$. Since f is injective, f(x_1) $\neq$ f(x_2), implies (x_1, f(x_2)) $\in$ (X $\times$ Y) $\setminus$ G(f). Since f is g$^*\omega\alpha$-closed graph there exist $U_1 \in$ G$^*\omega\alpha$O(X, x_1) and $V_1 \in$ O(Y, f(x_2)) such that f(U_1) $\cap$ cl(V_1) = ϕ. Since $x_1 \in U_1$, implies f(x_1) $\in$ f(U_1), so f(x_2) $\notin$ f(U_1) and $x_2 \notin U_1$.

Let us consider, (x_2, f(x_1)) $\in$ (X $\times$ Y) $\setminus$ G(f). Since f is g$^*\omega\alpha$-closed graph there exist $U_2 \in$ G$^*\omega\alpha$O(X, x_2) and $V_2 \in$ O(Y, f(x_1)) such that f(U_2) $\cap$ cl(V_2) = ϕ. Since $x_2 \in U_2$, implies f(x_2) $\in$ f(U_2), so f(x_1) $\notin$ f(U_2) and $x_1 \notin U_2$. Therefore, for each x_1, $x_2 \in$ X, there exists g$^*\omega\alpha$-open sets U_1 and U_2 in X such that $x_1 \in U_1$, $x_2 \notin U_1$ and $x_1 \notin U_2$, $x_2 \in U_2$. Hence X is g$^*\omega\alpha$-T_1-space. □

Corollary 6.2. *Let $f: X \to Y$ be bijective with $g^*\omega\alpha$-closed then both X and Y are $g^*\omega\alpha$-T_1-spaces.*

Theorem 6.4. *Let $f: X \to Y$ be surjective $g^*\omega\alpha$-closed graph then Y is T_2-space.*

Proof. Let y_1, $y_2 \in$ Y with $y_1 \neq y_2$. Since f is surjective, for each $x_1 \in$ X, f(x_1) = y_1. Now (x_1, y_2) $\in$ (X $\times$ Y) $\setminus$ G(f). Since f is g$^*\omega\alpha$-closed graph there exist U $\in$ G$^*\omega\alpha$O(X, x_1), V $\in$ O(Y, y_2), such that f(U) $\cap$ cl(V) = ϕ. Now $x_1 \in$ U, implies f(x_1) = $y_1 \in$ f(U). So $y_1 \neq$ cl(V) as f(U) $\cap$ cl(V) = ϕ. Therefore there exists W $\in$ O(Y, y_1) such that W $\cap$ V = ϕ. Hence, Y is T_2-space. □

Corollary 6.3. *Let $f: X \to Y$ be surjective $g^*\omega\alpha$-closed graph, then Y is $g^*\omega\alpha$-T_2-space.*

Acknowledgements

The authors are grateful to the University Grants Commission, New Delhi, India for its financial support by UGC SAP DRS-III, 2016-2021:F.510/3/DRS-III/2016(SAP-I) dated 29th Feb2016 to the Department of Mathematics, Karnatak University, Dharwad, India. Also this research was supported by the Karnatak University, Dharwad, India under No.KU/Sch/UGC-UPE/2014-15/893 dated 24th November, 2014.

References

[1] K. Balachandran, P. Sundaram and H. Maki. On generalized continuous maps in topological spaces. Mem. Fac. Sci. Kochi Univ. Ser. A Math., 1991, 12: 5–13.

[2] S. S. Benchalli, P. G. Patil and T. D. Rayanagaudar. $\omega\alpha$-closed sets in topological spaces. The Global. J. Appl. Math. and Math. Sci., 2009, 2: 53–63.

[3] S. S. Benchalli and P. G. Patil. Some new continuous maps in topological spaces. J. Adv. Studies in Topology, 2010, 1(2): 16–21.

[4] S. S. Benchalli, P. G. Patil and P. M. Nalwad. Generalized $\omega\alpha$-closed sets in topological spaces. J. New Results in Science, 2014, 7: 7–19.

[5] S. S. Benchalli, P. G. Patil and P. M. Nalwad. Some weaker forms of continuous functions in topological spaces. J. of Advanced Studies in Topology, 2016, 7: 101–109.

[6] R. Devi, K. Balachandran and H. Maki. On generalized α-continuous maps and α-generalized continuous maps. Far East J. Math. Sci., 1997, Special Volume, part I: 1–15.

[7] R. Devi, K. Balachandran and H. Maki. Generalized α-closed maps and α-generalized closed maps. Indian J. Pure and Applied Math., 1998, 29(1): 37–49.

[8] N. Levine. Semi-open sets and semi continuity in topological spaces. Amer. Math. Monthly, 1963, 70: 36–41.

[9] N. Levine. Generalized closed sets in topology. Rend. Circ. Mat. Palermo, 1970, 19(2): 89–96.

[10] P. E. Long. Functions with closed graph. Amer Math. Monthly, 1969, 76: 930-932.

[11] H. Maki, J. Umehara and T. Noiri. Every topological space is pre-$T_{1/2}$. Mem. Fac. Sci. Kochi Univ. Math., 1996, 17: 33–42.

[12] S. R. Malghan. Generalized closed maps. J. Karnatak Univ. Sci., 1982, 27: 82–88.

[13] A. S. Mashour, M. E. Abd El-Monsef and S. N. El-Deeb. α-open mappings. Acta. Math. Hung., 1983, 41: 213–218.

[14] O. Njastad. On some classes of nearly open sets. Pacific. J. Math., 1965, 15: 961–970.

[15] P. G. Patil, S. S. Benchalli and P. S. Mirajakar. Generalized star $\omega\alpha$-closed sets in topological spaces. J. New Results in Science, 2015, 9: 37–45.

[16] P. G. Patil, S. S. Benchalli and P. S. Mirajakar. Generalized star $\omega\alpha$-spaces in topological spaces. Int. J. Scientific and Innovative Mathematical Research, 2015, 3: 388–391.

[17] P. Sundaram and M. Sheik John. On ω-closed sets in topology, Acta Cienc. Indica Math., 2000, 4: 389–392.

Scientia Magna
Vol. 11 (2016), No. 2, 97-104

Hamming index of some thorn graphs with respect to adjacency matrix

Harishchandra S. Ramane, Gouramma A. Gudodagi and Ashwini S. Yalnaik

Department of Mathematics, Karnatak University
Dharwad - 580003, Karnataka, India
E-mails: hsramane@yahoo.com, gouri.gudodagi@gmail.com, ashwiniynaik@gmail.com

Abstract Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $A(G)$ be the adjacency matrix of a graph G. The rows of $A(G)$ corresponding to a vertex v of G, denoted by $s(v)$ is the string. The Hamming index of a graph G is the sum of the Hamming distances between all pairs of vertices of G. In this paper we obtain Hamming index generated by adjacency matrix of some thorn graphs.

Keywords Hamming distance, Hamming index, adjacency matrix, thorn graphs.
2010 Mathematics Subject Classification 05C99.

§1. Introduction

In information theory, the Hamming distance between two strings of equal length is the number of positions at which the corresponding symbols are different. In another way, it measures the minimum number of substitutions required to change one string into the other, or the minimum number of errors that could have transformed one string into the other.

The Hamming distance is named after Richard Hamming, who introduced it in his fundamental paper on Hamming codes Error detecting and error correcting codes in 1950 [4]. It is used in telecommunication to count the number of flipped bits in a fixed-length binary word as an estimate of error, and therefore is sometimes called the signal distance. Hamming weight analysis of bits is used in several disciplines including information theory, coding theory, and cryptography. However, for comparing strings of different lengths, or strings where not just substitutions but also insertions or deletions have to be expected. For q-array strings over an alphabet of size $q \geq 2$. The Hamming distance is applied in case of orthogonal modulation and is also used in systematics as a measure of genetic distance.

Let $\mathbb{Z}_2 = \{0, 1\}$. The set $\mathbb{Z}_2$ is a group under binary operation $\oplus$ with addition modulo 2. Therefore for any positive integer n, $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors) is a group under the operation $\oplus$ defined by

$$(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n).$$

Element of $\mathbb{Z}_2^n$ is an n-tuple $(x_1, x_2, \ldots, x_n)$ written as $x = x_1 x_2 \ldots x_n$, where every x_i is either 0 or 1 and is called a *string* or *word*. The number of 1's in $x = x_1 x_2 \ldots x_n$ is called the *weight* of x and is denoted by $wt(x)$.

Let $x = x_1x_2\dots x_n$ and $y = y_1y_2\dots y_n$ be the elements of $\mathbb{Z}_2^n$. Then the sum $x \oplus y$ is computed by adding the corresponding components of x and y under addition modulo 2. That is, $x_i + y_i = 0$ if $x_i = y_i$ and $x_i + y_i = 1$ if $x_i \neq y_i$, $i = 1, 2, \dots, n$.

The *Hamming distance* $H_d(x, y)$ between the strings $x = x_1x_2\dots x_n$ and $y = y_1y_2\dots y_n$ is the number of i's such that $x_i \neq y_i$, $1 \leq i \leq n$.

Thus $H_d(x, y) =$ Number of positions in which x and y differ $= wt(x \oplus y)$.

Example: Let $x = 01001$ and $y = 11010$. Therefore $x \oplus y = 10011$. Hence $H_d(x, y) = wt(x \oplus y) = 3$.

A graph G with vertex set $V(G)$ is called a *Hamming graph* [1, 4 - 7] if each vertex $v \in V(G)$ can be labeled by a string $s(v)$ of a fixed length such that $H_d(s(u), s(v)) = d_G(u, v)$ for all $u, v \in V(G)$, where $d_G(u, v)$ is the length of shortest path joining u and v in G. Here we denote $H_d(s(u_i), s(v_j)) = Hd_G(u_i, v_j)$.

§2. Preliminaries

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G and $E(G) = \{e_1, e_2, \dots, e_m\}$ be the edge set of G.

The *distance* between two vertices u and v in G is the length of shortest path joining u and v and is denoted by $d_G(u, v)$. The *adjacency matrix* of G is a matrix $A(G) = [a_{ij}]$ of order n, in which $a_{ij} = 1$ if the vertex v_i is adjacent to the vertex v_j and $a_{ij} = 0$, otherwise. Denote by $s(v)$, the row of the adjacency matrix corresponding to the vertex v. It is a string in the set $\mathbb{Z}_2^n$ of all n-tuples over the field of order two.

Sum of *Hamming distances* [3, 9] between all pairs of strings generated by the adjacency matrix of a graph G is denoted by $H_A(G)$. Thus,

$$H_A(G) = \sum_{1 \leq i < j \leq n} Hd_G(v_i, v_j).$$

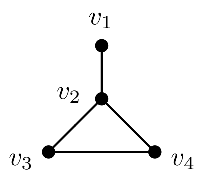

Figure 1: Graph G

For a graph G of Figure 1, the adjacency matrix is

$$A(G) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array},$$

and the strings are $s(v_1) = 0100$, $s(v_2) = 1011$, $s(v_3) = 0101$, $s(v_4) = 0110$.

$$Hd_G(v_1, v_2) = 4, \quad Hd_G(v_1, v_3) = 1, \quad Hd_G(v_1, v_4) = 1,$$
$$Hd_G(v_2, v_3) = 3, \quad Hd_G(v_2, v_4) = 3, \quad Hd_G(v_3, v_4) = 2.$$

Therefore $H_A(G) = 4 + 1 + 1 + 3 + 3 + 2 = 14$.

§3. Hamming distance between pair of vertices

The vertices which are adjacent to both u and v are called the common neighbours of u and v. The vertices which are neither adjacent to u nor adjacent to v are called non-common neighbours of u and v.

Theorem 3.1. [3] *Let G be a graph with n vertices. Let the vertices u and v of G have k common neighbours and l non common neighbours.*
(i) If u and v are adjacent vertices, then

$$Hd_G(u, v) = n - k - l.$$

(ii) If u and v are nonadjacent vertices, then

$$Hd_G(u, v) = n - k - l - 2.$$

Theorem 3.2. *Let G be a graph with n vertices. Let the vertices u and v of G have k common neighbours and l non common neighbours. Let w be another vertex of G.*

(i) If u and v are non adjacent vertices in G and $G^{'}$ is a graph obtained from G by joining u and v, then

$$Hd_{G'}(u, v) = Hd_G(u, v) + 2.$$

(ii) If w is vertex adjacent to both u and v in $G^{'}$, then

$$Hd_{G'}(u, w) = n - k - l - 1.$$

(iii) If w is vertex non adjacent to both u and v in $G^{'}$, then

$$Hd_{G'}(u, w) = n - k - l - 2 + 1.$$

(iv) If w is vertex adjacent to u but not v(vice-versa) in $G^{'}$, then

$$Hd_{G'}(u, w) = n - k - l - 2 - 1.$$

Proof. (i) If u and v are non adjacent in G, then from Theorem 3.1 (ii),

$$Hd_G(u, v) = n - k - l - 2. \tag{1}$$

$G^{'}$ is a graph obtained from G by joining u and v, then from Theorem 3.1 (i),

$$Hd_{G'}(u, v) = n - k - l. \tag{2}$$

Therefore, from Eq. (1) and Eq. (2), we get

$$Hd_{G'}(u,v) = Hd_G(u,v) + 2.$$

(ii) If w is vertex adjacent to both u and v, then from Theorem 3.1 (i),

$$Hd_G(u,v) = n - k - l. \tag{3}$$

Since w is vertex adjacent to both u and v, then the number of common neighbour in G' is $(k+1)$. Therefore Eq. (3) becomes,

$$Hd_{G'}(u,w) = n - k - l - 1.$$

(iii) If w is vertex non-adjacent to both u and v, then from Theorem 3.1 (ii),

$$Hd_G(u,v) = n - k - l - 2. \tag{4}$$

Since w is vertex not-adjacent to both u and v, then the number of non common neighbour in G' is $(l-1)$. Hence Eq. (4) becomes

$$Hd_{G'}(u,w) = n - k - l - 2 + 1.$$

(iv) If w is vertex adjacent to u but not v (vice-versa), then from Theorem 3.1 (i),

$$Hd_G(u,v) = n - k - l - 2. \tag{5}$$

Since w is adjacent to u but not v, then the number of common neighbours is $(k+1)$ and hence Eq. (5) becomes

$$Hd_{G'}(u,w) = n - k - l - 2 - 1.$$

□

§4. Hamming index of some thorn graphs

Definition. [2] *The thorn graph of a graph G denoted by G^{+k} is the graph obtained from G by attaching k pendent vertices to each vertex of G.*

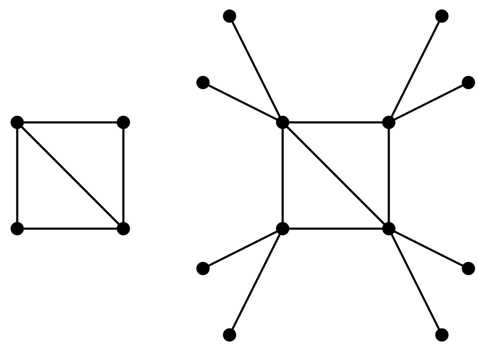

Figure 2: G and G^{+2}

Theorem 4.1. *Let C_n be a cycle on n vertices. Then Hamming index of C_n^{+k} is given by*

$$H_A(C_n^{+k}) = H_A(C_n) + 2k\binom{n}{2}(1+k) + [k^2n^2 - 4nk + 3n^2k].$$

Proof. Let C_n be a cycle on n vertices. Then adjecency matrix of C_n^{+k} is

$$A(C_n^{+k}) = \begin{pmatrix} A(C_n) & I \cdots & I \\ I & O \cdots & O \\ \vdots & \vdots & \\ I & O \cdots & O \end{pmatrix},$$

where $A(C_n)$is the adjacency matrix of C_n and I is the identity matrix of order n and O is the null matrix.

$$\begin{aligned} H_A(C_n^{+k}) &= \sum_{1\le i<j\le (k+1)n} Hd_G(u_i,v_j) \\ &= \sum_{1\le i<j\le n} Hd_G(u_i,v_j) + \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) \\ &= \sum_{(u,v)\in C_n} 2k + Hd_G(u,v) + \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j). \quad (6) \end{aligned}$$

$$(i) \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) = 2k^2\binom{n}{2}. \quad (7)$$

$$\begin{aligned} (ii)\sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) &= \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j)\text{for a pair of } (u_i,v_j) \text{ adjacent pairs} \\ &+ \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j)\text{for a pair of } (u_i,v_j) \text{ non-adjacent pairs.} \end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) = \text{Hamming distance between } kn \text{ adjacent pairs} = k(k+3)n. \quad (8)$$

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) &= \text{Hamming distance between}\, k(n^2-n)\, \text{non-adjacent pairs} \\ &= \text{Hamming distance between } 2nk \text{ pairs with common neighbour} \\ &+ \text{Hamming distance between } [k(n^2-n)-2nk] \text{ pairs with non} \\ &\quad \text{common neighbour} = (k+1)(2nk) + (k+3)(kn^2-3nk). \quad (9) \end{aligned}$$

Substituting Eq. (7), Eq. (8) and Eq. (9) in Eq. (6), we get

$$H_A(C_n^{+k}) = H_A(C_n) + 2k\binom{n}{2}(1+k) + [k^2n^2 - 4nk + 3kn^2].$$

□

Theorem 4.2. *Let K_n be a complete graph on n vertices. Then Hamming index of K_n^{+k} is given by*

$$H_A(K_n^{+k}) = H_A(K_n) + 2k\binom{n}{2}(1+k) + kn(n+k) + [(n-1)+(k-1)]k(n^2-n).$$

Proof. Let K_n be a complete graph on n vertices. Then adjecency matrix of k_n^{+k} is

$$A(K_n^{+k}) = \begin{pmatrix} A(K_n) & I\cdots & I \\ I & O\cdots & O \\ \vdots & \vdots & \\ I & O\cdots & O \end{pmatrix},$$

where $A(K_n)$ is the adjacency matrix of K_n, I is the identity matrix of order n and O is the null matrix.

$$\begin{aligned} H_A(K_n^{+k}) &= \sum_{1\le i<j\le (k+1)n} Hd_G(u_i,v_j) \\ &= \sum_{1\le i<j\le n} Hd_G(u_i,v_j) + \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) \\ &= \sum_{(u,v)\in K_n} 2k + Hd_G(u,v) + \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) + \\ &\quad \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j). \qquad (10) \end{aligned}$$

$$i) \sum_{n+1<=i<j<=(k+1)n} Hd_G(u_i,v_j) = 2k^2\binom{n}{2}. \qquad (11)$$

$$ii) \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) = \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j)\text{for a pair of } (\mathrm{u_i},\mathrm{v_j}) \text{ adjacent pairs}$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) = \text{Hamming distance between } kn \text{ adjacent pairs} = kn(k+n). \qquad (12)$$

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) &= \text{Hamming distance between } k(n^2-n) \text{ non}-\text{adjacent pairs} \\ &= [(n-1)+(k-1)]k(n^2-n). \qquad (13) \end{aligned}$$

Substituting Eq. (11), Eq. (12) and Eq. (13) in Eq. (10), we get

$$H_A(K_n^{+k}) = H_A(K_n) + 2k\binom{n}{2}(1+k) + kn(n+k) + [(n-1)+(k-1)]k(n^2-n).$$

□

Theorem 4.3. *Let P_n be a path on n vertices. Then Hamming index of P_n^{+k} is given by*

$$H_A(P_n^{+k}) = H_A(P_n) + 2k\binom{n}{2}(1+k) + k[n(3+k)-2] + 2k^2 + (n-2)[2k(k+1)]$$
$$+k(n-2)[n(k+3)-k-5].$$

Proof. Let P_n be a path on n vertices. Then adjecency matrix of P_n^{+k} is

$$A(P_n^{+k}) = \begin{pmatrix} A(P_n) & I\cdots & I \\ I & O\cdots & O \\ \vdots & \vdots & \\ I & O\cdots & O \end{pmatrix},$$

where $A(P_n)$is the adjacency matrix of P_n, I is the identity matrix of order n and O is the null matrix.

$$\begin{aligned} H_A(P_n^{+k}) &= \sum_{1\le i<j\le (k+1)n} Hd_G(u_i,v_j) \\ &= \sum_{1\le i<j\le n} Hd_G(u_i,v_j) + \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) + \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) \\ &= \sum_{(u,v)\in P_n} 2k + Hd_G(u,v) + \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) \\ &+ \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j). \end{aligned} \tag{14}$$

$$(i) \sum_{n+1\le i<j\le (k+1)n} Hd_G(u_i,v_j) = 2k^2\binom{n}{2}. \tag{15}$$

$$\begin{aligned} (ii)\sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) &= \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j)\text{for a pair of } (u_i,v_j)\text{adjacent pairs} \\ &+ \sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j)\text{for pair of } (u_i,v_j)\text{non adjacent pairs.} \end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) = \text{Hamming distance between } kn \text{ adjacent pairs} = k[n(k+3)-2]. \tag{16}$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=n+1}^{(k+1)n} Hd_G(u_i,v_j) &= \text{Hamming distance between } k(n^2-n) \text{ non-adjacent pairs}\\ &= \text{Hamming distance between } 2(n-1)k \text{ pairs with common neighbour}\\ &+ \text{Hamming distance between } [k(n^2-n)-2(n-1)k] \text{ pairs with}\\ &\quad \text{non-common neighbour} = (n-2)k[n(k+3)-k-5]. \qquad (17)\end{aligned}$$

Substituting Eq. (15), Eq. (16), and Eq. (17) in Eq. (14), we get

$$\begin{aligned}H_A(P_n^{+k}) = H_A(P_n) + 2k\binom{n}{2}(1+k) + k[n(3+k)-2] + 2k^2 + (n-2)[2k(k+1)]\\ +k(n-2)[n(k+3)-k-5].\end{aligned}$$

□

Acknowledgement

The author H. S. Ramane is thankful to UGC, Govt. of India for support through research grant under UGC-SAP DRS-III for 2016-2021: F. 510/3/DRS-III/2016(SAP-I) Dated: 29th Feb. 2016. Another author G. A. Gudodagi is thankful to the Karnatak University for financial support through the UGC-UPE scholarship No: KU/SCH/UGC-UPE/2014-15/901. And the third author Ashwini Yalnaik is thankful to UGC, Govt. of India for support through Rajiv Gandhi National Fellowship No. F1- 17. 1/2014-15-SC-KAR-74909.

References

[1] S. Bang, E. R. van Dam and J. H. Koolen. Spectral characterization of the Hamming graphs. Linear Algebra Appl., 2008, (429): 2678–2686.

[2] D. M. Cvetković, M. Doob and H. Sachs. Spectra of graphs. New York: Academic Press, 1980.

[3] A. B. Ganagi and H. S. Ramane. Hamming distance between the strings generated by adjacency matrix of a graph and their sum. Alg. Discr. Math, 2016, 22: 82–93.

[4] R. W. Hamming. Error detecting and error correcting codes. Bell System Tech. J., 1950, 29(2): 147–160.

[5] W. Imrich and S. Klavžar. A simple $O(mn)$ algorithm for recognizing Hamming graphs. Bull. Inst. Combin. Appl., 1993, 9: 45–56.

[6] W. Imrich and S. Klavžar. On the complexity of recognizing Hamming graphs and related classes of graphs. European J. Combin., 1996, 17: 209–221.

[7] W. Imrich and S. Klavžar. Recognizing Hamming graphs in linear time and space. Inform. Process. Lett., 1997, 63: 91–95.

[8] S. Klavžar and I. Peterin. Characterizing subgraphs of Hamming graphs. J. Graph Theory, 2005, 49: 302–312.

[9] H. S. Ramane and A. B. Ganagi. Hamming index of class of graphs. Int. J. Curr. Engg. Tech., 2013, 205–208.

SCIENTIA MAGNA

An international journal

ISSN 1556- 6706