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Some identities involving the Laguerre polynomials [1]

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Abstract Orthogonal polynomials play a very important role in analysis, mainly because functions belonging to very general classes can be expanded in series of orthogonal polynomials. The main purpose of this paper is using the elementary method and the properties of power series to study the calculating problem of one kind summation involving the Laguerre polynomials, and give some interesting identities.

Keywords Laguerre polynomials, power series, generating function, elementary method, orthogonal polynomials, identity.

§1. Introduction

For any real number x, the famous Laguerre polynomials $L_n(x)$ are defined by the coefficients of the power series expansion of the function $\frac{1}{1-t}\,e^{-\frac{xt}{1-t}}$. That is,

$$\frac{1}{1-t}\,e^{-\frac{xt}{1-t}}=\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}\,t^n\,.$$

For example, the first several polynomials are: $L_0(x)=1,\ L_1(x)=-x+1,\ L_2(x)=x^2-4x+2,\cdots$. It is well known that $L_n(x)$ is an orthogonal polynomial. And it play a very important rule in the theories and applications of mathematics. So there are many people had studied its properties, some results had related papers see references [2], [3], [4], [5] and [6].

In this paper, we shall study the calculating problem of the summation

$$\sum_{a_1+a_2+\cdots+a_k=n}\frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!},\tag{1}$$

and give some interesting calculating formula for it. About this problem, Professor Liu Duansen, Li Chao, Yan Chundian [2] and [3] obtained some simple conclusions, but there exist many calculating errors and typographical errors in their paper. In this paper, we consider the integral calculating problem of (1), and give an exact calculating formula for it. That is, we shall prove the following two conclusions:

Theorem 1. Let n and k are two positive integer with $k\geq 2$, then we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n}\frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!}=\sum_{a=0}^{n}\binom{a+k-2}{a}\frac{L_{n-a}(kx)}{(n-a)!},$$

[1]This work is supported by the Shaanxi Provincial Education Department Foundation 08JK433.

where $\sum_{a_1+a_2+\cdots+a_k=n}$ denotes the summation over all nonnegative integers $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

Theorem 2. For any positive integer n, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \int_0^\infty \frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!} \cdot e^{-x}\ dx = \sum_{a=0}^{n} (-1)^a (k-1)^a \binom{n-a+k-1}{k-1}.$$

It is clear that our methods can also be used to deal with the other orthogonal polynomials, such as the Legendre polynomials, the Chebyshev polynomials, Jacobi polynomials and the Hermite polynomials, etc.

§2. Proof of the theorems

In this section, we shall use the elementary method and the properties of the power series to prove our Theorems directly. First we prove Theorem 1. Let $f(x,\ t) = \frac{1}{1-t} e^{-\frac{xt}{1-t}}$, then from the definition of partial derivative we have

$$\frac{\partial f(kx,\ t)}{\partial x} = \left(\frac{1}{1-t}\, e^{-\frac{kxt}{1-t}}\right)' = (-1)^1 \frac{kt}{(1-t)^2}\, e^{-\frac{kxt}{1-t}},$$

$$\frac{\partial^2 f(kx,\ t)}{\partial x^2} = \left(\frac{1}{1-t}\, e^{-\frac{kxt}{1-t}}\right)'' = (-1)^2 \frac{k^2t^2}{(1-t)^3}\, e^{-\frac{kxt}{1-t}},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial^{(k-1)} f(kx,\ t)}{\partial x^{k-1}} = \left(\frac{1}{1-t}\, e^{-\frac{kxt}{1-t}}\right)^{(k-1)} = (-1)^{k-1} \frac{k^{k-1}t^{k-1}}{(1-t)^k}\, e^{-\frac{kxt}{1-t}}. \tag{2}$$

So from (2) and the generating function of the Laguerre polynomial $L_n(x)$ we may get

$$\begin{aligned} \frac{\partial^{(k-1)} f(kx,\ t)}{\partial x^{(k-1)}} &= (-1)^{k-1} k^{k-1} \cdot \frac{t^{k-1}}{(1-t)^{k-1}} \cdot \frac{1}{1-t}\, e^{-\frac{kxt}{1-t}} \\ &= (-1)^{k-1} k^{k-1} \frac{t^{k-1}}{(1-t)^{k-1}} \sum_{n=0}^{\infty} \frac{L_n(kx)}{n!} t^n. \end{aligned} \tag{3}$$

Note that the expansion of the power series

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

we have

$$\left(\frac{1}{1-t}\right)^{(k-2)} = \frac{(k-2)!}{(1-t)^{k-1}} = \sum_{n=0}^{\infty} (n+k-2)\cdots(n+1)t^n$$

or

$$\frac{1}{(1-t)^{k-1}} = \frac{1}{(k-2)!}\sum_{n=0}^{\infty}(n+k-2)\cdots(n+1)t^n = \sum_{n=0}^{\infty}\binom{n+k-2}{k-2}\ t^n. \tag{4}$$

Combining (3) and (4) we may get

$$\begin{aligned}\frac{\partial^{(k-1)}f(kx,\ t)}{\partial x^{k-1}} &= (-1)^{k-1}k^{k-1}t^{k-1}\left(\sum_{n=0}^{\infty}\binom{n+k-2}{k-2}\ t^n\right)\left(\sum_{n=0}^{\infty}\frac{L_n(kx)}{n!}t^n\right)\\ &= (-1)^{k-1}k^{k-1}t^{k-1}\sum_{n=0}^{\infty}\left(\sum_{a+b=n}\binom{a+k-2}{k-2}\frac{L_b(kx)}{b!}\right)t^n\\ &= (-1)^{k-1}k^{k-1}t^{k-1}\sum_{n=0}^{\infty}\left(\sum_{a=0}^{n}\binom{a+k-2}{k-2}\frac{L_{n-a}(kx)}{(n-a)!}\right)t^n. \end{aligned}\tag{5}$$

On the other hand, from the generating function of $L_n(x)$ and the properties of the power series we also have

$$\begin{aligned}\frac{1}{(1-t)^k}\,e^{-\frac{kxt}{1-t}} &= \left(\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}\cdot t^n\right)^k\\ &= \sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!}\right)t^n. \end{aligned}\tag{6}$$

Combining (2), (5) and (6) we may get

$$\sum_{n=0}^{\infty}\left(\sum_{a=0}^{n}\binom{a+k-2}{k-2}\frac{L_{n-a}(kx)}{(n-a)!}\right)t^n = \sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!}\right)t^n. \tag{7}$$

Then comparing the coefficients of t^n in (7) we may immediately deduce the identity

$$\sum_{a_1+a_2+\cdots+a_k=n}\frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!} = \sum_{a=0}^{n}\binom{a+k-2}{a}\frac{L_{n-a}(kx)}{(n-a)!}.$$

This proves Theorem 1.

Now we prove Theorem 2. Multiplicative (6) by e^{-x} and integral, we may get

$$\begin{aligned}&\int_0^{\infty}\frac{1}{(1-t)^k}e^{-\frac{kxt}{1-t}}\cdot e^{-x}\ dx\\ =\ &\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n}\int_0^{\infty}\frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!}\cdot e^{-x}\ dx\right)t^n. \end{aligned}\tag{8}$$

Note that the integral

$$\begin{aligned}&\int_0^{\infty}\frac{1}{(1-t)^k}e^{-\frac{kxt}{1-t}}\cdot e^{-x}\ dx = \frac{1}{(1-t)^k}\int_0^{\infty}e^{-\frac{kxt}{1-t}-x}\ dx\\ =\ &\frac{1}{(1-t)^k}\int_0^{\infty}e^{-x\left(\frac{kt}{1-t}+1\right)}\ dx = \frac{1}{(1-t)^k}\cdot\frac{1}{1+(k-1)t}. \end{aligned}\tag{9}$$

From the power series expansions of $\frac{1}{(1-t)^k} \cdot \frac{1}{1+(k-1)t}$ we have

$$\begin{aligned}\frac{1}{(1-t)^k} \cdot \frac{1}{1+(k-1)t} &= \left(\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} t^n\right)\left(\sum_{n=0}^{\infty}(-1)^n(k-1)^n t^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{a=0}^{n}(-1)^a(k-1)^a \binom{n-a+k-1}{k-1}\right) t^n. \qquad (10)\end{aligned}$$

Combining (8), (9) and (10), and comparing the coefficients of t^n we may get

$$\sum_{a_1+a_2+\cdots+a_k=n} \int_0^{\infty} \frac{L_{a_1}(x)\ L_{a_2}(x)\cdots L_{a_k}(x)}{a_1!\ a_2!\cdots a_k!} \cdot e^{-x}\ dx = \sum_{a=0}^{n}(-1)^a(k-1)^a \binom{n-a+k-1}{k-1}.$$

This completes the proof of Theorem 2.

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Elementary methods for solving equations of the third degree and fourth degree

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Abstract In this paper we will establish some new methods for solving third and fourth degree equations.

Keywords Cardano method, Ferrari method, equation.

§1. Introduction

In this paper, using new transformations, we will introduce new elementary methods for solving third and fourth degree equations, which essentially differ from Cardano's method and Ferrari's method. We will present the importance of these transformations through some examples.

§2. Equations of the third degree

Let

$$x^3 + px^2 + qx + r = 0 \tag{2.1}$$

be an equation of the third degree. Usually, it is solved by the means of Cardano's method. Let us start from the following definition:

Definition 1. The equation of form

$$X^3 + a = 0 \tag{2.2}$$

is called binomial equation of the third degree.

The following theorem gives the condition to be satisfied by the coefficients of equation (2.1) in order that (2.1) becomes in the form (2.2).

Theorem 2. Necessary and sufficient condition in order that equation (2.1) be an equation of form (2.2) is that its coefficients satisfy the condition

$$p^2 - 3q = 0. \tag{2.3}$$

Proof of Theorem 2. (Necessary Condition) Substituting $x = m + y$, in equation (2.1), where y is a new variable, we have

$$y^3 + (3m + p)y^2 + (3m^2 + 2pm + q)y + m^3 + pm^2 + qm + r = 0. \tag{2.4}$$

In order that equation (2.4) be a binomial equation of y, it is necessary that its coefficients satisfy conditions

$$3m + p = 0, 3m^2 + 2pm + q = 0. \tag{2.5}$$

From the first equation of (2.5) we have

$$m = -\frac{p}{3}. \tag{2.6}$$

Substituting the value of m from (2.6) in the second equation of (2.5) we obtain condition (2.3).

(Sufficient Condition). Let the coefficients of equation (2.1) satisfy condition (2.3). Then for equation (2.1) we have

$$x^3 + px^2 + \frac{p^2}{3}x + r = 0. \tag{2.7}$$

Substituting $x = y - \frac{p}{3}$ in equation (2.7), where y is a new variable, we obtain

$$y^3 + r - \frac{p^3}{27} = 0 \tag{2.8}$$

which is an equation of form (2.2) of y. This completes the proof.

Now the question is whether every equation of the form (2.1) can be transformed into equation of the form (2.2). This is ensured by the following Theorem.

Theorem 3. Every equation of the form (2.1) can be transformed to the form (2.2).

Proof of Theorem 3. If the coefficients of the equation (2.1) satisfy the condition (2.3) then by substitution $x = y - \frac{p}{3}$, where y is a new unknown, the equation (2.1) is transformed into (2.8), which presents equation on y of type (2.2).

If the coefficients of equation (2.1) do not satisfy in a direct way the condition (2.3), then in order to show that they also can be brought to the form (2.2), one should begin from equation (2.4).

Substituting $y = \frac{1}{t}$ in the equation (2.4), we obtain

$$\frac{1}{t^3} + (3m + p)\frac{1}{t^2} + (3m^2 + 2pm + q)\frac{1}{t} + m^3 + pm^2 + qm + r = 0$$

hence, multiplying the last equation by t^3, we have

$$(m^3 + pm^2 + qm + r)t^3 + (3m^2 + 2pm + q)t^2 + (3m + p)t + 1 = 0. \tag{2.9}$$

Thus, dividing the equation (2.9) by the leading coefficient, we have

$$t^3 + \frac{3m^2 + 2pm + q}{m^3 + pm^2 + qm + r}t^2 + \frac{3m + p}{m^3 pm^2 + qm + r}t + \frac{1}{m^3 + pm^2 + qm + r} = 0. \tag{2.10}$$

We determine the parameter m such that the coefficients of the equation (2.10) satisfy the condition (2.3). Hence

$$\Big(\frac{3m^2 + 2pm + q}{m^3 + pm^2 + qm + r}\Big)^2 - 3\frac{3m + p}{m^3 + pm^2 + qm + r} = 0$$

implying

$$(p^2 - 3q)m^2 + (pq - 9r)m + q^2 - 3pr = 0. \tag{2.11}$$

Equation (2.11) is a quadratic equation of the variable m. Considering Theorem 1, its solution transforms equation (1.10) to an equation of form (2.2) of the variable t. Equation (2.10) is obtained from equation (2.1) by the substitution $x = m + \frac{1}{t}$. The proof is completed.

Theorem 2 implies the following method of solving equation (2.1).

If equation (2.1) does not satisfy condition (2.3), then it is transformed first to form (2.10) by the substitution $x = m + \dfrac{1}{t}$. The parameter m is determined such that the coefficients of equation (2.10) satisfy condition (2.3). Thus equation (2.11) of m is obtained (the quadratic resolvent of equation (1.1)). A value of m is then substituted in (2.10), which yields an equation of form (2.7) of t. Substituting $t = z - \dfrac{P}{3}$ in equation (2.10), it is transformed to an equation of form (2.8) of z, namely

$$z^3 + R - \frac{P^3}{27} = 0 \tag{2.12}$$

where

$$P = \frac{3m^2 + 2pm + q}{m^3 + pm^2 + qm + r}, Q = \frac{3m + p}{m^3 + pm^2 + qm + r}, R = \frac{1}{m^3 + pm^2 + qm + r}.$$

Equation (2.12) is then solved. Put $a = R - \dfrac{P^3}{27}$, by (2.12) we have

$$(z + \sqrt[3]{a})(z^2 - z \cdot \sqrt[3]{a} + \sqrt[3]{a^2}) = 0. \tag{2.13}$$

Hence

$$z_1 = -\sqrt[3]{a}, z_2 = \frac{\sqrt[3]{a}}{2} + \frac{\sqrt[3]{a} \cdot \sqrt{3}}{2} i, z_3 = \frac{\sqrt[3]{a}}{2} - \frac{\sqrt[3]{a} \cdot \sqrt{3}}{2} i \tag{2.14}$$

From (2.14) we derive

$$t_1 = z_1 - \frac{P}{3}, t_2 = z_2 - \frac{P}{3}, t_3 = z_3 - \frac{P}{3} \tag{2.15}$$

hence, from (2.15) we obtain

$$x_1 = m + \frac{1}{t_1}, x_2 = m + \frac{1}{t_2}, x_3 = m + \frac{1}{t_3}. \tag{2.16}$$

Finally, (2.16) are the solutions of equation (1.1).

Example 4. Let us solve the equation

$$x^3 - 3x^2 + 3x + 8 = 0.$$

The coefficients are $p = -3$, $q = 3$ and $r = 8$. Obviously, condition (2.3) is satisfied. Substituting $x = y - \dfrac{p}{3}$, the equation is transformed to form (2.8), that is

$$y^3 + 9 = 0.$$

The solutions of last equation are

$$y_1 = -\sqrt[3]{9}, y_2 = \frac{\sqrt[3]{9}}{2} + \frac{3\sqrt[3]{6}}{2} i, y_3 = \frac{\sqrt[3]{9}}{2} - \frac{3\sqrt[3]{6}}{2} i$$

Substituting the values of y_1, y_2 and y_3 in $x = y + 1$, we have

$$x_1 = -\sqrt[3]{9} + 1, x_2 = \frac{2 + \sqrt[3]{9}}{2} + \frac{3\sqrt[3]{6}}{2}i, x_3 = \frac{2 + \sqrt[3]{9}}{2} - \frac{3\sqrt[3]{6}}{2}i$$

which are the required solutions.

Example 5. Let us solve the equation

$$15x^3 + 30x^2 + 15x + 4 = 0.$$

Dividing the equation by the leading coefficient, we have

$$x^3 + 2x^2 + x + \frac{4}{15} = 0. \tag{2.17}$$

The coefficients of equation (2.17) are $p = 2, q = 1$ and $r = \frac{15}{4}$. Obviously, condition (2.3) is not satisfied. Substituting $x = m + y$, the equation is transformed to form (2.4), namely

$$y^3 + (3m + 2)y^2 + (3m^2 + 4m + 1)y + m^3 + 2m^2 + m + \frac{4}{15} = 0. \tag{2.18}$$

Substituting $y = \frac{1}{t}$ in equation (2.18), we have

$$\frac{1}{t^3} + (3m + 2)\frac{1}{t^2} + (3m^2 + 4m + 1)\frac{1}{t} + m^3 + 2m^2 + m + \frac{4}{15} = 0. \tag{2.19}$$

Multiplying equation (2.19) by t^3, we obtain

$$(m^3 + 2m^2 + m + \frac{4}{15})t^3 + (3m^2 + 4m + 1)t^2 + (3m + 2)t + 1 = 0. \tag{2.20}$$

Dividing equation (2.20) by the leading coefficient, we have

$$t^3 + \frac{3m^2 + 4m + 1}{m^3 + 2m^2 + m + \frac{4}{15}}t^2 + \frac{3m + 2}{m^3 + 2m^2 + m + \frac{4}{15}}t + \frac{1}{m^3 + 2m^2 + m + \frac{4}{15}} = 0. \tag{2.21}$$

We calculate the value of parameter m in order that the coefficients of equation (2.21) satisfy condition (2.3). Thus

$$m^2 - \frac{2}{5}m - \frac{3}{5} = 0. \tag{2.22}$$

The solutions of equation are $m_1 = 1$ and $m_2 = -\frac{3}{5}$. We substitute $m = 1$ in equation (2.21), and obtain

$$t^3 + \frac{15}{8}t^2 + \frac{75}{64}t + \frac{15}{64} = 0. \tag{2.23}$$

The coefficients of equation (2.23) satisfy condition (2.3), and substituting $t = z - \frac{5}{8}$, we have

$$z^3 - \frac{5}{512} = 0 \tag{2.24}$$

The solutions of equation (2.24) are

$$z_1 = \frac{\sqrt[3]{5}}{8}, z_2 = -\frac{\sqrt[3]{5}}{16} + \frac{\sqrt[3]{5}\sqrt{3}}{16}i, z_3 = -\frac{\sqrt[3]{5}}{16} - \frac{\sqrt[3]{5}\sqrt{3}}{16}i.$$

Substituting the values of z_1, z_2 and z_3 in $t = z - \frac{5}{8}$, we have

$$t_1 = \frac{\sqrt[3]{5} - 5}{8}, t_2 = -\frac{\sqrt[3]{5} + 10}{16} + \frac{\sqrt[3]{5}\sqrt{3}}{16}i, t_3 = -\frac{\sqrt[3]{5} + 10}{16} - \frac{\sqrt[3]{5}\sqrt{3}}{16}i.$$

and hence, considering that $m = 1$ and $x = m + \frac{1}{t}$, we obtain

$$x_1 = \frac{\sqrt[3]{5} + 3}{\sqrt[3]{5} - 5}, x_2 = \frac{6 - \sqrt[3]{5} + i\sqrt[3]{5}\sqrt{3}}{-(\sqrt[3]{5} + 10) + i\sqrt[3]{5}\sqrt{3}}, x_3 = \frac{\sqrt[3]{5} - 6 + i\sqrt[3]{5}\sqrt{3}}{\sqrt[3]{5} + 10 + i\sqrt[3]{5}\sqrt{3}}$$

which are the solutions of equation (2.17).

§3. Equations of fourth degree

Let

$$x^4 + px^3 + qx^2 + rx + s = 0 \tag{3.1}$$

be an equation of the fourth degree.

There are three methods known for its solution. These are

1) Ferrari's method,

2) Euler's method, and

3) Descartes' method

In this paper, we introduce a new elementary method solving equation (3.1). Let us start from the symmetrical form of equation (3.1), namely

$$\left(x + \frac{a}{x}\right)^2 + b \cdot \left(x + \frac{a}{x}\right) + c = 0 \tag{3.2}$$

where a, b, c are arbitrary coefficients with $a \neq 0, b \neq 0$.

Equation (3.2) can be rearranged in form

$$x^4 + bx^3 + (2a + c)x^2 + abx + a^2 = 0. \tag{3.3}$$

In order to establish the conditions to be satisfied by the coefficients of equation (3.1) so that (3.1) can be transformed to the form (3.2), comparing the left sides of equations (3.1) and (3.3), we obtain

$$b = p, 2a + c = q, ab = r, a^2 = s. \tag{3.4}$$

Finding a, b and c from equations (3.4), we have

$$a = \frac{r}{p}, b = p, c = \frac{pq - 2r}{p} \tag{3.5}$$

and the following condition

$$r^2 - sp^2 = 0. \tag{3.6}$$

Obviously, assuming $p \neq 0$, (3.6) is necessary and sufficient condition for equation (3.1) to be transformed to form (3.2). The following Theorem proofs that every equation (3.1) can be brought to form (3.2).

Theorem 6. Every equation of the form (3.1) can be transformed to the form (3.2).

Proof of Theorem 6. Let us assume that coefficients of the following equation

$$x^4 + Px^3 + Qx^2 + Rx + S = 0 \tag{3.7}$$

do not satisfy condition (3.6).

Substituting in equation (3.7), we obtain

$$y^4+(4m+P)y^3+(6m^2+3mP+Q)y^2+(4m^3+3Pm^2+2Qm+R)y+m^4+Pm^3+Qm^2+Rm+S=0. \tag{3.8}$$

We determine the parameter m such that the coefficients of equation (3.8) satisfy the condition (3.6). Hence

$$(4m^3 + 3Pm^2 + 2Qm + r)^2 - (m^4 + Pm^3 + Qm^2 + Rm + S)(4m + P)^2 = 0$$

implying

$$(P^3-4PQ+8R)m^3+(P^2Q-4Q^2+2PR+16S)m^2+(RP^2+8PS-4QR)m+SP^2-R^2=0. \tag{3.9}$$

Equation (3.7) is a cubic equation of the variable m. Its solution transforms equation (3.8) to an equation of symmetrical form (3.2) of the variable y. The proof is completed.

Theorem 6 implies the following method of solving equation (3.1):

If equation (3.7) does not satisfy condition (3.6), then it is transformed first to form (3.8) by the substitution $x = m + y$. The parameter m is determined such that the coefficients of equation (3.8) satisfy condition (3.8). Thus equation (3.9) of m is obtained (the cubic resolvent of equation (3.1)). A value of m is then substituted in (3.8), which yields an equation of form (3.3) of y. Considering (4.5), we calculate the values of a, b and c, and thus obtain an equation of form

$$\left(y + \frac{a}{y}\right)^2 + b \cdot \left(y + \frac{a}{y}\right) + c = 0 \tag{3.10}$$

Substituting

$$y + \frac{a}{y} = t \tag{3.11}$$

in equation (3.10) we have

$$t^2 + bt + c = 0. \tag{3.12}$$

For two values of t in (3.12) we obtain four values of y in (3.11), and substituting them in $x = m + y$, we obtain

$$x_1 = m + y_1, x_2 = m + y_2, x_3 = m + y_3, x_4 = m + y_4. \tag{3.13}$$

Finally, (3.13) are the solutions of equation (3.7).

Remark 1. When $p = 0$ after elementary transformations $x = y + m$ the equation is transformed to the case (3.1), which can be solved, using previous theorem.

Remark 2. This method is especially efficient when all the coefficients of equation (2.1) are non-zero.

Remark 3. If in resolvent (3.9) holds $P^3 - 4PQ + 8R = 0$, then (3.9) is a quadratic equation with two equal real roots. For these values of m, (3.8) is a biquadratic equation of y, namely

$$y^4 + \alpha y^2 + \beta = 0.$$

Example 7. Let us solve the equation

$$x^4 + x^3 + 2x^2 - 5x + 1 = 0$$

The coefficients are $p = 5, q = 2, r = -5$ and $s = 1$.Obviously, condition (3.6) is satisfied. By (5) we have $a = -1, b = 5$ and $c = 4$. Substituting the values of a, b and c in (2.2), we obtain

$$\left(x - \frac{1}{x}\right)^2 + 5 \cdot \left(x - \frac{1}{x}\right) + 4 = 0 \tag{3.14}$$

Substituting $x - \frac{1}{x} = t$ in equation (3.14), we have

$$t^2 + 5t + 4 = 0 \tag{3.15}$$

and

$$x^2 - tx - 1 = 0 \tag{3.16}$$

The solutions of equation (3.15) are $t_1 = -1$ and $t_2 = -4$. Substituting these values in (3.16), we have

$$x^2 + x - 1 = 0 \tag{3.17}$$

and

$$x^2 + 4x - 1 = 0. \tag{3.18}$$

The solutions of equation (3.17) are $x_1 = \dfrac{-1 + \sqrt{5}}{2}$ and $x_2 = -\dfrac{1 + \sqrt{5}}{2}$ and solutions of (3.18) are $x_3 = -2 + \sqrt{5}, x_4 = -2 - \sqrt{5}$.

The values x_1, x_2, x_3 and x_4 are the required solutions.

Example 8. Let us solve the equation

$$x^4 - 4x^2 + 16x + 32 = 0.$$

The coefficients of this equation are $p = 0, q = -4, r = 16, s = 32$. Obviously, condition (3.6) is not satisfied. Substituting $x = m + y$, the equation is transformed to form (3.8), namely

$$y^4 + 4my^3 + (6m^2 - 4)y^2 + (4m^3 - 8m + 16)y + m^4 - 4m^2 + 16m + 32 = 0. \tag{3.19}$$

We calculate the value of parameter m in order that the coefficients of equation (3.19) satisfy condition (3.6). Thus

$$3m^3 + 7m^2 + 4m - 4 = 0. \tag{3.20}$$

The solutions of equation (3.20) are $m_{1,2} = -2, m_3 = \dfrac{1}{2}$. We substitute the value $m = \dfrac{1}{2}$ in equation (3.19), and obtain

$$y^4 + 2y^3 - \frac{5}{2}y^2 + \frac{25}{2}y + \frac{625}{16} = 0. \tag{3.21}$$

The coefficients of equation (3.21) satisfy condition (3.6), and by (3.5) we have $a = \frac{25}{4}, b = 2, c = -15$, which values we substitute in (3.7), obtaining

$$\left(y + \frac{25}{4y}\right)^2 + 2 \cdot \left(y + \frac{25}{4y}\right) - 15 = 0. \tag{3.22}$$

By (3.8) and (3.9) we have

$$y + \frac{25}{4y} = t \tag{3.23}$$

and

$$t^2 + 2t - 15 = 0. \tag{3.24}$$

From (3.23) we obtain the following equation

$$4y^2 - 4ty + 25 = 0. \tag{3.25}$$

The solutions of equation (3.24) are $t_1 = 3, t_2 = -5$. . Substituting these values of the variable t in (3.22), we have the following equations

$$4y^2 - 12y + 25 = 0 \tag{3.26}$$

and

$$4y^2 + 20y + 25 = 0 \tag{3.27}$$

The solutions of equation (3.26) are $y_1 = \frac{3}{2} + 2i, y_2 = \frac{3}{2} - 2i$, and those of (3.27) are $y_3 = y_4 = -\frac{5}{2}$. Substituting these values of y in (3.10), we obtain $x_{1,2} = 2 \pm 2i, x_{3,4} = -2$.

Remark 4. We can see that this equation has two solutions identical with these of resolvent (3.20). Of course, we could have solve this equation by dividing with $(x+2)(x+2) = x^2+4x+4$, i.e by factoring.

Example 9. Let us find the solutions of the equation

$$x^4 + 2x^3 + 3x^2 + 2x - 6 = 0.$$

The coefficients of this equation are $p = 2, q = 3, r = 2, s = -6$. Obviously, condition (3.6) is not satisfied. Substituting $x = m + y$, the equation is transformed to form (3.8), that is

$$y^4+(4m+2)y^3+(6m^2+6m+3)y^2+(4m^3+6m^2+6m+2)y+m^4+2m^3+3m^2+2m-6 = 0. \tag{3.28}$$

We calculate the value of parameter m in order that the coefficients of equation (3.25) satisfy condition (3.6). Whence

$$4m^2 + 4m + 1 = 0.$$

This equation has two equal real solutions $m = -\frac{1}{2}$. As we noted before, for the value equation (3.28) is transformed to a biquadratic equation of y, namely

$$y^4 + \frac{3}{2}y^2 - \frac{103}{16} = 0. \tag{3.29}$$

Substituting $y^2 = t$ in equation (3.29), we have

$$t^2 + \frac{3}{2}t - \frac{103}{16} = 0. \tag{3.30}$$

The solutions of equation (3.30) are

$$t_1 = -\frac{3}{4} + \sqrt{7}, t_2 = -\Big(\frac{3}{4} + \sqrt{7}\Big).$$

Substituting these values of t_1, t_2 in $y^2 = t$, we have

$$y^2 = -\frac{3}{4} + \sqrt{7} \tag{3.31}$$

and

$$y^2 = -\Big(\frac{3}{4} + \sqrt{7}\Big) \tag{3.32}$$

By (3.31) and (3.32) we have

$$y_1 = \sqrt{-\frac{3}{4} + \sqrt{7}}, y_2 = -\sqrt{-\frac{3}{4} + \sqrt{7}}, y_3 = i\sqrt{\frac{3}{4} + \sqrt{7}}, y_4 = -i\sqrt{\frac{3}{4} + \sqrt{7}}.$$

Substituting these values of y_1, y_2, y_3, y_4 and m in $x = m + y$, we obtain

$$x_1 = -\frac{1}{2} + \sqrt{-\frac{3}{4} + \sqrt{7}}, x_2 = -\frac{1}{2} - \sqrt{-\frac{3}{4} + \sqrt{7}},$$

$$x_3 = -\frac{1}{2} + i\sqrt{\frac{3}{4} + \sqrt{7}}, x_4 = -\frac{1}{2} - i\sqrt{\frac{3}{4} + \sqrt{7}},$$

which are the required solutions.

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On the mean value of the function $(\frac{\varrho(n)}{\phi(n)})^r$ [1]

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Abstract An integer a is called regular (mod n) if there is an integer x such that $a^2x \equiv a(\text{mod } n)$. Let $\varrho(n)$ denote the number of regular integers $a(\text{mod } n)$ such that $1 \leq a \leq n$, $\phi(n)$ is the Euler function. In this paper we investigate the mean value of the function $(\frac{\varrho(n)}{\phi(n)})^r$, where $r \geq 1$ is a fixed integer.

Keywords Regular integers (mod n), Euler's function, average order, convolution method, Euler product.

§1. Introduction

Let $n > 1$ be an integer. Consider the integers a for which there exists an integer x such that $a^2x \equiv a(\text{mod } n)$. Properties of these integers were investigated by J. Morgado [1], [2], who called them regular (mod n).

Let $Reg_n = \{a : 1 \leq a \leq n, \text{ a is regular (mod } n)\}$ and let $\varrho(n) = \sharp Reg_n$ denote the number of regular integers a (mod n) such that $1 \leq a \leq n$. This function is multiplicative and $\varrho(p^v) = \phi(p^v) + 1 = p^v - p^{v-1} + 1$ for every prime power $p^v (v \geq 1)$, where ϕ is the Euler function.

László Tóth [3] proved that

$$\sum_{n \leq x} \frac{\varrho(n)}{\phi(n)} = Bx + O(\log^2 x), \tag{1.1}$$

where $B = \frac{\pi^2}{6} \approx 1.6449$.

Let $r \geq 1$ be a fixed integer. The aim of the short paper is to establish the following asymptotic formula for the mean value of the function $\left(\frac{\varrho(n)}{\phi(n)}\right)^r$, which generalizes (1.1).

Theorem. Suppose $r \geq 1$ is a fixed integer, then

$$\sum_{n \leq x} \left(\frac{\varrho(n)}{\phi(n)}\right)^r = C_r x + O(\log^{2r} x), \tag{1.2}$$

where C_r is a constant.

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§2. Proof of the theorem

In order to prove our theorem, we need the following Lemmas, which can be found in Ivić [4]. From now on, suppose $\zeta(s)$ denotes the Riemann-zeta function.

Lemma 1. Suppose $t \geq 2$, then uniformly for σ we have

$$\zeta(\sigma+it) \ll \begin{cases} 1 & \text{for } \sigma \geq 2, \\ \log t & \text{for } 1 \leq \sigma \leq 2, \\ t^{(1-\sigma)/2}\log t & \text{for } 0 \leq \sigma \leq 1, \end{cases}$$

$$\zeta^{-1}(\sigma+it) \ll \begin{cases} 1 & \text{for } \sigma \geq 2, \\ \log t & \text{for } 1 \leq \sigma \leq 2. \end{cases}$$

Lemma 2. There exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for $\sigma > 1 - c/\log(|t|+2)$.

Proof of the Theorem.

Let $f(s) := \sum_{n=1}^{\infty} \frac{(\frac{\varrho(n)}{\phi(n)})^r}{n^s}, Res > 1$. It is easy to see that $(\frac{\varrho(n)}{\phi(n)})^r$ is multiplicative, so by the Euler product formula, for $Res > 1$ we have

$$\begin{aligned}
f(s) &= \prod_p (1 + \frac{(\frac{\varrho(p)}{\phi(p)})^r}{p^s} + \frac{(\frac{\varrho(p^2)}{\phi(p^2)})^r}{p^{2s}} + \cdots) \\
&= \prod_p (1 + \frac{(1+\frac{1}{p-1})^r}{p^s} + \frac{(1+\frac{1}{p^2-p})^r}{p^{2s}} + \cdots) \\
&= \prod_p (1 + \frac{(1+\frac{1}{p}(1+\frac{1}{p}+\cdots))^r}{p^s} + \frac{(1+\frac{1}{p^2}(1+\frac{1}{p}+\cdots))^r}{p^{2s}} + \cdots) \\
&= \prod_p (1 + \frac{1}{p^s} + \frac{\frac{r}{p}(1+\frac{1}{p}+\cdots)}{p^s} + \cdots + \frac{1}{p^{2s}} + \frac{\frac{r}{p^2}(1+\frac{1}{p}+\cdots)}{p^{2s}} + \cdots) \\
&= \zeta(s)\prod_p (1-\frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{\frac{r}{p}(1+\frac{1}{p}+\cdots)}{p^s} + \cdots \\
&\quad + \frac{1}{p^{2s}} + \frac{\frac{r}{p^2}(1+\frac{1}{p}+\cdots)}{p^{2s}} + \cdots) \\
&= \zeta(s)\prod_p (1 + \frac{r}{p^{s+1}} + \frac{r}{p^{s+2}} + \cdots - \frac{r}{p^{2s+1}} - \frac{r}{p^{2s+2}} - \cdots) \\
&= \zeta(s)\zeta^r(s+1)\prod_p (1-\frac{1}{p^{s+1}})^r (1 + \frac{r}{p^{s+1}} + \frac{r}{p^{s+2}} + \cdots - \frac{r}{p^{2s+1}} - \cdots) \\
&= \zeta(s)\zeta^r(s+1)\prod_p (1 - \frac{r}{p^{2s+1}} + \frac{r}{p^{s+2}} - \cdots) \\
&= \frac{\zeta(s)\zeta^r(s+1)}{\zeta^r(2s+1)}\prod_p (1-p^{-2s-1})^{-r}(1 - \frac{r}{p^{2s+1}} + \frac{r}{p^{s+2}} - \cdots) \\
&= \frac{\zeta(s)\zeta^r(s+1)}{\zeta^r(2s+1)}\prod_p (1 + \frac{r}{p^{s+2}} + \cdots). \qquad (1)
\end{aligned}$$

Write

$$V(s,r) =: \prod_p (1 + \frac{r}{p^{s+2}} + \cdots), Res > 1.$$

It is easy to check that the Dirichlet series $\sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $Res \geq -\frac{1}{10}$. So we have

$$\sum_{n\leq x} |v(n)| \ll 1, \ \sum_{n\leq x} |v(n)| n^{\frac{1}{10}} \ll x^{\varepsilon}, \tag{2}$$

where ε is a small positive real number.

Let $\frac{\zeta^r(s+1)}{\zeta^r(2s+1)} = \sum_{l=1}^{\infty} \frac{b_r(l)}{l^s}$, then according to the Dirichlet convolution, we obtain

$$\begin{aligned}
\sum_{n\leq x} (\frac{\varrho(n)}{\phi(n)})^r &= \sum_{mkl\leq x} b_r(l) v(k) \\
&= \sum_{k\leq x} v(k) \sum_{l\leq x/k} b_r(l) \sum_{m\leq x/kl} 1 \\
&= \sum_{k\leq x} v(k) \sum_{l\leq x/k} b_r(l) (\frac{x}{kl} + O(1)) \\
&= x \sum_{k\leq x} \frac{v(k)}{k} \sum_{l\leq x/k} \frac{b_r(l)}{l} + O(\sum_{k\leq x} |v(k)| \sum_{l\leq x/k} |b_r(l)|).
\end{aligned} \tag{3}$$

So the problem now is reduced to compute $\sum_{l\leq x} b_r(l)$ and $\sum_{l\leq x} |b_r(l)|$.

Similar to the proof of the prime number theorem, with the help of Lemma 1, Lemma 2 and Perron's formula we get

$$\sum_{l\leq x} b_r(l) = 1 + O(e^{-C\sqrt{\log x}}), \tag{4}$$

where $C > 0$ is some positive constant. We omit the proof of (4). By the partial summation, we get from (4) that

$$\sum_{l>x} \frac{b_r(l)}{l} \ll x^{-1}, \tag{5}$$

$$\sum_{l\leq x} \frac{b_r(l)}{l} = \sum_{l=1}^{\infty} \frac{b_r(l)}{l} - \sum_{l>x} \frac{b_r(l)}{l} = C_1 + O(x^{-1}). \tag{6}$$

Now we go on to bound the sum $\sum_{l\leq x} |b_r(l)|$. Since for $Res > 1$,

$$\begin{aligned}
\sum_{l=1}^{\infty} \frac{b_r(l)}{l^s} &= \frac{\zeta^r(s+1)}{\zeta^r(2s+1)} \\
&= \sum_{m=1}^{\infty} \frac{d_r(m)}{m^{s+1}} \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^{2s+1}} \\
&= \sum_{m,n} \frac{d_r(m)\mu_r(n)}{(mn^2)^s mn},
\end{aligned}$$

where $d_r(m) = \sum_{m=m_1\cdots m_r} 1, \mu_r(n) = \sum_{n=n_1\cdots n_r} \mu(n_1)\cdots\mu(n_r)$, we obtain

$$b_r(l) = \sum_{l=mn^2} \frac{d_r(m)\mu_r(n)}{mn}.$$

So

$$|b_r(l)| \leq \sum_{l=mn^2} \frac{d_r(m)d_r(n)}{mn},$$

which combining the well-known estimate

$$\sum_{n\leq x} d_r(m) \ll x \log^{r-1} x$$

gives

$$\sum_{l\leq x} |b_r(l)| \ll \log^{2r} x. \tag{7}$$

From (3)-(7), we obtain

$$\begin{aligned}
\sum_{n\leq x}(\frac{\varrho(n)}{\phi(n)})^r &= C_1 x \sum_{k\leq x} \frac{v(k)}{k} + O(\sum_{k\leq x} |v(k)| + \sum_{k\leq x} |v(k)| \log^{2r} \frac{x}{k}) \\
&= C_1 x \sum_{k=1}^{\infty} \frac{v(k)}{k} + O(x \sum_{k>x} \frac{|v(k)|}{k} + \sum_{k\leq x} |v(k)| \\
&\quad + \sum_{k\leq x} |v(k)| \log^{2r} \frac{x}{k}) \\
&= C_r x + O(\log^{2r} x) \qquad (8)
\end{aligned}$$

by recalling (2), where $C_r = C_1 \sum_{k=1}^{\infty} \frac{v(k)}{k}$ is a constant.

So our proof of the theorem is completed.

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Euler-Savary formula for the planar homothetic motions

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Abstract The homothetic motion in 2-dimensional Euclidean space E^2, the relation between the velocities of this motion and geometric results for the pole curves were studied in ref [1]. In this paper a canonical relative system of any plane with respect to other planes are given. Therefore in a homothetic motion E/E', Euler-Savary formula giving the relation between the curvature of trajectory curves drawn in the fixed plane E' by the points of the moving plane E is obtained. In the special case of homothetic scale h identically equal to 1, we get the Euler-Savary formula which was given by Muller [5]. Finally some geometrical results are reached using Euler-Savary formula.

Keywords Euler-Savary formula, homothetic motion, kinematics.

§1. Introduction

This study deals with instantaneous geometric plane kinematics. Which is the study, for a certain instant during a continuous motion, of the differential geometric properties? So if we are interested in, for instance, the path of a point we study its tangent, its curvature, and so on. The best way to deal with this subject analytically is to introduce canonical coordinate systems and to make use of the concept of instantaneous invariants. The circumstance of the motion being restricted to a plane simplifies considerably the general theory.

To investigate to geometry of the motion of a line or a point in the motion of plane is important in the study of planar kinematics or planar mechanisms or in physics. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of the mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces, [2]-[4].

Muller considered one and two parameter planar motions and gave the relation $\vec{V}_a = \vec{V}_f + \vec{V}_r$ between these motions' absolute, sliding and relative velocities [5]. Mathematicians had worked widely the curvature problems in the planar motion in the 18th and 19th centuries. At the end of these works the radius of the arc's curvature was calculated by using the Euler-Savary formula

$$\left(\frac{1}{a'} - \frac{1}{a}\right)\sin\alpha = \frac{1}{r'} - \frac{1}{r} = \frac{d\theta}{ds}$$

where the quantities of r and r' denotes the radii of the pole curves (P) and (P'), whereas ds stands for the arc element of the pole curve, $d\theta$ is the infinitesimal rotation angle of the motion. Furthermore a and a' are the distances from the points X and X' to rotation pole P, respectively, [5].

The Euler-Savary theorem is a well-known theorem and studied systematically in two and three dimensional Euclidean space E^2 and E^3 by [5]-[9]. This theorem is used in serious fields of study in engineering and mathematics. For each mechanism type a simple graphical procedure is outlined to determine the circles of inflections and cusps, which are useful to compute the curvature of any point of the mobile plane through the Euler-Savary equation.

Let the coordinate systems of moving plane E and fixed plane E' be $\{O;\vec{e}_1,\vec{e}_2\}$ and $\{O';\vec{e}_1',\vec{e}_2'\}$, respectively. In this case a one-parameter homothetic motion in 2-dimensional Euclidean plane defined by transformation [10]

$$\vec{x}' = h\vec{x} - \vec{u}. \tag{1}$$

In this equation h is homothetic scale and the vector $\overrightarrow{OO'} = \vec{u}$ is a vector connecting the initial point of the systems and the vectors $\vec{X}$, $\vec{X}'$ denote the position vector for the point $X \in E$ with respect to moving and fixed systems, respectively. In addition the relation between the absolute, sliding and relative velocities for one-parameter homothetic motion was expressed by the relation $\vec{V}_a = \vec{V}_f + h\vec{V}_r$, [10].

In this work we have defined canonical relative system of one-parameter planar homothetic motion. With the aid of this system we have obtained the Euler-Savary formula giving the relation between the curvature for the trajectory curves drawn by the points of moving plane E in fixed plane E' in one parameter planar homothetic motion E/E'. Finally we have obtained some geometrical results using Euler-Savary formula.

§2. Moving coordinate systems and theirs velocities

Let E_1 and E be moving planes and E' be a fixed plane. The perpendicular coordinate systems of the planes E_1, E and E' are $\{B;\vec{a}_1,\vec{a}_2\}$, $\{O;\vec{e}_1,\vec{e}_2\}$ and $\{O';\vec{e}_1',\vec{e}_2'\}$, respectively. Therefore, in one-parameter homothetic motion of E_1 with respect to E the following relations are hold

$$\begin{aligned} \vec{a}_1 &= \cos\theta\vec{e}_1 + \sin\theta\vec{e}_2, \\ \vec{a}_2 &= -\sin\theta\vec{e}_1 + \cos\theta\vec{e}_2, \end{aligned} \tag{2}$$

and

$$\overrightarrow{OB} = \vec{b} = b_1\vec{a}_1 + b_2\vec{a}_2, \tag{3}$$

where θ denotes the rotation angle of motion [1]. Similarly, in the one parameter homothetic motion of E_1 with respect to E', the following relations are also

$$\begin{aligned} \vec{a}_1 &= \cos\theta'\vec{e}_1' + \sin\theta'\vec{e}_2', \\ \vec{a}_2 &= -\sin\theta'\vec{e}_1' + \cos\theta'\vec{e}_2', \end{aligned} \tag{4}$$

and

$$\overrightarrow{O'B} = \vec{b}' = b_1'\vec{a}_1 + b_2'\vec{a}_2, \tag{5}$$

where θ' is angle of the motion [1]. From the equations (2)-(3) and (4)-(5) the differential equations for the motions E_1/E and E_1/E' are as follows, respectively [1]

$$d\vec{a}_1 = d\theta\vec{a}_2, \quad d\vec{a}_2 = -d\theta\vec{a}_1, \quad d\vec{b} = (db_1 - b_2 d\theta)\,\vec{a}_1 + (db_2 + b_1 d\theta)\,\vec{a}_2$$

and

$$d'\vec{a}_1 = d\theta'\vec{a}_2, \quad d'\vec{a}_2 = -d\theta'\vec{a}_1, \quad d'\vec{b}' = (db_1' - b_2' d\theta')\,\vec{a}_1 + (db_2' + b_1' d\theta')\,\vec{a}_2.$$

If one chooses

$$\begin{aligned} d\theta &= \lambda, & d\theta' &= \lambda', \\ db_1 - b_2 d\theta &= \sigma_1, & db_2 - b_1 d\theta &= \sigma_2, \\ db_1' - b_2' d\theta' &= \sigma_1', & db_2' - b_1' d\theta' &= \sigma_2', \end{aligned}$$

then the differential equations for E_1/E and E_1/E' become

$$d\vec{a}_1 = \lambda\vec{a}_2, \qquad d\vec{a}_2 = -\lambda\vec{a}_1, \qquad d\vec{b} = \sigma_1\vec{a}_1 + \sigma_2\vec{a}_2 \tag{6}$$

and

$$d'\vec{a}_1 = \lambda'\vec{a}_2, \qquad d'\vec{a}_2 = -\lambda'\vec{a}_1, \qquad d'\vec{b} = \sigma_1'\vec{a}_1 + \sigma_2'\vec{a}_2 \tag{7}$$

respectively. Here the quantities σ_j, σ_j', λ and λ' are called Pfaffian forms of the motion [1].

For the point X with the coordinates of x_1 and x_2 in the plane E_1 the following equations are hold

$$\begin{aligned} \overrightarrow{BX} &= x_1\vec{a}_1 + x_2\vec{a}_2 \\ \vec{x} &= (hx_1 + b_1)\,\vec{a}_1 + (hx_2 + b_2)\,\vec{a}_2 \\ \vec{x}' &= (hx_1 + b_1')\,\vec{a}_1 + (hx_2 + b_2')\,\vec{a}_2. \end{aligned} \tag{8}$$

Thus one obtains

$$d\vec{x} = (dhx_1 + hdx_1 + \sigma_1 - hx_2\lambda)\,\vec{a}_1 + (dhx_2 + hdx_2 + \sigma_2 + hx_1\lambda)\,\vec{a}_2 \tag{9}$$

$$d'\vec{x} = (dhx_1 + hdx_1 + \sigma_1' - hx_2\lambda')\,\vec{a}_1 + (dhx_2 + hdx_2 + \sigma_2' + hx_1\lambda')\,\vec{a}_2 \tag{10}$$

where $\vec{V}_r = \frac{d\vec{x}}{dt}$ and $\vec{V}_a = \frac{d'\vec{x}}{dt}$ are called relative and absolute velocities [1]. From equations (9) and (10) the condition for the point X to be fixed in the planes E and E' was written to be

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma_1 + hx_2\lambda \\ hdx_2 &= -dhx_2 - \sigma_2 + hx_1\lambda \end{aligned} \tag{11}$$

and

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma_1' + hx_2\lambda' \\ hdx_2 &= -dhx_2 - \sigma_2' + hx_1\lambda' \end{aligned} \tag{12}$$

respectively. Substituting equation (1) into equation (10) one reaches the sliding velocity $\vec{V}_f = \frac{d_f\vec{x}}{dt}$ to be

$$d_f\vec{x} = [(\sigma_1' - \sigma_1) - hx_2\,(\lambda' - \lambda)]\,\vec{a}_1 + [(\sigma_2' - \sigma_2) + hx_1\,(\lambda' - \lambda)]\,\vec{a}_2 \tag{13}$$

Therefore the pole point $P = (p_1, p_2)$ of the motion is, [1].

$$x_1 = p_1 = -\tfrac{\sigma_2' - \sigma_2}{h(\lambda' - \lambda)}, \qquad x_2 = p_2 = \tfrac{\sigma_1' - \sigma_1}{h(\lambda' - \lambda)} \tag{14}$$

§3. Canonical relative system and Euler-Savary formula

Now we choose the relative system $\{B;\vec{a}_1,\vec{a}_2\}$ proving the following conditions:

(i) Let the initial B of the system coincides with the instantaneous rotation P , i.e. $B = P$.

(ii) Let the axis $\{B;\vec{a}_1\}$ coincides with the tangent of the pole, i.e. with the common tangent of the pole curves (P), (P'). (see figure 1).

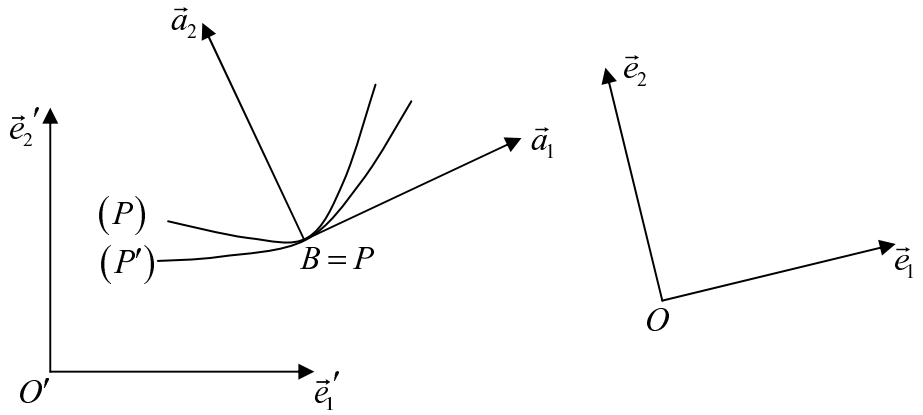

Figure1. Pole curves of (P) and (P')

Therefore, from the condition (i) we see that the coordinates of the rotation pole are $p_1 = p_2 = 0$. Thus, from equation (14) we reach that $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$. From these results we reach that

$$d\vec{b} = d\vec{p} = \sigma_1\vec{a}_1 + \sigma_2\vec{a}_2 = d'\vec{p} = d'\vec{b}.$$

We have, therefore, given the tangent of the pole and constructed the rolling for the pole curves (P) and (P'). If we consider condition (ii), then we see that $\sigma_2 = \sigma_2' = 0$. With the help of equations (6) and (7) we get the following equations for the differential equations related to the canonical relative system $\{P;\vec{a}_1,\vec{a}_2\}$ of the plane denoted by E_{1p}

$$d\vec{a}_1 = \lambda\vec{a}_2, \quad d\vec{a}_2 = -\lambda\vec{a}_1, \quad d\vec{p} = \sigma\vec{a}_1 \tag{15}$$

and

$$d'\vec{a}_1 = \lambda'\vec{a}_2, \quad d'\vec{a}_2 = -\lambda'\vec{a}_1, \quad d'\vec{p} = \sigma\vec{a}_1 \tag{16}$$

where we have chosen $\sigma_1 = \sigma_1' = \sigma$ for brevity. In the last equations $\sigma = ds$ is the scalar arc element for (P) and (P'); λ is cotangent angle, i.e. the angle between two neighboring tangents of (P). Thus, the curvature of pole (P) at the point P is $\frac{\lambda}{\sigma} = \frac{d\theta}{ds}$. Similarly, λ', cotangent angle of (P') and the curvature of pole curve (P') at the point (P) is $\frac{\lambda'}{\sigma} = \frac{d\theta'}{ds}$.

Therefore, $r = \frac{\sigma}{\lambda}$ and $r' = \frac{\sigma}{\lambda'}$ indicate the curvature radii for the pole curves (P) and (P'), respectively.

Moving plane E rotates the infinitesimal instantaneous angle of $d\Phi = \lambda' - \lambda$ around the rotation pole P within the time scale dt with respect to fixed plane E'. Therefore, the angular

velocity of rotational motion of E with respect to E' becomes

$$\frac{\lambda' - \lambda}{dt} = \frac{d\Phi}{dt} = \dot{\Phi}.$$

Let us suppose that the direction of the unit tangent vector $\vec{a}_1$ is same as direction of pole curves (P) and (P') $\left(\text{i.e.}\frac{ds}{dt} > 0\right)$. In this case curvature radii become $r > 0$ and $r' > 0$.

Now, we search the velocity of point X with the coordinates x_1 and x_2 with respect to canonical relative system. Considering equations (9) and (10) we write

$$d\vec{x} = (dhx_1 + hdx_1 + \sigma - hx_2\lambda)\,\vec{a}_1 + (dhx_2 + hdx_2 + hx_1\lambda)\,\vec{a}_2 \tag{17}$$

$$d'\vec{x} = (dhx_1 + hdx_1 + \sigma - hx_2\lambda')\,\vec{a}_1 + (dhx_2 + hdx_2 + hx_1\lambda')\,\vec{a}_2. \tag{18}$$

Therefore the condition for X to be fixed in E is

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma + hx_2\lambda \\ hdx_2 &= -dhx_2 + hx_1\lambda \end{aligned} \tag{19}$$

whereas in E' it is

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma + hx_2\lambda' \\ hdx_2 &= -dhx_2 + hx_1\lambda'. \end{aligned} \tag{20}$$

From these considerations we reach that the sliding velocity $\vec{V}_f$ of the motion is

$$d_f\vec{x} = h\,(-x_2\vec{a}_1 + x_1\vec{a}_2)\,(\lambda' - \lambda)\,.$$

Let us consider a point X which is on the moving plane E. This point X draws a trajectory on fixed plane E' during one-parameter planar homothetic motion E/E'. Now considering the canonical relative system we would like to find the curvature centre X' at the time t for this trajectory. Thus, the points X and X' have the coordinates (x_1, x_2) and (x_1', x_2') in the canonical relative system and they stay on a line (i.e. a instantaneous trajectory normal related to X) with instantaneous rotation pole P. In general a curvature centre at a point of any planar curve stays on the normal at this point. Furthermore, this curvature center is thought to be the limit of the meeting point of the two neighboring point that are on curve. (see figure 2).

Therefore the vectors $\overrightarrow{PX} = x_1\vec{a}_1 + x_2\vec{a}_2, \quad \overrightarrow{PX'} = x_1'\vec{a}_1 + x_2'\vec{a}_2$

have the same direction at the point P. Hence for the coordinates (x_1, x_2) and (x_1', x_2') of X and X' we write

$$x_1x_2' - x_2x_1' = 0. \tag{21}$$

Differentiating the equation (20) gives us

$$dx_1x_2' + x_1dx_2' - dx_1'x_2 - x_1'dx_2 = 0. \tag{22}$$

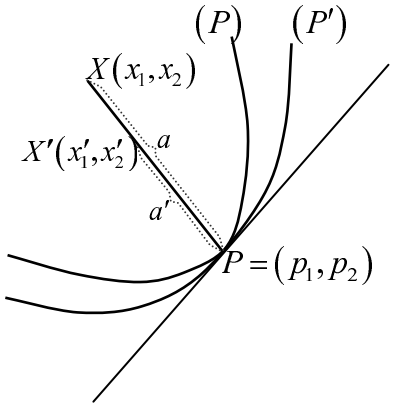

Figure 2. The vectors $\overrightarrow{PX}$ and $\overrightarrow{PX'}$

The condition of being fixed of X in the plane E had given in equations (19). Moreover, the condition of being fixed of X' in the plane E' is

$$\begin{aligned} hdx_1' &= -dhx_1' - \sigma + hx_2'\lambda' \\ hdx_2' &= -dhx_2' + hx_1'\lambda'. \end{aligned} \tag{23}$$

Considering equation (22) with equations (19) and (23), we find

$$(x_2' - x_2) + h\,(x_1x_1' + x_2x_2')\,\tfrac{(\lambda'-\lambda)}{\sigma} = 0. \tag{24}$$

If we switch to the spherical coordinates i.e.
$x_1 = a\cos\alpha, \quad x_2 = a\sin\alpha, \quad x_1' = a'\cos\alpha, \quad x_2' = a'\sin\alpha$
we get $(a'-a)\sin\alpha + haa'\frac{(\lambda'-\lambda)}{\sigma} = 0$. Therefore we obtain $\frac{\lambda'-\lambda}{\sigma} = \frac{d\Phi}{ds}$ and from thee last equation

$$\left(\tfrac{1}{a'} - \tfrac{1}{a}\right)\sin\alpha = h\left(\tfrac{1}{r'} - \tfrac{1}{r}\right) = h\tfrac{d\Phi}{ds}. \tag{25}$$

The last equation is called Euler-Savary formula for the planar homothetic motions.

Thus, we can give the following theorem.

Theorem 1. In one-parameter planar homothetic motion E/E', a point X in the moving plane E draws a trajectory with the instantaneous curvature centre X' in the fixed plane E'. In reverse motion, a point X' in the fixed plane E' draws a trajectory for which the curvature centre is the initial point X in the moving plane E. Interconnection between these two points X and X' is given by the Euler-Savary formula (25).

Special case. In the case of the homothetic scale h identically equal to 1, we get

$$\left(\frac{1}{a'} - \frac{1}{a}\right)\sin\alpha = \left(\frac{1}{r'} - \frac{1}{r}\right) = \frac{d\Phi}{ds}$$

which was given by Muller [5].

Differentiating equation (24) with respect to t and using equations (19) and (23) we reach

$$\left[\tfrac{dh}{h}\,(x_2 - x_2') + (2x_1 + x_1')\,\lambda - (2x_1' + x_1)\,\lambda'\right] + (x_1x_1' + x_2x_2')\,d\left(h\tfrac{d\Phi}{ds}\right) = 0. \tag{26}$$

Switching to polar coordinates we rewrite the last equations as follows

$$\left[\frac{dh}{h}\left(\frac{1}{a'}-\frac{1}{a}\right)\sin\alpha+(2\lambda-\lambda')\frac{\cos\alpha}{a'}-(2\lambda'-\lambda)\frac{\cos\alpha}{a}\right]+d\left(h\frac{d\Phi}{ds}\right)=0. \tag{27}$$

Considering Euler-Savary formula we get

$$\frac{1}{a'}=\frac{1}{a}+\frac{h\left(\lambda'-\lambda\right)}{\sigma\sin\alpha}.$$

Substituting the last equation in to the equation (27) and dividing it to angle $\frac{d\Phi}{ds}=\frac{\lambda'-\lambda}{\sigma}$ we obtain

$$\tfrac{1}{a}=\tfrac{dh}{3\sigma\cos\alpha}+\tfrac{\left(2\lambda-\lambda'\right)h}{3\sigma\sin\alpha}+\tfrac{d\left(h\frac{d\Phi}{ds}\right)}{3\cos\alpha d\Phi}=0. \tag{28}$$

If we choose

$$\frac{\left(2\lambda-\lambda'\right)h}{3\sigma}=A,\quad \frac{dh}{3\sigma}=B,\quad \frac{d\left(h\frac{d\Phi}{ds}\right)}{3d\Phi}=C$$

the equation (28) becomes

$$\tfrac{1}{a}=\tfrac{A}{\sin\alpha}+\tfrac{B+C}{\cos\alpha} \tag{29}$$

Reswitching to the Cartesian coordinate system and considering the last equation we can easily see that

$$x_1x_2=\left[Ax_1+\left(B+C\right)x_2\right]\left(x_1^2+x_2^2\right). \tag{30}$$

If $A\neq 0$, $B+C\neq 0$ then the equation (30) is third order rational equation of a curve. This curve. This is the locus of instantaneous centre for the curvature circles of point X at the moving plane E and denoted by k_3.

In a very similar manner, consider Euler-Savary formula we reach

$$\frac{1}{a}=\frac{1}{a'}-\frac{h\left(\lambda'-\lambda\right)}{\sigma\sin\alpha}.$$

Substituting the last equation into equation (27) and choosing

$$\frac{\left(2\lambda'-\lambda\right)h}{3\sigma}=A',\quad \frac{dh}{3\sigma}=B,\quad \frac{d\left(h\frac{d\Phi}{ds}\right)}{3d\Phi}=C$$

we find

$$\frac{1}{a'}=\frac{A'}{\sin\alpha}+\frac{B+C}{\cos\alpha}.$$

Thus switching to the Cartesian coordinates gives us to third order rational curve

$$x_1'x_2'=\left[Ax_1'+\left(B+C\right)x_2'\right]\left((x_1')^2+(x_2')^2\right). \tag{31}$$

This curve is the locus of the instantaneous centres for the curvature circles of point X' in the fixed plane E' and denoted by k_3'.

To sum, we can easily see that the locus of the curve k_3' in the plane E' and the curve k_3 in the plane E are similar. These two curves are tangents at two fold point through the pole tangent and the tangents at the three fold point through the pole curve. Hence the curves k_3 and k_3' are called circle-pointed and central pointed curve, respectively.

Therefore we can give the following theorem.

Theorem 2.

(i) The points drawing the curvature centers of the pole trajectories form a third order rational curve at time t in the moving plane E .

(ii) This curve crosses from the absolute point of plane E.

(iii) Pole tangent at the point P has a dual point with the pole normal.

(iv) The locus of the centres of the curvature circles X in the plane E is a curve of same type.

Conclusion

What is the relation between the curvatures of polar curves have been widely studied by mathematicians during the eighteenth and nineteenth centuries in the Euclidean space. Today the relation of the curvatures is called as the Euler-Savary's formula. The Euler-Savary equation is used in a consistent manner and a series of kinematic inversions are investigated. In addition the direct graphical technique can be used and applied to the analysis and synthesis of planar mechanisms in general. In this paper, Euler-Savary formula giving the relation between the curvature of trajectory curves drawn in the fixed plane E' by the points of the moving plane E is obtained in a homothetic motion E/E'. Hence we conclude that if the homothetic scale identically equal to 1, then the results which have been obtained in this paper correspond to the results in [5]. Finally some geometrical results are reached using Euler-Savary formula. This concept will be investigated further in a future paper on curvature theory.

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On the mean value of $\log \varrho(n)$ [1]

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Abstract Let $n > 1$ be an integer, $\varrho(n)$ denote the number of regular integers $m(mod\ n)$ such that $1 \leq m \leq n$. In this paper we shall investigate the mean value of the function $\log \varrho(n)$ by the convolution method.

Keywords Regular integers $(mod\ n)$, mean value, convolution method, Perron's formula.

§1. Introduction and main results

Let $n > 1$ be an integer. Consider the integers m for which there exists an integer x such that $m^2 x \equiv m(mod\ n)$. Let $\varrho(n)=\{m : 1 \leq m \leq n, m \text{ is regular}(mod\ n)\}$. This function is multiplicative and $\varrho(p^\gamma) = \varphi(p^\gamma) + 1 = p^\gamma - p^{\gamma-1} + 1$ for every prime power $p^\gamma (\gamma \geq 1)$, where $\varphi(n)$ is the Euler function (see [1]).

The mean value of the function $\varrho(n)$ was considered in [2], [4]. One has,

$$\lim_{x\to\infty} \frac{1}{x^2} \sum_{n\leq x} \varrho(n) = \frac{1}{2} A \approx 0.4407,$$

where $A = \prod_p (1 - \frac{1}{p^2(p+1)}) = \zeta(2) \prod_p (1 - \frac{1}{p^2} - \frac{1}{p^3} + \frac{1}{p^4}) \approx 0.8815$ is the so called quadratic class-number constant.

More exactly, V. S. Joshi [2] proved

$$\sum_{n\leq x} \varrho(n) = \frac{1}{2} A x^2 + R(x), \tag{1}$$

where $R(x) = O(x \log^3 x)$. This was improved into $R(x) = O(x \log^2 x)$ in [3], and into $R(x) = O(x \log x)$ in [6]. The Ω-estimate $R(x) = \Omega_\pm(x\sqrt{\log\log x})$ was also proved in [6].

László Tóth [1] proved the following three results:

$$\sum_{n\leq x} \frac{\varrho(n)}{\varphi(n)} = \frac{3}{\pi^2} x + O(\log^2 x), \tag{2}$$

$$\sum_{n\leq x} \frac{\varphi(n)}{\varrho(n)} = Bx + O((\log x)^{5/3} (\log\log x)^{4/3}), \tag{3}$$

$$\sum_{n\leq x} \frac{1}{\varrho(n)} = C_1 \log x + C_2 + O(\frac{\log^9 x}{x}), \tag{4}$$

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where C_1 and C_2 are constants,

$$C_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_p (1 - \frac{p(p-1)}{p^2-p+1} \sum_{\gamma=1}^{\infty} \frac{1}{p^\gamma - p^{\gamma-1}+1}).$$

In this paper, we shall prove a result about the mean value of $\log \varrho(n)$. Our main result is the following

Theorem. We have

$$\sum_{n \le x} \log \varrho(n) = x \log x + Ex + O(x^{1/2} \log^{3/2} x), \tag{5}$$

where

$$E = \sum_p (1-p^{-1}) \sum_{\alpha=2}^{\infty} p^{-\alpha} \log(1 - p^{-1} + p^{-\alpha}).$$

Notations. Throughout this paper, $\varepsilon > 0$ denotes a small positive constant.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas, which can be found in Ivić [5]. From now on, $\zeta(s)$ denotes the Riemann-zeta function.

Lemma 1. Let $T \ge 2$ be a real number , then we have

$$\int_0^T |\ \zeta(\frac{1}{2} + it)\ |^2\ dt = T \log T + (2\gamma - 1 - \log 2\pi)T + O(T^{1/2}),$$

where γ is the Euler constant.

Lemma 2. For $t \ge t_0 \ge 2$, we have uniformly for σ that

$$\zeta(\sigma + it) \ll \begin{cases} \log t, & \text{for } 1 \le \sigma \le 2, \\ t^{(1-\sigma)/2} \log t, & \text{for } 0 \le \sigma \le 1. \end{cases}$$

Proof of Theorem.

Let $f(n) := \varrho^u(n)$, where u is a fixed complex number with $|\ u\ | \le \frac{1}{4}$. Then for every prime power p^α,

$$f(p^\alpha) = \varrho^u(p^\alpha) = (p^\alpha - p^{\alpha-1} + 1)^u = p^{\alpha u}(1 - p^{-1} + p^{-\alpha})^u. \tag{6}$$

Since $f(n)$ is multiplicative, by the Euler product we get for $\Re s > 1$ that

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) n^{-s} &= \prod_p (1 + \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)}{p^{\alpha s}}) \\ &= \prod_p (1 + \frac{p^u}{p^s} + \sum_{\alpha=2}^{\infty} \frac{p^{\alpha u}(1 - p^{-1} + p^{-\alpha})^u}{p^{\alpha s}}) \\ &= \zeta(s-u) \prod_p (1 - \frac{p^u}{p^s})(1 + \frac{p^u}{p^s} + \sum_{\alpha=2}^{\infty} \frac{p^{\alpha u}(1 - p^{-1} + p^{-\alpha})^u}{p^{\alpha s}}) \\ &= \zeta(s-u) G(s,u), \end{aligned} \tag{7}$$

where

$$G(s,u)=\prod_p(1-\frac{p^u}{p^s})(1+\frac{p^u}{p^s}+\sum_{\alpha=2}^{\infty}\frac{p^{\alpha u}(1-p^{-1}+p^{-\alpha})^u}{p^{\alpha s}}). \tag{8}$$

Write $G(s,u)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ $(\Re s>1)$. It is easy to see that this infinite series is absolutely convergent in the range $\Re s>\Re u$, which implies that

$$G(s,u)=O\varepsilon(1),\qquad \Re s\geq\Re u+\varepsilon. \tag{9}$$

Taking $T=x^2$. By Perron's formula we obtain

$$\sum_{n\leq x}\varrho^u(n)=\frac{1}{2\pi i}\int_{1+\Re u+\varepsilon-iT}^{1+\Re u+\varepsilon+iT}\zeta(s-u)G(s,u)\cdot\frac{x^s}{s}ds+O(\frac{x^{1+\Re u+\varepsilon}}{T}).$$

Then by the residue theorem we obtain

$$\sum_{n\leq x}\varrho^u(n)=\frac{G(1+u,u)}{1+u}\cdot x^{1+u}+\int_1+\int_2+\int_3+O(\frac{x^{1+\Re u+\varepsilon}}{T}), \tag{10}$$

where

$$\int_1=\frac{1}{2\pi i}\int_{1+\Re u+\varepsilon-iT}^{\frac{1}{2}+\Re u-iT}\zeta(s-u)G(s,u)\cdot\frac{x^s}{s}ds, \tag{11}$$

$$\int_2=\frac{1}{2\pi i}\int_{\frac{1}{2}+\Re u-iT}^{\frac{1}{2}+\Re u+iT}\zeta(s-u)G(s,u)\cdot\frac{x^s}{s}ds, \tag{12}$$

$$\int_3=\frac{1}{2\pi i}\int_{\frac{1}{2}+\Re u+iT}^{1+\Re u+\varepsilon+iT}\zeta(s-u)G(s,u)\cdot\frac{x^s}{s}ds. \tag{13}$$

Denote $s=\sigma+it, u=\Re u+iv$, where $\mid u\mid\leq\frac{1}{4}$. Form (9) we get

$$\begin{aligned}\int_2 &= \frac{1}{2\pi}\int_{-T}^{T}\zeta(\frac{1}{2}+i(t-v))G(\frac{1}{2}+\Re u+it,u)\frac{x^{\frac{1}{2}+\Re u+it}}{\frac{1}{2}+\Re u+it}dt\\ &\ll x^{\frac{1}{2}+\Re u}\int_0^{T+|v|}\mid\zeta(\frac{1}{2}+it)\mid\frac{1}{\sqrt{(1/2+\Re u)^2+(t+v)^2}}dt\\ &\ll x^{\frac{1}{2}+\Re u}\int_0^{T}\mid\zeta(\frac{1}{2}+it)\mid\frac{1}{1+t}dt.\end{aligned} \tag{14}$$

With the partial summation and (14), we obtain

$$\begin{aligned}\int_2 &\ll x^{\frac{1}{2}+\Re u}\int_0^T\frac{1}{1+t}dB(t)\\ &\ll x^{\frac{1}{2}+\Re u}\left(\frac{B(T)}{T}+\int_0^T\frac{B(t)}{(1+t)^2}dt\right).\end{aligned} \tag{15}$$

where $B(t)=\int_0^t\mid\zeta(\frac{1}{2}+iw)\mid dw$.

With Lemma 1 and Cauchy's inequality, we get (for $t\geq 2$)

$$\begin{aligned}B(t) &\ll \left(\int_0^t 1dw\right)^{1/2}\left(\int_0^t\mid\zeta(\frac{1}{2}+iw)\mid^2dw\right)^{1/2}\\ &\ll t\log^{1/2}t.\end{aligned} \tag{16}$$

With (15) and (16), we get

$$\int_2 \ll x^{\frac{1}{2}+\Re u} \log^{3/2} T \ll x^{\frac{1}{2}+\Re u} \log^{3/2} x \tag{17}$$

With Lemma 2, (9) and (13), we get

$$\begin{aligned}
\int_3 &= \frac{1}{2\pi i} \int_{\frac{1}{2}+\Re u}^{1+\Re u+\varepsilon} \zeta(\sigma+iT-u) G(\sigma+iT,u) \cdot \frac{x^{\sigma+iT}}{\sigma+iT} d\sigma \\
&\ll \int_{\frac{1}{2}+\Re u}^{1+\Re u+\varepsilon} |\, \zeta(\sigma+iT-u)\,| \cdot \frac{x^\sigma}{T} d\sigma \\
&\ll \frac{1}{T}\Big(\int_{\frac{1}{2}+\Re u}^{1} T^{\frac{1-\sigma-\Re u}{2}} \log T \cdot x^\sigma d\sigma + \int_1^{1+\Re u+\varepsilon} \log T \cdot x^\sigma d\sigma\Big) \\
&= \frac{\log T}{T^{\frac{1}{2}(1+\Re u)}} \cdot \int_{\frac{1}{2}+\Re u}^{1} \Big(\frac{x}{\sqrt{T}}\Big)^\sigma d\sigma + \frac{\log T}{T} \int_1^{1+\Re u+\varepsilon} x^\sigma d\sigma.
\end{aligned}$$

Since the integrands in the above integrals above are monotone, we get

$$\int_3 \ll T^{-\frac{1}{2}(1+\Re u)} \log T \cdot \Big(\frac{x}{\sqrt{T}} + \Big(\frac{x}{\sqrt{T}}\Big)^{\frac{1}{2}+\Re u}\Big) + \frac{\log T}{T} x^{1+\Re u+\varepsilon} \ll 1. \tag{18}$$

Similarly, we have

$$\int_1 \ll 1. \tag{19}$$

By (10)-(13) and (17)-(19), we obtain

$$\sum_{n\le x} \varrho^u(n) = \frac{G(1+u,u)}{1+u} \cdot x^{1+u} + O(x^{1/2+\Re u} \log^{3/2} x). \tag{20}$$

By differentiating (20) term by term, we derive

$$\sum_{n\le x} \varrho^u(n) \log \varrho(n) = H'(u) x^{1+u} + H(u) x^{1+u} \log x + O(x^{\frac{1}{2}+\Re u} \log^{\frac{3}{2}} x), \tag{21}$$

where $H(u) := \frac{G(1+u,u)}{1+u}$.

Letting $u = 0$ in (21), we get

$$\sum_{n\le x} \log \varrho(n) = H(0) x \log x + H'(0) x + O(x^{1/2} \log^{3/2} x). \tag{22}$$

Now we evaluate $H(0)$ and $H'(0)$. According to (8), we have

$$H(u) = \prod_p (1-\frac{1}{p})(1+\frac{1}{p}+\sum_{\alpha=2}^{\infty} \frac{(1-p^{-1}+p^{-\alpha})^u}{p^\alpha}), \tag{23}$$

which implies immediately that $H(0) = 1$.

Taking the logarithm derivative from both sides of (23) we get that

$$\frac{H'(u)}{H(u)} = \sum_p \frac{\sum_{\alpha=2}^{\infty} p^{-\alpha}(1-p^{-1}+p^{-\alpha})^u \log(1-p^{-1}+p^{-\alpha})}{1+\frac{1}{p}+\sum_{\alpha=2}^{\infty} \frac{(1-p^{-1}+P^{-\alpha})^u}{p^\alpha}}, \tag{24}$$

which together with $H(1) = 1$ gives

$$H'(0) = \sum_{p}(1 - p^{-1}) \sum_{\alpha=2}^{\infty} p^{-\alpha} \log(1 - p^{-1} + p^{-\alpha}). \tag{25}$$

Now Theorem follows from (22), (24) and (25).

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Semigroup of continuous functions and Smarandache semigroups [1]

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Abstract The main aim of the following text is to study the semigroup of continuous functions from a topological space to itself under the operation of composition of maps, in the point of view of Smarandache semigroup's approach.

Keywords Continuous map, Smarandache semigroup, topological space.

§1. Introduction

As it has been mentioned [3, page 29], a Smarandache semigroup S is a semigroup which is not a group and has a proper subset A with at least two elements such that A under the operation of S is a group. One may want to restrict the elements of S under particular properties, e.g., one may ask about $\overline{G}$ in X^X where X is a compact Hausdorff topological space and G is a semigroup of continuous functions on X, this case has been studied in [1]. In this paper our main interest is on semigroup of continuous functions on X, i.e., $C(X, X)$.

Let X be a topological space. By $C(X, X)$ we mean the set of all continuous maps like $f : X \to X$, which is clearly a semigroup under the composition of maps. In the next section which is the main section of this paper, we want to study the conditions under which $C(X, X)$ is a Smarandache semigroup.

Remark. If $f : X \to Y$ is a map and $D \subseteq X$, by $f|_D : D \to Y$ we mean the restriction of f to D. Moreover $id_X : X \to X$ denotes the identity function on X, i.e., $id_X(x) = x(x \in X)$.

§2. $C(X, X)$ and Smarandache semigroup's concept

In this section we want to be as close as possible to the cases in which $C(X, X)$ is a Smarandache semigroup (under the composition of maps). From now on suppose X is a topological space with at least two elements and consider $C(X, X)$ under the composition of maps operation (so $C(X, X)$ is a semi-group).

Lemma 2.1. $C(X, X)$ is not a group.

[1] With thanks to Prof. A. Mamourian for his helpful comments, ideas and encouragement.

Proof. Let x_1, x_2 be two distinct elements of X and $f_i(x) = x_i$, then f_1, f_2 are to idempotents of $C(X, X)$ but $f_1, f_2 = id_X$.

Corollary 2.2. $C(X, X)$ contains a group with at least two elements if and only if it is a Smarandache semigroup.

Proof. Use Lemma 2.1.

Theorem 2.3. If X satisfies one of the following conditions, then $C(X, X)$ is a Smarandache semigroup:

(1) There exists a homeomorphism $f : X \to X$ such that $f = id_X$;

(2) There exists nonempty disjoint topological spaces Y and Z such that $X = Y \cup Z$ and X is topological disjoint union of Y and Z, i.e., $\{U \cup V : U$ is an open subset of Y and V is an open subset of $Z\}$ is the topology of X.

Proof. (1) $f_n : n \in Z$ is a subgroup of $C(X, X)$ with at least two elements ($id_X (= f^0)$ and $f (= f^1)$), now use Corollary 2.2.

(2) Let $x_1 \in Y$ and $x_2 \in Z$. Define $g : X \to X$ with $g(x) = x_2$ for $x \in Y$ and $g(x) = x_1$ for $x \in Z$. $g, g \circ g$ is a subset of $C(X, X)$ with two elements and it is a group under the composition of maps. Corollary 2.2 completes the proof.

Counterexample 2.4.

(1) Consider $Y := \mathbb{N}$ with topology $\{\{1, \cdots, n\} : n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\}$ and $Z := \{0\}$ with topology $\{Z, \emptyset\}$. Then $X = Y \cup Z$ as topological disjoint union of Z and Y satisfies item (2) in Theorem 2.3, so $C(X, X)$ is a Smarandache semigroup, however X does not satisfy item (1) in Theorem 2.3.

(2) Consider $X := \digamma$ with topology $\{\{-n, -n+1, \cdots, n\} : n \in \mathbb{N}\} \cup \{X, \emptyset\}$ satisfies item (1) in Theorem 2.3 ($-id_X : X \to X$ is a homeomorphism and $-id_X \neq id_X$) so $C(X, X)$ is a Smarandache semigroup, however X does not satisfy item (2) in Theorem 2.3 since for any two nonempty open subsets U, V of X, $\{0\} \subseteq U \cap V$ and $U \cap V \neq \emptyset$.

(3) Consider $X := \{1, 2\}$ with topology $\{\{1\}, X, \emptyset\}$, then $C(X, X)$ is the set consisting of three elements: id_X, constant function 1, and constant function 2, so it is clear that all of the elements of $C(X, X)$ are idempotents, thus $C(X, X)$ does contain any group with at least two elements and by Corollary 2.2 it is not a Smarandache semigroup.

Corollary 2.5. $C(X \times X, X \times X)$ is a Smarandache semigroup.

Proof. $f : X \times X \to X \times X$ with $f(x, y) = (y, x)((x, y) \in X \times X)$ is a homeomorphism and $f \neq id_{X \times X}$, so by Theorem 2.3, $C(X \times X, X \times X)$ is a Smarandache semigroup.

Note 2.6. Comparing Corollary 2.5 and item (3) in Counterexample 2.4, leads us to the fact that there are cases in which $C(X, X)$ is not a Smarandache semigroup but $C(X \times X, X \times X)$ is a Smarandache semigroup.

Proposition 2.7. Let Y be a topological space. If $C(X, X)$ is a Smarandache semigroup, then $C(X \times Y, X \times Y)$ is a Smarandache semigroup too.

Proof. Since $C(X, X)$ is a Smarandache semigroup, thus there exists a group $G \subseteq C(X, X)$ with more than two elements. For $K := \{(f, id_Y) : f \in G\}$ (where $(f, id_Y) : X \times Y \to X \times Y$ is $(f, id_Y)(x, y) = (f(x), y)$) is a group and subset of $C(X \times Y, X \times Y)$ with more than one element, so by Corollary 2.2, $C(X \times Y, X \times Y)$ is a Smarandache semigroup.

Note 2.8. If X is discrete, then it is topological disjoint union of two spaces, so $C(X, X)$

is a Smarandache semigroup by Theorem 2.3.

Theorem 2.9. If (X, X) is a Smarandache semigroup and $G \subseteq C(X, X)$ with at least two elements is a group, then there exists $W \subseteq X$ with at least two elements such that for all $f \in G$:

(1) $f|_W : Y \to W$ is a homeomorphism.

(2) $f|_W = id_W$ if and only if f is the identity of G.

Proof. Suppose k is the identity of G and $f \in G$. For all $x \in X, k(f(x)) = f(k(x)) = f(x)$, therefore $f(x) \in \{z \in X : k(z) = z\} =: W$. Therefore $f|_W : W \to W$ is a continuous function. It is clear that $k|_W = id_W$. Moreover there exists $g \in G$ such that $f \circ g = g \circ f = k$, also $g|_W : W \to W$ is continuous. Using $f|_W \circ g|_W = g|_W \circ f|_W = k|_W = id_W$ leads us to the fact that $f|_W : W \to W$ is a homeomorphism.

Since $f(X) \subseteq W$, thus $W \neq \emptyset$. We claim that W has at least two elements, otherwise $W = a$, f is constant function a (note that $f(X) \subseteq W = \{a\}$), and it is idempotent, since $f \in G$ is arbitrary, thus all of the elements of G are idempotent, which is a contradiction since G is a group with more than one element and in a group there exists just one idempotent element.

Moreover suppose $f \in G$ is such that $f|_W = id_W$. There exists $g \in G$ such that $f \circ g = k$. In addition we know $g(X) \subseteq W$, thus for all $x \in X$ we have $k(x) = f(g(x)) = f|_W(g(x)) = id_W(g(x)) = g(x)$ which shows $g = k$, thus $k = f \circ g = f \circ k = f$.

Considering Theorem 2.3 and Theorem 2.9 make us to ask:

Problem 2.10. If $C(X, X)$ is a Smarandache semigroup, does at least one of the conditions:

(1) There exists a homeomorphism $f : X \to X$ such that $f = id_X$;

(2) There exists nonempty disjoint topological spaces Y and Z such that $X = Y \cup Z$ and X is topological disjoint union of Y and Z. hold?

§3. A short glance to other topological properties else continuity

In this section we deal with examples of semigroups of resp. close, open, clopen, and proper maps from topological space X to itself, with the operation of composition of maps.

Remark. In topological space X:

(1) $f : X \to X$is called open if for any open subset U of X, $f(U)$ is open;

(2) $f : X \to X$ is called close if for any close subset C of X, $f(C)$ is close;

(3) $f : X \to X$ is called clopen if it is both close and open;

(4) $f : X \to X$ is called proper if for any compact subset K of X, $f^{-1}(K)$ is compact.

Remark. If X has at least two elements, then X^X, the set of all functions $f : X \to X$, is a Smarandache semigroup.

Lemma 3.1. If X has at least two elements, then $S = \left\{f \in X^X : \forall x \in X(f^{-1}(x) \text{ is finite})\right\}$ is a Smarandache semigroup.

Proof. Let a, b be two distinct elements of X and X has at least three elements. Define $f : X \to X$ with $f(x) = a$ for $x = a, b$ and $f(x) = x$ for $x \in X - \{a, b\}$. f and id_X are two different idempotent elements of S, so S is not a group. Define $g : X \to X$ with $g(a) = b$,

$g(b) = a$ and $g(x) = x$for $x \in X - \{a, b\}$, $g, g \circ g$ is a subset of S with two elements and it is a group, thus S is a Smarandache semigroup.

Example 3.2. Let S be the semigroup of all closed maps like $f : X \to X$, under composition of maps:

(i) If X with at least two elements is discrete, then $S = X^X$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z}$ with topology $\{X, \emptyset\} \cup \{X - \{-n, \cdots, -1, 0, 1, \cdots, n\} : n \in \mathbb{N} \cup \{0\}\}$, then X is not discrete and S is a Smarandache semigroup; since:
$-id_X$ and constant function 0 are two different idempotent elements of S, so S is not a group, $-\{id_X, -id_X\}$ is a subgroup of S with two elements;

(iii) If $X = \{1, 2\}$ with topology $\{X, \{1\}, \emptyset\}$, then S has two elements and it is not a Smarandache semigroup.

Example 3.3. Let S be the semigroup of all open maps like $f : X \to X$, under composition of maps:

(i) If X with at least two elements is discrete, then $S = X^X$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z}$ with topology $\{X, \emptyset\} \cup \{\{-n, \cdots, -1, 0, 1, \cdots, n\} : n \in \mathbb{N} \cup \{0\}\}$, then X is not discrete and S is a Smarandache semigroup (use a similar method described in Example 3.2);

(iii) If $X = \{1, 2\}$ with topology $\{X, \{1\}, \emptyset\}$, then S has two elements and it is not a Smarandache semigroup.

Example 3.4. Let S be the semigroup of all clopen maps like $f : X \to X$, under composition of maps:

(i) If X with at least two elements is discrete, then $S = X^X$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z} \cup \pi\mathbb{Z}$ with topological basis $\{\{-n, \cdots, -1, 0, 1, \cdots, n\} \cup A : n \in \mathbb{N} \cup \{0\}, A \subseteq \pi\mathbb{Z}\}$, then X is not discrete and S is a Smarandache semigroup (use a similar method described in Example 3.2);

(iii) If $X = \{1, 2\}$ with topology $\{X, \{1\}, \emptyset\}$, then S has one element and is not a Smarandache semigroup.

Example 3.5. Let S be the set of all proper maps like $f : X \to X$, under composition of maps (use Lemma 3.1):

(i) If X with at least two elements is discrete, then $S = \{f \in X^X : \forall x \in X(f^{-1}(x) \text{ is finite})\}$ is a Smarandache semigroup;

(ii) If $X = \mathbb{Z} \cup \pi\mathbb{Z}$ with topological basis $\{\{-n, \cdots, -1, 0, 1, \cdots, n\} \cup A : n \in \mathbb{N} \cup \{0\}, A \subseteq \pi\mathbb{Z}\}$, then S is a Smarandache semigroup, and X is not discrete (in this case we have $S = \{f \in X^X : \forall x \in X(f^{-1}(x) \text{ is finite})\}$ too).

§4. More examples

In this section suppose G (resp. $\overline{G}$) is a domain (resp. closed domain) in $\mathbb{C}$, also suppose it is bounded with nonempty interior. We have the following examples which deal with product operation (not composition of maps) on complex valued maps with a paticular property.

(1) $C(\overline{G})$: the set of continuous complex valued functions on closed domain $\overline{G}$ [4, page 3]. $C(\overline{G})$ under product operation is a Smarandache semigroup.

(2) $C^m(G)$: the set of continuous complex valued functions on domain G, with continuous partial derivations up to the m-th order [4, page 3]. $C^m(G)$ under product operation is a Smarandache semigroup.

(3) $C_\alpha(\overline{G})$: the set of all bounded functions $f : \overline{G} \to \mathbb{C}$ for closed domain $\overline{G}$ such that there exists $0 < H < +\infty$ with

$$\forall z_1, z_2 \in \overline{G}(|f(z_1) - f(z_2)| \leq H|z_1 - z_2|^\alpha)$$

for $0 < \alpha \leq 1$ [4, page 7]. $C_\alpha(\overline{G})$ under product operation is a Smarandache semigroup.

(4) $C^m_\alpha(\overline{G}) = \left\{ f \in C^m(\overline{G}) : \forall k \in \{0, \cdots, m\} \frac{\partial^m f}{\partial x^{m-k} - \partial y^k} \in C_\alpha(\overline{G}) \right\}$ for $0 < \alpha \leq 1$ and a closed domain $\overline{G}$ [4, page 7]. $C^m_\alpha(\overline{G})$ under product operation is a Smarandache semigroup.

(5) $A(G)$: the set of all analytic functions on G. $A(G)$ under product operation a Smarandache semigroup.

(6) $\ell^\infty(X)$: the set of all bounded functions like $f : X \to C$ [2, Chapter 6] where X has more than one element. $\ell^\infty(X)$ under product operation is a Smarandache semigroup.

(7) $\ell^p(X)$: the set of all functions like $f : X \to C$ where X at least two elements, with $\sum\limits_{x \in X} |f(x)|^p < +\infty$, for fixed $p \in [1, +\infty)$ [2, Chapter 6] where X has more than one element. $\ell^p(X)$ under product operation is a Smarandache group.

Proof. For items $(1), \cdots, (6)$: let S denotes the related semigroup. For $h(x) = 0$, $f(x) = 1$ we have $h, f \in S$ and it is clear that under product operation S is not a group, moreover $f, -f$ is a proper subset of S with two elements which is group under product operation, so S is a Smarandache semigroup.

(7): Suppose x_1, x_2 be two distinct elements of X, for $i = 1, 2$ let $f_i(x_i) = 1$ and $f_i(x) = 0$ for $x \in X - \{x_i\}$, then $0 = f_1 f_2, f_1, f_2 \in \ell^p(X)$ and $\ell^p(X)$ under product operation is not group, but it contains $\{f_1, -f_1\}$ which is a group, so $\ell^p(X)$ under product operation is a Smarandache semigroup.

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On the mean value of the e-squarefree e-divisior function [1]

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Abstract Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n. The aim of the present paper is to establish an asymptotic formula for the mean value of the function $(t^{(e)})^r$, where $r \geq 1$ is a fixed integer.

Keywords E-squarefree e-divisor function, the generalized divisor function, convolution method.

§1. Introduction

An integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an e-divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$ if $b_i | a_i$ for every $i \in \{1, 2, ..., s\}$, notation: $d|_e n$. By convention $1|_e 1$. The integer $n > 1$ is called e-squarefree if all exponents $a_1, \cdots a_s$ are squarefree. The integer 1 is also considered to be e-squarefree.

Consider now the esponential squarefree exponential divisor (e-squarefre e-divisor)of n. Here $d = \prod_{i=1}^{s} p_i^{b_i}$ is an e-squarefree e-divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$, if $b_1 \mid a_1, \ \cdots \ b_s \mid a_s$ and $b_1, \ \cdots \ b_s$ are squarefree. Note that the integer 1 is e-squarefree but is not an e-divisor of $n > 1$.

Let $t^{(e)}(n)$ denote the number of e-squarefree e-divior of n. The function $t^{(e)}$ is called the e-squarefre e-divisor function, which is multiplicative and if $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s} > 1$, then (see [1])

$$t^{(e)}(n) = 2^{\omega(\alpha_1)} \cdots 2^{\omega(\alpha_s)},$$

where $\omega(\alpha) = s$ denotes the number of distinct prime factors of α.

László Tóth [2] proved that the estimate

$$\sum_{n \leq x} t^{(e)}(n) = c_1 x + c_2 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\epsilon}) \tag{1}$$

holds for every $\epsilon > 0$, where

$$c_1 := \prod_p (1 + \sum_{\alpha=6}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)}}{p^{\alpha}}),$$

$$c_2 := \zeta(\frac{1}{2}) \prod_p (1 + \sum_{\alpha=4}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)} - 2^{\omega(\alpha-2)} + 2^{\omega(\alpha-4)}}{p^{\frac{\alpha}{2}}}).$$

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It is very difficult to improve the exponent 1/4 in (1.1) unless there has substantial progress in the study of the zero-free region of the Riemann zeta-function $\zeta(s)$. Therefore it is reasonable to get better improvements by assuming the truth of the Riemann Hypothesis (RH). The following Theorem 1 is such an result.

Theorem 1. Suppose RH is true. Then we have

$$\sum_{n\leq x} t^{(e)}(n) = c_1 x + c_2 x^{\frac{1}{2}} + O(x^{\frac{7}{29}+\epsilon}).$$

Now suppose $r \geq 2$ is a fixed integer. We shall study the mean value of the function $(t^{(e)})^r$, which is closely related to the general divisor function $d_k(n) = \sum_{n=n_1\cdots n_k} 1$, where $k \geq 2$ is a fixed integer. Let $\Delta_k(x)$ denote the error term defined by

$$\Delta_k(x) := \sum_{n\leq x} d_k(n) - xP_{k-1}(\log x),$$

where $P_{k-1}(t)$ is a polynomial of degree $k-1$ in t. Let $\alpha_k > 0$ is a real number such that the estimate

$$\Delta_k(x) \ll x^{\alpha_k+\epsilon} \tag{2}$$

holds. For example, one can take $\alpha_3 = 43/96$, $\alpha_4 = 1/2$, $\alpha_7 = 17/28$, see Chapter 13 of Ivić [4].

We have the following:

Theorem 2. Suppose $r \geq 2$ is a fixed integer, then the asymptotic formula

$$\sum_{n\leq x} (t^{(e)}(n))^r = C_r x + x^{\frac{1}{2}} P_{2^r-2}(\log(x)) + O(x^{\frac{1}{3-\alpha_{2^r-1}}+\epsilon}) \tag{3}$$

for every $\epsilon > 0$, where P_{2^r-2} is a polynomial of degree $2^r - 2$ and C_r is a positive constant.

Corollary. When $r = 2, 3$ we have

$$\sum_{n\leq x} (t^{(e)}(n))^2 = C_2 x + x^{\frac{1}{2}} P_2(\log(x)) + O(x^{\frac{96}{245}+\epsilon}), \tag{4}$$

$$\sum_{n\leq x} (t^{(e)}(n))^3 = C_3 x + x^{\frac{1}{2}} P_6(\log(x)) + O(x^{\frac{28}{67}+\epsilon}). \tag{5}$$

§2. Some lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 1. Let $d(1,2;n) = \sum_{ab^2=n} 1$, then

$$\sum_{n\leq x} d(1,2;n) = \zeta(2)x + \zeta(\frac{1}{2})x^{\frac{1}{2}} + O(x^{\frac{2}{9}+\epsilon}). \tag{6}$$

Proof. See for example, Krätzel [3].

Lemma 2. Suppose RH is true, then we have

$$\sum_{mn^4\leq x} d(1,2;m)\mu(n) = d_1 x + d_2 x^{\frac{1}{2}} + O(x^{\frac{7}{29}+\epsilon}), \tag{7}$$

where $\mu(n)$ is the Möbius function, d_1 and d_2 are computable constants.

Proof. Suppose $1 < y < x^{\frac{1}{4}}$ is a parameter to be determined. Then

$$\begin{aligned}&\sum_{mn^4\le x} d(1,2;m)\mu(n)\\ =&\sum_{mn^4\le x;n\le y} d(1,2;m)\mu(n)+\sum_{mn^4\le x;n>y} d(1,2;m)\mu(n)\\ =&\sum_1+\sum_2.\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}\sum_1 &= \sum_{mn^4\le x;n\le y} d(1,2;m)\mu(n)\\ &= \sum_{n\le y}\mu(n)\sum_{m\le\frac{x}{n^4}} d(1,2;m)\\ &= \sum_{n\le y}\mu(n)(\zeta(2)\frac{x}{n^4}+\zeta(\frac{1}{2})\frac{x^{\frac{1}{2}}}{n^2}+O((\frac{x}{n^4})^{2/9+\epsilon}))\\ &= x\zeta(2)\sum_{n\le y}\frac{\mu(n)}{n^4}+x^{\frac{1}{2}}\zeta(\frac{1}{2})\sum_{n\le y}\frac{\mu(n)}{n^2}+O(x^{2/9+\epsilon}y^{1/9}).\end{aligned}$$

For $\sum_2$, be a familiar argument (see for example, Zhai [5]) we can get

$$\begin{aligned}\sum_2 &= \sum_{mn^4\le x;n>y} d(1,2;m)\mu(n)\\ &= x\zeta(2)\sum_{n>y}\frac{\mu(n)}{n^4}+x^{\frac{1}{2}}\zeta(\frac{1}{2})\sum_{n>y}\frac{\mu(n)}{n^2}+O(\frac{x^{\frac{1}{2}+\epsilon}}{y^{\frac{3}{2}}}).\end{aligned}$$

The above three formulas complete the proof of Lemma 2 by taking $y = x^{\frac{5}{29}}$.

Lemma 3. Suppose $k \ge 2$ is a fixed integer. Then we have

$$\sum_{nm^2\le x} d_k(m) = \zeta^k(2)x + x^{\frac{1}{2}}Q_{k-1}(\log x) + O(x^{\frac{1}{3-\alpha_k}+\epsilon}), \tag{8}$$

where $Q_{k-1}(u)$ is a polynomial of degree $k-1$in u.

Proof. Lemma 3 follows from (2) via the well-known convolution method. See, for example, Chapter 14 of Ivić [4]. We omit the details of the proof.

§3. Proof of theorems

We first prove Theorem 1. By the Euler product formula and the multiplicative property of $t^{(e)}(n)$, we get for $\Re s > 1$

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{t^{(e)}(n)}{n^s} &= \prod_p(1+\sum_{\alpha=1}^{\infty}t^e(n)p^{-\alpha s}) \\
&= \prod_p(1+\sum_{\alpha=1}^{\infty}2^{\omega(\alpha)}p^{-\alpha s}) \\
&= \prod_p(1+p^{-s}+2p^{-2s}+2p^{-3s}+2p^{-4s}+2p^{-5s}+4p^{-6s}+\cdots) \\
&= \frac{\zeta(s)\zeta(2s)}{\zeta(4s)}\prod_p(1+2p^{-6s}+\cdots) \\
&= \frac{\zeta(s)\zeta(2s)}{\zeta(4s)}G(s).
\end{aligned}$$

Write $G(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$. It is easy to see that this Dirichlet series is absolutely convergent for $Res > \frac{1}{6}$. Thus we have

$$\sum_{n\le x}|g(n)| \ll x^{1/6+\epsilon},$$

which combining the convolution approach and Lemma 2 gives

$$\begin{aligned}
\sum_{n\le x}t^{(e)}(n) &= \sum_{ab^4c\le x}d(1,2;a)\mu(b)g(c) \\
&= \sum_{c\le x}g(c)\sum_{ab^4\le\frac{x}{c}}d(1,2;a)\mu(b) \\
&= \sum_{c\le x}g(c)(d_1\frac{x}{c}+d_2(\frac{x}{c})^{\frac{1}{2}}+O((\frac{x}{c})^{\frac{7}{29}+\epsilon})) \\
&= c_1x+c_2x^{\frac{1}{2}}+O(x^{\frac{7}{29}+\epsilon}).
\end{aligned}$$

Now we prove Theorem 2. Let $f(s)=\sum_{n=1}^{\infty}\frac{(t^{(e)}(n))^r}{n^s}, r\ge 2, \Re s>1$. By the Euler product formula and the multiplicative property of $t^e(n)$ again, we have

$$\begin{aligned}
f(s) &= \prod_p(1+\sum_{\alpha=1}^{\infty}(t^e(n))^rp^{-\alpha s}) \\
&= \prod_p(1+\sum_{\alpha=1}^{\infty}(2^{\omega(\alpha)})^rp^{-\alpha s}) \\
&= \prod_p(1+p^{-s}+2^rp^{-2s}+2^rp^{-3s}+2^rp^{-4s}+2^rp^{-5s}+4^rp^{-6s}+\cdots) \\
&= \prod_p(1-p^{-s})^{-1}\prod_p(1-p^{-s})(1+p^{-s}+2^rp^{-2s}+2^rp^{-3s} \\
&\quad +2^rp^{-4s}+2^rp^{-5s}+4^rp^{-6s}+\cdots) \\
&= \zeta(s)\prod_p(1+(2^r-1)p^{-2s}+(4^r-2^r)p^{-6s}-4^rp^{-7s}+\cdots) \\
&= \zeta(s)\zeta^{2^r-1}(2s)V(s,r),
\end{aligned}$$

where $V(s,r)$ can be written as the form

$$\begin{aligned} V(s,r) &= \prod_p (1-p^{-2s})^{2^r-1}(1+(2^r-1)p^{-2s}+(4^r-2^r)p^{-6s}-4^r p^{-7s}+\cdots) \\ &= \prod_p (1+C_r p^{-4s}+C_{r+1}p^{-6s}+\cdots) \end{aligned}$$

($C_r, C_{r+1}, \cdots$ are constants). Write

$$V(s,r)=\sum_{n=1}^{\infty}\frac{v(n)}{n^s}.$$

It is easy to see that this Dirichlet series is absolutely convergent for $Res > 1/4$. Thus

$$\sum_{n\le x}|v(n)| \ll x^{1/4+\epsilon},$$

which combining the convolution approach and Lemma 3 gives

$$\begin{aligned} \sum_{n\le x}(t^{(e)}(n))^r &= \sum_{nm_1^2\cdots m_{2^r-1}^2 b\le x} v(b) \\ &= \sum_{b\le x} v(b) \sum_{nm^2\le x/b} d_{2^r-1}(m) \\ &= \sum_{b\le x} v(b)(\zeta^{2^r-1}(2)\frac{x}{b}+(\frac{x}{b})^{\frac{1}{2}}Q_{2^r-2}(\log x)+O(x^{\frac{1}{3-\alpha_{2^r-1}}+\epsilon})) \\ &= C_r x + x^{\frac{1}{2}}P_{2^r-2}(\log(x))+O(x^{\frac{1}{3-\alpha_{2^r-1}}+\epsilon}). \end{aligned}$$

This completes the proof of our Theorem 2.

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An equation involving the Pseudo-Smarandache function and F. Smarandache LCM function

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Abstract For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as $Z(n) = \min\left\{m:\ m \in N,\ n \mid \frac{m(m+1)}{2}\right\}$. And the F.Smarandache function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. In this paper, we use the elementary methods and congruences to study the solvability of the equation $Z(n) + SL(n) = n$, and give its all positive integer solutions.

Keywords Pseudo-Smarandache function, F. Smarandache LCM function, equation, solution, congruences.

§1. Introduction and results

For any positive integer $n \geq 1$, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $1 + 2 + \cdots + m$ is divisible by n. That is,

$$Z(n) = \min\left\{m:\ m \in N,\ n \mid \frac{m(m+1)}{2}\right\}.$$

In reference [1], Professor F. Smarandache introduced this function, and asked us to study its various properties. About this problem, some authors had studied it, and obtained a series interesting results, see references [1]-[4] and [10]. For example, A.A.K Majumdar [2], [3] and [4] studied this function both theoretically and computationally, and got the following conclusions:

For any prime $p \geq 3$, $Z(p) = p - 1$.

For any prime $p \geq 3$, and $k \in N$, $Z(p^k) = p^k - 1$.

For any $k \in N$, $Z(2^k) = 2^{k+1} - 1$.

On the other hand, for any positive integer n, the famous F. Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where$[1,\ 2,\ \cdots, k]$ denotes the least common multiple of $1,\ 2,\ \cdots, k$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers, then

$$SL(n) = \max\left\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\right\}.$$

About the elementary properties of $SL(n)$, some people had also studied it, and obtained some important conclusions.

In this paper, we use the elementary methods and congruences to study the solvability of the equation $Z(n) + SL(n) = n$, and give its all positive integer solutions. That is, we will prove the following:

Theorem. Let n be a positive integer, then every positive integer solution of the equation $Z(n) + SL(n) = n$ can be expressed as $n = 2^k p^\alpha$, where $p > 2$ be a prime, k and α are positive integers satisfying the following conditions:

1. If $2^k > p^\alpha$, then $p^\alpha \mid (2^k - 1)$.
2. If $2^k < p^\alpha$, then $2^k \mid (p^\alpha - 1)$, $2^{k+1} \nmid (p^\alpha - 1)$.

§2. Proof of the theorem

In this section, we shall use the elementary method and congruences to complete the proof of our Theorem.

It is clear that $n = 6$ is a solution of the equation $Z(n) + SL(n) = n$. Now we suppose that $n = 2^k \cdot s$, where s is an odd integer, we discuss the solutions in following several cases:

(a). If n be an odd integer, then $k = 0$ and $n = s$.

(1) Let $s = 1$, then $Z(1) = 1, SL(1) = 1$. So we can get $Z(1) + SL(1) = 2 \neq 1$.

(2) Let $s = p$, p be an odd prime, then $SL(p) = p,\ Z(p) = p - 1$. So we get

$$Z(p) + SL(p) = 2p - 1 \neq p.$$

(3) Let $s = p^\alpha$, p be an odd prime, α be a positive integer, then $SL(p^\alpha) = p^\alpha,\ Z(p^\alpha) = p^\alpha - 1$. So we get $Z(p^\alpha) + SL(p^\alpha) = 2p^\alpha - 1 \neq p^\alpha$.

(4) Let $s = p^\alpha \cdot p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p,\ p_1,\ p_2,\ \cdots,\ p_r$ are odd primes, α be a positive integer, p^α is the greatest prime power divisor of s. That is,

$$p^\alpha = \max\left\{p^\alpha,\ p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\right\}.$$

Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = t$, then $s = p^\alpha \cdot t$. So $SL(n) = p^\alpha$.

If $Z(n) = n - SL(n) = p^\alpha(t-1)$, according to the definition of $Z(n)$, we have

$$p^\alpha \cdot t \mid \frac{p^\alpha(t-1)[p^\alpha(t-1)+1]}{2}.$$

So that $t \mid (p^\alpha - 1)$. But in this case, if we take $m = p^\alpha - 1$, then we also have $n = p^\alpha \cdot t$ divide $\frac{m(m+1)}{2}$. Note that $p^\alpha - 1 < p^\alpha(t-1)$. So in this case, the equation has also no positive integer solutions.

From the cases (1)-(4) we know that the equation has no odd positive integer solution.

(b). If n be an even integer, then $k \neq 0$.

(1) Let $s = 1$, so $n = 2^k$, then $Z(2^k) = 2^{k+1} - 1,\ SL(2^k) = 2^k$. Hence

$$Z(2^k) + SL(2^k) = 3 \cdot 2^k - 1 \neq 2^k.$$

(2) Let $s = p$, so $n = 2^k \cdot p$, p is an odd prime, while k is a positive integer.

In this case, if $2^k > p$, then $SL(n) = 2^k$, if $Z(n) + SL(n) = n$, then

$$Z(n) = m = n - SL(n) = 2^k p - 2^k = 2^k(p-1).$$

According to the definition of $Z(n)$. We have

$$2^k \cdot p \mid \frac{2^k(p-1)(2^k(p-1)+1)}{2}.$$

So we can get $p \mid (2^k - 1)$. Now we prove that $m = 2^k(p-1)$ is the minimum value which satisfy the definition of $Z(n)$. From the properties of $Z(n)$, we know that $Z(n) \geq 2^{k+1} - 1$, and the possible value of $Z(n)$ between $2^{k+1} - 1$ and $2^{k+1} \cdot \frac{p-1}{2}$ are as follows:

A. $2^{k+1} - 1,\ 2^{k+1} \cdot 2 - 1,\ \cdots,\ 2^{k+1} \cdot \frac{p-1}{2} - 1$.

We mark this group of integers as $2^{k+1} \cdot s_1 - 1, s_1 \in \{1, 2, \cdots, \frac{p-1}{2}\}$.

$$2^{k+1} \cdot s_1 - 1 \equiv 2s_1 - 1(\bmod p).$$

Then we can get $1 \leq 2s_1 - 1 \leq p - 2$. So $p \nmid (2^{k+1} \cdot s_1 - 1)$.

B. $2^{k+1},\ 2^{k+1} \cdot 2,\ \cdots,\ 2^{k+1} \cdot (\frac{p-1}{2} - 1)$.

We mark this group of integers as $2^{k+1} \cdot s_2, s_2 \in \{1, 2, \cdots, \frac{p-1}{2} - 1\}$.

$$2^{k+1} \cdot s_2 \equiv 2s_2(\bmod p).$$

Then we can get $2 \leq 2s_2 \leq p - 3$. So $p \nmid 2^{k+1} \cdot s_2$.

If $2^k < p$, then $SL(n) = p$, and if $Z(n) + SL(n) = n$, then

$$Z(n) = m = n - SL(n) = p(2^k - 1).$$

So we can get

$$n = 2^k \cdot p \mid \frac{p(2^k-1)(p(2^k-1)+1)}{2}.$$

Therefore, $2^{k+1} \mid [(2^k - 1)p + 1]$. That is, $2^k \mid (p-1)$, $2^{k+1} \nmid (p-1)$.

Now we prove that $m = p(2^k - 1)$ is the minimum value which satisfy the definition of $Z(n)$. From the properties of $Z(n)$, we know that $Z(n) \geq p - 1$, and the possible value of $Z(n)$ between $p - 1$ and $p(2^k - 1)$ are as follows:

C. $p - 1,\ p \cdot 2 - 1,\ \cdots,\ p \cdot (2^k - 1) - 1$.

We mark this group of integers as $p \cdot s_1 - 1, s_1 \in \ \{1, 2, \cdots, 2^k - 1\}$.

$$p \cdot s_1 - 1 \equiv s_1 - 1(\bmod 2^k).$$

Then we can get $0 \leq s_1 - 1 \leq 2^k - 2$. When $s_1 - 1 = 0$, so $s_1 = 1$, then we can get $m = p - 1$, $2^{k+1} \mid (p-1)$. Here we obtain contradiction. So $2^k \nmid (p \cdot s_1 - 1)$.

D. $p,\ p \cdot 2,\ \cdots,\ p \cdot (2^k - 2)$.

We mark this group of integers as $p \cdot s_2, s_2 \in \ \{1, 2, \cdots, 2^k - 2\}$.

$$p \cdot s_2 \equiv s_2(\bmod 2^k).$$

Then we can get $1 \leq s_2 \leq 2^k - 2$. So $2^k \nmid p \cdot s_2$.

(3) Let $s = p^\alpha,\ n = 2^k \cdot p^\alpha$, p be an odd prime while k and α are two positive integers.

In this case, if $2^k > p^\alpha$, then $SL(n) = 2^k$. When $Z(n) + SL(n) = n$, we find that $Z(n) = m = n - SL(n) = 2^k(p^\alpha - 1)$. According to the definition of $Z(n)$, we have

$$2^k \cdot p^\alpha \mid \frac{2^k(p^\alpha - 1)(2^k(p^\alpha - 1) + 1)}{2}.$$

So that $p^\alpha \mid (2^k - 1)$.

Now we prove that $m = 2^k(p^\alpha - 1)$ is the minimum value which satisfy the definition of $Z(n)$. From the properties of $Z(n)$, we know that $Z(n) \geq 2^{k+1} - 1$, and the possible value of $Z(n)$ between $2^{k+1} - 1$ and $2^{k+1} \cdot \frac{p^\alpha - 1}{2}$ are as follows:

A. $2^{k+1} - 1,\ 2^{k+1} \cdot 2 - 1,\ \cdots,\ 2^{k+1} \cdot \frac{p^\alpha - 1}{2} - 1$.

We mark this group of integers as $2^{k+1} \cdot s_1 - 1, s_1 \in \{1, 2, \cdots, \frac{p^\alpha - 1}{2}\}$.

$$2^{k+1} \cdot s_1 - 1 \equiv 2s_1 - 1(\bmod\ p^\alpha).$$

Then we can get $1 \leq 2s_1 - 1 \leq p^\alpha - 2$. So $p^\alpha \nmid (2^{k+1} \cdot s_1 - 1)$.

B. $2^{k+1},\ 2^{k+1} \cdot 2,\ \cdots,\ 2^{k+1} \cdot (\frac{p^\alpha - 1}{2} - 1)$.

We mark this group of integers as $2^{k+1} \cdot s_2, s_2 \in \{1, 2, \cdots, \frac{p^\alpha - 1}{2} - 1\}$.

$$2^{k+1} \cdot s_2 \equiv 2s_2(\bmod\ p^\alpha).$$

Then we can get $2 \leq 2s_2 \leq p - 3$. So $p^\alpha \nmid 2^{k+1} \cdot s_2$.

If $2^k < p^\alpha$, then $SL(n) = p^\alpha$, from $Z(n) + SL(n) = n$ we may deduce that $Z(n) = m = n - SL(n) = p^\alpha \left(2^k - 1\right)$. Thus,

$$n = 2^k \cdot p^\alpha \mid \frac{p^\alpha \left(2^k - 1\right) \left(p^\alpha \left(2^k - 1\right) + 1\right)}{2}.$$

From this we may immediately deduce that $2^k \mid (p^\alpha - 1)$ and $2^{k+1} \nmid (p^\alpha - 1)$.

Now we prove that $m = p^\alpha(2^k - 1)$ is the minimum value which satisfy the definition of $Z(n)$. From the properties of $Z(n)$, we know that $Z(n) \geq p^\alpha - 1$, and the possible value of $Z(n)$ between $p^\alpha - 1$ and $p^\alpha(2^k - 1)$ are as follows:

C. $p^\alpha - 1,\ p^\alpha \cdot 2 - 1,\ \cdots,\ p^\alpha \cdot (2^k - 1) - 1$.

We mark this group of integers as $p^\alpha \cdot s_1 - 1, s_1 \in \{1, 2, \cdots, 2^k - 1\}$.

$$p^\alpha \cdot s_1 - 1 \equiv s_1 - 1(\bmod\ 2^k).$$

Then we can get $0 \leq s_1 - 1 \leq 2^k - 2$. When $s_1 - 1 = 0$, so $s_1 = 1$, and $m = p^\alpha - 1$, $2^{k+1} \mid (p^\alpha - 1)$. Here we obtain contradiction. So $2^k \nmid (p^\alpha \cdot s_1 - 1)$.

D. $p^\alpha,\ p^\alpha \cdot 2,\ \cdots,\ p^\alpha \cdot (2^k - 2)$.

We mark this group of integers as $p^\alpha \cdot s_2, s_2 \in \{1, 2, \cdots, 2^k - 2\}$.

$$p^\alpha \cdot s_2 \equiv s_2(\bmod\ 2^k).$$

Then we can get $1 \leq s_2 \leq 2^k - 2$. So $2^k \nmid p^\alpha \cdot s_2$.

(4) Let $n = 2^k \cdot s$, where $s = p^\alpha \cdot p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, $p,\ p_1,\ p_2,\ \cdots,\ p_r$ are odd primes, and α be a positive integer, p^α is the greatest prime power divisor of s. That is,

$$p^{\alpha} = \max\left\{p^{\alpha},\ p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\right\}.$$

In this case, we prove that n can not satisfy the equation $Z(n) + SL(n) = n$. It means that if n has at least three different prime factors, then n is not the solution of the equation.

Let $a = 2^{k-1} \cdot p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then $n = 2a \cdot p^{\alpha}$, $\alpha \geq 1$, $(2a,\ p^{\alpha}) = 1$, p is a prime and $p \geq 3$.

Now we discuss the solution of the equation in the following two cases:

If $2^k > p^{\alpha}$, then $SL(n) = 2^k$. Only if $Z(n) = n - 2^k$, the equation has positive integer solution.

From $(2a,\ p^{\alpha}) = 1$, we know that the congruent equation

$$4ax \equiv 1(\bmod p^{\alpha})$$

has positive integer solution, so the congruent equation

$$16a^2x^2 \equiv 1(\bmod p^{\alpha})$$

has positive integer solution. We assume that the solution is y, taking $1 \leq y \leq p^{\alpha} - 1$, then $p^{\alpha} - y$ is also its positive integer solution.

So we can take $1 \leq y \leq \frac{p^{\alpha}-1}{2}$. From $16a^2y^2 \equiv 1(\bmod p^{\alpha})$, we can get $p^{\alpha} \mid (4ay - 1)$ or $p^{\alpha} \mid (4ay + 1)$.

A. If $p^{\alpha} \mid (4ay - 1)$, then

$$n = 2a \cdot p^{\alpha} \mid \frac{4ay(4ay-1)}{2}.$$

So we can get

$$Z(n) = m \leq 4ay - 1 \leq \frac{4a(p^{\alpha}-1)}{2} - 1 = n - 2a - 1.$$

B. If $p^{\alpha} \mid (4ay + 1)$, then

$$n = 2a \cdot p^{\alpha} \mid \frac{4ay(4ay+1)}{2}.$$

So we can get

$$Z(n) = m \leq 4ay \leq \frac{4a(p^{\alpha}-1)}{2} = n - 2a.$$

And it is obviously that $2^k < a$, when $Z(n) = n - 2^k$, $Z(n) > n - a$. So the equation has no solution.

If $2^k < p^{\alpha}$, then $SL(n) = p^{\alpha}$. Only if $Z(n) = n - p^{\alpha} = p^{\alpha}(2a - 1)$, the equation has positive integer solution.

From $(2a,\ p^{\alpha}) = 1$, we know that the congruent equation

$$p^{\alpha}x \equiv 1(\bmod 2a)$$

has positive integer solution, so the congruent equation

$$p^{2\alpha}x^2 \equiv 1(\bmod 2a)$$

has positive integer solution. We assume that the solution is y, taking $1 \leq y \leq 2a-1$, then $2a-y$ is also a positive integer solution of the equation. So we can take $1 \leq y \leq \frac{2a-1}{2}$.

From $p^{2\alpha}x^2 \equiv 1(\bmod 2a)$, we can get $2a \mid (p^\alpha y - 1)$ or $2a \mid (p^\alpha y + 1)$.

C. If $2a \mid (p^\alpha y - 1)$, then

$$n = 2a \cdot p^\alpha \mid \frac{p^\alpha y(p^\alpha y - 1)}{2}.$$

y is an even integer.

So we can get

$$Z(n) = m \leq p^\alpha y - 1 \leq p^\alpha \cdot \frac{2a-1}{2} - 1.$$

D. If $2a \mid (p^\alpha y + 1)$, then

$$n = 2a \cdot p^\alpha \mid \frac{p^\alpha y(p^\alpha y + 1)}{2}.$$

y is an even integer.

So we can get

$$Z(n) = m \leq p^\alpha y \leq p^\alpha \cdot \frac{2a-1}{2}.$$

Then $Z(n) = n - p^\alpha$ is not the minimum value which satisfy the definition of $Z(n)$. In this case the equation has no positive integer solution. This completes the proof of Theorem.

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Complete monotonicity properties for the gamma function and Barnes G-function

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Abstract In this paper, some complete monotonicity properties of functions related to the gamma function and Barnes G-function are proved.

Keywords Gamma function, double gamma function, Barnes G-function, complete monotonicity, asymptotic expansion.

§1. Introduction

A function f is said to be completely monotonic over (a, b), where $-\infty \leq a < b \leq \infty$, if

$$(-1)^k f^{(k)}(x) \geq 0, \tag{1}$$

for $a < x < b$ and $k = 0, 1, 2, \cdots$. If, in addition, f is continuous at $x = a$, then it is called completely monotonic over $[a, b)$, with similar definitions for $(a, b]$ and $[a, b]$.

Dubourdien [1, p. 98] pointed out that if a non-constant function f is completely monotonic on (a, ∞), then strict inequality holds in (1). See also [2] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$, see [3, p. 161]. The main properties of completely monotonic functions are given in [3, Chapter IV]. We also refer to [4], where a detailed list of references on completely monotonic functions can be found.

In this paper, some complete monotonicity properties of functions related to the gamma function and Barnes G-function are proved, see sections 2 and 3.

§2. Gamma function

Euler's gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (\Re(z) > 0).$$

For the various properties and characteristics of the gamma function, the reader may be referred (for example) to the works [5, 6, 7, 8, 9].

Inspired by the asymptotic expansion

$$\ln\Gamma(z) \sim (z-\frac{1}{2})\ln z - z + \ln\sqrt{2\pi} + \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)z^{2i-1}} \quad (z \to \infty \text{ in } |\arg z < z|),$$

where $B_i (i = 1, 2, \cdots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=1}^{\infty} B_i \frac{t^i}{i!} = 1 - \frac{1}{2}t + \sum_{j=1}^{\infty} B_{2j} \frac{t^{2j}}{(2j)!}, \quad |t| < 2\pi,$$

H. Alzer [6] proved the following theorem.

Theorem 1. For all $n = 0, 1, 2, \cdots$, the functions

$$F_n(x) = \ln\Gamma(x) - (x-\frac{1}{2})\ln x + x - \ln\sqrt{2\pi} - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

and

$$G_n(x) = -\ln\Gamma(x) + (x-\frac{1}{2})\ln x - x + \ln\sqrt{2\pi} + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

are completely monotonic on $(0, \infty)$, which is equivalent to the function

$$R_n(x) = (-1)^n \left[\ln\Gamma(x) - (x-\frac{1}{2})\ln x + x - \ln\sqrt{2\pi} - \sum_{i=1}^{n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right]$$

being completely monotonic on $(0, \infty)$.

For $n = 0$, Theorem 1 was proved by Muldoon [10]. Applying the Euler-Maclaurin summation formula, S. Koumandos [11] proved that for all $m = 1, 2, \cdots$ and $x > 0$,

$$1 - \frac{x}{2} + \sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!}x^{2j} < \frac{x}{e^x - 1} < 1 - \frac{x}{2} + \sum_{j=1}^{2m-1} \frac{B_{2j}}{(2j)!}x^{2j},$$

and then used it to give a new and simpler proof of Theorem 1.

We here presented a very short proof of Theorem 1 by using Binet's first formula for the $\ln\Gamma(x)$ [3, p. 127]

$$\ln\Gamma(x) = (x-\frac{1}{2})\ln x - x + \ln\sqrt{2\pi} + \theta(x), \quad (x > 0), \tag{2}$$

where the remainder $\theta(x)$ is given by

$$\theta(x) = \int_0^{\infty} (\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2})\frac{e^{-xt}}{t}dt. \tag{3}$$

Proof of Theorem 1. It is known in [12, p. 64] that

$$(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2})\frac{1}{t} = \sum_{i=1}^{n} \frac{B_{2i}}{(2i)!}t^{2i-2} + (-1)^n t^{2n}\nu_n(t), \quad (n \geq 0), \tag{4}$$

where

$$\nu_n(t) = \sum_{k=1}^{\infty} \frac{2}{(t^2 + 4\pi^2 k^2)(2\pi k)^{2n}} > 0.$$

Thus (3) and (4) yield

$$\theta(x) = \sum_{i=1}^{n} \frac{B_{2i}}{(2i)!} \int_0^{\infty} t^{2i-2} e^{-xt} dt + (-1)^n \int_0^{\infty} t^{2n} \nu_n(t) e^{-xt} dt$$

$$= \sum_{i=1}^{n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} + (-1)^n \int_0^{\infty} t^{2n} \nu_n(t) e^{-xt} dt. \tag{5}$$

Combining (2) with (5), we immediately obtain

$$R_n(x) = \int_0^{\infty} t^{2n} \nu_n(t) e^{-xt} dt.$$

It is easy to see that the function $x \mapsto R_n(x)$ is completely monotonic on $(0, \infty)$ for all $n = 0, 1, 2, \cdots$. The proof is complete.

§3. Barnes G-function

The multiple gamma function Γ_n is defined as a generalization of Γ-function by the following recurrence functional equation for all complex numbers z and all positive integers n:

$$\Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \quad \Gamma_1(z) = \Gamma(z) \quad \text{and} \quad \Gamma_n(1) = 1$$

by Barnes [13, 14] and others about 100 years ago. Barnes [13] gave several explicit Weierstrass canonical product forms of the double Gamma function (or Barnes G-function) $\Gamma_2 := 1/G$, one of which is recalled here in the form:

$$\begin{aligned} [\Gamma_2(z+1)]^{-1} &= G(z+1) \\ &= (2\pi)^{z/2} \exp(-\tfrac{1}{2}z - \tfrac{1}{2}(\gamma+1)z^2) \prod_{k=1}^{\infty} [(1+\tfrac{z}{k})^k \cdot \exp(-z + \tfrac{z^2}{2k})], \end{aligned} \tag{6}$$

where $\gamma = 0.5772156649\cdots$ denotes the Euler-Mascheroni constant. The double gamma function satisfies $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$. The theory of the double Gamma function has indeed found interesting applications in many other recent investigations (see, for details, [15]).

V. S. Adamchik [16, Proposition 3] derived the Binet integral representation for the Barnes G-function

$$\begin{aligned} \ln G(z+1) = z \ln \Gamma(z) + \frac{z^2}{4} - \frac{\ln z}{2} B_2(z) - \ln A \\ - \int_0^{\infty} \frac{e^{-zt}}{t^2} (\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} - \frac{t}{12}) dt, \Re(z) > 0, \end{aligned} \tag{7}$$

where $B_2(z) = z^2 - z + 1/6$ is the second Bernoulli polynomial, A is Glaisher's constant defined as

$$\ln A = \lim_{n \to \infty} \left[\ln(\prod_{k=1}^{n} k^k) - (\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}) \ln n + \frac{n^2}{4} \right],$$

the numerical value of A being $1.282427\cdots$.

A result similar to Theorem 1 holds.

Theorem 2. For all $n = 1, 2, \cdots$, the functions

$$\begin{aligned} U_n(x) &= \ln G(x+1) - x\ln\Gamma(x) - \frac{x^2}{4} + \frac{\ln x}{2}(x^2 - x + \frac{1}{6}) \\ &\quad + \ln A + \sum_{k=2}^{2n-1} \frac{B_{2k}}{2k(2k-1)(2k-2)x^{2k-2}} \end{aligned}$$

and

$$\begin{aligned} V_n(x) &= -\ln G(x+1) + x\ln\Gamma(x) + \frac{x^2}{4} - \frac{\ln x}{2}(x^2 - x + \frac{1}{6}) \\ &\quad - \ln A - \sum_{k=2}^{2n} \frac{B_{2k}}{2k(2k-1)(2k-2)x^{2k-2}} \end{aligned}$$

are completely monotonic on $(0, \infty)$, which is equivalent to the function

$$\begin{aligned} P_n(x) &= (-1)^n \Big[-\ln G(x+1) + x\ln\Gamma(x) + \frac{x^2}{4} - \frac{\ln x}{2}(x^2 - x + \frac{1}{6}) \\ &\quad - \ln A - \sum_{k=2}^{n} \frac{B_{2k}}{2k(2k-1)(2k-2)x^{2k-2}} \Big] \end{aligned}$$

being completely monotonic on $(0, \infty)$.

Proof. Write (4) as

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} - \frac{t^2}{12} = \sum_{k=2}^{n} \frac{B_{2k}}{(2k)!} t^{2k} + (-1)^n t^{2n+2} \nu_n(t), \quad n \geq 1. \tag{8}$$

Thus (7) and (8) yield

$$\begin{aligned} \ln G(x+1) &= x\ln\Gamma(x) + \frac{x^2}{4} - \frac{\ln x}{2}(x^2 - x + \frac{1}{6}) \\ &\quad - \ln A - \int_0^\infty \frac{e^{-xt}}{t^3}(\frac{t}{e^t - 1} - 1 + \frac{t}{2} - \frac{t^2}{12})dt \\ &= x\ln\Gamma(x) + \frac{x^2}{4} - \frac{\ln x}{2}(x^2 - x + \frac{1}{6}) - \ln A \\ &\quad - \sum_{k=2}^{n} \frac{B_{2k}}{(2k)!} \int_0^\infty t^{2k-3} e^{-xt} dt + (-1)^{n+1} \int_0^\infty t^{2n-1} \nu_n(t) e^{-xt} dt \\ &= x\ln\Gamma(x) + \frac{x^2}{4} - \frac{\ln x}{2}(x^2 - x + \frac{1}{6}) - \ln A \\ &\quad - \sum_{k=2}^{n} \frac{B_{2k}}{2k(2k-1)(2k-2)x^{2k-2}} + (-1)^{n+1} \int_0^\infty t^{2n-1} \nu_n(t) e^{-xt} dt, \end{aligned}$$

and therefore,

$$P_n(x) = \int_0^\infty t^{2n-1} \nu_n(t) e^{-xt} dt.$$

Clearly, the function $x \mapsto P_n(x)$ is completely monotonic on $(0, \infty)$ for all $n = 1, 2, \cdots$. The proof is complete.

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A short interval result for the exponential divisor function [1]

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Abstract The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$ if $b_i|a_i$ for every $i \in 1, 2, \cdots, s$. Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n, where $\tau^{(e)}(1) = 1$ by convention. In this paper we shall establish a short interval result for r-th power of the function $\tau^{(e)}$, where $r \geq 1$ is a fixed integer.

Keywords The exponential divisor function, generalized divisor function, short interval.

§1. Introduction

The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$ if $b_i|a_i$ for every $i \in$\{1,2,...,s\}, denoted by $d|_e n$. By convention $1|_e 1$.

Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n, which is called the exponential divisor function. The properties of the function $\tau^{(e)}$ were investigated by many authors, see example, [1], [2], [4].

Suppose $r \geq 1$ is fixed integer. Let

$$A(x) := \sum_{n \leq x} (\tau^{(e)}(n))^r.$$

Recently László Tóth [3] proved that

$$A(x) = A_r x + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\epsilon}), \tag{1.1}$$

where P_{2^r-2} is a polynomial of degree $2^r - 2$, $u_r = \frac{2^r-1}{2^r+1}$, and

$$A_r := \prod_p (1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^r - (\tau(a-1))^r}{p^a}). \tag{1.2}$$

In this short note, we shall prove the following short interval result.

Theorem. If $x^{\frac{1}{5}+2\epsilon} \leq y \leq x$, then

$$\sum_{x < n \leq x+y} (\tau^{(e)}(n))^r = A_r y + O(yx^{-\epsilon/2} + x^{1/5+\epsilon}). \tag{1.3}$$

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Notations. Throughout this paper, ϵ denotes a fixed but sufficiently small positive constant. If $1 \leq a \leq b$ are fixed integers, we define

$$d(a,b;n) = \sum_{n=n_1^a n_2^b} 1.$$

We have the estimate $d(a,b;k) \ll n^{\epsilon^2}$.

§2. Proof of theorem

In order to prove our theorem, we need the following lemmas.

Lemma 1. Suppose s is a complex number for with $\Re s > 1$, then

$$F(s) := \sum_{n=1}^{\infty} \frac{(\tau^{(e)}(n))^r}{n^s} = \zeta(s)\zeta^{2^r-1}(2s)\zeta^{-C_r}(4s)G(s,r),$$

where $C_r = 2^{r-1} + 2^{2r-1} - 3^r \geq 0$, the function $G(s,r)$ can be written as a Dirichlet series $G(s,r) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$, which is absolutely convergent for $\mathrm{Re}s > 1/5$.

Proof. The function $\tau^{(e)}(n)$ is multiplicative. So by the Euler product formula, we have for $\sigma > 1$ that

$$\begin{aligned}
\sum_{n\leq x}(\tau^{(e)}(n))^r &= \prod_p (1 + \frac{(\tau^{(e)}(p))^r}{p^s} + \frac{(\tau^{(e)}(p^2))^r}{p^{2s}} + \frac{(\tau^{(e)}(p^3))^r}{p^{3s}} + \cdots) \\
&= \prod_p (1 + \frac{1}{p^s} + \frac{2^r}{p^{2s}} + \frac{2^r}{p^{3s}} + \frac{3^r}{p^{4s}} + \frac{2^r}{p^{5s}} + \cdots) \\
&= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{2^r}{p^{2s}} + \cdots) \\
&= \zeta(s)\zeta^{2^r-1}(2s) \prod_p (1 - \frac{1}{p^s})^{2^r-1}(1 + \frac{2^r-1}{p^{2s}} + \frac{3^r-2^r}{p^{4s}} + \cdots) \\
&= \zeta(s)\zeta^{2^r-1}(2s)\zeta^{-C_r}(4s)G(s,r).
\end{aligned}$$

Now we write $C_r = 2^{r-1} + 2^{2r-1} - 3^r$ and $G(s,r) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. It is easily seen the Dirichlet series is absolutely convergent for $Res > 1/5$.

Lemma 2. Let $l \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers. Then

$$\sum_{\substack{x < nm^l \leq x+y \\ m > x^{\epsilon}}} 1 \ll yx^{-\epsilon} + x^{\frac{1}{2l+1}} \log x.$$

Proof. This Lemma is very important when studying the short interval distribution of l-free numbers, see for example, [5].

Let $a(n), b(n)$ and $c(n)$ be arithmetic functions defined by the following Dirichlet series (for

$Res > 1$):

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)G(s,r), \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta^{2^r-1}(2s), \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \zeta^{-C_r}(4s). \tag{3}$$

Lemma 3. Let $a(n)$ be the arithmetic function defined by (1), then we have

$$\sum_{n\leq x} a(n) = Ax + O(x^{\frac{1}{5}+\epsilon}), \tag{4}$$

where $A = Res_{s=1}\zeta(s)G(s,r)$.

Proof. From Lemma 1 the infinite series $\sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ converges absolutely for $\sigma > 1/5$, it follows that that

$$\sum_{n\leq x} |g(n)| \ll x^{\frac{1}{5}+\epsilon}.$$

Therefore from the definition of $g(n)$ and (1), it follows that

$$\begin{aligned}\sum_{n\leq x} a(n) &= \sum_{mn\leq x} g(n) \\ &= \sum_{n\leq x} g(n) \sum_{m\leq \frac{x}{n}} 1 \\ &= \sum_{n\leq x} g(n)[\frac{x}{n}] \\ &= Ax + O(x^{\frac{1}{5}+\epsilon}),\end{aligned}$$

and $A = Res_{s=1}\zeta(s)G(s,r)$.

Now we prove our theorem. From Lemma 3 and the definition of $a(n)$, $b(n)$ and $c(n)$, we obtain

$$(\tau^{(e)}(n))^r = \sum_{n=n_1n_2^2n_3^4} a(n_1)b(n_2)c(n_3),$$

and

$$a(n) \ll n^{\epsilon^2}, b(n) \ll n^{\epsilon^2}, c(n) \ll n^{\epsilon^2}. \tag{5}$$

We consider two cases. If $r = 1$, then $C_r = 0$. In this case, we have

$$\begin{aligned}A(x+y) - A(x) &= \sum_{x<n_1n_2^2\leq x+y} a(n_1)b(n_2) \\ &= \sum_{1} + O(\sum_{2}),\end{aligned} \tag{6}$$

where

$$\sum_1 = \sum_{n_2 \le x^\epsilon} b(n_2) \sum_{\frac{x}{n_2^2} < n_1 \le \frac{x+y}{n_2^2}} a(n_1),$$
$$\sum_2 = \sum_{\substack{x < n_1 n_2^2 \le x+y \\ n_2 > x^\epsilon}} |a(n_1)b(n_2)|.$$

In view of Lemma 3,

$$\begin{aligned}\sum_1 &= \sum_{n_2 \le x^\epsilon} b(n_2)\left(\frac{Ay}{n_2^2} + O\left(\left(\frac{x}{n_2^2}\right)^{\frac{1}{5}+\epsilon}\right)\right) \\ &= A_1 y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}), \end{aligned} \tag{7}$$

where $A_1 = Res_{s=1}F(s)$. By (5) we have and Lemma 2 with $l = 2$ we have

$$\begin{aligned}\sum_2 &\ll \sum_{x < n_1 n_2^2 \le x+y} (n_1 n_2^2)^{\epsilon^2} \\ &\ll x^{\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon}) \\ &= yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \end{aligned} \tag{8}$$

Now suppose $r \ge 2$, we have

$$\begin{aligned} A(x+y) - A(x) &= \sum_{x < n_1 n_2^2 n_3^4 \le x+y} a(n_1)b(n_2)c(n_3) \\ &= \sum_3 + O\left(\sum_4 + \sum_5\right), \end{aligned} \tag{9}$$

where

$$\sum_3 = \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} b(n_2)c(n_3) \sum_{\frac{x}{n_2^2 n_3^4} < n_1 \le \frac{x+y}{n_2^2 n_3^4}} a(n_1),$$
$$\sum_4 = \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^\epsilon}} |a(n_1)b(n_2)c(n_3)|,$$
$$\sum_5 = \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_3 > x^\epsilon}} |a(n_1)b(n_2)c(n_3)|.$$

By Lemma 3 again, we get

$$\begin{aligned}\sum_3 &= \sum_{\substack{n_2 \le x^\epsilon \\ n_3 \le x^\epsilon}} b(n_2)c(n_3)\left(\frac{A_1 y}{n_2^2 n_3^4} + O\left(\left(\frac{x}{n_2^2 n_3^4}\right)^{\frac{1}{5}+\epsilon}\right)\right) \\ &= A_r y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}), \end{aligned} \tag{10}$$

where $A_r = Res_{s=1}F(s)$.

From Lemma 2, (5) and the estimate $d(1,4;m) \ll m^{\epsilon^2}$, we get

$$\begin{aligned}\sum_4 &\ll \sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_2>x^\epsilon}} (n_1n_2n_3)^{\epsilon^2} \ll x^{\epsilon^2} \sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_2>x^\epsilon}} 1\\ &= x^{\epsilon^2} \sum_{\substack{x<mn_2^2\le x+y\\ n_2>x^\epsilon}} d(1,4;m) \ll x^{2\epsilon^2} \sum_{\substack{x<mn_2^2\le x+y\\ n_2>x^\epsilon}} 1\\ &\ll x^{2\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon})\\ &\ll yx^{-\epsilon/2} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \end{aligned} \tag{11}$$

Similarly we have

$$\sum_5 \ll yx^{-\epsilon/2} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \tag{12}$$

Now our theorem for the case $r \ge 2$ follows from (9)-(12).

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Multiplicative mappings at unit operator on $B(H)$ [1]

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Abstract Let $\mathcal{A}$ be a subalgebra of $B(H)$. We say that a linear mapping φ from $\mathcal{A}$ into itself is a multiplicative mapping at $Z(Z \in \mathcal{A})$ if $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S,\ T \in \mathcal{A}$ with $ST = Z$. Let H be an infinite dimensional complex Hilbert space, and let φ be a surjective linear map on $B(H)$. In this paper, we prove that if φ is a multiplicative mapping at I and continuous in the weak operator topology, then φ is an automorphism. We also prove that if φ is a weak continuous multiplicative mapping at any invertible operator with $\varphi(I) = I$ then φ is an automorphism.

Keywords Operator algebra, multiplicative mappings at unit operator, automorphism.

§1. Introduction and preliminaries

Let H be an infinite dimensional complex Hilbert space. We denote by $B(H)$ the algebra of all bounded linear operators on H. The purpose of this paper is to show the following theorem.

Theorem 1.1. Let $\varphi : B(H) \to B(H)$ be a continuous linear surjective mapping in the weak operator topology. Then the following statements are equivalent:

(1) φ is a multiplicative mapping at I from $B(H)$ into itself, i.e.

$$\varphi(ST) = \varphi(S)\varphi(T) \quad (S,\ T \in B(H),\ ST = I).$$

(2) φ is an automorphism, i.e. there exists an invertible operator $A \in B(H)$ such that

$$\varphi(T) = ATA^{-1} \quad (T \in B(H)).$$

Characterizing linear maps on operator algebras is one of the most active and fertile research topics in the theory of operator algebras during the past one hundred years. Recently, some authors have paid their attention to the study of automorphisms and derivations. Many profound results have been obtained in these domains, which helps us to understand operator algebras from a new aspect. We describe some of the results related to ours. Let $\varphi : \mathcal{A} \to B(H)$ be linear. We say that φ is a multiplicative mapping at Z if $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S,\ T \in \mathcal{A}$

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with $ST = Z$. In 1993, Šemrl [11] proved that a linear bijective mapping φ with $\varphi(I) = I$ from $B(H)$ into itself is a spatial isomorphism if and only if φ is preserving zero product. In 2002, Cui and Hou [4] proved that a zero product preserving bounded linear bijective mapping φ with $\varphi(I) = I$ between atomic nest algebras on Hilbert spaces is an isomorphism. Note that $\varphi(0) = 0$, thus, φ is preserving zero product if and only if φ is multiplicative at 0. It follows that a linear mapping is an automorphism if it is multiplicative at 0 under some conditions. In 2007, Zhu and Xiong [6] proved that every strongly operator topology continuous derivable mapping at I on a nest algebra is an inner derivation. For other results, see [7-10,12,14]. In this paper, the operator space $B(H)$ is given the weak operator topology. It is the aim of us to prove that every weak continuous linear surjective mapping is an automorphism if and only if it is multiplicative at I.

The plan of this paper is as follows. In section 2, we introduce some preliminary lemmas relating to the proof of Theorem 1.1. In section 3, we give the proof of Theorem 1.1 and get Corollary 3.1 in which the identity operator I in Theorem 1.1 is replaced by any invertible operator $G \in B(H)$.

Throughout this paper, we use H to denote an infinite dimension complex Hilbert space. The symbols $B(H)$ and $F(H)$ stand for the set of all bounded linear operators and the set of all finite rank operators on H, respectively. $< \cdot, \cdot >$ denotes the inner product on H. $x \otimes y$ and I denote the rank one operator $< \cdot, y > x$ and the identity operator on H, respectively. Denote by $ran(T)$ and $N(T)$ the range space and the kernel space of $T (T \in B(H))$. Also T^* denotes the adjoint of T. The weak operator topology on $B(H)$ is the topology induced by the seminorms $P_{x,y}(T) = | < Tx, y > |$ for all $x, y \in H$.

§2. Preliminary lemmas

Lemma 2.1 is obtained from [3]. We only state its content and omit its proof. In Lemma 2.1, X is a Banach space over the field of real numbers or the field of complex numbers. The symbol $B(X)$ stands for the set of all bounded linear operators on X.

Lemma 2.1.[3] Let φ be a weak continuous linear mapping which preserve the rank of rank-one operators non-increasing. Then one of the following holds:

(i) There exist linear mappings $A \in B(X)$ and $C \in B(X)$ such that $\varphi(T) = ATC$ for any $T \in B(X)$;

(ii) There exist linear mappings $A \in B(X)$ and $C \in B(X)$ such that $\varphi(T) = AT^*C$ for any $T \in B(X)$;

(iii) There exist weak-weak continuous linear mapping $\delta(\cdot) : B(X) \to X$ and $f_0 \in X^*$ such that $\varphi(T) = \delta(T) \otimes f_0$ for any $T \in B(X)$;

(iv) There exist weak-weak* continuous linear mapping $\lambda(\cdot) : B(X) \to X$ and $x_0 \in X$ such that $\varphi(T) = x_0 \otimes \lambda(T)$ for any $T \in B(X)$.

Lemma 2.2. Let $\varphi : B(H) \to B(H)$ be a weak operator topology continuous linear surjective mapping. If φ is multiplicative at I, then $\varphi(I) = I$.

Proof. For any idempotent operator $P \in B(H)$, we have

$$I = (P - \frac{1+\sqrt{3}i}{2}I)(P - \frac{1-\sqrt{3}i}{2}I),$$

then

$$\begin{aligned} \varphi(I) &= \varphi(P - \frac{1+\sqrt{3}i}{2}I)\varphi(P - \frac{1-\sqrt{3}i}{2}I) \\ &= \varphi(P)^2 - \frac{1+\sqrt{3}i}{2}\varphi(I)\varphi(P) - \frac{1-\sqrt{3}i}{2}\varphi(P)\varphi(I) + \varphi(I)^2. \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} \varphi(I) &= \varphi(P - \frac{1-\sqrt{3}i}{2}I)\varphi(P - \frac{1+\sqrt{3}i}{2}I) \\ &= \varphi(P)^2 - \frac{1-\sqrt{3}i}{2}\varphi(I)\varphi(P) - \frac{1+\sqrt{3}i}{2}\varphi(P)\varphi(I) + \varphi(I)^2. \end{aligned} \tag{2}$$

It follows from equations (1) and (2) that

$$\varphi(I)\varphi(P) = \varphi(P)\varphi(I).$$

The result of [1] implies that every operator in $B(H)$ can be written as the sum of five idempotents in $B(H)$. As φ is surjective, we have

$$\varphi(I)T = T\varphi(I) \quad (T \in B(H)).$$

It follows that $\varphi(I) \in \mathbb{C}I$. Namely, there exists $\lambda \in \mathbb{C}$ such that $\varphi(I) = \lambda I$. By $\varphi(I) = \varphi(I^2) = \varphi(I)^2$, we get that $\lambda I = \lambda^2 I$. Hence, either $\lambda = 0$ or $\lambda = 1$.

If $\varphi(I) = 0$, according to Equation (1), we have $\varphi(P)^2 = 0$ for any idempotent $P \in B(H)$. Let $S \in F(H)$ be any self-adjoint operator. Then $S = \sum_{k=1}^{n} \alpha_i P_i$ for some orthogonal projections $P_i \in B(H)$ and $\alpha_i \in \mathbb{R}$ $(1 \leq i \leq n)$. Since $P_i + P_j$ is also a projection $(i \neq j)$, it follows from $\varphi(P_i + P_j)^2 = 0$ that

$$\varphi(P_i)\varphi(P_j) + \varphi(P_j)\varphi(P_i) = 0.$$

This further yields $\varphi(S)^2 = 0$. For any $F \in F(H)$, there exist two self-adjoint operators $S_1, S_2 \in F(H)$ such that $F = S_1 + iS_2$. Since $\varphi(S_1 + S_2)^2 = 0$, we have

$$\varphi(S_1)\varphi(S_2) + \varphi(S_2)\varphi(S_1) = 0.$$

It follows that

$$\begin{aligned} \varphi(F)^2 &= \varphi(S_1 + iS_2)^2 \\ &= \varphi(S_1)^2 + i(\varphi(S_1)\varphi(S_2) + \varphi(S_2)\varphi(S_1)) - \varphi(S_2)^2 = 0. \end{aligned}$$

As the ideal of all finite-rank operators in $B(H)$ is a dense subset of $B(H)$ in the weak operator topology, we obtain that $\varphi(T)^2 = 0$ for any $T \in B(H)$. This shows that the range of φ is a set consisting of nilpotent operators. However, $I \in B(H)$ is not a nilpotent operator, which contradicts the assumption that φ is surjective. Hence $\lambda = 1$ and so $\varphi(I) = I$.

Lemma 2.3. Let $\varphi : B(H) \to B(H)$ be a weak operator topology continuous linear surjective mapping. If φ is multiplicative at I, then

(i) For every idempotent operator $P \in B(H)$, we have $\varphi(P)^2 = \varphi(P)$;

(ii) For every $P \in B(H)$ with $P^2 = 0$, we have $\varphi(P)^2 = 0$.

Proof. (i) It is an immediate consequence of Equation (1) and Lemma 2.2.

(ii) For every operator $P \in B(H)$ with $P^2 = 0$, we have

$$\begin{aligned} \varphi(I) &= \varphi((I-P)(I+P)) \\ &= \varphi(I-P)\varphi(I+P) \\ &= \varphi(I)^2 - \varphi(P)^2 \\ &= \varphi(I) - \varphi(P)^2. \end{aligned}$$

It follows that $\varphi(P)^2 = 0$.

§3. The proof of the main theorem and a corollary

Proof of Theorem 1.1. We divide the proof into the following three steps.

Step 1. We will prove that $rank(\varphi(P)) = 1$ for any idempotent $P \in B(H)$ with $rank(P) = 1$. Set

$$X_1 = PB(H)P, \qquad X_2 = PB(H)(I-P),$$
$$X_3 = (I-P)B(H)P, \qquad X_4 = (I-P)B(H)(I-P),$$

and

$$Y_1 = \varphi(P)B(H)\varphi(P), \qquad Y_2 = \varphi(P)B(H)(I-\varphi(P)),$$
$$Y_3 = (I-\varphi(P))B(H)\varphi(P), \qquad Y_4 = (I-\varphi(P))B(H)(I-\varphi(P)).$$

Then $B(H) = X_1 \oplus X_2 \oplus X_3 \oplus X_4 = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$. For any $T \in X_2$, we have $(T+P)^2 = T+P$ and $T^2 = 0$. It follows from Lemma 2.3 that $\varphi(T+P)^2 = \varphi(T+P)$ and $\varphi(T)^2 = 0$. Namely,

$$\varphi(T) = \varphi(P)\varphi(T) + \varphi(T)\varphi(P). \tag{3}$$

According to equation (3), we have $\varphi(P)\varphi(T)\varphi(P) = 0$. Hence

$$\begin{aligned} \varphi(T) &= \varphi(P)\varphi(T) + \varphi(T)\varphi(P) \\ &= \varphi(P)\varphi(T) - \varphi(P)\varphi(T)\varphi(P) + \varphi(T)\varphi(P) - \varphi(P)\varphi(T)\varphi(P) \\ &= \varphi(P)\varphi(T)(I-\varphi(P)) + (I-\varphi(P))\varphi(T)\varphi(P). \end{aligned}$$

It follows that $\varphi(X_2) \subseteq Y_2 + Y_3$. Similarly, we can show that $\varphi(X_3) \subseteq Y_2 + Y_3$.

Since P is an idempotent operator of rank one, we have $X_1 = \mathbb{C}P$. It follows that $\varphi(X_1) \subseteq Y_1$. Next, let us prove that $\varphi(X_4) \subseteq Y_4$. Since X_4 is ismorphic to $B(Ker(P))$, and the result

of [1] says every operator in $B(H)$ can be expressed as the sum of five idempotents in $B(H)$, we need only prove that $\varphi(Q) \in Y_4$ for every idempotent operator $Q \in X_4$. In fact, for every idempotent operator $Q \in X_4$ we have $PQ = QP = 0$. Then $(P+Q)^2 = P+Q$. By Lemma 2.3, we get that

$$\varphi(P)\varphi(Q) + \varphi(Q)\varphi(P) = 0. \tag{4}$$

Note that $\varphi(P)^2 = \varphi(P)$. So

$$\varphi(P)\varphi(Q) + \varphi(P)\varphi(Q)\varphi(P) = \varphi(Q)\varphi(P) + \varphi(P)\varphi(Q)\varphi(P) = 0. \tag{5}$$

According to equations (4) and (5), we get that

$$\varphi(P)\varphi(Q) = \varphi(Q)\varphi(P) = 0,$$

which further yields

$$\begin{aligned} \varphi(Q) &= \varphi(Q)(I - \varphi(P)) \\ &= (I - \varphi(P))\varphi(Q) \\ &= (I - \varphi(P))\varphi(Q)(I - \varphi(P)) \in Y_4. \end{aligned}$$

Furthermore, by $X_1 = \mathbb{C}P$, we have $\varphi(X_1) = \mathbb{C}\varphi(P)$. So, $\varphi(B(H)) \subseteq \mathbb{C}\varphi(P) \oplus Y_2 \oplus Y_3 \oplus Y_4$. However, φ is a surjective mapping, that is, $\mathbb{C}\varphi(P) = Y_1 = \varphi(P)B(H)\varphi(P)$. This means that $\varphi(P)$ has rank one.

Step 2. We will prove that $rank(\varphi(x \otimes y)) \leq 1$ for any rank one operator $x \otimes y \in B(H)$. In fact, if $< x,\ y > \neq 0$, then $(< x,\ y >)^{-1}x \otimes y$ is a rank one idempotent operator, that is, $rank(\varphi(x \otimes y)) = 1$ by step 1. Assume that $< x,\ y >= 0$. For $x \in H$, it follows from Hahn-Banach theorem that there exists $y_1 \in H$ such that $< x,\ y_1 >= 1$. Let $y_2 = y_1 - y$. Then $< x,\ y_2 >= 1$. Hence $x \otimes y_1$, $x \otimes y_2$ are rank one idempotent operators and $x \otimes y = x \otimes y_1 - x \otimes y_2$. According to step 1, $\varphi(x \otimes y_i) = s_i \otimes t_i$ with $< s_i,\ t_i >= 1$, $i = 1, 2$. For any $m \in [0, 1]$, we have $< x,\ my_1 + (1-m)y_2 >= 1$. Thus, there exist $s_m, t_m \in H$ with $< s_m,\ t_m >= 1$ such that

$$\varphi(mx \otimes y_1 + (1-m)x \otimes y_2) = \varphi(x \otimes (my_1 + (1-m)y_2)) = s_m \otimes t_m,$$

i.e.

$$ms_1 \otimes t_1 + (1-m)s_2 \otimes t_2 = s_m \otimes t_m.$$

This means that either s_1, s_2 are linearly dependent or t_1, t_2 are linearly dependent. So

$$\begin{aligned} rank(\varphi(x \otimes y)) &= rank(\varphi(x \otimes y_1 - x \otimes y_2)) \\ &= rank(s_1 \otimes t_1 - s_2 \otimes t_2) \leq 1. \end{aligned}$$

Step 3. We will prove that there exists invertible operator $A \in B(H)$ such that $\varphi(T) = ATA^{-1}$ or $\varphi(T) = AT^*A^{-1}$ for every $T \in B(H)$. By step 2, φ satisfies the conditions of Lemma 2.1. Applying the assumption that φ is a surjective mapping, we get that there exists

$T_0 \in B(H)$ such that $rank(\varphi(T_0)) > 1$. This means that φ has only one of the forms (i) and (ii) in Lemma 2.1.

Suppose that the case (i) of Lemma 2.1 occurs. Then there exist linear mappings $A, C \in B(H)$ such that

$$\varphi(T) = ATC \quad (T \in B(H)). \tag{6}$$

Next, let us prove that A is bijective. Since φ is surjective, equation(6) shows that A is surjective. Assume that there exists nonzero vector $x_0 \in H$ such that $Ax_0 = 0$. Then there exists $z \in H$ such that $< x_0, z >= 1$ by the Hahn-Banach theorems. From step 1, $\varphi(x_0 \otimes z)$ is a rank one operator, which contradicts $\varphi(x_0 \otimes z) = Ax_0 \otimes Cz = 0$. This shows that A is bijective. According to $I = \varphi(I) = AIC = AC$ we have $C = A^{-1}$. So $\varphi(T) = ATA^{-1}$.

Similarly, if the case (ii) of Lemma 2.1 occurs, we can prove that $\varphi(T) = AT^*A^{-1}$.

Lastly, we will prove that φ is an automorphism. Let $M \subseteq H$ be a separable subspace and $\{e_n\}_{n=0}^{\infty}$ is a basis of M. We define the operator S on M by $S(e_n) = e_{n+1}(n = 0, 1, \cdots)$. Notice that S has a left inverse but no right inverse. So, for any invertible operator $T \in B(M^{\perp})$, $S \oplus T \in B(H)$ is a left invertible operator. Namely, there exists $Z \in B(H)$ such that $Z(S \oplus T) = I$ but $(S \oplus T)Z \neq I$. Suppose that $\varphi(T) = AT^*A^{-1}$ for every $T \in B(H)$, then

$$\begin{aligned} I = \varphi(I) &= \varphi(Z(S \oplus T)) \\ &= \varphi(Z)\varphi(S \oplus T) \\ &= AZ^*(S \oplus T)^*A^{-1} \\ &= A((S \oplus T)Z)^*A^{-1} \\ &= \varphi((S \oplus T)Z). \end{aligned}$$

It contradicts $\varphi(S \oplus T)Z) \neq \varphi(I)$. Hence, φ is an automorphism.

Corollary 3.1. Let $\varphi : B(H) \to B(H)$ be a weak operator topology continuous linear surjective mapping with $\varphi(I) = I$. For any invertible operator $G \in B(H)$, if φ is multiplicative at G, then φ is an automorphism.

Proof. For every idempotent operator $P \in B(H)$, we have

$$G = IG = (P - \frac{1 + \sqrt{3}i}{2}I)(P - \frac{1 - \sqrt{3}i}{2}I)G.$$

It follows that

$$\begin{aligned} \varphi(G) &= \varphi(I - \frac{1 + \sqrt{3}i}{2}P)\varphi((I - \frac{1 - \sqrt{3}i}{2}P)G) \\ &= (\varphi(I) - \frac{1 + \sqrt{3}i}{2}\varphi(P))(\varphi(G) - \frac{1 - \sqrt{3}i}{2}\varphi(PG)) \\ &= \varphi(I)\varphi(G) - \frac{1 - \sqrt{3}i}{2}\varphi(I)\varphi(PG) - \frac{1 + \sqrt{3}i}{2}\varphi(P)\varphi(G) \\ &\quad + \varphi(P)\varphi(PG). \end{aligned} \tag{7}$$

Similary,

$$\begin{aligned} \varphi(G) &= \varphi(I)\varphi(G) - \frac{1 + \sqrt{3}i}{2}\varphi(I)\varphi(PG) - \frac{1 - \sqrt{3}i}{2}\varphi(P)\varphi(G) \\ &\quad + \varphi(P)\varphi(PG). \end{aligned} \tag{8}$$

Comparing equations (7) and (8), we have

$$\varphi(PG) = \varphi(P)\varphi(G).$$

Notice that every rank one operator in $B(H)$ may be denoted as a linear combination of at most four idempotents in $B(H)$ (see [13]), and every finite rank operator in $B(H)$ may be represented as a sum of rank one operator in $B(H)$. Thus we get that

$$\varphi(FG) = \varphi(F)\varphi(G) \quad (F \in F(H)).$$

Since $F(H)$ is a dense subset of $B(H)$ in the weak operator topology, we have

$$\varphi(TG) = \varphi(T)\varphi(G) \quad (T \in B(H)).$$

Using the same method, we can prove

$$\varphi(GT) = \varphi(G)\varphi(T) \quad (T \in B(H)).$$

For any $S,\ T \in B(H)$ with $ST = I$, we have $G = STG$. So

$$\varphi(G) = \varphi(S)\varphi(TG) = \varphi(S)\varphi(T)\varphi(G).$$

This further yields

$$(\varphi(S)\varphi(T) - I)\varphi(G) = 0. \tag{9}$$

Note that G is an invertible operator, it follows from

$$I = \varphi(I) = \varphi(GG^{-1}) = \varphi(G)\varphi(G^{-1})$$

and

$$I = \varphi(I) = \varphi(G^{-1}G) = \varphi(G^{-1})\varphi(G)$$

that $\varphi(G)$ is also invertible. Hence, we get that $\varphi(S)\varphi(T) = I = \varphi(I)$ by equation (9), i.e. φ is a multiplicative mapping at I. It follows from Theorem 1.1 that φ is an automorphism.

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The minimum number of polychromatic $\mathcal{C}$-hyperedges of the complete uniform mixed hypergraphs under one special condition [1]

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Abstract The upper and lower chromatic number of uniform mixed hypergraphs have inevitable relation with $\mathcal{C}$-hyperedge and $\mathcal{D}$-hyperedge. In general, the increase in the $\mathcal{C}$-hyperedge will increase lower chromatic number $\chi_{\mathcal{H}}$, increasing $\mathcal{D}$-hyperedge will decrease upper chromatic number $\overline{\chi}_{\mathcal{H}}$. This papers take complete uniform mixed hypergraphs for example, revealed further the relationship of $\mathcal{C}$-hyperedge with between the upper chromatic number and lower chromatic number. We give a few conclusions for mixed hypergraph $\mathcal{K}(n,l,m) = (X, \binom{X}{l}, \binom{X}{m})$ whose some $\mathcal{C}$-hyperedge are deleted.

Keywords A complete uniform mixed hypergraph, polychromatic $\mathcal{C}$-hyperedges, the minimum, upper chromatic number, lower chromatic number.

§1. Lemma and the basic concepts

Definition 1.1.[1] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, $\mathcal{C} = \{C_1, C_2, \cdots, C_l\}$, $\mathcal{D} = \{D_1, D_2, \cdots, D_m\}$ are two subset clusters of X, all $C_i \in \mathcal{C}$ meet with $|C_i| \geq 2$, and all $D_j \in \mathcal{D}$ meet with $|D_j| \geq 2$. Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ called as a mixed hypergraph from X, and each $C_i \in \mathcal{C}$ called as the $\mathcal{C}-$hyperedges, and each $D_j \in \mathcal{D}$ called as the $\mathcal{D}-$hyperedges. In particular, that $\mathcal{H}_D = (X, \mathcal{D})$ called as a $\mathcal{D}$-hypergraph, the $\mathcal{H}_C = (X, \mathcal{C})$ for $\mathcal{C}-$hypergraph.

Definition 1.2.[2] For $2 \leq l, m \leq n = |X|$, let

$$\mathcal{K}(n,l,m) = (X, \mathcal{C}, \mathcal{D}) = (X, \binom{X}{l}, \binom{X}{m}),$$

where $|\mathcal{C}| = \binom{n}{l}$, $|\mathcal{D}| = \binom{n}{m}$. Then $\mathcal{K}(n,l,m)$ is called as a complete (l,m)-uniform mixed hypergraph with n vertices.

It is clear that for a given n, l, m, in a sense of the isomorphic existence just one $\mathcal{K}(n,l,m)$.

Definition 1.3.[3,4] For Mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the largest i among all strict i-coloring of $\mathcal{H}$ call as the upper chromatic number $\mathcal{H}$, said that for $\bar{\chi}_{\mathcal{H}}$.

Definition 1.4.[2] For mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if a i partition $X = \{X_1, X_2, \cdots, X_i\}$ of vertex sets X satisfy with follow condications:

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1) For each $\mathcal{C}$-hyperedge at least two vertices be allocated into the same block;

2) For each $\mathcal{D}$-hyperedge at least two vertices be allocated into different blocks.

Then the partition is called as a feasible partition of $\mathcal{H}$.

Obviously, any strict i coloring of $\mathcal{H}$ corresponds with a strict i feasible partition, and vice versa. They are equivalent. Therefore, we write one feasible partition of $\mathcal{H}$ or a strict i-coloring c as: $c = X_1 \bigcup X_2 \bigcup \cdots \bigcup X_i$, and $r_i(\mathcal{H}) = r_i$ is the sum total of all feasible i partition .

Definition 1.5.[2] Let S be a subset of the vertice set X of mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if the set does not contain any $\mathcal{C}$-hyperedge ($\mathcal{D}$-hyperedge) as a subset, then it is called $\mathcal{C}$ stable or $\mathcal{C}$ independent ($\mathcal{D}$ stable or $\mathcal{D}$ independent).

Lemma 1.1.[4,5] Let mixed hypergraph $\mathcal{H} = (X, \binom{X}{r}, \mathcal{D})$, where $2 \leq r \leq n = n(\mathcal{H})$, then arbitrary a coloring of $\mathcal{H}$ meet condition

$$\overline{\chi}(\mathcal{H}) = r - 1.$$

Definition 1.6.[6] For mixed hypergraph for $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if there is a mapping $c : Y \to \{1, 2, \cdots, \lambda\}$ that between subset $Y \in X$ and λ colors $\{1, 2, \cdots, \lambda\}$, and it meet following conditions:

1) For each $\mathcal{C}$-hyperedge $C \in \mathcal{C}$, at least two vertice are the same color;

2) For each $\mathcal{D}$-hyperedge $D \in \mathcal{D}$, at least two vertice are different colors.

Then mapping c is called as one λ colors normal coloring of the mixed hypergraph $\mathcal{H}$.

Definition 1.7.[2] In a normal $i-$coloring of $\mathcal{H}$, if i colors are used, then the coloring is called as a strict i-coloring.

It is clear that a normal $\chi(\mathcal{H})$ coloring of mixed hypergraph $\mathcal{H}$ must be a strict coloring.

Definition 1.8.[2] For any coloring c of the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, let Y is a subset of X, then if Y satisfied: arbitrary $y_1 \in Y$ and $y_2 \in Y$, there is $c(y_1) = c(y_2)$, then we call the subset Y as monochromatic, if each of two is different colors, that is $c(y_1) \neq c(y_2)$, then we call subset Y as the polychromatic.

By the definition of the normal coloring of the mixed hypergraph, we know that for any normal coloring of hypergraph, a $\mathcal{D}$-hyperedge not is a subset of monochromatic, a $\mathcal{C}$-hyperedge not is a subset of polychromatic.

Definition 1.9.[2] In arbitrary a strictly i-coloring of $\mathcal{H}$, the vertex set X of $\mathcal{H}$ is divided into i partitions, and each partition is a non-empty subset of monochromatic, we call it as the color class.

Lemma 1.2.[2] Let mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, and $n = |X|$, then regardless $\mathcal{H}$ can normally coloring or not can, but the coloring of its sub-hypergraph $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{D}}$ is always available and there is $\chi(\mathcal{H}_{\mathcal{C}}) = 1$, $r_1(\mathcal{H}_{\mathcal{C}}) = 1$, $\overline{\chi}(\mathcal{H}_{\mathcal{D}}) = n(\mathcal{H}), r_n(\mathcal{H}_{\mathcal{D}}) = 1$.

Lemma 1.3.[2] For mixed hypergraph $\mathcal{H}' = (X, \mathcal{C}, \binom{X}{m})$, if $\forall C \in \mathcal{C}$, where $|C| = k$ and $n(\mathcal{H}') \leq (k-1)(m-1)$, then $\overline{\chi}(\mathcal{H}') \geq k - 1$.

Lemma 1.4. For mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, let $\mathcal{H}^{'}$ is arbitrary a subhypergraph of $\mathcal{H}$, the $\chi(\mathcal{H}) \geq \chi(\mathcal{H}^{'})$, $\bar{\chi}(\mathcal{H}) \leq \bar{\chi}(\mathcal{H}^{'})$.

Lemma 1.5.[7] For a colorable mixed hypergraph $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$, where $2 \leq l, m \leq n$, then

1) $\chi(\mathcal{H}) = \lceil \frac{n(\mathcal{H})}{m-1} \rceil$, $\overline{\chi}(\mathcal{H}) = l - 1$;

2) $\mathcal{H}$ is the uncolorable if and only if $\lceil \frac{n(\mathcal{H})}{m-1} \rceil \geq l$.

Lemma 1.6.[7] For $\mathcal{C}$-hypergraph $\mathcal{H} = (X, \mathcal{C}, \emptyset)$, if $\forall C \in \mathcal{C}$ with $|C| \geq k$, then $\overline{\chi}(\mathcal{H}) \geq k-1$, and all $(k-1)$-coloring of $\mathcal{H}$ are normal.

Lemma 1.7. Set $X = (X_1, X_2, \cdots, X_i)$ as the partition which correspond to an i-coloring of the mixed hypergraph $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$ (It not necessarily is normal coloring), and $|X_1| = n_1, |X_2| = n_2, \cdots, |X_i| = n_i$, k is the number of the polychromatic $\mathcal{C}$-hyperedges from this partition, then

1) when $i < l$, have $k = 0$;

2) when $i \geq l$, have

$$k = \sum_{\{n_1', n_2', \cdots, n_l'\} \subseteq \{n_1, n_2, \cdots, n_i\}} n_1' n_2' \cdots n_l'.$$

Proof. 1) Clearly, when $i < l$, a arbitrary $\mathcal{C}$-hyperedges have at least two vertices whose color is the same, then $k = 0$.

2) When $i \geq l$, it is clear that $k > 0$. Because the vertices in the polychromatic $\mathcal{C}$-hyperedges have various colors from different classes, and for the uniform mixed hypergraph $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$, have $|C| = l$, so the $\mathcal{C}$-hyperedges which accoding to following steps are agained have to be the polychromatic:

i) From the i color classes arbitrarily selected out l color classes $X_1', X_2', \cdots, X_l'$;

ii) From the X_i', $i = 1, 2, \cdots, l$ arbitrarily selection out l vetices x_i, $i = 1, 2, \cdots, l$, this l vetices formation a polychromatic $\mathcal{C}$-hyperedges.

It is clear that all polychromatic $\mathcal{C}$-hyperedges are included in which are gained through the above-mentioned methods.

Therefore

$$k = \sum_{\{n_1', n_2', \cdots, n_l'\} \subseteq \{n_1, n_2, \cdots, n_i\}} n_1' n_2' \cdots n_l'.$$

§2. The main results

Theorem. For the complete (l, m)-uniform mixed hypergraphs $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$, let $n = |X|$, c be a i coloring of $\mathcal{H}$, and each of $\mathcal{D}$-hyperedges of $\mathcal{H}$ is normally colored by it (c not necessarily is a normal coloring of $\mathcal{H}$), k is the number of polychromatic $\mathcal{C}$-hyperedges under c, then

1) When $n < (l-1)(m-1)+1$, $k_{min} = 0$;

2) When $n \geq (l-1)(m-1)+1$, let $n = q(m-1) + r(0 \leq r < m-1)$, for a fixed n, have $k_{min} = \binom{q}{l-1}(m-1)^{l-1} r + \binom{q}{l}(m-1)^l$.

In particular, when $n = (l-1)(m-1) + r(0 < r \leq m-1)$, have $k_{min} = r(m-1)^{l-1}$, to all $n \geq (l-1)(m-1)+1$ and to all coloring c which all $\mathcal{D}$-hyperedges are normally colored by it, have $k_{min} = (m-1)^{(l-1)}$.

Proof. 1) Clearly, when $n < (l-1)(m-1)+1$, $k \geq 0$. The following we only can prove that the existence of a coloring i which meeting with $k = 0$, then this conclusion is true. It is clear, for each of $\mathcal{D}$-hyperedges coloring is normal, the vertex number in each of color classes no more

than $m-1$. Thus, the following method may be use for coloring the complete (l,m)-uniform mixed hypergraph $\mathcal{H}=(X,\binom{X}{l},\binom{X}{m})$.

Select out arbitrary $m-1$ vertices from the vetex set X of $\mathcal{H}$ and give color 1 to them, then select out $m-1$ vertices from the remaining $n-m+1$ vertices and give color 2 to them, $\cdots$, followed by and so on, until when the number of remaining vertices less than $m-1$, give all of remaining vertices with a new color which differented from above all colores.

Since $n<(l-1)(m-1)+1$, so that the color number which yielded in the coloring process no more than $l-1$. Thus, in this coloring process, coloring of all $\mathcal{C}$-hyperedges are normal. Therefore, $k_{min}=0$.

2) Because all $\mathcal{D}$-hyperedges are normally colored by c so that the number of vertices which contained in each of color classes not more than $m-1$, if $n\geq(l-1)(m-1)+1$ and $n=q(m-1)+r(0\leq r<m-1)$, then the number of colores used in coloring c is at least q (when $r=0$) or $q+1$ (when $0<r<m-1$) and not less than l, that is $i\geq q\geq l$.

When the number of vertices n is the same, among all coloring programs which each of $\mathcal{D}$-hyperedges of $\mathcal{H}=(X,\binom{X}{l},\binom{X}{m})$ is normally colored by they, exception for that those the color number is q (when $r=0$) and $q+1$ (when $r>0$), the each of the rest coloring program can see as that it is gained through moving vertices from some color classes of the coloring program of the color number is the q (when $r=0$) or $q+1$ (when $r>0$) into the other color classes. And by Lemma 1.7, we can see that for two different coloring programs c_1 and c_2, if the number of colors both used are the same and the vertex number included in each of color classes of c_1 equal to that of c_2 when appropriate exchange the order of color classes. Then, the number of the polychromatic $\mathcal{C}$-hyperedges from both must are equal. Thereby we as long as prove that the number of the polychromatic $\mathcal{C}$-hyperedges from coloring program $c=\{x_{11},x_{12},\cdots,x_{1(m-1)}\}\bigcup\{x_{21},x_{22},\cdots,x_{2(m-1)}\}\bigcup\cdots\bigcup\{x_{q1},x_{q2},\cdots,x_{q(m-1)}\}\bigcup\{x_{(q+1)1},x_{(q+1)2},\cdots,x_{(q+1)r}\}$, where vertex number $n=q(m-1)+r(0<r<m-1)$ and the color number $i=q+1$ is less than that from the other coloring programes, then the conclusion is true. Follow is the proof.

Use the mathematical induction on vertex number moved.

Let the number of vertices moved is p, the division Corresponding with this coloring program is $c=\{x_{11},x_{12},\cdots,x_{1(m-1)}\}\bigcup\{x_{21},x_{22},\cdots,x_{2(m-1)}\}\bigcup\cdots\bigcup\{x_{q1},x_{q2},\cdots,x_{q(m-1)}\}\bigcup\{x_{(q+1)1},x_{(q+1)2},\cdots,x_{(q+1)r}\}$.

Its graph representation is as follow (see Fig. 1)

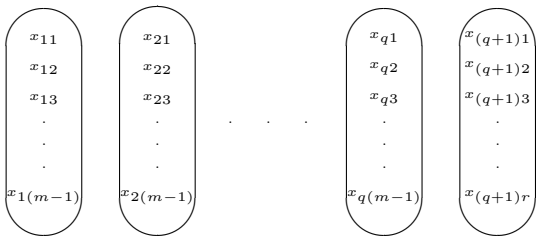

Fig.1

First of all, we prove that when $p=1$, the conclusions is true. By Lemma 1.7, we know that

the number of the polychromatic $\mathcal{C}$-hyperedges that correspond with the coloring $c = \{x_{11}, x_{12}, \cdots, x_{1(m-1)}\} \bigcup \{x_{21}, x_{22}, \cdots, x_{2(m-1)}\} \bigcup \cdots \bigcup \{x_{q1}, x_{q2}, \cdots, x_{q(m-1)}\} \bigcup \{x_{(q+1)1}, x_{(q+1)2}, \cdots, x_{(q+1)r}\}$ is

$$k_c = \sum_{\{X_1', X_2', \cdots, X_l'\} \subseteq \{X_1, X_2, \cdots, X_{q+1}\}} |X_1'||X_2'| \cdots |X_l'|,$$

where $X_1 = \{x_{11}, x_{12}, \cdots, x_{1(m-1)}\}, X_2 = \{x_{21}, x_{22}, \cdots, x_{2(m-1)}\}, \cdots, X_q = \{x_{q1}, x_{q2}, \cdots, x_{q(m-1)}\}, X_{q+1} = \{x_{(q+1)1}, x_{(q+1)2}, \cdots, x_{(q+1)r}\}$. Because $|X_1| = |X_2| = \cdots = |X_q| = m - 1, |X_{q+1}| = r$, then,

$$k_c = \binom{q}{l-1}(m-1)^{l-1}r + \binom{q}{l}(m-1)^l.$$

Clearly, to take out a vertex from any color class and to put into other a color class, only include the following situations:

Case 1. Take out a vertex from one of color classes $X_1, X_2, \cdots, X_q$ and put it into other a color classes;

Case 2. Take out a vertex from color class X_{q+1} and put this vertex into the new color class X_{q+2}.

First we prove that Case 1.

Where Case 1 was divided into the following two circumstances:

Case 1.1. The vertex took out from one of color class $X_1, X_2, \cdots, X_q$ is put into X_{q+1};

Case 1.2. The vertex took out from one of color class $X_1, X_2, \cdots, X_q$ is put into X_{q+2}.

For Case 1.1, let the number of polychromatic $\mathcal{C}$-hyperedges corresponding with the coloring program obtained by moving this vertex for k_1, and we may let that take out the vertex x_{js} from color Class $X_j (j \in \{1, 2, \cdots, q\})$ and put x_{js} into the color class X_{q+1}. It is clear that after make such a movement of vertex x_{js} comparison with the mobile ago, only two color classes X_j and X_{q+1} their vertex number is changed, and the vertex number of that included in the remaining color classes still is the same before mobile. Let that the division after moving vertex x_{js} as follows (see Fig. 2)

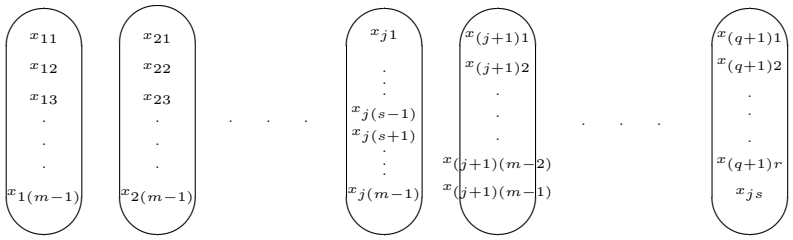

Fig.2

By Lemma 1.7, we know

$$\begin{aligned} k_1 - k_c &= \binom{q-1}{l-1}(m-1)^{l-1}(r+1) + \binom{q-1}{l}(m-1)^l \\ &+ \binom{q-1}{l-1}(m-1)^{l-1}(m-2) + \binom{q-1}{l-2}(m-1)^{l-2}(m-2)(r+1) \\ &- [\binom{q}{l-1}(m-1)^{l-1}r + \binom{q}{l}(m-1)^l]. \end{aligned}$$

For the calculation of value of $k_1 - k_2$, we do not use conventional methods of algebraic deformation, but consider about the essence of the calculation of Lemma 1.7 gain the values of k_1 and k_c with follow methods: by Lemma 1.7, we know that the polychromatic $\mathcal{C}$-hyperedges corresponding to someone coloring program of $\mathcal{H}$ can be seen as obtained through method of that as follows: the first select out l color classes from all-color classes, and then select out a vertex different from everyone of l color classes. All of that selected out l vertices are a polychromatic $\mathcal{C}$- hyperedges. The number of this kind selecting methods above is equivalent to the sum total of polychromatic $\mathcal{C}$-hyperedges. Therefore, according to this we known that all of the polychromatic $\mathcal{C}$-hyperedges can be divided into two types: one is that its some vertices are from the color class X_j or X_{q+1}; The other is that all of vertices not are from X_j and X_{q+1}. Because for the coloring programs corresponding to k_1 and k_c, among all their color classes, exception for X_j or X_{q+1}, others all color classes did not make any changement. Therefore, only the polychromatic $\mathcal{C}$-hyperedges which adjacented to the color class X_j or X_{q+1} for $k_1 - k_c$'s contribution is not necessarily zero, and the rest of the polychromatic $\mathcal{C}$-hyperedges are zero contribution to $k_1 - k_c$.
So

$$\begin{aligned} k_1 - k_c &= \binom{q-1}{l-1}(m-1)^{l-1}(m-2) + \binom{q-1}{l-1}(m-1)^{l-1}(r+1) \\ &\quad + \binom{q-1}{l-2}(m-1)^{l-2}(m-2)(r+1) - [\binom{q-1}{l-1}(m-1)^{l} \\ &\quad + \binom{q-1}{l-1}(m-1)^{l-1}r + \binom{q-1}{l-2}(m-1)^{l-1}r] \\ &= \binom{q-1}{l-1}(m-1)^{l-1}[m-2+r+1-m+1-r] \\ &\quad + \binom{q-1}{l-2}(m-1)^{l-2}[mr-2-2r+m-mr+r] \\ &= \binom{q-1}{l-2}(m-1)^{l-2}[m-(2+r)]. \end{aligned} \tag{2}$$

As a prerequisite condition of that proved this result is $n \geq (l-1)(m-1)+1$, and $n = q(m-1)+r(0 < r < m-1)$, therefore, $m > r+1$, so $m-(2+r) \geq 0$, and then by formula (2), we can see $k_1 \geq k_c$. Therefore, the conclusion is true.

For Case 1.2, let the number of the polychromatic $\mathcal{C}$-hyperedges producted from coloring program that corresponding with the division obtained by move the vertex for k_1, and let that we take out x_{js} from color class $X_j(j \in \{1, 2, \cdots, q\})$, and put it into color class X_{q+2}. Clearly, the color classes of that after vertex moved compared to that before vertex moved, only X_j and X_{q+2} have the vertices changed, and the vertic number of the remaining color classes still is the same before. The new division obtained by moving vertex is as follows (see Fig. 3)

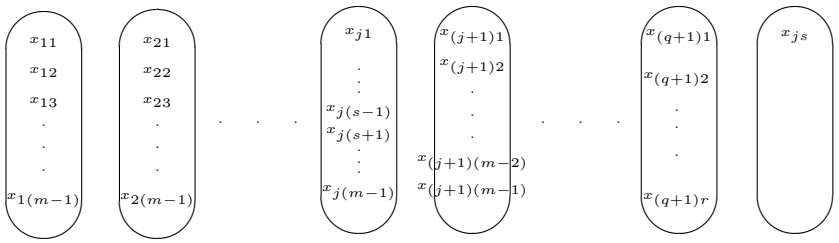

Fig.3

Similar with Case 1.1, we can obtain that

$$\begin{aligned} k_1 - k_c &= \binom{q-1}{l-1}(m-1)^{l-1}(m-2) + \binom{q-1}{l-2}(m-1)^{l-2}(m-2)r \\ &\quad + \binom{q-1}{l-1}(m-1)^{l-1} + \binom{q-1}{l-2}(m-1)^{l-2}(m-2) \\ &\quad + \binom{q-1}{l-2}(m-1)^{l-2}r + \binom{q-1}{l-3}(m-1)^{l-3}(m-2)r \\ &\quad - [\binom{q-1}{l-1}(m-1)^{l} + \binom{q-1}{l-2}(m-1)^{l-1}r] \\ &= \binom{q-1}{l-1}(m-1)^{l-1}[m-2+1-(m-1)] \\ &\quad + \binom{q-1}{l-2}(m-1)^{l-2}[(m-2)r + (m-2) + r - (m-1)r] \\ &\quad + \binom{q-1}{l-3}(m-1)^{l-3}(m-2)r \\ &= \binom{q-1}{l-2}(m-1)^{l-2}(m-2) + \binom{q-1}{l-3}(m-1)^{l-3}(m-2)r \\ &> 0. \end{aligned}$$

Therefore, $k_1 > k_c$. Conclusions also is true.

Next to prove Case 2.

Similar with the provement of the Case 1, we let the number of the polychromatic $\mathcal{C}$-hyperedges producted from coloring program that corresponding with the division obtained by move the vertex for k_1, and let that we take $x_{(q+1)r}$ out from color class X_{q+1}, and put it into color class X_{q+2}. Clearly, the color classes of after vertex moved compared to that before vertex moved, only X_{q+1} and X_{q+2} have that vertices changed, and the vertic number of the remaining color classes still is the same before. The new division obtained by moving vertex is as follows (see Fig. 4):

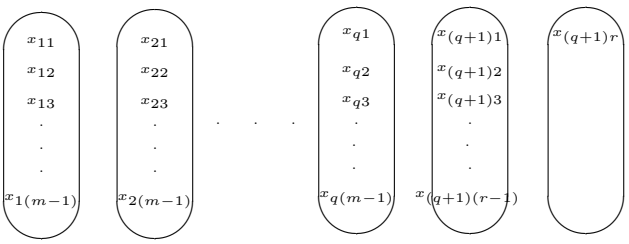

Fig.4

By Lemma 1.7 we gain that

$$\begin{aligned} k_1 - k_c &= \binom{q}{l-1}(m-1)^{l-1}(r-1) + \binom{q}{l-1}(m-1)^{l-1} \\ &\quad + \binom{q}{l-2}(m-1)^{l-2}(r-1) - \binom{q}{l-1}(m-1)^{l-1}r \\ &= \binom{q}{l-1}(m-1)^{l-1}(r-1+1-r) + \binom{q}{l-2}(m-1)^{l-2}(r-1) \\ &= \binom{q}{l-2}(m-1)^{l-2}(r-1) \\ &\geq 0. \end{aligned}$$

Thus, $k_1 \geq k_c$. Then the conclusions of this theorem is true.

Synthesis over provement, we know that the conclusions of this theorem is true when $p = 1$.

We let that when $p = t$, the conclusion also is true. Next to we prove that when $p = t + 1$ the conclusion also is true.

Because the division corresponding with $p = t$ is obtained by moving t vertices of the division (indicated in Fig. 1) from its some color classes to other some color classes, and in this a process of moving vertex, we can always keep that the number of vertices in each of color classes are descending order from left to right.

By Lemma 1.7, we can see that this does not affect the value of k. This is very obvious, because even if the descending order of the number of vertices in each of all color classes from left to right is destroyed by moving vertices, as long as appropriate exchange location of each of color classes, all of color classes are re-ordered with descending order from left to right and this exchange will not affect the value of k. Thus, we can let that when $p = t$, the corresponding division for that indicated in Fig. 5.

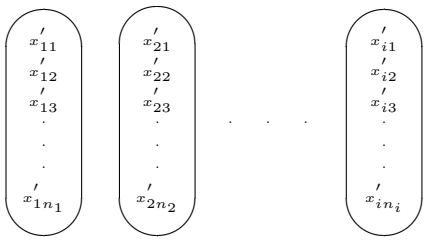

Fig.5

And the vertices numbers of all of the color classes meet:

$$n_1 \geq n_2 \geq \cdots \geq n_i.$$

Obviously $i \geq q + 1$. At the same time, we let that the total number of the polychromatic $\mathcal{C}$-hyperedges that producted by coloring program which corresponding $p = t$ for k_t, then when $p = t + 1$, the corresponding division is obtained by moving a vertex of someone color class of the division for $p = t$ to other one color class. Therefore, when $p = t + 1$, that corresponding division may is the following two situations:

1. Choose a vertex and put it into other color classes, but does not increase the chromatic number;

2. Choose a vertex and put it into the new color class, namely, the chromatic number is increased.

In both cases, following we give out the proof of this problem.

First we prove that Case 1.

Let that the division corresponding with $p = t+1$ be obtained by taking out $x_{j_1 s}$ from the j_1-the color class of the division that indicated in Fig. 5 and put it into the j_2-th color class of this division. At the same time, we may let $j_1 \leq j_2 \leq i$, as for the case of $j_1 \geq j_2$, we can similarly give out the proof. Then similar with above proof, we have that

$$\begin{aligned}
k_{t+1} \quad & -k_t \\
= \quad & n_{j_1} \sum_{\{j'_1,j'_2,\cdots,j'_{l-1}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-1}} \\
& +n_{j_2} \sum_{\{j'_1,j'_2,\cdots,j'_{l-1}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-1}} \\
& +n_{j_1} n_{j_2} \sum_{\{j'_1,j'_2,\cdots,j'_{l-2}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-2}} \\
& -(n_{j_1}-1) \sum_{\{j'_1,j'_2,\cdots,j'_{l-1}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-1}} \\
& -(n_{j_2}+1) \sum_{\{j'_1,j'_2,\cdots,j'_{l-1}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-1}} \\
& -(n_{j_1}-1)(n_{j_2}+1) \sum_{\{j'_1,j'_2,\cdots,j'_{l-2}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-2}} \\
= \quad & \sum_{\{j'_1,j'_2,\cdots,j'_{l-1}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-1}} (n_{j_1}-1+n_{j_2}+1-n_{j_1}-n_{j_2}) \\
& + \sum_{\{j'_1,j'_2,\cdots,j'_{l-2}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-2}} [(n_{j_1}-1)(n_{j_2}+1)-n_{j_1} n_{j_2}] \\
= \quad & \sum_{\{j'_1,j'_2,\cdots,j'_{l-2}\}\subseteq\{1,2,\cdots,i\}\setminus\{j_1,j_2\}} n_{j'_1} n_{j'_2} \cdots n_{j'_{l-2}} [n_{j_1}-(n_{j_2}+1)]. \qquad (3)
\end{aligned}$$

If that we move each vertex by follow method: to maintain the vertices numbers of all of color classes are in descending order from left to right throughout the entire process of moving vertices. Then we want to achieve the above the movement of that for $p = t+1$, the movement must meet with the $n_(j_1) \geq n_(j_2) + 1$. In fact, easy to know from Lemma 1.7, the value of k corresponding to arbitrary coloring i must be equal to the number of polychromatic $\mathcal{C}$-hyperedges from the coloring program corresponding to someone division obtained through making a vertices special movement that according to above rules. Therefore, for the range of the value of k, it is enough to only make the special movement as above. Thus, by formula (3) and $n_{j_1} \geq n_{j_2} + 1$, we know that $k_{t+1} \geq k_t$. Also from the assumption we can see: $k_t \geq k_c$, thus $k_{t+1} \geq k_c$.

Next to we prove that Case 2.

Let that we take vertex x_{js} out from the j-th color class of the division in graph 5 and put it into the $i+1$-th color class of this division, then similar with above proof we gain:

$$\begin{aligned}
k_{t+1}-k_t &= (n_j-1)\sum_{\{j_1,j_2,\cdots,j_{l-1}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-1}} \\
&\quad+\sum_{\{j_1,j_2,\cdots,j_{l-1}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-1}} \\
&\quad+(n_j-1)\sum_{\{j_1,j_2,\cdots,j_{l-2}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-2}} \\
&\quad-n_j\sum_{\{j_1,j_2,\cdots,j_{l-1}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-1}} \\
&= \sum_{\{j_1,j_2,\cdots,j_{l-1}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-1}}(n_j-1+1-n_j) \\
&\quad+(n_j-1)\sum_{\{j_1,j_2,\cdots,j_{l-2}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-2}} \\
&= (n_j-1)\sum_{\{j_1,j_2,\cdots,j_{l-2}\}\subseteq\{1,2,\cdots,j-1,j+1,\cdots,i\}} n_{j_1}n_{j_2}\cdots n_{j_{l-2}} \\
&\geq 0.
\end{aligned}$$

That is, $k_{t+1}\geq k_t$. By inductive assumption, we can see $k_t\geq k_c$, therefore, $k_{t+1}\geq k_c$.

So, when $p=t+1$, the conclusions are true.

Integrated above, we can see that the conclusions are true for the arbitrary $p\in N$.

Namely, to certain $n\geq(l-1)(m-1)+1$, if let $n=q(m-1)+r(0<i\leq m-1)$, then for arbitrary a coloring program c which with all $\mathcal{D}$-hyperedges are normally colored through it, k (the number of the polychromatic $\mathcal{C}$-hyperedges from it) not less than that k_c, where k_c is from coloring $c=\{x_{11},x_{12},\cdots,x_{1(m-1)}\}\bigcup\{x_{21},x_{22},\cdots,x_{2(m-1)}\}\bigcup\cdots\bigcup\{x_{q1},x_{q2},\cdots,x_{q(m-1)}\}\bigcup\{x_{(q+1)1},x_{(q+1)2},\cdots,x_{(q+1)r}\}$.

Clearly,

$$k_c=\sum_{\{X_1',X_2',\cdots,X_l'\}\subseteq\{X_1,X_2,\cdots,X_{q+1}\}}|X_1'||X_2'|\cdots|X_l'|,$$

where $X_1=\{x_{11},x_{12},\cdots,x_{1(m-1)}\},X_2=\{x_{21},x_{22},\cdots,x_{2(m-1)}\},\cdots,X_q=\{x_{q1},x_{q2},\cdots,x_{q(m-1)}\},X_{q+1}=\{x_{(q+1)1},x_{(q+1)2},\cdots,x_{(q+1)r}\}$.

Consequently, when $n\geq(l-1)(m-1)+1$, if let $n=q(m-1)+r(0\leq r<m-1)$, then to a certain n, have $k_{min}=\binom{q}{l-1}(m-1)^{l-1}r+\binom{q}{l}(m-1)^l$.

Through the above-mentioned proved process of and the last conclusion above we easy to know that, if $n=(l-1)(m-1)+r(0<r\leq m-1)$, to all of coloring programe which with all of the $\mathcal{D}$-hyperedges are normally colored through they, have $k_{min}=r(m-1)^{l-1}$.

Follow we prove the second half of the conclusion (2) of the theorem.

First prove the following assertion:

Assertion. The minimum of k above is increase with n (n is the number of vertices of $\mathcal{H}$).

Proof of this assertion. We know from the above proof that the minimum of k also is equal when the vertex number is equal. Here we think two the complete uniform mixed hypergraph $\mathcal{H}_1 = (X_1, \binom{X_1}{l}, \binom{X_1}{m})$ and $\mathcal{H}_2 = (X_2, \binom{X_2}{l}, \binom{X_2}{m})$, the number of their vertices different are n_1 and n_2, and $n_1 > n_2 \geq (l-1)(m-1)+1$. Then exist $r > 0$ so that $n_1 = n_2 + r$. We will divide the set of vertices of $\mathcal{H}_1$ into two parts: take n_2 vertices out from $|X|$ as the first part; Remaining r vertices as the other part. For anyone coloring c of $\mathcal{H}_1$ which all $\mathcal{D}$-hyperedges are normally colored through it, then the polychromatic $\mathcal{C}$-hyperedges from it also can divide into two kinds: one kind is the their vertices are all from the first part above, namely, they do't include the vertices from the second part above; Other kind is that at least include one vertex from the second part above.

Let that the sum total of the polychromatic $\mathcal{C}$-hyperedges from the first kind be k_{n_2} and that from the second kind is k_r where coloring program c is one that for k reach to the minimum. At same time, we let that the minimum of the polychromatic $\mathcal{C}$-hyperedges from coloring program c which all $\mathcal{D}$-hyperedges are normally colored through it different is k^1_{min} for $\mathcal{H}_1$ and k^2_{min} for $\mathcal{H}_2$, then $k^1_{min} = k_{n_2} + k_r \geq k^2_{min} + k_r > k^2_{min}$. This is because $r > 0$, so have $k_r > 0$.

Therefore, the assertion is proved.

By the assertion and the proof of the preceding theorem, we can see that for all $n \geq (l-1)(m-1)+1$ and all coloring c which each of $\mathcal{D}$-hyperedges is normally colored, k reach the minimum if and only if $n = (l-1)(m-1)+1$ and when the vertex numbers of all color classes different are $\underbrace{m-1, m-1, \cdots, m-1}_{(l-1)}, 1$, by Lemma 1.7, we can see that $k_{min} = (m-1)^{l-1}$.

Theorem is proved.

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AC-algebras

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Abstract In this paper we introduce an AC-algebra as an algebra $(X, *, 0)$ with a binary operation $*$ which satisfies the conditions; $x*(y*z) = (x*y)*z$, $x*y = y*x$ and $x*y = 0$ if and only if $x = y$ for all x, y, z in X. We show that an AC-algebra is an Abelian group and that a subset of an AC-algebra is a subalgebra if and only if it is an ideal. We also present an algebraic structure called a quotient AC-algebra relative to a subalgebra of the AC-algebra and show that the quotient algebra is also an AC-algebra. Moreover,we define functions on an AC-algebra and prove that a set of finite compositions of these functions is also an AC-algebra and show the properties of this set.

Keywords AC-algebra, quotient AC-algebra, functions on AC-algebra.

§1. Introduction

By an algebra $X = (X, *, 0)$ we mean a non-empty set X together with a binary operation $*$ and some distinguished element 0.

Kondo [1] studied an algebraic structure called a BCI-algebra which is an algebra $(X, *, 0)$ with a binary operation $*$ which, for all $x, y, z \in X$, satisfies the four properties:

1. $((x*y)*(x*z))*(z*y) = 0$;
2. $(x*(x*y))*y = 0$;
3. $x*x = 0$;
4. $x*y = y*x = 0$ implies that $x = y$.

In 2003, Roh et al. [2] introduced a difference algebra as an algebraic structure which, for all $x, y, z \in X$, satisfies the five properties:

1. $(X, \leq)$ is a poset;
2. $x \leq y$ implies $x*z \leq y*z$;
3. $(x*y)*z \leq (x*z)*y$;
4. $0 \leq x*x$;

5. $x \leq y$ if and only if $x * y = 0$.

In this paper we first introduce an algebraic structure called an AC-algebra and study its properties. We then define a quotient AC-algebra and show that it is also an AC-algebra. Finally, we define functions on an AC-algebra and prove that a finite composition of these functions is an AC-algebra.

§2. Definition and properties of an AC-algebra

We define an AC-algebra as follows:

Definition 2.1. An algebra $(X, *, 0)$ with a binary operation $*$ is called an AC-Algebra if it satisfies the conditions:

[AC-1] $x * (y * z) = (x * y) * z$;

[AC-2] $x * y = y * x$;

[AC-3] $x * y = 0$ if and only if $x = y$, for all x, y, z in X.

It is easy to show that the following properties are true for an AC-algebra. For all x, y, z in X:

1. $(x * y) * z = (x * z) * y$;
2. $(x * (x * y)) * y = 0$;
3. $x * 0 = x$;
4. $0 * (x * y) = (0 * x) * (0 * y)$;
5. $((x * z) * (y * z)) * (x * y) = 0$;
6. $((x * y) * (x * z)) * (z * y) = 0$;
7. $x * y = 0$ if and only if $(x * z) * (y * z) = 0$;
8. $x * y = 0$ if and only if $(z * x) * (z * y) = 0$;
9. $x * y = x$ if and only if $y = 0$;
10. $x * (((x * y) * y) * x) * x) = y * (((y * x) * x) * y) * y)$;
11. $x = y * (y * x)$;
12. $(x * (x * y)) * (x * y) = y * (y * x)$.

Example 2.1. Let $X = \{(0,0), (0,1), (1,0), (1,1)\}$ and let $*$ be defined by

$*$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

Then $(X, *, (0, 0))$ is an AC-algebra.

Example 2.2. Let $B = \{0, 1\}$ be the set of binary digits. Let $B^n = \{(x_1, x_2, \cdots, x_n) | x_j \in B,\ 1 \leq j \leq n < \infty\}$ be an ordered n-tuple of binary digits. Let $*$ be the operation $+$ for binary digits (i.e., $0+0=0,\ 0+1=1,\ 1+0=1,\ 1+1=0$). Then $(B^n, +, \mathbf{0})$, where $\mathbf{0} = (0, 0, \cdots, 0)$, is an AC-algebra. Note that Example 2.1 is the special case of $n = 2$.

Example 2.3. Let $B = \{F, T\}$, where F means *false* and T means *true*. Let $B^n = \{(x_1, x_2, \cdots, x_n) | x_j \in B,\ 1 \leq j \leq n < \infty\}$ be an ordered n-tuple of *false* or *true*. Let $*$ be the logical operation *xor* (exclusive or), i.e., $F\ xor\ F = F,\ F\ xor\ T = T,\ T\ xor\ F = T,\ T\ xor\ T = F$. Let $\mathbf{F}$ be the n-tuple with every entry *false*. Then $(B^n, xor, \mathbf{F})$ is an AC-algebra.

Definition 2.2. A non-empty subset I of X is called an ideal of X if

[I_1] $0 \in I$;

[I_2] $x * y \in I$ *and* $y \in I$ *imply* $x \in I$ for all x, y in X.

Definition 2.3. A non-empty subset S of X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

Theorem 2.1. S is a subalgebra of an AC-algebra X if and only if S is an ideal of X.

Proof. Let S be a subalgebra of X. For every $x, y \in X$ we have $0 = x * x \in S$. We suppose that $y, x * y \in S$ and we will show that $x \in S$. Since S is a subalgebra of X, we have $(x * y) * y \in S$. By [AC-1] and [AC-3], we obtain $x = x * 0 = x * (y * y) = (x * y) * y \in S$. Hence S is an ideal of X.

Conversely, suppose that S is an ideal of X. For every $x, y \in S$ we have $(x * y) * x = (x * x) * y = 0 * y = y * 0 = y \in S$. By the definition of an ideal, we conclude that $x * y \in S$. Hence S is a subalgebra of X.

Theorem 2.2. An AC-algebra is an Abelian group and each element is its own inverse.

Proof. An AC-algebra $(X, *, 0)$ has the following four properties under the operation $*$. For all $x, y, z \in X$,

1. $x * y \in X$.
2. $x * (y * z) = (x * y) * z$ [AC-1].
3. There exists an identity 0 such that $x * 0 = x$. Property 3 in section 2.
4. $x * x = 0$ and $x * y = 0$ if and only if $x = y$ [AC-3]. Therefore x is the unique inverse of x.

Therefore an AC-algebra is a group. Further, from [AC-2], $x * y = y * x$ and therefore an AC-algebra is an Abelian group.

Next we construct a quotient AC-algebra and study its properties.

§3. Definition and properties of a quotient AC-algebra

Definition 3.1. Let I be a subalgebra of an AC-algebra $(X, *, 0)$. For every $x, y \in X$, we define $x \sim_I y$ if and only if $x * y \in I$.

Theorem 3.1. Let I be a subalgebra of an AC-algebra $(X, *, 0)$. The relation $\sim_I$ is a congruence relation on X.

Proof. We first prove $\sim_I$ is a reflexive, symmetric and transitive relation, i.e., it is an equivalence relation.

1. Reflexive property. From [AC-3], $x * x = 0$ and $0 \in I$. Therefore $x \sim_I x$.

2. Symmetric property. From [AC-2], $x*y = y*x$. Therefore, $x*y \in I$ if and only if $y*x \in I$ and hence $x \sim_I y$ if and only if $y \sim_I x$.

3. Transitive property. If $x \sim_I y$ and $y \sim_I z$ then $x * y \in I$ and $y * z \in I$. Therefore, since I is a subalgebra, $(x * y) * (y * z) \in I$. Then, from [AC-1], [AC-3] and property 3 of section 2, we have

$$(x * y) * (y * z) = ((x * y) * y) * z = (x * (y * y)) * z = (x * 0) * z = x * z,$$

and therefore $x * z \in I$ and hence $x \sim_I z$.

We next prove the congruence property that $x \sim_I y$ and $u \sim_I v$ implies $(x * u) \sim_I (y * v)$ for all $x, y, u, v \in X$.
If $x \sim_I y$ and $u \sim_I v$, we have $x * y \in I$ and $u * v \in I$. Therefore, since I is a subalgebra, we have $(x * y) * (u * v) \in I$. Then, from [AC-1] and [AC-2], we have

$$(x * u) * (y * v) = (x * y) * (u * v) \in I.$$

Hence $(x * u) \sim_I (y * v)$.

Definition 3.2. Let I be a subalgebra of an AC-algebra $(X, *, 0)$ and let

$$[x]_I = \{y \in X | x \sim_I y\}.$$

We define the set X/I to be $X/I = \{[x]_I | x \in X\}$ and a binary operation "$\odot$" on X/I by $[x]_I \odot [y]_I = [x * y]_I$.

Theorem 3.2. Let I be a subalgebra of an AC-algebra $(X, *, 0)$ and $x, y \in X$. Then $[x]_I = [y]_I$ if and only if $x \sim_I y$.

Proof.

1. Assume $[x]_I = [y]_I$. Then $x \in [x]_I$ since $0 \in I$ and $x * x = 0$. Therefore, $x \in [y]_I$ and hence $x \sim_I y$.

2. Assume $x \sim_I y$. Let $z \in [x]_I$, then $z \sim_I x$. Therefore, using the transitive property of $\sim_I$ we have $z \sim_I y$ and hence $z \in [y]_I$. Similarly, we can prove if $z \in [y]_I$ then $z \in [x]_I$. That is, $[x]_I = [y]_I$.

Note that the operation $\odot$ is well-defined since $\sim_I$ is a congruence relation on X.

Theorem 3.3. Let I be a subalgebra of an AC-algebra $(X, *, 0)$ and let $[x]_I \odot [y]_I = [x*y]_I$. Then $(X/I, \odot, [0]_I)$ is an AC-algebra, which we call the quotient AC-algebra by subalgebra I.

Proof. Let $[x]_I, [y]_I, [z]_I \in X/I$. We will show that $(X/I, \odot, [0]_I)$ satisfies [AC-1], [AC-2] and [AC-3].

[AC-1]

$$\begin{aligned} ([x]_I \odot [y]_I) \odot [z]_I &= [x * y]_I \odot [z]_I = [(x * y) * z]_I \\ &= [x * (y * z)]_I = [x]_I \odot [y * z]_I = [x]_I \odot ([y]_I \odot [z]_I). \end{aligned}$$

[AC-2] $[x]_I \odot [y]_I = [x * y]_I = [y * x]_I = [y]_I \odot [x]_I$.

[AC-3] Suppose $[x]_I \odot [y]_I = [0]_I$. By definition of operation " $\odot$ ", we have $[x * y]_I = [0]_I$ and therefore from Theorem 3.2 we have $x * y \sim_I 0$. Then $(x * y) * 0 \in I$. But, from properties of AC-algebra, $(x * y) * 0 = x * y$ and hence $x * y \in I$. Therefore $x \sim_I y$ and then from Theorem 3.2 $[x]_I = [y]_I$.

Conversely, suppose $[x]_I = [y]_I$, then from Theorem 3.2, we have $x \sim_I y$ and therefore $x * y \in I$. By properties of AC-algebra we have $x * y = (x * y) * 0$ and therefore $(x * y) * 0 \in I$ and $x * y \sim_I 0$. Therefore, from Theorem 3.2, $[x * y]_I = [0]_I$ and thus $[x]_I \odot [y]_I = [0]_I$.

Hence $(X/I, \odot, [0]_I)$ is an AC-algebra.

§4. Definition and properties of functions

Let $(X, *, 0)$ be an AC-algebra. For any $a \in X$, we define a function $f_a : X \to X$ by $f_a(x) = x * a$ for all x in X.

Remark 4.1. $f_0(x) = x * 0 = x$ for all x in X.

Definition 4.1. Let $(X, *, 0)$ be an AC-algebra. We define composition of two functions on X by $f_a \circ f_b(x) = f_a(f_b(x)) = f_a(x * b) = (x * b) * a$ for all $a, b, x \in X$

Theorem 4.1. Composition of two functions on an AC-algebra is a commutative and associative binary operation.

Proof. If f_a, f_b, and f_c are three functions on an AC-algebra then, for all $x \in X$:

$$\begin{aligned} f_a \circ f_b(x) &= (x * b) * a = x * (b * a) = x * (a * b) = (x * a) * b = f_b \circ f_a(x) \\ (f_a \circ f_b) \circ f_c(x) &= (f_a \circ f_b)(x * c) = ((x * c) * b) * a = (x * c * b) * a = f_a(x * c * b) \\ &= f_a \circ (f_b \circ f_c)(x). \end{aligned}$$

Remark 4.2. From Theorem 4.1, we can omit the brackets in a product of functions and write:

$$(f_a \circ f_b) \circ f_c(x) = f_a \circ (f_b \circ f_c)(x) = f_a \circ f_b \circ f_c(x).$$

Remark 4.3. Because of the associativity of the $*$ operation [AC-1], we can write the composition of any number of functions in the form:

$$f_a \circ f_b \circ f_c \circ \cdots \circ f_m(x) = x * m * \cdots * c * b * a$$

i.e., all groupings of the functions in the composition will give the same function values and therefore brackets can be omitted. Also, because of the commutativity of the $*$ operation [AC-2], every ordering of composition of the functions will give the same value.

Theorem 4.2. Let $(X, *, 0)$ be an AC-algebra and $a, b \in X$. If $f_a \circ f_b(x) = f_0(x)$ for all $x \in X$ then $f_a(x) = f_b(x)$ for all $x \in X$.

Proof. $f_a \circ f_b(x) = x * b * a = f_0(x) = x$ for all x. Then from property 9 of section 2, we have $b * a = 0$ and therefore by [AC-2] and [AC-3], we have $a = b$. Therefore $f_a(x) = f_b(x)$ for all $x \in X$.

Next we define a set of finite products of functions under composition on an AC-algebra and show that this set is an AC-algebra.

Definition 4.2. Let $(X, *, 0)$ be an AC-algebra. We define

$$M(X) = \{f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_n} \mid a_j \in X;\ j = 1, 2, \ldots, n;\ 1 \leq n < \infty\}.$$

Theorem 4.3. If $(X, *, 0)$ is an AC-algebra, then $(M(X), \circ, f_0)$ is an AC-algebra.

Proof. Let $(X, *, 0)$ be an AC-algebra and $M(X)$ be defined as in Definition 4.2. Let

$$f_a \circ f_b \circ ... \circ f_k \in M(X),$$
$$f_m \circ f_n \circ ... \circ f_r \in M(X),$$
$$f_u \circ f_v \circ ... \circ f_z \in M(X).$$

Then, using the fact that brackets in products of functions can be omitted, we have for all $x \in X$

$$\begin{aligned} & [(f_a \circ f_b \circ \cdots \circ f_k) \circ (f_m \circ f_n \circ \cdots \circ f_r)] \circ (f_u \circ f_v \circ \cdots \circ f_z)(x) \\ = \ & [f_a \circ f_b \circ \cdots \circ f_k \circ f_m \circ f_n \circ \cdots \circ f_r] \circ (f_u \circ f_v \circ \cdots \circ f_z)(x) \\ = \ & f_a \circ f_b \circ \cdots \circ f_k \circ f_m \circ f_n \circ \cdots \circ f_r \circ f_u \circ f_v \circ \cdots \circ f_z(x) \\ = \ & (f_a \circ f_b \circ \cdots \circ f_k) \circ [(f_m \circ f_n \circ \cdots \circ f_r) \circ (f_u \circ f_v \circ \cdots \circ f_z)](x). \end{aligned}$$

Therefore $M(X)$ satisfies [AC-1]. Also, by a similar argument and using the properties of AC-algebras listed in section 2, we can show that $M(X)$ satisfies [AC-2]. Finally, property [AC-3] follows by applying Theorem 4.2 to products of functions.

In the following, we let $f_a f_b \cdots f_k$ denote $f_a \circ f_b \circ \cdots \circ f_k$ and f_a^n denote $f_a \circ f_a \circ \cdots \circ f_a$ for n terms.

Remark 4.4. Note that $f_a \circ f_b \circ \cdots \circ f_l \circ f_m(x) = f_{m*l*\cdots*b*a}(x)$ and also $f_a^n(x) = f_{a^n}(x)$.

Definition 4.3. Let $(X, *, 0)$ be an AC-algebra. We define the relation $\equiv$ on $(M(X), \circ, f_0)$ by $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$ if and only if $(f_a f_b \cdots f_k) \circ (f_m f_n \cdots f_r)(x) = f_0(x)$ for all $x \in X$.

Remark 4.5. Note that from Theorem 4.2, $(f_a f_b \cdots f_k) \circ (f_m f_n \cdots f_r)(x) = f_0(x)$ *for all* $x \in X$ if and only if $f_a f_b \cdots f_k(x) = f_m f_n \cdots f_r(x)$ *for all* $x \in X$. Alternative definitions of $\equiv$ are therefore $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$ if and only if $f_a f_b \cdots f_k(x) = f_m f_n \cdots f_r(x)$ for all $x \in X$, *or, using [AC-3] if and only if* $(f_a f_b \cdots f_k)(x) * (f_m f_n \cdots f_r)(x) = 0$ *for all* $x \in X$.

Remark 4.6. As a special case, note that $f_a \equiv f_b$ if and only if $f_a \circ f_b(x) = f_0(x)$, i.e., if and only if $f_a(x) = f_b(x)$ for all $x \in X$.

Theorem 4.4. The relation $\equiv$ defined in Definition 4.3 is an equivalence relation.

Proof.

1. The relation $\equiv$ is reflexive. From [AC-3], $f_a f_b \cdots f_k(x) * f_a f_b \cdots f_k(x) = 0$ for all $x \in X$ and therefore $f_a f_b \cdots f_k \equiv f_a f_b \cdots f_k$.

2. The relation $\equiv$ is symmetric. If $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$ then $f_a f_b \cdots f_k(x) * f_m f_n \cdots f_r(x) = 0$ for all $x \in X$. Then from commutativity $f_m f_n \cdots f_r(x) * f_a f_b \cdots f_k(x) = 0$ for all $x \in X$. Therefore $f_m f_n \cdots f_r \equiv f_a f_b \cdots f_k$.

3. The relation $\equiv$ is transitive. If $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$ then from Remark 4.5 $f_a f_b \cdots f_k(x) = f_m f_n \cdots f_r(x)$ for all $x \in X$. If $f_m f_n \cdots f_r \equiv f_u f_v \cdots f_z$ then $f_m f_n \cdots f_r(x) = f_u f_v \cdots f_z(x)$ for all $x \in X$. Therefore, from Remark 4.5 $f_a f_b \cdots f_k \equiv f_u f_v \cdots f_z$.

Theorem 4.5. Let $(X, *, 0)$ be an AC-algebra. Then $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$ if and only if $a * b * \cdots * k = m * n * \cdots * r$.

Proof. ($\Rightarrow$) Let $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$. Then, from Remark 4.5, we have $f_a f_b \cdots f_k(x) = f_m f_n \cdots f_r(x)$ for all $x \in X$.

Therefore, from Remark 4.3, $x * a * b * \cdots * k = x * m * n * \cdots * r$ for all $x \in X$. Then, using property 8 of section 2 and [AC-3], we have

$$a * b * \cdots * k = m * n * \cdots * r.$$

($\Leftarrow$) Let $a * b * \cdots * k = m * n * \cdots * r$. Then, for all $x \in X$,

$$x * a * b * \cdots * k = x * m * n * \cdots * r.$$

and therefore from Remarks 4.3 and 4.5, we have $f_a f_b \cdots f_k \equiv f_m f_n \cdots f_r$

Lemma 4.1. For all $f_{a_j} \in M(X),\ j = 1, 2, \ldots, m$, we have

$$(f_{a_1} f_{a_2} \cdots f_{a_m})^k \equiv f_{a_1}^{\,k} f_{a_2}^{\,k} \cdots f_{a_m}^{\,k}$$

for any positive integers m and k .

Proof. The proof is by induction on k. The lemma is obviously true for $k = 1$.
We assume the lemma is true for k and it is sufficient to show that the lemma is true for $k+1$. Then,

$$\begin{aligned}
(f_{a_1} f_{a_2} \cdots f_{a_m})^{k+1} &\equiv (f_{a_1} f_{a_2} \cdots f_{a_m})^k (f_{a_1} f_{a_2} \cdots f_{a_m}) \\
&\equiv (f_{a_1}^{\,k} f_{a_2}^{\,k} \cdots f_{a_m}^{\,k})(f_{a_1} f_{a_2} \cdots f_{a_m}) \\
&\equiv f_{a_1}^{\,k} f_{a_1} f_{a_2}^{\,k} f_{a_2} \cdots f_{a_m}^{\,k} f_{a_m} \\
&\equiv f_{a_1}^{\,k+1} f_{a_2}^{\,k+1} \cdots f_{a_m}^{\,k+1}
\end{aligned}$$

Lemma 4.2. Let $(X, *, 0)$ be an AC-algebra. For each positive integer k we have $\sigma^k \equiv \sigma \quad \forall\ \sigma \in M(X)$ if and only if $f_a^k \equiv f_a \quad \forall\ a \in X$

Proof. Let k be a positive integer. Suppose that $\sigma^k \equiv \sigma$ for all $\sigma \in M(X)$ and $a \in X$ so that $f_a \in M(X)$. Thus, choosing $\sigma \equiv f_a$, we have $f_a^k \equiv f_a$.

Conversely, suppose that for each $a \in X,\ f_a^k \equiv f_a$.
Then we have:

$$\begin{aligned}
\sigma^k &\equiv (f_{a_1} f_{a_2} \cdots f_{a_m})^k \\
&\equiv f_{a_1}^{\,k} f_{a_2}^{\,k} \cdots f_{a_m}^{\,k} \qquad \text{by Lemma 4.1} \\
&\equiv f_{a_1} f_{a_2} \cdots f_{a_m} \\
&\equiv \sigma.
\end{aligned}$$

Theorem 4.8. For any AC-algebra $(X, *, 0)$ and any positive integer m, the following conditions hold:

1. $f_a^{2m} \equiv f_0$ for all $a \in X$.

2. $f_a^{2m+1} \equiv f_a$ for all $a \in X$.

3. $X = \{0\}$ if and only if $f_a^{2m} \equiv f_a$ for all $a \in X$.

Proof.

1. From Remark 4.4 and the fact that $a^{2m} = 0$ for all $a \in X$, we have $f_a^{2m} \equiv f_{a^{2m}} \equiv f_0$.

2. For all $a \in X$ and by the properties of AC-algebra, we have
$a^{2m+1} = a * (a^{2m}) = a * (a^2)^m = a * 0 = a$.
Hence $f_{a^{2m+1}} \equiv f_a$ and therefore, from Remark 4.4, $f_a^{2m+1} \equiv f_a$.

3. Suppose $X = \{0\}$. Then if $a, x \in X$ we have $a = 0$ and $x = 0$ and therefore $f_a(x) = x * a = 0 * 0 = 0$ for all $a, x \in X$. Also ${f_a}^{2m}(x) = x * (a)^{2m} = 0 * (0)^{2m} = 0$ for all $a, x \in X$. Therefore $f_a \equiv f_a^{2m}$.

   Conversely, for each $a \in X$,suppose $f_a^{2m} \equiv f_a$. From Remark 4.4, $f_a^{2m}(x) = f_{a^{2m}}(x)$. Then $f_{a^{2m}} \equiv f_a^{2m} \equiv f_a$. By Theorem 4.5 we have $a^{2m} = a$ and since $a^{2m} = 0$ therefore $a = 0$. This means that $X = \{0\}$.

Corollary 4.9. Let $(X, *, 0)$ be an AC-algebra and $(M(X), \circ, f_0)$ be the corresponding AC-algebra for function compositions. Then, for any positive integer m, the following conditions hold:

1. $\sigma^{2m} \equiv f_0$ for all $\sigma \in M(X)$.

2. $\sigma^{2m+1} \equiv \sigma$ for all $\sigma \in M(X)$.

3. $M(X) = \{f_0\}$ if and only if $\sigma^{2m} \equiv \sigma$ for all $\sigma \in M(X)$.

Proof. Apply Lemma 4.2 to Theorem 4.8.

Definition 4.4. Let $(X, *, 0)$ be an AC-algebra and k be any positive integer. We define $L_k = \{\sigma \in M(x) | \sigma^k \equiv \sigma\}$.

Theorem 4.10. If, for positive integer m, $k = 2m$ then $L_k = \{f_0\}$ and if $k = 2m+1$ then $L_k = M(X)$.

Proof.

1. If $\sigma \in L_{2m}$ then $\sigma^{2m} \equiv \sigma$. But, from Corollary 4.9, $\sigma^{2m} \equiv f_0$ for all $\sigma \in M(X)$. Therefore, if $\sigma \in L_{2m}$, then $\sigma \equiv f_0$. Conversely, if $\sigma \in \{f_0\}$, then $\sigma^{2m} \equiv f_0 \equiv \sigma$ and therefore $\sigma \in L_{2m}$. Therefore $L_{2m} = \{f_0\}$.

2. If $\sigma \in M(X)$, then from Corollary 4.9, we have $\sigma^{2m+1} \equiv \sigma$. Therefore $\sigma \in L_{2m+1}$. That is, $M(X) \subseteq L_{2m+1}$. But, by definition $L_{2m+1} \subseteq M(X)$, and therefore $L_{2m+1} = M(X)$.

§5. Discussion and conclusions

We have defined:

An AC-algebra as an algebra $(X, *, 0)$ with a binary operation $*$ which satisfies the conditions:

[AC-1] $x * (y * z) = (x * y) * z$.
[AC-2] $x * y = y * x$.
[AC-3] $x * y = 0$ if and only if $x = y$ for all $x, y, z \in X$.
We have shown that:

1. S is a subalgebra of an AC-algebra X if and only if S is an ideal of X.

2. An AC-algebra is an Abelian group and each element is its own inverse.

3. The algebra of the logical *exclusive or* (xor) operation applied to ordered n-tuples of true (T) and false (F) is an example of an AC-algebra.

4. Let I be a subalgebra of an AC-algebra $(X, *, 0)$.
If we define $[x]_I$ by $[x]_I = \{y \in X | x * y \in I\}$ and an operation $\odot$ by $[x]_I \odot [y]_I = [x * y]_I$, then $(X/I, \odot, [0]_I)$ is an AC-algebra, which we have called a quotient AC-algebra by the subalgebra I.

5. We have defined functions on an AC-algebra by $f_a(x) = x * a$, $a, x \in X$ and a composition operation $\circ$ for these functions. If $M(X)$ is the set of finite compositions of functions, then we have shown that $(M(x), \circ, f_0)$ is an AC-algebra, where $f_0(x) = x * 0 = x$ for all $x \in X$.

6. We have defined the equivalence relation $\equiv$ on $(M(X), \circ, f_0)$ by $f_a \equiv f_b$ if and only if $f_a \circ f_b(x) = f_0(x)$ for all $x \in X$.

7. For any AC-algebra $(X, *, 0)$ and any positive integer m, the following conditions hold:

   (a) $f_a^{2m} \equiv f_0$ for all $a \in X$.
   (b) $f_a^{2m+1} \equiv f_a$ for all $a \in X$.
   (c) $X = \{0\}$ if and only if $f_a^{2m} \equiv f_a$ for all $a \in X$.

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A negative order result for the exponential divisor function [1]

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Abstract In this paper we shall study the mean value of the exponential divisor function involving a negative r-th power by the convolution method. This gives a comparison with the result of László Tóth.

Keywords The exponential divisor function, the general divisor function, Euler product, convolution method.

§1. Introduction

The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$ if $b_i|a_i$ for every $i \in\{1,2,...,s\}$, notation: $d|_e n$. By convention $1|_e 1$.

Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n. The function $\tau^{(e)}$ is called the exponential divisor function. The properties of the function $\tau^{(e)}$ is investigated by many authors, see, for example, [1], [4], [5], [6].

Suppose $r \geq 1$ is a fixed integer. M. V. Subbarao [3] remarked that

$$\sum_{n\leq x}(\tau^{(e)}(n))^r \sim A_r x, \tag{1.1}$$

where

$$A_r := \prod_p (1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^r - (\tau(a-1))^r}{p^a}). \tag{1.2}$$

László Tóth [4] improved the result (1.1) and established a more precise asymptotic formula for the r-th power of the function $\tau^{(e)}$

$$\sum_{n\leq x}(\tau^{(e)}(n))^r = A_r x + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\epsilon}), \tag{1.3}$$

where A_r is given by (1.2), P_{2^r-2} is a polynomial of degree $2^r - 2$ and $u_r = \frac{2^r-1}{2^r+1}$.

The aim of this short note is to prove the following:

Theorem. Suppose $r \geq 1$ and $N \geq 1$ are fixed integers, then

$$\sum_{n\leq x}(\tau^{(e)}(n))^{-r} = C_r x + x^{\frac{1}{2}} \log^{2^{-r}-2}(\sum_{j=0}^{N} d_j(r)\log^{-j} x + O(\log^{-N-1} x)), \tag{1.4}$$

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where $d_0(r), d_1(r), \cdots, d_N(r)$ are computable constants, and

$$C_r := \prod_p (1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^{-r} - (\tau(a-1))^{-r}}{p^a}).$$

In order to prove our theorem, we define for an arbitrary complex number z the general divisor function $d_z(n)$ by

$$\sum_{n=1}^{\infty} d_z(n) n^{-s} = \zeta^z(s) = \prod_p (1 - p^{-s})^{-z} \quad (\sigma > 1),$$

where a branch of $\zeta^z(s)$ is defined by

$$\zeta^z(s) = exp\{z \log \zeta(s)\} = exp(-z \sum_p \sum_{j=1}^{\infty} j^{-1} p^{-js}) \quad (\sigma > 1).$$

This definition shows that $d_z(n)$ is multiplicative function of n which generalizes $d_k(n)$. The divisor function $d_k(n)$ ($k \geq 2$ a fixed integer) may be defined by

$$\sum_{n=1}^{\infty} d_k(n) n^{-s} = \zeta^k(s) = \prod_p (1 - p^{-s})^{-k} \quad (\sigma > 1).$$

Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Proof of theorem

The proof of the theorem is based on the following lemmas.

Lemma 1. Suppose s is a complex number for with $\Re s > 1$, $r \geq 1$ is a fixed integer, then

$$F(s) := \sum_{n=1}^{\infty} \frac{(\tau^{(e)}(n))^{-r}}{n^s} = \zeta(s) \zeta^{2^{-r}-1}(2s) G(s, r), \tag{2.1}$$

where the Dirichlet series $G(s,r) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $Res > 1/4$.

Proof. Since $\tau^{(e)}(n)$ is multiplicative, by the Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
f(s) &= \prod_p (1 + \frac{(\tau^{(e)}(p))^{-r}}{p^s} + \frac{(\tau^{(e)}(p^2))^{-r}}{p^{2s}} + \frac{(\tau^{(e)}(p^3))^{-r}}{p^{3s}} + \cdots) \\
&= \prod_p (1 + \frac{1}{p^s} + \frac{2^{-r}}{p^{2s}} + \frac{2^{-r}}{p^{3s}} + \cdots) \\
&= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{2^{-r}}{p^{2s}} + \cdots) \\
&= \zeta(s) \prod_p (1 + \frac{2^{-r} - 1}{p^{2s}} + \frac{3^{-r} - 2^{-r}}{p^{4s}} + \cdots) \\
&= \zeta(s) \zeta^{2^{-r}-1}(2s) G(s, r),
\end{aligned}$$

where

$$G(s,r)=\prod_p(1-\frac{1}{p^{2s}})^{2^{-r}-1}(1+\frac{2^{-r}-1}{p^{2s}}+\frac{3^{-r}-2^{-r}}{p^{4s}}+\cdots).$$

Write $G(s,r):=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$. It is easily seen the Dirichlet series is absolutely convergent for $Res>1/4$.

Lemma 2. Let $A>0$ be arbitrary but fixed real number, and let $N_1\geq 1$ be an arbitrary but fixed integer. If $|z|\leq A$, then uniformly in z

$$\begin{aligned}\sum_{n\leq x}d_z(n) &= C_1(z)xlog^{z-1}x+C_2(z)xlog^{z-2}x+\cdots\\ &\quad +C_{N_1}(z)xlog^{z-N_1}x+O(xlog^{Rez-N_1-1}x),\end{aligned}$$

where $C_j(z)=B_j(z)/\Gamma(z-j-1)(j=1,\cdots,N_1)$ and each $B_j(z)$ is regular for $|z|\leq A$.

Proof. See Ivić [2], Theorem 14.9.

Lemma 3. Let $A>0$ be arbitrary but fixed real number, and let $M\geq 1$ be an arbitrary but fixed integer. If $|z|\leq A$, then uniformly in z

$$\begin{aligned}\sum_{mn^2\leq x}d_z(n) &= \zeta^z(2)x+x^{\frac{1}{2}}(K_1(z)log^{z-1}x+K_2(z)log^{z-2}x+\cdots\\ &\quad +K_M(z)log^{z-M}x)+O(x^{\frac{1}{2}}log^{Rez-M-1}x),\end{aligned}$$

where the functions $K_j(z)(j=1,\cdots,M)$ are regular in $|z|\leq A$.

Proof. Suppose $1\leq y\leq x$ is a parameter to be determined later. We have

$$\begin{aligned}\sum_{mn^2\leq x}d_z(n) &= \sum_{n\leq y}d_z(n)\sum_{m\leq\frac{x}{n^2}}1+\sum_{m\leq\frac{x}{y^2}}\sum_{n^2\leq\frac{x}{m}}d_z(n)-\sum_{m\leq\frac{x}{y^2}}\sum_{n\leq y}d_z(n)\\ &= \sum_1+\sum_2-\sum_3.\end{aligned}$$

For $\sum_1$, we have

$$\begin{aligned}\sum_1 &= \sum_{n\leq y}d_z(n)[\frac{x}{n^2}]\\ &= x\sum_{n\leq y}\frac{d_z(n)}{n^2}+O(\sum_{n\leq y}|d_z(n)|).\end{aligned}$$

We see that $|d_z(n)|\leq d_k(n)$, if $k=[A]+1$ and $|z|\leq A$. If we use the weak asymptotic formula (see, Ivić [2])

$$\sum_{n\leq x}d_k(n)=xP_{k-1}(\log x)+O(x^{\frac{k}{k+1}}),$$

the error term in Σ_1 is bounded by $O(y\log^{k-1}y)$. So by Lemma 2 and the partial summation, we have

$$\begin{aligned}\sum_1 &= x\sum_{n=1}^{\infty}\frac{d_z(n)}{n^2} - x\sum_{n>y}\frac{d_z(n)}{n^2} + O(y\log^{k-1}y)\\ &= \zeta^z(2)x + \frac{x}{y}\sum_{j=1}^{N_1}C_j(z)log^{z-j}y + \frac{2x}{y}\sum_{j=1}^{N_1}(z-j)C_j(z)log^{z-j-1}y\\ &\quad +\frac{2x}{y}\sum_{j=1}^{N_1}(z-j)(z-j-1)C_j(z)log^{z-j-2}y + \cdots\\ &\quad +O(\frac{x}{y}log^{Rez-N_1-1}y) + O(y\log^{k-1}y).\end{aligned}$$

Using Lemma 2, it is seen that

$$\begin{aligned}\sum_3 &= \sum_{m\le\frac{x}{y^2}}\sum_{n\le y}d_z(n)\\ &= \sum_{n\le y}d_z(n)(\frac{x}{y^2}+O(1))\\ &= \frac{x}{y}\sum_{j=1}^{N_1}C_j(z)log^{z-j}y + O(\frac{x}{y}log^{Rez-N_1-1}y) + O(ylog^{k-1}y).\end{aligned}$$

By similar computation, we can obtain

$$\begin{aligned}\sum_2 &= \sum_{m\le\frac{x}{y^2}}[\sum_{j=1}^{N_1}C_j(z)\sqrt{\frac{x}{m}}log^{z-j}(\frac{x}{m})^{\frac{1}{2}} + O(\sqrt{\frac{x}{m}}log^{Rez-L-1}(\frac{x}{m}))]\\ &= \sqrt{x}\sum_{j=1}^{N_1}C_j(z)\sum_{m\le\frac{x}{y^2}}m^{-\frac{1}{2}}log^{z-j}(\frac{x}{m})^{\frac{1}{2}} + O(\sum_{m\le\frac{x}{y^2}}\sqrt{\frac{x}{m}}log^{Rez-N_1-1}(\frac{x}{m}))\\ &= \sqrt{x}\sum_{j=1}^{N_1}C_j(z)(\frac{1}{2})^{z-j}log^{z-j}x\sum_{m\le\frac{x}{y^2}}m^{-\frac{1}{2}}(1-\frac{\log m}{\log x})^{z-j}\\ &\quad +O(\sqrt{x}log^{Rez-N_1-1}x\sum_{m\le\frac{x}{y^2}}m^{-\frac{1}{2}})\\ &= \sum_{2,1} +O(\frac{x}{y}log^{Rez-N_1-1}x),\end{aligned}$$

where we define

$$\sum_{2,1} = \sqrt{x}\sum_{j=1}^{N_1}C_j(z)(\frac{1}{2})^{z-j}log^{z-j}x\sum_{m\le\frac{x}{y^2}}m^{-\frac{1}{2}}(1-\frac{\log m}{\log x})^{z-j}.$$

Using Taylor formula and foregoing method, we have

$$\begin{aligned}\sum_{2,1} &= \sqrt{x}\sum_{j=1}^{N_1} C_j(z)(\frac{1}{2})^{z-j} log^{z-j}x \sum_{m\le \frac{x}{y^2}} m^{-\frac{1}{2}}(1-(z-j)\frac{\log m}{\log x}\\ &\quad +\frac{(z-j)(z-j-1)}{2!}(\frac{\log m}{\log x})^2+\cdots)\\ &= x^{\frac{1}{2}}\sum_{j=1}^{N_1} K_j(z) log^{z-j}x - \frac{2x}{y}\sum_{j=1}^{N_1}(z-j)C_j(z)log^{z-j-1}y\\ &\quad -\frac{2x}{y}\sum_{j=1}^{N_1}(z-j)(z-j-1)C_j(z)log^{z-j-2}y+\cdots\\ &\quad +O(y\log^{Rez-1})+O(\frac{x}{y}log^{Rez-N_1-1}y),\end{aligned}$$

where $K_1(z),\cdots,K_M(z)$ are regular functions. So by choosing $y=\sqrt{x}\log^C x$, $C=Rez-M-k$ and $N_1=2M+k-Rez$ completes the proof of the Lemma 3.

Now we go on with the proof of our main Theorem. Combining Lemma 1 and Lemma 3, we get

$$\begin{aligned}\sum_{n\le x}(\tau^{(e)}(n))^{-r} &= \sum_{n_1 n_2^2 n_3\le x} d_z(n_2)g(n_3)\\ &= \sum_{n_3\le x} g(n_3)\sum_{n_1 n_2^2\le x/n_3} d_z(n_2)\\ &= x\zeta^{2^{-r}-1}(2)\sum_{n_3\le x} g(n_3){n_3}^{-1}\\ &\quad +x^{\frac{1}{2}}\sum_{n_3\le x} g(n_3){n_3}^{-1/2}\sum_{j=1}^{M} K_j\log^{z-j}(\frac{x}{n_3})\\ &\quad +O(\sum_{n_3\le x} g(n_3)(\frac{x}{n_3})^{\frac{1}{2}}log^{Rez-M-1}(\frac{x}{n_3}))\\ &= S_1(x)+S_2(x)+O(S_3(x)),\end{aligned}\tag{3}$$

say, where we choose $z=2^{-r}-1$.

In the following procedure, we just need to calculate the three sums separately. That is

$$\begin{aligned}S_1(x) &= x\zeta^{2^{-r}-1}(2)\sum_{n_3=1}^{\infty} g(n_3){n_3}^{-1} - x\zeta^{2^{-r}-1}(2)\sum_{n_3>x} g(n_3){n_3}^{-1}\\ &= C_r x+O(x^{\frac{1}{4}+\epsilon}).\end{aligned}\tag{4}$$

Analogously to $\sum_{2,1}$, here we can again use the Taylor formula to obtain

$$\begin{aligned} S_2(x) &= x^{\frac{1}{2}} \sum_{n_3 \le x} g(n_3) {n_3}^{-1/2} \sum_{j=1}^{M} K_j \log^{z-j}(\frac{x}{n_3}) \\ &= x^{\frac{1}{2}} \sum_{j=1}^{M} K_j \log^{z-j} x \sum_{n_3 \le x} g(n_3) {n_3}^{-1/2} (1 - \frac{\log n_3}{\log x})^{z-j} \\ &= x^{\frac{1}{2}} \log^{2^{-r}-2} x \sum_{j=0}^{M-1} E_j(r) \log^{-j} x + O(x^{\frac{1}{2}} \log^{2^{-r}-M-2} x) \\ &= x^{\frac{1}{2}} \log^{2^{-r}-2} x \sum_{j=0}^{N} d_j(r) \log^{-j} x + O(x^{\frac{1}{2}} \log^{2^{-r}-N-3} x), \end{aligned} \tag{5}$$

where $E_1(r), \cdots, E_N(r)$ are computable constants depending on r, and we set $N = M - 1$. Similarly,

$$\begin{aligned} S_3(x) &= \sum_{n_3 \le x} g(n_3) (\frac{x}{n_3})^{\frac{1}{2}} log^{Rez-M-1}(\frac{x}{n_3}) \\ &= x^{\frac{1}{2}} log^{Rez-M-1} x \sum_{n_3 \le x} g(n_3) {n_3}^{-1/2} (1 - \frac{\log n_3}{\log x})^{Rez-M-1} \\ &\ll x^{\frac{1}{2}} \log^{2^{-r}-M-2} x = x^{\frac{1}{2}} \log^{2^{-r}-N-3} x. \end{aligned} \tag{6}$$

So our theorem follows from (3)-(6).

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Harnark extention principle of Henstock integral for Banach-valued function [1]

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Abstract The paper mainly gives Harnark extention principle of Henstock integral on banach-valued function, promotes Harnark extention principle of Henstock integral on Real-valued function.

Keywords Henstock integral, Harnark extention principle.

§1. Introductions and basic definitions

Throughout this paper X denotes a real Banach Space and A partition of $[a,b]$ is a finite collection of interval-point pairs $\{I,\xi\}$ with the intervals non-overlapping and their union $[a,b]$, here ξ is the associated point of I, we write $D=\{I,\xi\}$. It is said to be δ-fine partition of $[a,b]$ if for each interval-point $\{I,\xi\}$ we have $\xi\in I\subset(\xi-\delta(x),\xi+\delta(x))$.

Let f is Banach-valued function defined on $[a,b]$ and we use to $(D)\sum f(\xi)|I|$ represent the Riemann sum of f corresponding to the δ-fine partition $D=\{I,\xi\}$.

Definition 1.1. The function $f:[a,b]\to X$ is Henstock integrable on $[a,b]$ and $A\in X$ is its Henstock integral if for each $\varepsilon>0$ there is gauge $\delta(t)$ on $[a,b]$ such that for any δ-fine partition $D=\{I,\xi\}$ of $[a,b]$ we have

$$\|(D)\sum f(\xi)|I|-A\|<\varepsilon.$$

The function f is Henstock integrable on a set $E\subset[a,b]$ if the function $f\cdot\chi_E$ is Henstock integrable on $[a,b]$ and we denote $(H)\int_a^b f\chi_E=(H)\int_E f$, where χ_E denotes the characteristic function of E and we denote $(H)\int_a^b f\chi_E=(H)\int_E f$.

The properties of Henstock integral of Banach-valued functions are similar to real-valued functions, the reader is referred to [1] [2] [3] for the details.

§2. Main results

First we give Harnark extention principle of Henstock integral on Real-valued function.

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Theorem 2.1.[1][2] Let $f : [a,b] \to R$ is Henstock integrable on close set $E \subset [a,b]$, $[a,b] \setminus E = \bigcup_{i=1}^{\infty}(c_i,d_i)$, if $f(x)$ is Henstock integrable on each $[c_i,d_i]$, and $\sum_{i=1}^{\infty}\omega(F_i,[c_i,d_i]) < \infty$, then the function f is Henstock integrable on $[a,b]$

$$(H)\int_a^b f = (H)\int_E f + \sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i} f.$$

In order to prove Theorem 2.2, we need the following lemmas and these notes, Let $E_0 = E, E_1 = (c_1,d_1), E_2 = (c_2,d_2), \cdots, E_i = (c_i,d_i), \cdots, E_i \bigcap E_j = \phi(i \neq j)$, such that $[a,b] = \bigcup_{i=0}^{\infty} E_i$. Define function

$$f_n(x) = \begin{cases} f(x), & x \in \bigcup_{i=0}^{n} E_i \\ 0, & x \in [a,b] \setminus \bigcup_{i=0}^{n} E_i \end{cases} \tag{1}$$

Obviously, $f_n(x) \to f(x), x \in [a,b]$.

Definition 2.1.[4] A family $\{f_n\}$ of Henstock integrable functions is said to be uniformly Henstock integrable on $[a,b]$ if for each $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any $\delta-$fine partition $D = \{I,\xi\}$ of $[a,b]$ and all $n \in \mathbb{N}$ we have

$$\|(D)\sum f_n(\xi)|I| - (H)\int_a^b f_n\| < \varepsilon.$$

Lemma 2.1.[4] Let $f_n : [a,b] \to X$ are Henstock integrable on $[a,b]$, $n \in \mathbb{N}$ and satisfies:

i) $f_n(x) \to f(x), x \in [a,b]$;

ii) $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$.

Then f is Henstock integrable on $[a,b]$ and

$$(H)\int_a^b f = \lim_{n\to\infty}(H)\int_a^b f_n.$$

Definition 2.2. Let $f_n : [a,b] \to X, n \in \mathbb{N}$, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ and positive function $\delta(\xi)$, such that for any $\delta-$fine partition $D = \{I,\xi\}$ of $[a,b]$ for$m,n \geq N$, we have

$$\|(D)\sum f_m(\xi)|I| - (D)\sum f_n(\xi)|I|\| < \varepsilon,$$

then $\{f_n\}$ is a $\delta-$Cauchy sequence.

Lemma 2.2. Let $f : [a,b] \to X$ is Henstock integrable on close set E and each interval $[c_i,d_i], i = 1,2,\cdots$, and $\sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i} f$ is unconditionally convergent, define the function $\{f_n\}$ such as (1), then $\{f_n\}$ is a $\delta-$Cauchy sequence.

Proof. Because $\sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i} f$ is unconditionally convergent, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\|\sum_{i=N}^{\infty}(H)\int_{c_i}^{d_i} f\| < \frac{\varepsilon}{2}, \tag{2}$$

because $f\chi_{E_n}$ is Henstock integrable on $[a,b]$, for $\varepsilon>0$ and each n there is $\delta_n(t)>0, \delta_{n+1}(t)<\delta_n(t), t\in[a,b]$ such that for any δ_n-fine partition $D_n=\{I,\xi\}$ of $[a,b]$, we have

$$\|(D_n)\sum f\chi_{E_n}(\xi)|I|-(H)\int_{E_n}f\|<\frac{\varepsilon}{2^{n+2}}. \tag{3}$$

Define function $\delta:[a,b]\to(0,+\infty)$ as follows

$$\delta(t)=\delta_n(t), t\in[c_i,d_i], n=0,1,2,\cdots$$

Let $D=\{I,\xi\}$ is $\delta-$ fine partition of $[a,b]$, it must be δ_n-fine partition, when $m,n\geq N(m>n)$, by (2) and (3), we have,

$$\begin{aligned}
&\|(D)\sum f_n(\xi)|I|-(D)\sum f_m(\xi)|I|\|\\
&=\|\sum_{i=0}^{n}(D)\sum f\chi_{E_i}(\xi)|I|-\sum_{i=0}^{m}(D)\sum f\chi_{E_i}(\xi)|I|\|\\
&=\|\sum_{i=n+1}^{m}(D)\sum f\chi_{E_i}(\xi)|I|\|\\
&\leq\|\sum_{i=n+1}^{m}[(D)\sum f\chi_{E_i}(\xi)|I|-(H)\int_{c_i}^{d_i}f]\|+\|\sum_{i=n+1}^{m}(H)\int_{c_i}^{d_i}f\|\\
&<\sum_{i=n+1}^{m}\cdot\frac{\varepsilon}{2^{i+2}}+\|\sum_{i=N}^{\infty}(H)\int_{c_i}^{d_i}f\|\\
&<\frac{\varepsilon}{2^{N+1}}+\frac{\varepsilon}{2}<\varepsilon,
\end{aligned}$$

then $\{f_n\}$ is a $\delta-$Cauchy sequence.

Lemma 2.3. Let $f:[a,b]\to X$ is Henstock integrable on close set E and each interval $[c_i,d_i], i=1,2,\cdots$, and $\sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i}f$ is unconditionally convergent, define the function $\{f_n\}$ such as (1), then $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$.

Proof. By Lemma 2.2, $\{f_n\}$ is a $\delta-$Cauchy sequence, then for any $\varepsilon>0$ there is $N_0\in\mathbb{N}$ and $\delta_0(t)>0$ such as for any δ_0 fine partition $D_0=\{I,\xi\}$ of $[a,b]$, and when $m,n\geq N_0$ we can obtain

$$\|(D_0)\sum f_n(\xi)|I|-(D_0)\sum f_m(\xi)|I|\|<\frac{\varepsilon}{3}. \tag{4}$$

Since $\sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i}f$ is unconditionally convergent, there is $N_1\in\mathbb{N}$ such that

$$\|\sum_{i=N_1}^{\infty}(H)\int_{c_i}^{d_i}f\|<\frac{\varepsilon}{3}. \tag{5}$$

Each f_n is Henstock integrable on $[a,b]$ and $(H)\int_a^b f_n=\sum_{i=0}^{n}(H)\int_{E_i}f$, when $m,n\geq N_1(m>n)$, according to (5) ,

$$\|(H)\int_a^b f_n-(H)\int_a^b f_m\|=\|\sum_{i=n+1}^{m}(H)\int_{c_i}^{d_i}f\|<\frac{\varepsilon}{3}, \tag{6}$$

and for each $n \in \mathbb{N}$, there is gauge function $\delta_n(t)$ of $[a,b]$, for any δ_n-fine partition $D_n = \{I, \xi\}$ of $[a,b]$, the inequality

$$\|(D_n)\sum f_n(\xi)|I| - \sum_{i=0}^{n}(H)\int_{E_i} f\| < \frac{\varepsilon}{3} \tag{7}$$

holds. Choose $N = \max\{N_0, N_1\}$,and let $H_N = \bigcup_{i=0}^{N} E_i$ and $H_n = (c_n, d_n), n > N$, because (c_n, d_n) are disjoint, the sets H_N and $H_n(n > N)$ are also disjoint and $H_N \bigcup (\bigcup_{n>N} H_n) = [a,b]$. Let us define $\delta : [a,b] \to (0, +\infty)$ such that

$$\delta(t) = \min\{\delta_0(t), \delta_1(t), \delta_2(t), \cdots, \delta_N(t)\}, t \in H_N$$

and

$$\delta(t) = \min\{\delta_0(t), \delta_1(t), \delta_2(t), \cdots, \delta_n(t)\}, t \in H_n, n > N.$$

The following we will prove that for any $\delta-$ fine partition $D = \{I, \xi\}$ of $[a,b]$ and for all $n \in \mathbb{N}$, the following inequality holds

$$\|(D)\sum f_n(\xi)|I| - (H)\int_a^b f_n\| < \varepsilon.$$

Let $D = \{I, \xi\}$ is a $\delta-$ fine partition of $[a,b]$, then D is δ_n- fine partition, using (4), for $0 \leq n \leq N$, we have

$$\|(D)\sum f_n(\xi)|I| - (H)\int_a^b f_n\| < \frac{\varepsilon}{3} < \varepsilon,$$

by (4),(7) and (6), for $n > N$ we have

$$\begin{aligned}
&\|(D)\sum f_n(\xi)|I| - (H)\int_a^b f_n\| \\
&\leq \|(D)\sum f_n(\xi)|I| - (D)\sum f_N(\xi)|I|\| + \|(D)\sum f_N(\xi)|I| - (H)\int_a^b f_N\| \\
&+\|(H)\int_a^b f_N - (H)\int_a^b f_n\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.
\end{aligned}$$

For any $\delta-$fine partition $D = \{I, \xi\}$, $\|\sum_D f_n(\xi)|I| - (H)\int_a^b f_n\| < \varepsilon$, that is, for all $n \in \mathbb{N}$ this equality $\|(D)\sum f_n(\xi)|I| - \sum_{i=0}^{n}(H)\int_{E_i} f\| < \varepsilon$ also holds, so $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$.

Now we give Harnark extention principle of Henstock integral for Bnanch-valued function as bellow.

Theorem 2.2. Let $f : [a,b] \to X$ is Henstock integrable on close set $E \subset [a,b]$, $[a,b] \setminus E = \bigcup_{i=1}^{\infty}(c_i, d_i)$. Let $f(x)$ is Henstock integrable on each $[c_i, d_i]$, and $\sum_{i=1}^{\infty}\omega(F_i, [c_i, d_i])$ is unconditionally convergent, then The function f is Henstock integrable on $[a,b]$

$$(H)\int_a^b f = (H)\int_E f + \sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i} f.$$

Proof. Define function $\{f_n\}$ such as (1), then $f_n(x) \to f(x), x \in [a,b]$, using above two lemmas, $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$, and by Lemma 2.1 we have known that f must be Henstock integrable on $[a,b]$ and

$$\begin{aligned}(H)\int_a^b f &= \lim_{n\to\infty}(H)\int_a^b f_n = \lim_{n\to\infty}\sum_{i=0}^{n}(H)\int_{E_i} f \\ &= \sum_{i=0}^{\infty}(H)\int_{E_i} f = (H)\int_E f + \sum_{i=1}^{\infty}(H)\int_{c_i}^{d_i} f.\end{aligned}$$

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The ruin problem for a class of correlated aggregated claims model

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Abstract In this paper, the author studies a risk model. Under this model the two claim processes are correlated. Claim occurrence relate to Poisson and Erlang processes. The formulae is derived for the distribution of the surplus immediately before ruin, for the distribution of the surplus after ruin and the joint distribution of the surplus immediately before and after ruin. The asymptotic property of these ruin functions is also discussed.

Keywords Double risk model, Poisson process, Erlang(2) process, correlated aggregate claim, ruin function.

§1. Introduction

Resently, many authors studied various correlated aggregate claims models. In this model, the two claim number processes are correlated. Ambagaspitiya(1998) considered a general method of constructing a vector of p dependent claim numbers from a vector of independent random variables, derived formulae to get the correlated claims distribution.

On the other hand, Erlang(2) distribution is also one of the mostly commonly used distributions in risk theory, for example, Diskson and Hipp(1998)considered the infinite time survival probability as a compound geometric random variable under the Erlang(2) risk model. Sun and Yang(2004) derived the integro-differential equation and Laplace transform of the joint distributions of the surplus immediately before and after ruin for Erlang(2) risk processes.

In this paper we consider a correlated risk model. Under the assumed risk model the claim number processed involve Poisson and Erlang(2) process. We derived the formulae for the distribution of the surplus immediately before ruin, for the distribution after ruin and the joint distribution of the surplus immediately before and after ruin. The asymptotic property of these ruin functions is also studied.

§2. Model set-up and model transformation

We define the surplus process

$$U(t) = u + ct - \sum_{i=1}^{N_1(t)} X_i - \sum_{i=1}^{N_2(t)} Y_i, \quad t \geq 0, \tag{1}$$

where u is the initial surplus, $c > 0$ is the premium rate. $\{X_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$are independent random variables with distributions $F_X(x)$, $F_Y(y)$, density functions $f_X(x)$, $f_Y(y)$.

$$\begin{cases} N_1(t) = M_1(t) + M_2(t), \\ N_2(t) = M(t) + M_2(t), \end{cases} \tag{2}$$

here $\{M_1(t), t \geq 0\}, \{M_2(t), t \geq 0\}$ are Poisson processes with parameters λ_1 and λ_2 respectively. $\{M(t), t \geq 0\}$ is a Erlang(2) process with parameter β. We also assume $\{M_1(t), t \geq 0\}, \{M_2(t), t \geq 0\}$ and $\{M(t), t \geq 0\}$ are three independent renewal processes.

Let T denote the time of ruin, so that

$$T = \inf\{t \geq 0 : U(t) < 0\}.$$

Then the probability of ultimate ruin with initial surplus is defined as

$$\psi(u) = P(T < \infty | U(0) = u), \quad u \geq 0.$$

The probability that the surplus immediately prior to ruin is smaller than $x > 0$ with the initial surplus u as

$$B(u, x) = P(T < \infty, U(T_-) \leq x | U(0) = u).$$

The probability that the deficit (negative surplus immediately after ruin) is smaller than $y > 0$ with the initial surplus u as

$$G(u, y) = P(T < \infty, |U(T)| \leq y | U(0) = u).$$

And the joint probability that the surplus immediately prior to ruin is smaller than $x > 0$ and the surplus after ruin is larger than $-y$ with the initial surplus u as

$$J(u; x, y) = P(T < \infty, U(T_-) \leq x, |U(T)| \leq y | U(0) = u).$$

We also define the following auxiliary probability distribution:

$$A(u; x, y) = P(T < \infty, |U(T)| \geq x, U(T_-) > y | U(0) = u).$$

It is easy to obtain that

$$\begin{cases} \psi(u) = A(u; 0, 0) \\ B(u, x) = A(u; 0, 0) - A(u; 0, x) \\ G(u, y) = A(u; 0, 0) - A(u; y, 0) \\ J(u; x, y) = A(u; 0, 0) - A(u; 0, x) - A(u; y, 0) + A(u; y, x) \end{cases} \tag{3}$$

Hence all the four ruin functions can be get from $A(u; x, y)$.

From (1) and (2), the model (1) can be rewritten as

$$\widetilde{U}(t) = u + ct - \sum_{i=1}^{\widetilde{N}_1(t)} X_i^{'} - \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0,$$

here $\widetilde{N}_1(t) = M_1(t) + M_2(t) + M_2(t)$, is a Poisson process with parameter $(\lambda_1 + 2\lambda_2)$. and its distribution function is

$$F_{X'} = \frac{\lambda_1 + \lambda_2}{\lambda_1 + 2\lambda_2} F_X(x) + \frac{\lambda_2}{\lambda_1 + 2\lambda_2} F_Y(x).$$

Since $\widetilde{U}(t)$ is transformed from $U(t)$,the process $U(t)$ can be examined via $\widetilde{U}(t)$.

Let $T_1, T_2, \cdots$ be the inter-arrival times for $\{X_i^{'}, i \geq 1\}$, then T_i are independent and exponentially distributed with parameter $(\lambda_1 + 2\lambda_2)$.

Let $L_1, L_2, \cdots$ be the inter-arrival times for $\{Y_i, i \geq 1\}$, then L_i are independent and have Erlang(2) distribution with parameter β ,by the define of Erlang(2, β)

$$L_i = L_{i1} + L_{i2}, \quad i \geq 1,$$

here $L_{i1}, \quad i \geq 1, j \geq 1$ are independent exponential random variables with β. In order to derive the formulae for the above ruin functions, we consider a delayed renewal process $\widetilde{M}(t)$ corresponding to Erlang(2, β), i.e

$$L_1 = L_{12}, \quad L_i = L_{i1} + L_{i2}, \quad i \geq 2.$$

We denote the corresponding ruin functions by $\widetilde{\varphi}(u), A_1(u; x, y), B_1(u, x), J_1(u; x, y), G_1(u, y)$.

§3. Main results

Lemma 3.1. For any $u > y > 0$, $u > x > 0$,

$$\begin{aligned} A(u; x, y) &= \frac{\lambda_1 + 2\lambda_2}{c} \left[\int_{u+x}^{\infty} \overline{F}_{X'}(z) dz + \int_0^u A(u - z; x, y) \overline{F}_{X'}(z) dz \right] \\ &\quad + \frac{\beta}{c} \int_u^{\infty} (A_1(z; x, y) - A(z; x, y) dz. \end{aligned}$$

Proof. Consider the time $V = \min\{T_1, L_{11}\}$, we obtain

$$\begin{aligned} A(u; x, y) &= \int_0^{\infty} A(\widetilde{U}(t); x, y) dt = \int_0^{\infty} E\left[A(\widetilde{U}(t); x, y) | V \right] dt \\ &= \int_0^{\infty} P(T_1 > t, V = L_{11} = t) A_1(u + ct; x, y) dt \\ &\quad + \int_0^{\infty} P(T_1 \leq t, V = T_1 \leq t) E\left[A(u + ct - X_1^{'}; x, y) \right] dt, \end{aligned}$$

here

$$\begin{cases} P(T_1 > L_{11}) = P(V = L_{11}) = \dfrac{\beta}{\lambda}, \\ P(T_1 < L_{11}) = P(V = T_1) = \dfrac{\lambda_1 + 2\lambda_2}{\lambda}, \\ P(V > t | V = T_1) = P(V > t | V = L_{11}) = e^{-\lambda t}, \end{cases}$$

here $\lambda = \lambda_1 + 2\lambda_2 + \beta$.

$$E[A(u+ct-X_1^{'};x,y)] = \int_0^{u+ct} A(u-z;x,y)dF_{X'}(z) + \int_{u+ct+x}^{\infty} I_{\{u+ct>y\}}dF_{X'}(z),$$

then

$$\begin{aligned} A(u;x,y) &= \frac{\beta}{\lambda}\int_0^{\infty} \lambda e^{-\lambda t}A_1(u+ct;x,y)dt \\ &\quad + \frac{\lambda_1+2\lambda_2}{\lambda}\int_0^{\infty}\lambda e^{-\lambda t}\int_0^{u+ct} A(u+ct-z;x,y)dF_{X'}(z)dt \\ &\quad + \frac{\lambda_1+2\lambda_2}{\lambda}\int_0^{\infty}\lambda e^{-\lambda t}\int_{u+ct+x}^{\infty} I_{\{u+ct>y\}}dF_{X'}(z)dt. \end{aligned}$$

Putting $s = u + ct$ yields

$$\begin{aligned} cA(u;x,y) &= \beta\int_u^{\infty} e^{-\frac{\lambda(s-u)}{c}t}A_1(s;x,y)dt \\ &\quad + (\lambda_1+2\lambda_2)\int_u^{\infty} e^{-\frac{\lambda(s-u)}{c}t}\int_0^s A(s-z;x,y)dF_{X'}(z)ds \\ &\quad + (\lambda_1+2\lambda_2)\int_0^{\infty} e^{-\frac{\lambda(s-u)}{c}t}\int_{s+x}^{\infty} I_{\{s>y\}}dF_{X'}(z)dt. \end{aligned}$$

Differentiating with respect to u

$$\begin{aligned} & cA'(u;x,y) \\ = \; & -\beta A_1(u;x,y) + \frac{\lambda\beta}{c}\int_u^{\infty} e^{-\frac{\lambda(s-u)}{c}t}A_1(s;x,y)dt - (\lambda_1+2\lambda_2)\int_0^u A(u-z;x,y)dF_{X'}(z) \\ & + \frac{\lambda(\lambda_1+2\lambda_2)}{c}\int_u^{\infty} e^{-\frac{\lambda(s-u)}{c}}\int_0^s A(s-z;x,y)dF_{X'}(z)ds \\ & -(\lambda_1+2\lambda_2)I_{\{u>y\}}\overline{F}_{X'}(u+x) + \frac{\lambda(\lambda_1+2\lambda_2)}{c}\int_u^{\infty} e^{-\frac{\lambda(s-u)}{c}}I_{\{s>y\}}\overline{F}_{X'}(s+x)ds \\ = \; & -\beta A_1(u;x,y) - (\lambda_1+2\lambda_2)\int_0^u A(u-z;x,y)dF_{X'}(z) \\ & -(\lambda_1+2\lambda_2)I_{\{u>y\}}\overline{F}_{X'}(u+x) + \lambda A(u;x,y). \end{aligned} \tag{4}$$

Integrating (4) both sides from 0 to u gives

$$\begin{aligned} A(u;x,y) &= A(0;x,y) - \frac{\beta}{\lambda}\int_0^u A_1(;x,y)ds + \frac{\lambda}{c}\int_0^s A(s;x,y)ds \\ &\quad + \frac{\lambda_1+2\lambda_2}{c}\int_0^u\int_0^s A(s-z;x,y)d\overline{F}_{X'}(z)ds \\ &\quad - \frac{\lambda_1+2\lambda_2}{c}\int_0^u I_{\{s>y\}}\overline{F}_{X'}(s+x)ds. \end{aligned}$$

By interchanging integral signs and performing integration ,we get

$$\begin{aligned} A(u;x,y) &= A(0;x,y) - \frac{\beta}{c}\int_0^u (A_1(s;x,y) - A(s;x,y))ds \\ &\quad + \frac{\lambda_1+2\lambda_2}{c}\int_0^u A(u-z;x,y)\overline{F}_{X'}(z)dz - \frac{\lambda_1+2\lambda_2}{c}\int_{y+x}^{u+x}\overline{F}_{X'}(s)ds, \end{aligned} \tag{5}$$

here $u > y > 0$.
Let $u \to \infty$, then

$$A(0;x,y) = \frac{\beta}{c}\int_0^\infty (A_1(s;x,y) - A(s;x,y))ds + \frac{\lambda_1+2\lambda_2}{c}\int_{y+x}^\infty \overline{F}_{X'}(s)ds, \tag{6}$$

Putting (6) into (5), we get the following theorem.

Theorem 3.1. For any $u > y > 0$, $u > x > 0$, we have

$$\psi(u) = \frac{\lambda_1+2\lambda_2}{c}\left[\int_u^\infty \overline{F}_{X'}(z)dz + \int_0^u \psi(u-z)\overline{F}_{X'}(z)dz\right] + \frac{\beta}{c}\int_u^\infty (\psi_1(z) - \psi(z))dz,$$

$$B(u;x) = \frac{\lambda_1+2\lambda_2}{c}\int_0^u B(u-z;x)\overline{F}_{X'}(z)dz + \frac{\beta}{c}\int_u^\infty (B_1(z;x) - B(z;x))dz$$

$$G(u;y) = \frac{\lambda_1+2\lambda_2}{c}\left[\int_u^{u+y} \overline{F}_{X'}(z)dz + \int_0^u G(u-z,y)\overline{F}_{X'}(z)dz\right] + \frac{\beta}{c}\int_u^\infty (G_1(z,y) - G(z,y))dz$$

$$J(u;x,y) = \frac{\lambda_1+2\lambda_2}{c}\int_0^u J(u-z;x,y)\overline{F}_{X'}(z)dz + \frac{\beta}{c}\int_u^\infty (J_1(z;x,y) - J(z;x,y))dz$$

The above formulae can be derived by Lemma 3.1 and (3).

If we assume that there exists $r_1 > 0$ and $r_2 > 0$ such that

$$r \to r_1, h_1(r) = \int_0^\infty e^{rz}dF_{X'}(z) - 1 \to \infty,$$
$$r \to r_2, h_2(r) = \int_0^\infty e^{rz}dF_{Y'}(z) - 1 \to \infty.$$

We can get the result:

Theorem 3.2.

$$\begin{cases} \lim\limits_{u\to\infty} e^{Ru}(A(u;x,y) + A_1(u;x,y)) \le \dfrac{2\rho}{1+\rho}\dfrac{c}{(\lambda_1+2\lambda_2)h_1'(R) + 2^{-1}\beta h_2'(R) - c} \\ \lim\limits_{u\to\infty} e^{Ru}(\psi(u) + \psi_1(u)) \le \dfrac{2\rho}{1+\rho}\dfrac{c}{(\lambda_1+2\lambda_2)h_1'(R) + 2^{-1}\beta h_2'(R) - c} \\ \lim\limits_{u\to\infty} e^{Ru}(B(u,y) + B_1(u,y)) \le \dfrac{2\rho}{1+\rho}\dfrac{c}{(\lambda_1+2\lambda_2)h_1'(R) + 2^{-1}\beta h_2'(R) - c} \\ \lim\limits_{u\to\infty} e^{Ru}(G(u,y) + G_1(u,y)) \le \dfrac{2\rho}{1+\rho}\dfrac{c}{(\lambda_1+2\lambda_2)h_1'(R) + 2^{-1}\beta h_2'(R) - c} \\ \lim\limits_{u\to\infty} e^{Ru}(J(u;x,y) + J_1(u;x,y)) \le \dfrac{2\rho}{1+\rho}\dfrac{2c}{(\lambda_1+2\lambda_2)h_1'(R) + 2^{-1}\beta h_2'(R) - c} \end{cases}$$

where

$$\rho = \frac{c}{(\lambda_1+2\lambda_2)\mu_{X'} + 2^{-1}\beta\mu_Y} - 1$$

is the relative security loading, and R is the positive solution of the equation $(\lambda_1+2\lambda_2)h_1'(R) + 2^{-1}\beta h_2'(R) = cr$.

Proof. We only prove the first inequality

$$\begin{aligned} A_1(u;x,y) &= \beta\int_0^\infty e^{-\lambda t}\int_0^{u+ct} A(u+ct-z;x,y)dF_Y(z)dt \\ &\quad +\beta\int_0^\infty e^{-\lambda t}I_{\{u+ct>y\}}\int_{u+ct+x}^\infty A(u+ct-z;x,y)dF_Y(z)dt \\ &\quad +(\lambda_1+2\lambda_2)\int_0^\infty e^{-\lambda t}\int_0^{u+ct} A_1(u+ct-z;x,y)dF_{X'}(z)dt \\ &\quad +(\lambda_1+2\lambda_2)\int_0^\infty e^{-\lambda t}I_{\{u+ct>y\}}\int_{u+ct+x}^\infty dF_{X'}(z)dt. \end{aligned}$$

Putting $s=u+ct$, differentiating with respect to u yields

$$\begin{aligned} cA_1'(u;x,y) &= -\beta\int_0^u A(u-z;x,y)dF_Y(z)-\beta I_{\{u>y\}}\int_{x+u}^\infty dF_Y(z) \\ &\quad -(\lambda_1+2\lambda_2)\int_0^u A_1(u-z;x,y)dF_{X'}(z) \\ &\quad -(\lambda_1+2\lambda_2)I_{\{u>y\}}\int_{u+x}^\infty dF_{X'}(z)+\lambda A_1(u;x,y). \end{aligned} \tag{7}$$

Integrating (7) both sides from 0 to u gives

$$\begin{aligned} cA_1(u;x,y) &= cA_1(0;x,y)+\beta\int_0^u A(u-z;x,y)\overline{F}_Y(z)dz \\ &\quad +(\lambda_1+2\lambda_2)\int_0^u A_1(u-z;x,y)\overline{F}_{X'}(z)dz-\beta\int_{y+x}^{u+x}\overline{F}_Y(z)dz \\ &\quad -(\lambda_1+2\lambda_2)\int_{x+y}^{u+x}\overline{F}_{X'}(z)dz+\beta\int_0^u(A_1(s;x,y)-A(s;x,y))ds \end{aligned} \tag{8}$$

where $u>y>0$.

Let $u\to\infty$,we have

$$\begin{aligned} cA_1(0;x,y) &= (\lambda_1+2\lambda_2)\int_{x+y}^\infty\overline{F}_{X'}(z)dz+\beta\int_{y+x}^\infty\overline{F}_Y(z)dz \\ &\quad -\beta\int_0^\infty(A_1(s;x,y)-A(s;x,y))ds. \end{aligned} \tag{9}$$

From (6)and (8), (9), we have

$$\begin{aligned} & A(u;x,y)+A_1(u;x,y) \\ \le\ & \int_{x+u}^\infty(\frac{2(\lambda_1+2\lambda_2)}{c}\overline{F}_{X'}(z)+\frac{\beta}{c}\overline{F}_Y(z))dz \\ & +\int_0^u(\frac{\lambda_1+2\lambda_2}{c}\overline{F}_{X'}(z)+\frac{\beta}{2c}\overline{F}_Y(z))(A_1(u-z;x,y)-A(u-z;x,y))dz. \end{aligned}$$

By the property of security loading, we have

$$\int_0^\infty(\frac{\lambda_1+2\lambda_2}{c}\overline{F}_{X'}(z)+\frac{\beta}{2c}\overline{F}_Y(z))dz<1.$$

In view of the assumption, there exists R such that

$$\int_0^\infty e^{Rz}(\frac{\lambda_1+2\lambda_2}{c}\overline{F}_{X'}(z)+\frac{\beta}{2c}\overline{F}_Y(z))dz=1.$$

Therefore

$$\begin{aligned}&\frac{1}{2}e^{Ru}(A(u;x,y)+A_1(u;x,y))\\&\le e^{Ru}\int_0^\infty(\frac{\lambda_1+2\lambda_2}{c}\overline{F}_{X'}(z)+\frac{\beta}{2c}\overline{F}_Y(z))dz\\&+\frac{1}{2}\int_0^\infty e^{Rz}(\frac{\lambda_1+2\lambda_2}{c}\overline{F}_{X'}(z)+\frac{\beta}{2c}\overline{F}_Y(z))\\&e^{R(u-z)}(A_1(u-z;x,y)-A(u-z;x,y))dz.\end{aligned}$$

Denote

$$H(z)=\frac{\lambda_1+2\lambda_2}{c}\overline{F}_{X'}(z)+\frac{\beta}{2c}\overline{F}_Y(z).$$

Applying the renewal theorem to the right hand side of the above inequality, it follows that

$$\lim_{u\to\infty}e^{Ru}(A(u;x,y)+A_1(u;x,y))\le\frac{2\int_0^\infty e^{Ru}\int_u^\infty H(z)dzdu}{\int_0^\infty ze^{Rz}H(z)dz},$$

and

$$\int_0^\infty e^{Ru}\int_u^\infty\overline{F}_{X'}(z)dzdu=\frac{1}{R}(\frac{h_1(R)}{R}-\mu_{X'}),$$

$$\int_0^\infty ze^{Rz}\overline{F}_{X'}(z)dz=\frac{1}{R}(-\frac{h_1(R)}{R}+h_1'(R)),$$

then

$$\frac{2\int_0^\infty e^{Ru}\int_u^\infty H(z)dzdu}{\int_0^\infty ze^{Rz}H(z)dz}=\frac{2\rho}{1+\rho}\frac{c}{(\lambda_1+2\lambda_2)h_1'(R)+2^{-1}\beta h_2'(R)-c},$$

where

$$\rho=\frac{c}{(\lambda_1+2\lambda_2)\mu_{X'}+2^{-1}\beta\mu_Y}-1.$$

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Projective synchronization in autonomous chaotic system via tracking control [1]

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Abstract This paper presents the projective synchronization of chaos systems by designing tracking controller based on Lyapunov stability theory. Frist, this method is implemented in synchronization of a simple system, then we realize the synchronization of Lu hyper-chaotic system. Numerical simulations show the united synchronization method works well.

Keywords Projective synchronization, tracking control, chaotic system, Hyper-chaotic system.

§1. Introduction

Synchronization is a fundamental phenomenon that enables coherent behavior in coupled systems. In 1990, pecora and carroll proposed a successful method to synchronize two identical chaotic systems with different initial conditions [1]. Chaos synchronization has received a significant attention in the last few years due to its potential applications [3-14]. There exist many types of synchronization such as complete synchronization [2], anti-synchronization [4]. Mainieri and Rehacek [10] reported a new form of chaos synchronization, termed as projective synchronization, that the drive and response systems could be synchronized up to a scaling factor (a proportional relation), which is usually observable in a class of systems with partial linearity. In this regard, this paper we put forward tracking control method to achieve the projective synchronizaiton for chaotic systems. We prove the feasibility of the method from theoretic analysis on the basis of two chaotic systems. Numerical simulation are used to verify the effectiveness of the proposed scheme.

We organized this paper as follows. In section 2 we discuss the design of tracking controller. In section 3, we present an application of this approach to control of the system and numerical simulations demonstrate the effectiveness of the proposed synchronization scheme. Finally concluding remark and references close the paper.

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§2. Design of controller

Consider nonlinear chaotic system as follows:

$$\dot{x}(t) = f(x), \tag{1}$$

to achieve projective synchronization, we assume that system (1) is the drive system and the controlled system (2) is response system

$$\dot{y}(t) = f(y) + u, \tag{2}$$

where u is united controller. The projective synchronization errors between system (1) and (2) are defined as

$$e = (e_1, e_2, \cdots, e_n)^T = (y_1 - x_1\alpha, y_2 - x_2\alpha, \cdots, y_n - x_n\alpha)^T$$

where a is scaling factor. Then the error dynamical system can be described as:

$$\dot{e} = f(x) + \alpha[f(y) + u].$$

In order to make system (3) controllable, the feedback controller u will be appropriately chosen. Let

$$u = [f(x) + e]\alpha - f(y), \tag{3}$$

Theorem 1. If we choose the controller as equation (3), this guarantees the asymptotic stability of system (3).

Proof. Construct a Lyapunov function:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2), \tag{4}$$

The time derivative of V along the trajectory of the error dynamical system (2) is as follows

$$\dot{V} = -(e_1^2 + e_2^2 + e_3^2), \tag{5}$$

Since V is a positive definite function and $\dot{V}$ is a negative definite function, according to the Lyapunov's direct method, the system (1) and system (2) achieve the projective synchronization under the controller (3).

§3. Numerical simulation

In this section, an appropriate controller was designed to achieve projective synchronization based on Lyapunov's direct method. Lorenz chaotic system and hyper-chaotic Lu system are chosen to illustrate the effectiveness of the projective synchronization behavior and to illustrate the effectiveness of the proposed scheme.

§3.1 Projective synchronization of Lorenz system

The Lorenz system can be described by

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) \\ \dot{x_2} = cx_1 - x_2 - x_1x_3 \\ \dot{x_3} = x_1x_2 - bx_3, \end{cases} \tag{6}$$

$x = (x_1, x_2, x_3)^T$ is the state vector, and a, b, c are parameters of this system.

We define the response systems of (6) as follows

$$\begin{cases} \dot{y_1} = a(y_2 - y_1) + u_1 \\ \dot{y_2} = cy_1 - y_2 - y_1y_3 + u_2 \\ \dot{y_3} = y_1y_2 - by_3 + u_3, \end{cases} \tag{7}$$

where $u = (u_1, u_2, u_3)$ is the nonlinear controller to be designed for projective synchronization of the two Lorenz systems with the same parameters and the different initial conditions. Define the synchronization errors signal between the drive and response systems as $e(t) = y(t) - \alpha x(t)$. Thus, the error dynamical signal between the drive system (6) and the response system (7) is

$$\begin{cases} e_1 = y_1 - \alpha(x_1) \\ e_2 = y_2 - \alpha(x_2) \\ e_3 = y_3 - \alpha(x_3), \end{cases} \tag{8}$$

then the error dynamical system between the drive system (6) and the response system (7) is

$$\begin{cases} \dot{e_1} = a(y_2 - y_1) + u_1 - \alpha(a(x_2 - x_1)) \\ \dot{e_2} = cy_1 - y_2 - y_1y_3 + u_2 - \alpha(cx_1 - x_2 - x_1x_3) \\ \dot{e_3} = y_1y_2 - by_3 + u_3 - \alpha(x_1x_2 - bx_3) \end{cases} \tag{9}$$

The goal of control is to find a controller$u = (u_1, u_2, u_3)^T$ for system (9) such that (6) and (7) are in projective synchronization.

Let us now choose the control functions u_1,u_2 and u_3 as follows

$$\begin{cases} u_1 = \alpha(a(x_2 - x_1)) - a(y_2 - y_1) - e_1 \\ u_2 = \alpha(cx_1 - x_2 - x_1x_3) - cy_1 - y_2 - y_1y_3 - e_2 \\ u_3 = \alpha(x_1x_2 - bx_3) - y_1y_2 - by_3 - e_3, \end{cases} \tag{10}$$

If the Lyapunov function candidate is taken as:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2), \tag{11}$$

The time derivative of V along the trajectory of the error dynamical system (9) is as follows

$$\dot{V} = -(e_1^2 + e_2^2 + e_3^2), \tag{12}$$

according to the Lyapunov's direct method, the error variables become zero as time tends to infinity.

This means that the two Lorenz chaotic systems realize the projective synchronization under the controller (10).

For numerical simulations, the system parameters $a = 10$, $b = 8/3$, $c = 28$ are chosen to be such that the two systems display chaotic behavior. We assume that the initial states of the drive system and response system are $(x_1(0), x_2(0), x_3(0) = (1, 5, 2)$ and $(y_1(0), y_2(0), y_3(0) = (1, 5, 2)$ respectively, we set the scaling factors$\alpha = 2$, the state errors is in shown in Fig. 1, and the attractors between two chaotic systems is shown in Fig. 2.

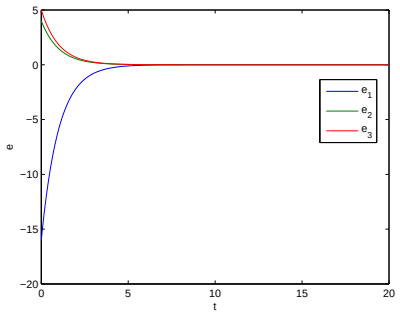

Fig. 1 The time evolution of the errors with the scaling factor $\alpha = 2$

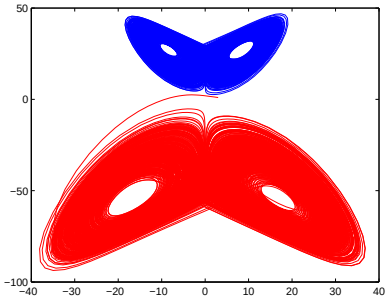

Fig. 2 Two chaotic attractor in three-dimensional phase space with the scaling factor $\alpha = 2$

§3.2 Projective synchronization of hyper chaotic Lu system

We choose hyper-chaotic Lu system as examples to illustrate our proposed synchronization scheme. The hyper- chaotic Lu system can be described by following nonlinear ordinary differential equation. The hyper-chaotic Lu system described as follows:

$$\begin{cases} \dot{x_1} = a(x_2 - x_4) + x_1 \\ \dot{x_2} = bx_2 - x_1x_3 \\ \dot{x_3} = -cx_3 + x_1x_2 \\ \dot{x_4} = dx_4 + x_1x_3, \end{cases} \tag{13}$$

$x = (x_1, x_2, x_3, x_4)^T$ is the state vector, and a, b, c, d are system parameters.

We take (13) as the drive system and the response system is given by:

$$\begin{cases} \dot{y_1} = a(y_2 - y_4) + y_1 + u_1 \\ \dot{y_2} = by_2 - y_1y_3 + u_2 \\ \dot{y_3} = -cy_3 + y_1y_2 + u_3 \\ \dot{y_4} = dy_4 + y_1y_3 + u_4, \end{cases} \tag{14}$$

$y = (y_1, y_2, y_3, y_4)^T$ is the state vector, and $u = u(x, y) = (u_1(x, y), u_2(x, y), u_3(x, y), u_4(x, y))^T$ is the controller to be determined for the purpose of projective synchronization. Let the vector error state be $e(t) = y(t) - \alpha x(t)$. Thus, the error dynamical system between the drive system (13) and the response system (14) is

$$\begin{cases} \dot{e_1} = a(x_2 - x_1) + x_4 - \alpha(a(y_2 - y_1)y_4 + u_1) \\ \dot{e_2} = bx_2 - x_1x_3 - \alpha(by_2 - y_1y_3 + u_2) \\ \dot{e_3} = -cx_3 + x_1x_2 - \alpha(-cy_3 + y_1y_2 + u_3) \\ \dot{e_4} = dx_4 + x_1x_3 - \alpha(dy_4 + y_1y_3 + u_4), \end{cases} \tag{15}$$

The goal of control is to find a controller $u = (u_1, u_2, u_3, u_4)^T$ for system (12) such that systems (9) and (10) are in projective synchronization. Let us now choose the control function as follows :

$$\begin{cases} \dot{u_1} = \frac{\dot{x_1}+e_1}{\alpha} - \dot{y_1} \\ \dot{u_2} = \frac{\dot{x_2}+e_2}{\alpha} - \dot{y_2} \\ \dot{u_3} = \frac{\dot{x_3}+e_3}{\alpha} - \dot{y_3} \\ \dot{u_4} = \frac{\dot{x_4}+e_4}{\alpha} - \dot{y_4}, \end{cases} \tag{16}$$

For the numerical simulations, fourth-order Runge-Kutta method is used to solve the systems of differential equations (13) and (14). The initial states of the drive system and response system are $(x_1(0), x_2(0), x_3(0), x_4(0) = (1, 5, 2, 2)$ and $(y_1(0), y_2(0), y_3(0), y_4(0) = (11, 15, 12, 8)$ The state errors between two hyper-chaotic Lu systems are shown in Fig. 3. Obviously, the synchronization errors converge asymptotically to zero and two systems are indeed achieved chaos synchronization.

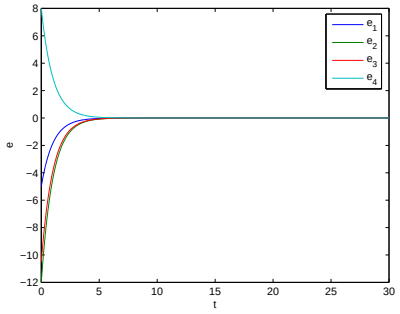

Fig. 3 The time evolution of the errors with the scaling factors $\alpha = 2$

§4. Conclusion and discussion

In this letter, we put forward tracking control method to realize the projective synchronization of the chaotic and hyper-chaotic system. We prove the feasibility of the method from theoretic analysis and numerical simulations are provide to further verify the effectiveness of the proposed scheme. The proposed method is simple and flexible.

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Continuous dependence of bounded $\Phi-$variation solutions on parameters for Kurzweil equations [1]

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Abstract The theorems of continuous dependence of bounded $\Phi-$variation solutions on parameters for Kurzweil equations are established through making use of the functions of bounded Φ-variation were introduced by Musiclak and Orlicz in paper [2]. The results are essential generalization of continuous dependence of bounded variation solutions on parameters for Kurzweil equations in paper [6].

Keywords Kurzweil equation, bounded $\Phi-$variation solution, continuous dependence on parameter.

§1. Introduction

Kurzweil generalized ordinary differential equation was introduced in 1957 by Kurzweil [1], the functions of bounded Φ-variation were introduced by Musiclak and Orlicz in paper [2]. The functions of bounded Φ-variation and Kurzweil generalized ordinary differential equations were connected originally in paper [3, 4], existence and uniqueness theorems of bounded Φ-variation solutions for generalized ordinary differential equations were established. In this paper, the theorems of continuous dependence of bounded $\Phi-$variation solutions on parameters for Kurzweil equations are established through making use of the functions of bounded Φ-variation. The results are stronger than that in paper [5], meanwhile the results are essential generalization of continuous dependence of bounded variation solutions on parameters for Kurzweil equations in paper [6].

§2. Preliminaries and definitions

Definition 2.1.[3−7] A function $U : [a, b] \times [a, b] \to R^n$ is called Kurzweil integrable over $[a, b]$, if there is a $A \in R^n$ such that given $\varepsilon > 0$, there is a positive function $\delta(\tau)$ such that for any $\delta(\tau)$–fine partition $D = \{(\tau_j, [\alpha_{j-1}, \alpha_j]), j = 1, \cdots, k\}$ satisfying $\tau_j \in [\alpha_{j-1}, \alpha_j] \subset$

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$[\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)]$, we have

$$\|S(U,D) - A\| = \|\sum_{j=1}^{k}[U(\tau_j,\alpha_j) - U(\tau_j,\alpha_{j-1})] - A\| < \varepsilon,$$

A is called the Kurzweil integral of U over $[a,b]$ and will be denoted by $\int_a^b DU(\tau,t)$, if $\int_a^b DU(\tau,t)$ exists then define $\int_b^a DU(\tau,t) = -\int_a^b DU(\tau,t)$, set $\int_b^a DU(\tau,t) = 0$ when $a = b$. We denote by $\mathcal{K}([a,b])$ the class of all functions U which are integrable over $[a,b]$.

Let $G \subset R^{n+1}$ be an open set, assume that $F : G \to R^n$ is a given R^n–valued function defined for $(x,t) \in G, x \in R^n, t \in R$.

Definition 2.2.[3−7] A function $x : [a,b] \to R^n$ is called a solution of the Kurzweil equation

$$\frac{dx}{d\tau} = DF(x,t) \tag{2.1}$$

on the interval $[\alpha,\beta] \subset R$, if $(x(t),t) \in G$ for all $t \in [\alpha,\beta]$, and if

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau),t)$$

holds for every pair $s_1, s_2 \in [\alpha,\beta]$.

Let $\Phi(u)$ denote a continuous and increasing function defined for $u \geq 0$ with $\Phi(0) = 0$, $\Phi(u) > 0$ for $u > 0$, and satisfying the following conditions:

(Δ_2) There exist $u_0 \geq 0$ and $a > 0$ such that $\Phi(2u) \leq a\Phi(u)$ for $0 < u \leq u_0$;

(c) $\Phi(u)$ is a convex function.

Let $[a,b] \subset R, -\infty < a < b < +\infty$. We consider the function $x : [a,b] \to R^n, x(t)$ is of bounded Φ–variation over $[a,b]$ if for any partition $\pi : a = t_0 < t_1 < \cdots < t_m = b$, we have

$$V_\Phi(x;[a,b]) = \sup_\pi \sum_{i=1}^{m} \Phi(\|x(t_i) - x(t_{i-1})\|) < +\infty,$$

$V_\Phi(x;[a,b])$ is called Φ–variation of $x(t)$ over $[a,b]$. We always assume $\Phi(u)$ satisfying (Δ_2) and (c).

Lemma 2.1.[2,5] **(Helly's extracting theorem)** Every sequence $x^k \in BV_\Phi^*$ bounded in variation includes converging to a function $x = x(t)$ of the class BV_Φ^* pointwise in $[a,b]$.

Given $c > 0$, we denote $B_c = \{x \in R^n; \|x\| < c\}$. Let $(a,b) \subset R$ be an interval with $-\infty < a < b < +\infty$, and set $G = B_c \times (a,b)$. Assume that $h : [a,b] \to R$ is a increasing function and continuous from the left on the interval $[a,b]$, $\omega : [0,+\infty) \to R$ is a continuous and increasing function with $\omega(0) = 0$.

Definition 2.3.[10] A function $F : G \to R^n$ belongs to the class $\mathcal{F}_\Phi(G,h,\omega)$, if

(1) The inequality

$$\| F(x,t_2) - F(x,t_1) \| \leq \Phi(|h(t_2) - h(t_1)|) \tag{2.2}$$

holds for all $(x,t_1),(x,t_2) \in G$;

(2) The inequality

$$\| F(y,t_2) - F(y,t_1) - F(x,t_2) + F(x,t_1) \| \leq \omega(\| y - x \|)\Phi(|h(t_2) - h(t_1)|)$$

holds for all $(x,t_1),(x,t_2),(y,t_1),(y,t_2)\in G$.

Lemma 2.2.[3−5,7] Assume that $F:G\to R^n$ satisfies the condition (2.2), if $x:[\alpha,\beta]\to R^n,[\alpha,\beta]\subset(a,b)$ is such that $(x(t),t)\in G$ for every $t\in[\alpha,\beta]$ and if the Kurzweil integral $\int_\alpha^\beta DF(x(\tau),t)$ exists, then for every pair $s_1,s_2\in[\alpha,\beta]$ the inequality

$$\| \int_\alpha^\beta DF(x(\tau),t) \| \le V_\Phi(h;[s_1,s_2]) \tag{2.3}$$

holds.

Corollary 2.1.[3−5,7] Assume that $F:G\to R^n$ satisfies the condition (2.2), if $x:[\alpha,\beta]\to R^n,[\alpha,\beta]\subset(a,b)$ is a solution of (2.1) then x is of bounded Φ–variation and $V_\Phi(x;[\alpha,\beta])\le\Phi(V_\Phi(h;[\alpha,\beta]))<+\infty$. Moreover every point in $[\alpha,\beta]$ at which the function h is continuous from the left is a left continuity point of the solution $x:[\alpha,\beta]\to R^n$.

Theorem 2.1.[3,4] Let $F:G\to R^n$ belongs to the class $\mathcal{F}_\Phi(G,h,\omega)$ and let $(\tilde{x},t_0)\in G$ be such that $\tilde{x}^+=\tilde{x}+F(\tilde{x},t_0^+)-F(\tilde{x},t_0)\in B_c$ is satisfied. Then there exist $\triangle^-,\triangle^+>0$ such that on the interval $[t_0-\triangle^-,t_0+\triangle^+]$ there exist a bounded Φ–variation solution $x(t)$ of the Kurzweil equation (2.1) for which $x(t_0)=\tilde{x}$.

§3. Prime results

Theorem 3.1. Assume that $F_k:G\to R^n$ belongs to the class $\mathcal{F}_\Phi(G,h_k,\omega)$ for $k=0,1,2,\cdots$ where $H,h_k:(a,b)\to R$ are nondecreasing functions which are continuous from the left when $k=0,1,2,\cdots$. Assume further that

$$\Phi(h_k(t_2)-h_k(t_1))\le\Phi(H(t_2)-H(t_1)) \tag{3.1}$$

for every $a\le t_1\le t_2\le b$. Suppose that

$$\lim_{k\to\infty}F_k(x,t)=F_0(x,t) \tag{3.2}$$

for $(x,t)\in G$. Let $x:[\alpha,\beta]\to R^n$ is a solution of the generalized differential equation

$$\frac{dx}{d\tau}=DF_0(x,t) \tag{3.3}$$

on $[\alpha,\beta]\subset(a,b)$ which has the following uniqueness property: if $y:[\alpha,\gamma]\to R^n,[\alpha,\gamma]\subset[\alpha,\beta]$ is a solution of (3.3) such that $y(\alpha)=x(\alpha)$ then $y(t)=x(t)$ for every $t\in[\alpha,\gamma]$.

Assume further that there is a $\rho>0$ such that if $s\in[\alpha,\beta]$ and $\| y-x(s)\|<\rho$ then $(y,s)\in G$, and let $y_k\in R^n,k=1,2,\cdots$ satisfy $\lim_{k\to\infty}y_k=x(\alpha)$. Then for sufficiently large $k\in N$ there exists a solution x_k of the generalized differential equation

$$\frac{dx}{d\tau}=DF_k(x,t) \tag{3.4}$$

on $[\alpha,\beta]$ with $x_k(\alpha)=y_k$ and $\lim_{k\to\infty}x_k(s)=x(s),s\in[\alpha,\beta]$.

Proof. By assumption we have $(y,\alpha)\in G$ provided

$$\| y-x(\alpha)\|<\frac{\rho}{2}$$

or

$$\| y - x(\alpha+) \| = \| y - x(\alpha) - F_0(x(\alpha), \alpha+) + F_0(x(\alpha), \alpha) \| < \frac{\rho}{2},$$

Assume that $\varepsilon > 0$. By continuity of the function $\omega : [0, +\infty) \to R, \omega(0) = 0$, there exist a $\delta > 0$ such that

$$\omega(t) < \frac{\varepsilon}{V_\Phi(H; [s_1, s_2]) + 1},$$

whenever $0 < t < \delta$ and $[s_1, s_2] \subset [\alpha, \beta]$. Since $\lim_{k\to\infty} y_k = x(\alpha)$ then there exist $k_1 \in N$ such that for $k \geq k_1$ we have $\| y_k - x(\alpha) \| < \delta$. So whenever $k \geq k_1$, we have

$$\omega(\| y_k - x(\alpha) \|) < \frac{\varepsilon}{V_\Phi(H; [s_1, s_2]) + 1},$$

for $[s_1, s_2] \subset [\alpha, \beta]$.

Because

$$\begin{aligned} &\| F_k(y_k, \alpha+) - F_k(y_k, \alpha) - F_k(x(\alpha), \alpha+) + F_k(x(\alpha), \alpha) \| \\ &\leq \omega(\| y_k - x(\alpha) \|) V_\Phi(h_k; [\alpha+, \alpha]) \leq \omega(\| y_k - x(\alpha) \|) V_\Phi(H; [\alpha+, \alpha]) \\ &< \tfrac{\varepsilon}{V_\Phi(H; [\alpha+, \alpha]) + 1} V_\Phi(H; [\alpha+, \alpha]) < \varepsilon. \end{aligned} \tag{3.5}$$

By (3.2) we have

$$F_k(x(\alpha), \alpha+) - F_k(x(\alpha), \alpha) - F_0(x(\alpha), \alpha+) + F_0(x(\alpha), \alpha) \longrightarrow 0, \tag{3.6}$$

Since $y_k \to x(\alpha)$ for $k \to \infty$, we have by (3.5), (3.6) also

$$y_k + F_k(y_k, \alpha+) - F_k(y_k, \alpha) \longrightarrow x(\alpha) + F_0(x(\alpha), \alpha+) - F_0(x(\alpha), \alpha).$$

Hence $(y_k, \alpha) \in G$ and $((y_k + F_k(y_k, \alpha+) - F_k(y_k, \alpha)), \alpha) \in G$ for $k \geq k_1$.

Since the set B_c is open there exists $d > \alpha$ such that if $t \in [\alpha, d]$ and

$$\| x - (y_k + F_k(y_k, \alpha+) - F_k(y_k, \alpha)) \| \leq \Phi(H(t) - H(\alpha+))$$

then $(x, t) \in G$ for $k > k_1$. Using Theorem 2.1 we obtain that for $k > k_1$ there exists a solution $x_k : [\alpha, d] \to R^n$ of the generalized differential equation (3.4) on $[\alpha, d]$ such that $x_k(\alpha) = y_k, k > k_1$. We claim that $\lim_{k\to\infty} x_k(t) = x(t)$ for $t \in [\alpha, d]$.

Let us mention that the solution x_k of (3.4) exist on the interval $[\alpha, d]$ and that this interval is the same for all $k > k_1$. Indeed, looking at the proof of Theorem 2.1 it is easy to check that the value $d > \alpha$ depends on the function H which is common for all right hand sides F_k of (3.4).

By Lemma 2.1, if the sequence (x_k) contains a pointwise convergent subsequence on $[\alpha, d]$ then the limit of this subsequence is necessarily $x(t)$ for $t \in [\alpha, d]$ by the uniqueness assumption on the solution x of (3.2).

By Corollary 2.1 the sequence $(x_k), k > k_1$, of functions on $[\alpha, d]$ is equibounded and of uniformly bounded variation on $[\alpha, d]$. Therefore by Helly's Choice theorem the sequence (x_k) contains a pointwise convergent subsequence and (x_k) is therefore the only accumulation point of the sequence(x_k) for every $t \in [\alpha, d]$, i.e. $\lim_{k\to\infty} x_k(t) = x(t)$ for $t \in [\alpha, d]$.

In this way we have shown that the theorem holds on $[\alpha, d], d > \alpha$. Let us assume that the convergence result does not hold on the whole interval $[\alpha, \beta]$. Then there exists $d^\star \in (\alpha, \beta)$ such that for every $d < d^\star$ there is a solution x_k of (3.4) with $x_k(\alpha) = y_k$ on $[\alpha, d]$ provided $k \in N$ is sufficiently large and $\lim\limits_{k\to\infty} x_k(t) = x(t)$ for $t \in [\alpha, d]$ but this does not hold on $[\alpha, d]$ for $d > d^\star$. By Lemma 2.2 and Definition 2.2 we have

$$\| x_k(t_2) - x_k(t_1) \| \leq \Phi(| h_k(t_2) - h_k(t_1) |) \leq \Phi(| H(t_2) - H(t_1) |), \quad t_1, t_2 \in [\alpha, d^\star),$$

for $k \in N$ sufficiently large. Therefore the limits $x_k(d^\star-)$ exist and e also have

$$\lim_{k\to\infty} x_k(d^\star-) = x(d^\star-) = x(d^\star).$$

since the solution x is continuous form the left. Defining $x_k(d^\star) = x_k(d^\star-)$, we obtain $\lim\limits_{k\to\infty} x_k(d^\star) = x(d^\star)$ and this means that Theorem 3.1 holds on the closed interval $[\alpha, d^\star]$, too. Using now $d^\star < \beta$ as the starting point we can show in the same way as above that the theorem holds also on the interval $[d^\star, d^\star + \Delta]$ with some $\Delta > 0$ and this contradicts our assumption. Therefore the theorem holds also on the interval $[\alpha, \beta]$.

Corollary 3.1. Assume that $F_k : G \to R^n$ belongs to the class $\mathcal{F}_\Phi(G, h, \omega)$ for $k = 0, 1, 2, \cdots$ where $h : (a, b) \to R$ is nondecreasing functions which is continuous from the left and (3.2) holds. Let $x : [\alpha, \beta] \to R^n$, is a solution of (3.3) on $[\alpha, \beta] \subset (a, b)$ which has the uniqueness property same as Theorem 3.1.

Assume further that there is a $\rho > 0$ such that if $s \in [\alpha, \beta]$ and $\| y - x(s) \| < \rho$ then $(y, s) \in G$, and let $y_k \in R^n$, $k = 1, 2, \cdots$ satisfy $\lim\limits_{k\to\infty} y_k = x(\alpha)$. Then for sufficiently large $k \in N$ there exists a solution x_k of (3.4) on $[\alpha, \beta]$ with $x_k(\alpha) = y_k$ and $\lim\limits_{k\to\infty} x_k(s) = x(s), \quad s \in [\alpha, \beta]$.

Remark 1. This corollary is only a reformulation of Theorem 3.1 for the case when $h_k = h$ for $k = 0, 1, 2, \cdots$.

Remark 2. The assumption of Theorem 3.1 and Corollary 3.1 implies that all the right hand sides F_k of $(3.4), k = 0, 1, 2, \cdots$ have to belong to the same class $F_\Phi(G, H, \omega)$. In the sequel we will aim at weakening this assumption in order to obtain the following continuous dependence results that the right hand sides of (3.4) and the right hand sides of (3.3) do not belong to the same class $F_\Phi(G, H, \omega)$.

Theorem 3.2. Assume that $F_k : G \to R^n$ belongs to the class $\mathcal{F}_\Phi(G, h_k, \omega)$ for $k = 0, 1, 2, \cdots$ where $h_k : (a, b) \to R$, are nondecreasing functions which are continuous from the left when $k = 1, 2, \cdots$, and $h_0 : (a, b) \to R$ is nondecreasing functions which is continuous. Assume further that

$$\lim_{k\to\infty} \sup \Phi(h_k(t_2) - h_k(t_1)) \leq \Phi(h_0(t_2) - h_0(t_1)), \tag{3.7}$$

for every $a \leq t_1 \leq t_2 \leq b$. Suppose that (3.2) holds for $(x, t) \in G$.

Let $x : [\alpha, \beta] \to R^n$, is a solution of (3.3) on $[\alpha, \beta] \subset (a, b)$ which has the following uniqueness property: if $y : [\alpha, \gamma] \to R^n, [\alpha, \gamma] \subset [\alpha, \beta]$ is a solution of (3.3) such that $y(\alpha) = x(\alpha)$ then $y(t) = x(t)$ for every $t \in [\alpha, \gamma]$.

Assume further that there is a $\rho > 0$ such that if $s \in [\alpha, \beta]$ and $\| y - x(s) \| < \rho$ then $(y, s) \in G$, and let $y_k \in R^n,\ k = 1, 2, \cdots$ satisfy $\lim\limits_{k\to\infty} y_k = x(\alpha)$. Then for sufficiently large $k \in N$ there exists a solution x_k of (3.4) on $[\alpha, \beta]$ with $x_k(\alpha) = y_k$ and $\lim\limits_{k\to\infty} x_k(s) = x(s), \quad s \in [\alpha, \beta]$.

Proof. A similar result can be shown in the same way as in Theorem 3.1 with minor changes arising from the assumption (3.7).

Remark 3. Theorem 3.1 and Theorem 3.2 show that for a sufficiently close to the limit equation (3.3) the solution are pointwise close to the given solution of (3.3). The following theorem shows that for a sufficiently close to the limit equation (3.3) the solution are uniformly close to the given solution of (3.3).

Theorem 3.3. Assume that $F_k : G \to R^n$ belongs to the class $\mathcal{F}_\Phi(G, h_k, \omega)$ for $k = 0, 1, 2, \cdots$ where $h_k : (a, b) \to R$, are nondecreasing functions which are continuous from the left when $k = 1, 2, \cdots$, and $h_0 : (a, b) \to R$ is nondecreasing functions which is continuous. Assume further that (3.7) holds for every $a \le t_1 \le t_2 \le b$ and (3.2) holds for $(x, t) \in G$.

Let $x : [\alpha, \beta] \to R^n$ is a solution of (3.3) on $[\alpha, \beta] \subset (a, b)$ which has the uniqueness property same as Theorem 3.2.

Assume further that there is a $\rho > 0$ such that if $s \in [\alpha, \beta]$ and $\| y - x(s) \| < \rho$ then $(ys) \in G$, and let $y_k \in R^n,\ k = 1, 2, \cdots$ satisfy $\lim\limits_{k\to\infty} y_k = x(\alpha)$.

Then for every $\mu > 0$ there exists a $k_\star \in N$ such that for $k \in N, k > k_\star$ there exists a solution x_k of (3.4) on $[\alpha, \beta]$ with $x_k(\alpha) = y_k$ and

$$\| x_k(s) - x(s) \| < \mu, \quad s \in [\alpha, \beta]. \tag{3.8}$$

Proof. The existence of the solutions x_k of the equation (3.4) for sufficiently large $k \in N$ and the pointwise convergence $\lim\limits_{k\to\infty} x_k(s) = x(s), \quad s \in [\alpha, \beta]$. can be shown in the same way as in Theorem 3.2.

For showing (3.8) let us consider the difference $x_k(s) - x(s)$ for sufficiently large $k \in N$ for $s \in [\alpha, \beta]$. By the definition of a solution we have

$$x_k(s) - x(s) = y_k - x(\alpha) + \int_\alpha^s D[F_k(x_k(\tau), t) - F_0(x(\tau), t)]. \tag{3.9}$$

for every $s \in [\alpha, \beta]$.

Since $F_k : G \to R^n$ belongs to the class $\mathcal{F}_\Phi(G, h_k, \omega)$ for $k = 0, 1, 2, \cdots$ and x is a solution of (3.3) on $[\alpha, \beta]$, therefore by Corollary 2.1 $V_\Phi(x; [\alpha, \beta]) \le \Phi(V_\Phi(h_0; [\alpha, \beta])) < +\infty$, and by Definition 2.2 and Lemma 2.2 $\| x(t_2) - x(t_1) \| \le \Phi(| h_0(t_2) - h_0(t_1) |),\ t_1, t_2 \in [\alpha, \beta]$. Therefore, $x : [\alpha, \beta] \to R^n$ is a function of bounded Φ-variation which is continuous on $[\alpha, \beta]$. Then integral $\int_\alpha^\beta D[F_k(x(\tau), t)$ exists, and

$$\begin{aligned} \| \int_\alpha^s D[F_k(x_k(\tau), t) - F_0(x(\tau), t)] \| \quad &\le \quad \| \int_\alpha^s D[F_k(x_k(\tau), t) - F_k(x(\tau), t)] \| \\ &+ \quad \| \int_\alpha^s D[F_k(x(\tau), t) - F_0(x(\tau), t)] \| \end{aligned}$$

for $k = 1, 2, \cdots$ and $s \in [\alpha, \beta]$.

Assume that $\varepsilon > 0$. By continuity of the function $\omega : [0, +\infty) \to R$ and $\omega(0) = 0$, there exists a $\delta > 0$ such that

$$\omega(t) < \frac{\varepsilon}{V_\Phi(h_0; [\alpha, s]) + 1}$$

whenever $0 < t < \delta$. Let $\delta(\tau)$ be a positive function, since $\lim\limits_{k\to\infty} x_k(\tau) = x(\tau)$ for every $\tau \in [\alpha, \beta]$, there is a $k_1 \in N$ such that for $k \geq k_1$ we have $\| x_k(\tau) - x(\tau) \| < \delta(\tau)$. So whenever $k \geq k_1$, we have

$$\omega(\| x_k(\tau) - x(\tau) \|) < \frac{\varepsilon}{V_\Phi(h_0; [\alpha, s]) + 1}$$

for $[\alpha, s] \subset [\alpha, \beta]$.

By the definition of Kurzweil integral, for any division $D : \alpha = t_0 < t_1 < \cdots < t_m = s$ and any $\{\tau_1, \tau_2 \cdots, \tau_m\}$ satisfying $\tau_i - \delta(\tau_i) < t_{i-1} < t_i < \tau_i + \delta(\tau_i), i = 1, 2, \cdots, m$, we have

$$\begin{aligned}
&\| \textstyle\int_\alpha^s D[F_k(x_k(\tau), t) - F_k(x(\tau), t)] \| \\
\leq &\| \textstyle\int_\alpha^s D[F_k(x_k(\tau), t) - F_k(x(\tau), t)] \\
&- \textstyle\sum_{i=1}^m [F_k(x_k(\tau_i), t_i) - F_k(x_k(\tau_i), t_{i-1}) - F_k(x(\tau_i), t_i) + F_k(x(\tau_i), t_{i-1})] \| \\
&+ \textstyle\sum_{i=1}^m \| [F_k(x_k(\tau_i), t_i) - F_k(x_k(\tau_i), t_{i-1}) - F_k(x(\tau_i), t_i) + F_k(x(\tau_i), t_{i-1})] \| \\
&\leq \varepsilon + \max_{1\leq i\leq m} \omega(\| x_k(\tau) - x(\tau) \|) \textstyle\sum_{i=1}^m \Phi(h_k(t_i) - h_k(t_{i-1})) \\
&\leq \varepsilon + \tfrac{\varepsilon}{V_\Phi(h_0;[\alpha,s])+1} V_\Phi(h_k; [\alpha, s]) \leq \varepsilon + \tfrac{\varepsilon}{V_\Phi(h_0;[\alpha,s])+1} \Phi(h_k(s) - h_k(\alpha)) \\
&\leq \varepsilon + \tfrac{\varepsilon}{V_\Phi(h_0;[\alpha,s])+1} \Phi(h_0(s) - h_0(\alpha)) = \varepsilon + \tfrac{\varepsilon}{V_\Phi(h_0;[\alpha,s])+1} V_\Phi(h_0; [\alpha, s]) < 2\varepsilon. \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
&\| \textstyle\int_\alpha^s D[F_k(x(\tau), t) - F_0(x(\tau), t)] \| \\
\leq &\| \textstyle\int_\alpha^s D[F_k(x(\tau), t) - F_0(x(\tau), t)] \\
&- \textstyle\sum_{i=1}^m [F_k(x(\tau_i), t_i) - F_k(x(\tau_i), t_{i-1}) - F_0(x(\tau_i), t_i) + F_0(x(\tau_i), t_{i-1})] \| \\
&+ \textstyle\sum_{i=1}^m \| [F_k(x(\tau_i), t_i) - F_k(x(\tau_i), t_{i-1}) - F_0(x(\tau_i), t_i) + F_0(x(\tau_i), t_{i-1})] \| \\
&\leq \varepsilon + \textstyle\sum_{i=1}^m \| [F_k(x(\tau_i), t_i) - F_k(x(\tau_i), t_{i-1}) - F_0(x(\tau_i), t_i) + F_0(x(\tau_i), t_{i-1})] \|, \qquad 3.11)
\end{aligned}$$

By the assumption (3.2), for every $\mu > 0$ there is a $k_2 \in N,\ k_2 > k_1$ such that

$$\sum_{i=1}^m \| F_k(x(\tau_i), t_i) - F_k(x(\tau_i), t_{i-1}) - F_0(x(\tau_i), t_i) + F_0(x(\tau_i), t_{i-1}) \| \leq \frac{\mu}{4}. \qquad (3.12)$$

Since $\varepsilon > 0$ can be arbitrary, we choice it so that

$$\varepsilon < \frac{\mu}{12}, \qquad (3.13)$$

By the inequality $(3.10), (3.11), (3.12), (3.13)$ we then obtain for $s \in [\alpha, \beta]$

$$\| \int_\alpha^s D[F_k(x_k(\tau), t) - F_0(x(\tau), t)] \| \leq 2\varepsilon + \varepsilon + \frac{\mu}{4} \leq \frac{\mu}{2}.$$

Hence by (3.9) we have $\| x_k(s) - x(s) \| \leq \| y_k - x(\alpha) \| + \frac{\mu}{2}$ for every $s \in [\alpha, \beta]$, finally, we take $k_\star \in N$ such that $k_\star > k_2$ and $\| y_k - x(\alpha) \| < \frac{\mu}{2}$ for $k > k_\star$. Then we obtain $\| x_k(s) - x(s) \| < \mu$ for every $s \in [\alpha, \beta],\ k > k_\star$ and the theorem is proved.

Remark 4. If the function $\Phi(u)$ was defined in last section such that $0 < \frac{\Phi(u)}{u} < +\infty$ then by Theorem 1.15 in paper [2], we have $BV_\Phi[\alpha, \beta] = BV[\alpha, \beta]$ where $BV_\Phi[\alpha, \beta]$ and $BV[\alpha, \beta]$

denote the classes of the functions bounded Φ-variation and the functions bounded variation on $[\alpha, \beta]$ respectively in usual sense. Hence, the prime results of this paper are equivalent to the results of variational stability in paper [6].

If $\lim_{u\to 0^+} \frac{\Phi(u)}{u} = 0$ then by Theorem 1.15 in paper [2], we have $BV[\alpha, \beta] \subset BV_\Phi[\alpha, \beta]$. Such as if $\Phi(u) = u^p$ $(1 < p < +\infty)$ we have $\lim_{u\to 0^+} \frac{\Phi(u)}{u} = \frac{u^p}{u} = 0$. Therefore these results are essential generalization of continuous dependence of bounded variation solutions on parameters for Kurzweil Equations in paper [6].

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On Smarandache rings

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Abstract It is proved that a ring R in which for every $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is always a Smarandache Ring. Two examples are provided for justification.

Keywords Ring, Smarandache ring, field, partially ordered set, idempotent elements.

§1. Introduction

In [14], it is stated that, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset $B \subset A$ which is embedded with a stronger structure S. These types of structures occur in our every day's life.

The study of Smarandache Algebraic structures was initiated in the year 1998 by Raul Padilla following a paper written by Florentin Smarandache called "Special Algebraic Structures". Padilla treated the Smarandache Algebraic Structures mainly with associative binary operation.

In [11], [12], [13], [14], W. B. Vasantha Kandasamy has succeeded in defining around 243 Smarandache concepts by creating the Smarandache analogue of the various ring theoretic concepts.

The Smarandache notions are an excellent means to study local properties in Rings. The definitions of two levels of Smarandache rings, namely, S-rings of level I and S-rings of level II are given. S-ring level I, which by default of notion, will be called S-ring.

In [3] a ring R in which for every $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is introduced. In the literature such rings exist naturally, for instance, the rings Z_6 (modulo integers), Z_{10} (modulo integers), Boolean ring.

In this paper we prove that "A ring R in which for every $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$" is always a Smarandache ring. Two examples are provided for justification.

In section 2 we recall some definitions, examples and propositions pertaining to Smarandache Rings. In section 3 we prove our main theorem. In section 4, we give examples to justify our theorem. For basic definitions and concepts please refer [3].

§2. Preliminaries

Definition 2.1.([13]) A Smarandache ring (in short S-ring) is defined to be a ring A such that a proper subset of A is a field with respect to the operations induced. By a proper subset we understand a set included in A different from the empty set, from the unit element if any and from A.

Example 2.2. Let $F[x]$ be a polynomial ring over a field F. Then $F[x]$ is an S-ring.

Example 2.3. Let $Z_{12} = \{0, 1, 2, \ldots 11\}$ be a ring. Then Z_{12} is an S-ring as $A = \{0, 4, 8\}$ is a field with 4 acting as the unit element.

It is interesting to note that we do not demand the unit of the ring to be the unit of the field.

Definition 2.4. Let R be a ring. R is said to be a Smarandache ring of level II (S-ring II) if R contains a proper subset A ($A \neq \emptyset$) such that

(2.4.1) A is an additive abelian group;

(2.4.2) A is a semi group under multiplication;

(2.4.3) For a, $b \in A$, $a \cdot b = 0$ if and only if $a = 0$ or $b = 0$.

Proposition 2.5.([13]) Let R be an S - ring I, then R is an S - ring II.

Proposition 2.6.([4]) Any finite domain is a division ring.

§3. Proof of the theorem

In this section we show that the ring R in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is a Smarandache ring. For completeness, we write some lemmas from [1].

In [3], it is well known that the ring R in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is commutative and $x^{n(x)-1}$ is an idempotent element of R, i.e, for every $x \in R$,

$$(x^{n(x)-1})^2 = x^{n(x)-1}, \tag{i}$$

which implies that R has no nonzero nilpotent elements i.e., for every $x \in R$ and every natural number $k > 1$

$$x^k = 0 \Rightarrow x = 0. \tag{ii}$$

Lemma 3.1. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. The ring R is partially ordered by $\leq$ where for all elements x and y of R

$$x \leq y \iff xy = x^2. \tag{iii}$$

Proof. It is immediate that $\leq$ is reflexive as $xx = x^2$. Next , let $x \leq y$ and $y \leq x$ then $xy = x^2$ and $yx = y^2$.

Now

$$(x^2 - xy) - (yx - y^2) = 0$$

$$\Rightarrow x^2 - xy - yx + y^2 = 0$$

$$\Rightarrow x^2 - xy - xy + y^2 = 0$$

as R is commutative. This implies that

$$x^2 - 2xy + y^2 = 0$$

$$\Rightarrow (x - y)^2 = 0.$$

In view of (ii), we get

$$x - y = 0$$

or

$$x = y.$$

Hence, $\leq$ is anti-symmetric. Finally, let $x \leq y$ and $y \leq z$ i.e.,

$$xy = x^2$$

and

$$yz = y^2.$$

Now,

$$x^2z = xyz = xy^2 = x^2y = x^3.$$

So

$$x^2z = x^3 => x^2z^2 = x^3z$$

and

$$x^3z = x^4.$$

But, then

$$(x^2z^2 - x^3z) - (x^3z - x^4) = 0$$

$$\Rightarrow x^2z^2 - 2x^3z + x^4 = 0.$$

$$\Rightarrow (xz - x^2)^2 = 0.$$

In view of (ii), we get

$$xz = x^2$$

$$\Rightarrow x \leq z.$$

Therefore, $\leq$ is transitive. Hence, $(R, \leq)$ is a partially ordered set.

Lemma 3.2. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. Then for all elements x, y, z of R

$$y \leq z => xy \leq xz \qquad (iv)$$

and

$$x^{n(x)-1}y \leq y. \qquad (v)$$

Proof. Let x, y, z be any three elements of R. In view of (iii)

$$y \leq z => yz = y^2$$

$$\Rightarrow x^2(yz) = x^2y^2$$

$$\Rightarrow (xy)(xz) = (xy)^2$$

$$\Rightarrow xy \leq xz.$$

Further, in view of (i) we have

$$x^{n(x)-1}y^2 = (x^{n(x)-1}y)^2$$

$$\Rightarrow x^{n(x)-1}y \leq y.$$

Definition 3.3. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. A nonzero element a of R is called an atom of R provided for every $x \in R$,

$$x \leq a \text{ implies } x = a \text{ or } x = 0. \qquad (vi)$$

More over, R is called atomic provided for every nonzero element r of R there exists an atom a of R such that

$$a \leq r.$$

Lemma 3.4. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$, and let a be an atom of R. Then

$$r^{n(r)-1}a = a \text{ or } ra = 0,$$

for every element r of R.

Proof. By (v), we have

$$r^{n(r)-1}a \leq a$$

and since a is an atom by (vi) we have

$$r^{n(r)-1}a = a \text{ or } r^{n(r)-1}a = 0.$$

$$\Rightarrow r^{n(r)-1}a = a \text{ or } ra = 0, (\textit{ since } r^{n(r)} = r).$$

Definition 3.5. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. A subset S of R is called orthogonal provided

$$xy = 0$$

for distinct elements x and y of S.

Lemma 3.6. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$. Then the set $(e_i)_{i \in I}$ of all idempotent atoms of R is an orthogonal set.

Proof. Since for each $i \in I$, e_i is both an atom and an idempotent, from Lemma (3.4) it follows that $e_i e_j = e_j = e_i$ or $e_i e_j = 0$.

Lemma 3.7. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$, and let a be an atom of R. Then $a^{n(a)-1}$ is an idempotent atom of R.

Proof. From (i) it follows that $a^{n(a)-1}$ is idempotent. Now, let $x \leq a^{n(a)-1}$. But by (iv) we get $ax \leq a^{n(a)} = a$ i.e., $ax \leq a$. Since a is an atom by (vi) it follows that $ax = a$ or $ax = 0$.

If $ax = a$ then $a^{n(a)-1}x = a^{n(a)-1}$. By (iii) we get $a^{n(a)-1} \leq x$. Hence, $x = a^{n(a)-1}$.

If $ax = 0$ then $a^{n(a)-1}x = 0$, but $a^{n(a)-1}x = x^2$. Therefore $x^2 = 0$. By (ii) we get $x = 0$.

Lemma 3.8. Let R be a ring in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ and let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R, then for every $i \in I$ the ideal F_i of R given by

$$F_i = \{re_i / r \in R\} \qquad (vii)$$

is a subfield of R.

Proof. Since $e_i^2 = e_i$, it follows that e_i is an element of F_i and also the unit of F_i.

Now let re_i be a non zero element of F_i. We show that re_i has an inverse in F_i. If $n(r) > 2$ then by Lemma (3.4) we have $(re_i)(r^{n(r)-2}e_i) = e_i$. It follows that $r^{n(r)-2}e_i$ is the inverse of re_i in F_i. If $n(r) = 2$ then by Lemma (3.4) we have $(re_i)(re_i) = r^2 e_i^2 = re_i = e_i$. It shows that re_i has its own inverse in F_i.

Now, we are ready to prove the main theorem.

Theorem 3.9. The ring R in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$ is always a Smarandache ring.

Proof. Let $(e_i)_{i \in I}$ be the set of all idempotent atoms of R. In view of the Lemma (3.8), for every $i \in I$, the ideal F_i of R given by $F_i = \{re_i / r \in R\}$ is a field of R. Hence, the ring R is a Smarandache ring.

§4. Examples

In this section we give examples to justify our Theorem 3.9. Further, we show by an example that the condition ' for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$' satisfied by the ring R in our results is a sufficient condition but not a necessary condition.

Example 4.1. Consider the ring $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (modulo integers). It is obvious that $0^2 = 0$; $1^2 = 1$; $2^5 = 2$; $3^5 = 3$; $4^3 = 4$; $5^2 = 5$; $6^2 = 6$; $7^5 = 7$; $8^5 = 8$; $9^3 = 9$. Therefore the ring

$$R = Z_{10}$$

satisfies the condition ' for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$'. Further, in view of the relation table (see table I), and Lemma (3.1), $(Z_{10}, \leq)$ is a partially ordered set. The Hasse diagram (see [9]) of the p.o. set $(Z_{10}, \leq)$ is given (see fig. 1) for our use. From the Hasse diagram it is obvious that the elements

2, 4, 5, 6, 8 are atoms and the elements 5, 6 are idempotent atoms in $(Z_{10}, \leq)$. In view of Lemma (3.8), the ideals

$$F_1 = \{r \cdot 5 / r \in Z_{10}\} = \{0, 5\}$$

and

$$F_2 = \{r \cdot 6 / r \in Z_{10}\} = \{0, 2, 4, 6, 8\}$$

are fields. Hence the ring Z_{10} (modulo integers) is a Smarandache ring.

$\leq$	0	1	2	3	4	5	6	7	8	9
0	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
1		✓								
2			✓					✓		
3				✓						
4					✓					✓
5		✓		✓		✓		✓		✓
6		✓					✓			
7								✓		
8				✓					✓	
9										✓

Table 1

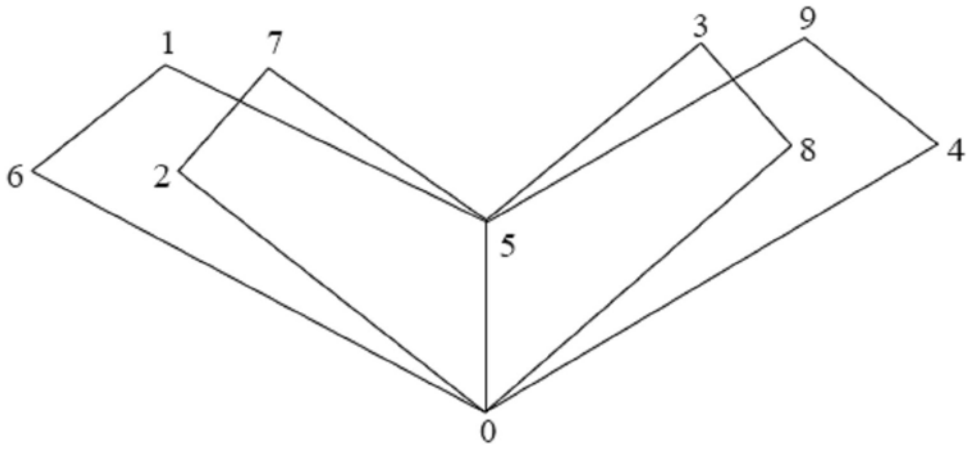

Figure 1:

Example 4.2. Consider the ring

$$Z_6 = \{0, 1, 2, 3, 4, 5\}$$

(modulo integers). It is obvious that $0^2 = 0$; $1^2 = 1$; $2^3 = 2$; $3^2 = 3$; $4^2 = 4$; $5^3 = 5$. Therefore, the ring $R = Z_6$ satisfies the condition ' for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$'. In view of the relation table (see table II), and Lemma (3.1), $(Z_6, \leq)$ is a partially ordered set. The Hasse diagram (see [9]) of the p.o. set $(Z_6, \leq)$ is given (see fig. 2) for our use. From the Hasse diagram, it is obvious that the elements 2, 3, 4, are atoms and the elements 3, 4 are idempotent atoms in $(Z_6, \leq)$. In view of Lemma (3.8), the ideals $F_1 = \{r \cdot 3 / r \in Z_6\} = \{0, 3\}$ and $F_2 = \{r \cdot 4 / r \in Z_6\} = \{0, 2, 4,\}$ are fields. Hence Z_6 (modulo integers) is a Smarandache ring.

$\leq$	0	1	2	3	4	5
0	✓	✓	✓	✓	✓	✓
1		✓				
2			✓			✓
3		✓		✓		✓
4		✓			✓	
5						✓

Table 2

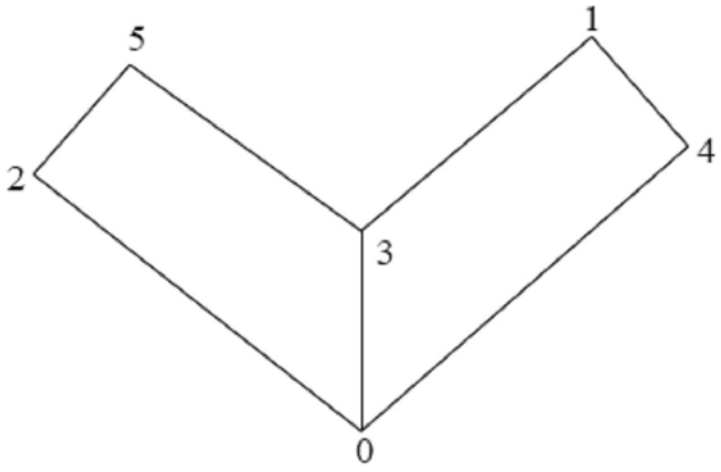

Figure 2:

Finally, we show by an example that the condition ' for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$' satisfied by the ring R in our results is a sufficient condition but not a necessary condition.

Example 4.3. In [13] Vasantha Kandasamy W. B. quoted the Example (2.3) for Smarandache ring. This ring $Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ (modulo integers) is a Smarandache ring but the condition ' for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x) > 1$ such that $x^{n(x)} = x$' fails in the ring Z_{12} as there does not exist an integer $n(2) > 1$ for the integer 2 in Z_{12} such that $2^{n(2)} = 2$. Hence, the condition is a sufficient condition but not a necessary condition.

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