Vol.7,No.2,2011

SCIENTIA MAGNA

An international journal

Edited by Department of Mathematics
Northwest University, P.R.China

Vol. 7, No. 2, 2011 ISSN 1556-6706

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Department of Mathematics
Northwest University
Xi'an, Shaanxi, P.R.China

Scientia Magna is published annually in 400-500 pages per volume and 1,000 copies.

It is also available in **microfilm format** and can be ordered (online too) from:

Books on Demand
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300 North Zeeb Road
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Printed in the United States of America
Price: US$ 69.95

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Weyl's theorem and Tensor product for m-quasi class A_k operators

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Abstract In this paper we generalize quasi class A operator and introduce m-quasi class A_k operators, where k and m are positive integers, which coincides with quasi-class A operator for $k = 1$ and $m = 1$. We prove that if T is a m-quasi class A_k operator, then T is of finite ascent, T is an isoloid and Weyl's theorem holds for T and f(T), where f is an analytic function in a neighborhood of the spectrum of T. We also show that m-quasi classA_k operators are closed under tensor product.

Keywords Class A_k, quasi-class A_k, m-quasi class A_k, Weyl's theorem, Tensor product.

§1. Introduction and preliminaries

Let $T \in B(H)$ be the Banach Algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator T, we mean an element from $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. An operator T is called paranormal if $\left\|T^2(x)\right\| \geq \|Tx\|^2$ for every unit vector $x \in H$. An operator T belongs to class A, if $\left|T^2\right| \geq |T|^2$. An operator T is called n-perinormal for positive integer n such that $n \geq 2$, if $T^{*n}T^n \geq (T^*T)^n$. An operator T is called k-paranormal for positive integer k, if $\left\|T^{k+1}x\right\| \geq \|Tx\|^{k+1}$ for every unit vector x in H. For $0 < p < 1$, an operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is called hyponormal. An operator T is called log-hyponormal if T is invertible and $log(T^*T) \geq log(TT^*)$. An operator T is said to be of class $A(k)$ for $k > 0$, if $(T^* |T|^{2k} T)^{\frac{1}{k+1}} \geq |T|^2$. An operator T is called quasi-class (A, k), if $T^{*k} \left|T^2\right| T^k \geq T^{*k} |T|^2 T^k$ and quasi-class A if $T^* \left|T^2\right| T \geq T^* |T|^2 T$. An operator T is called normaloid if $r(T) = \|T\|$, where $r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}$ and isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Furuta et al [4] have proved that every log-hyponormal is a class A operator and every class A operator is a paranormal operator. We define an operator [24] $T \in B(H)$ as class A_k for a positive integer k, if $\left|T^{k+1}\right|^{\frac{2}{k+1}} \geq |T|^2$. If $k = 1$, then class A_k coincides with class A operator.

We have also proved the following results.

Theorem 1.1.[24] If T is a p-hyponormal or a log-hyponormal operator, then T is class A_k operator, for each positive integer k.

Theorem 1.2.[24] Let T be an invertible and class A operator. Then

1. T is of class A_k operator for every positive integer k.

2. $\text{classA}_1 \subseteq \text{classA}_2 \subseteq \text{classA}_3 \subseteq ...$

3. For all positive integer n, T^n is of class A_k operator for every positive integer k.

4. T^{-1} is of class A_k operator for every positive integer k.

Theorem 1.3.[24] If T is of class A_k for some positive integer k, then T is k-paranormal.

An operator T is called a Fredholm operator if the range of T denoted by $ran(T)$ is closed and both $kerT$ and $kerT^*$ are finite dimensional. The index of a Fredholm operator is an integer defined as $index(T) = dimKerT - dimKerT^*$. The ascent of $T \in B(H)$, denoted by asc(T) is the least non-negative integer n such that $\ker T^n = \ker T^{n+1}$. We say that T is of finite ascent, if $asc(T - \lambda) < \infty$, for all $\lambda \in C$. An operator $T \in B(H)$ is called Weyl if it is Fredholm of index 0. The spectrum of T is denoted by $\sigma(T)$ and the set of all isolated eigenvalues of finite multiplicity is denoted by π_{00}. The essential spectrum of T is defined as $\sigma_e(T) = \{\lambda \in C : T - \lambda I \quad is \quad not \quad Fredholm\}$. The Weyl spectrum of T is defined as $w(T) = \{\lambda \in C : T - \lambda I \quad is \quad not \quad Weyl\}$. When the space is infinite dimensional $w(0) = 0$ and $w(T) = \{0\}$, if T is compact. H. Weyl has shown that $\lambda \in \sigma(T + K)$ for every compact operator k if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. We say that Weyl's theorem holds for T [3] if T satisfies the equality $\sigma(T) - \omega(T) = \pi_{00}(T)$. Let $H(\sigma(T))$ be the set of all analytic functions on an open neighbourhood of $\sigma(T)$. The spectral picture [19] of an operator $T \in B(H)$, denoted by $SP(A)$ consists of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

In this paper, we discuss the class of operators called m-quasi class A_k operators for positive integers k and m, which is a superclass of p-hyponormal, log-hyponormal and class A_k operators. We prove that if T is a m-quasi class A_k operator, then T is an isoloid, T is of finite ascent and Weyl's theorem holds for both T and $f(T)$, where f is an analytic function in a neighborhood of the spectrum of T. We also prove that the the class of m-quasi class A_k operator is closed under tensor product.

§2. Definition and examples

Definition 2.1. An operator $T \in B(H)$ is defined to be of m-quasi class A_k, if $T^{*m}(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)T^m \geq 0$, where k and m are positive integers. If $k = 1$ and $m = 1$, then m-quasi class A_k operators coincides with quasi-class A operators.

Obviously, $1 - quasiclassA_k \subseteq 2 - quasiclassA_k \subseteq 3 - quasiclassA_k \subseteq \cdots$

Example 2.2. Let H be the direct sum of a denumerable number of copies of two dimensional Hilbert space $R \times R$. Let A and B be two positive operators on $R \times R$. For any fixed positive integer n, define an operator $T = T_{A,B,n}$ on H as follows:

$T((x_1, x_2, \ldots)) = (0, A(x_1), A(x_2), \ldots, A(x_n), B(x_{n+1}), \ldots)$.

Its adjoint T^* is given by:

$T^*((x_1, x_2, \ldots)) = (A(x_2), A(x_3), \ldots, A(x_{n+1}), B(x_{n+2}), \ldots)$.

For $n \geq k$, $T_{A,B,n}$ is of m-quasi classA_k if and only if A and B satisfies:

$$A^m(A^{k-i+1}B^{2i}A^{k-i+1})^{\frac{1}{k+1}}A^m \geq A^{2+2m} \quad for \quad i = 1, 2, \ldots, k.$$

If $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $T_{A,B,n}$ is of m-quasi classA_k, for every positive integer k.

Since $S \geq 0$ implies $T^*ST \geq 0$, the following result is trivial.

Theorem 2.3. If T belongs to class A_k, for some positive integer $k \geq 1$, then T belongs to m-quasi classA_k, for every positive integer m.

From Theorems 1.2 and 2.3, we get the following results.

Theorem 2.4. Let T be an invertible and class A operator. Then for each positive integer m,

1. T is of m-quasi classA_k operator for every positive integer k.
2. $m - quasi\, classA_1 \subseteq m - quasi\, classA_2 \subseteq m - quasi\, classA_3 \subseteq ....$
3. For all positive integers n, T^n is of m-quasi classA_k operator for every positive integer k.
4. T^{-1} is of m-quasi classA_k operator for every positive integer k.

§3. Matrix representation

Matrix representation of operators is used to study various properties of an operator. Class A_k operators have the matrix representation [24] $T = \begin{pmatrix} A & S \\ 0 & 0 \end{pmatrix}$ with respect to direct sum of closure of range of T and kernel of T^*. The next theorem gives the matrix representation for m-quasi classA_k operators.

Proposition 3.1. (Hansen Inequality [6]) If $A, B \in B(H)$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then $(B^*AB)^\delta \geq B^*A^\delta B$ for all $\delta \in (0, 1]$.

Theorem 3.2. Assume that $T \in B(H)$ is a m-quasi class A_k operator for positive integers k and m, T has no dense and T has the following representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \, on\, H = \overline{ran(T^m)} \oplus ker(T^{m*}).$$

Then T_1 is class A_k operator on $\overline{ran(T^m)}$ and T_3 is nilpotent. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let P be the orthogonal projection onto $\overline{ran(T^m)}$. Then

$$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP.$$

Since T is m-quasi classA_k operator,

$$P(\left|T^{1+k}\right|^{\frac{2}{1+k}} - |T|^2)P \geq 0.$$

By Hansen's inequality,

$$P(\left|T^{1+k}\right|)^{\frac{2}{1+k}}P = P(T^{*1+k}T^{1+k})^{\frac{1}{1+k}}P \leq (PT^{*1+k}T^{1+k}P)^{\frac{1}{1+k}} = \begin{pmatrix} \left|T_1^{1+k}\right|^{\frac{2}{1+k}} & 0 \\ 0 & 0 \end{pmatrix}$$

and $P(|T|^2)P = PT^*TP = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, $\begin{pmatrix} \left|T_1^{1+k}\right|^{\frac{2}{1+k}} & 0 \\ 0 & 0 \end{pmatrix} \geq P(\left|T^{1+k}\right|)^{\frac{2}{1+k}}P \geq$ $P(|T|^2)P = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, $\left|T_1^{1+k}\right|^{\frac{2}{1+k}} \geq |T_1|^2$. Hence T_1 is class A_k operator on $\overline{ran(T^m)}$.

For any x $= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H$, $\langle T_3^m x_2, x_2\rangle = \langle T^m(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{m*}(I-P)x\rangle$ $= 0$. Hence $T_3^m = 0$. By [[5], corollary 7], $\sigma(T_1)\bigcup\sigma(T_3) = \sigma(T)\bigcup\tau$, where τ is the union of certain of the holes in $\sigma(T)$ which happen to be a subset of $\sigma(T_1)\bigcap\sigma(T_3)$, and $\sigma(T_1)\bigcap\sigma(T_3)$ has no interior points. Therefore $\sigma(T) = \sigma(T_1)\cup\sigma(T_3) = \sigma(T_1)\cup\{0\}$.

Since class A_k operators are isoloid [24], we immediately have the following corollary.

Corollary 3.3. Assume that $T \in B(H)$ is a m-quasi class A_k operator for positive integers k and m, T has no dense range and T has the following representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \; on\; H = \overline{ran(T^m)} \oplus ker(T^{*m}).$$

Then T_1 is isoloid and T_3 is nilpotent. Furthermore, $\sigma(T) = \sigma(T_1)\cup\{0\}$.

§4. Some properties of m-quasi class A_k

Theorem 4.1. If $T \in B(H)$ is m-quasi classA_k operator for some positive integers k and m, then T is isoloid.

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^m)} \oplus kerT^{*m}$. Let λ_0 be an isolated point of $\sigma(T)$. Then either $\lambda_0 = 0$ or $0 \neq \lambda_0 \in iso\sigma(T_1)$. Since T_1 is isoloid [Cor 3.3], if $\lambda_0 \in iso\sigma(T_1)$, then $\lambda_0 \in \sigma_p(T_1)$ and hence $\lambda_0 \in \sigma_p(T)$. On the contrary, if $\lambda_0 = 0$ and $0 \notin \sigma(T_1)$, then T_1 is invertible. Also $\dim kerT_3 \neq 0$. Hence there exists $x \neq 0$ in $kerT_3$ such that $T(-T_1^{-1}T_2x \oplus x) = 0$. Hence $-T_1^{-1}T_2x \oplus x \in kerT$. Hence in both cases, λ_0 is an eigenvalue of T. Therefore T is isoloid.

Theorem 4.2. If $T \in B(H)$ is m-quasi class A_k operator for positive integers k and m, $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \ker(T-\lambda)^{\perp}$, then

1. $T_2 = 0$ and
2. T_3 is m-quasi class A_k.

Proof. Let P be the orthogonal projection of H onto $\ker(T-\lambda)$. Since T is m-quasi classA_k, T satisfies

$$T^{*m}(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)T^m \geq 0,$$

where k and m are positive integers. Hence $P(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)P \geq 0$,
where $P\,|T|^2\,P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\left(P\left|T^{k+1}\right|^2 P\right) = \begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix}$.
Therefore,

$$\begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = \left(P\left|T^{k+1}\right|^2 P\right)^{\frac{1}{k+1}} \geq P\left|T^{k+1}\right|^{\frac{2}{k+1}} P \geq P\,|T|^2\,P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$P\left|T^{k+1}\right|^{\frac{2}{k+1}} P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} = P\,|T|^2\,P.$$

Hence $\left|T^{k+1}\right|^{\frac{2}{k+1}}$ is of the form $\left|T^{k+1}\right|^{\frac{2}{k+1}} = \begin{pmatrix} |\lambda|^2 & A \\ A^* & B \end{pmatrix}$.

Since $\begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix} = P\left(\left|T^{k+1}\right|^2\right)P = P\left(\left|T^{k+1}\right|^{\frac{2}{k+1}}\right)^{k+1} P$, we can easily show that $A = 0$.
Therefore, $\left|T^{k+1}\right|^{\frac{2}{k+1}} = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & B \end{pmatrix}$ and hence $\left|T^{k+1}\right|^2 = \begin{pmatrix} |\lambda|^{2(k+1)} & 0 \\ 0 & B^{(k+1)} \end{pmatrix}$.
This implies that $\lambda^k T_2 + \lambda^{k-1}T_2T_3 + \cdots + T_2T_3^k = 0$ and $B = \left|T_3^{k+1}\right|^{\frac{2}{k+1}}$. Therefore,

$$0 \leq T^{*m}\left(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2\right)T^m = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix},$$

where $X = 0, Y = -\overline{\lambda}^{m+1}T_2T_3^m$ and $Z = -(\overline{\lambda}^{m-1}T_2^* + \cdots + T_3^{*m-1}T_2^*)\overline{\lambda}T_2T_3^m - \lambda T_3^{*m}T_2^*(\lambda^{m-1}T_2 + \cdots + T_2T_3^{m-1}) - T_3^{*m}\,|T_2|^2\,T_3^m + T_3^*m\left(\left|T_3^{k+1}\right|^{\frac{2}{k+1}} - |T_3|^2\right)T_3^m$.

A matrix of the form $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X \geq 0$, $Z \geq 0$ and $Y = X^{1/2}WZ^{1/2}$, for some contraction W. Therefore, $T_2T_3^m = 0$. This together with $\lambda^k T_2 + \lambda^{k-1}T_2T_3 + \cdots + T_2T_3^k = 0$ gives that $T_2 = 0$ and T_3 is m-quasi class A_k.

Corollary 4.3. If $T \in B(H)$ is m-quasi class A_k operator for positive integers k and m and $(T-\lambda)x = 0$ for $\lambda \neq 0$ and $x \in H$, then $(T-\lambda)^*x = 0$.

Theorem 4.4. Let $T \in B(H)$ be a m-quasi class A_k operator for positive integers k and m, then T satisfies $\left\|T^{k+1+m}x\right\|\,\|T^m x\|^k \geq \left\|T^{m+1}\right\|^{k+1}$.

Proof. Using McCarthy inequality [18], for each $x \in H$,

$$\begin{aligned} 0 &\leq \left\langle T^{*m}(\left|T^{1+k}\right|^{\frac{2}{1+k}} - |T|^2)T^m x, x \right\rangle \\ &\leq \left\langle \left|T^{1+k}\right|^2 T^m x, T^m x \right\rangle^{\frac{1}{1+k}} \|T^m x\|^{2(1-\frac{1}{1+k})} - \left\langle |T|^2 T^m x, T^m x \right\rangle \\ &= \left\|T^{1+k+m}x\right\|^{\frac{2}{1+k}} \|T^m x\|^{\frac{2k}{1+k}} - \left\|T^{m+1}x\right\|^2. \end{aligned}$$

Hence $\left\|T^{k+1+m}x\right\| \|T^m x\|^k \geq \left\|T^{m+1}x\right\|^{k+1}$, for every $x \in H$. Hence the required result.

Theorem 4.5. If $T \in B(H)$ is m-quasi class A_k operator for positive integers m and k, then T is of finite ascent.

Proof. By Corollary 4.3, for $\lambda \neq 0$, $ker(T-\lambda) \subset ker(T-\lambda)^*$. Hence $ker(T-\lambda)^2 = ker(T-\lambda)$. If $\lambda = 0$, let $0 \neq x \in kerT^{k+1+m}$. By Theorem 4.4, $x \in kerT^{m+1} \subset kerT^{k+m}$. Hence $kerT^{k+1+m} = kerT^{k+m}$. Hence $asc(T-\lambda) < \infty$, for all $\lambda \in C$.

§5. Weyl's theorem

Theorem 5.1. If $T \in B(H)$ is m-quasi class A_k operator for some positive integers k and m, then Weyl's theorem holds for T.

Theorem 5.2. If $T \in B(H)$ is m-quasi class A_k operator for some positive integers k and m, then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

To prove these theorems, we need the following results.

Proposition 5.3.[15] (Theorem 6) For given operators $A, B, C \in B(H)$, there is an equality $w(A) \bigcup w(B) = w(M_C) \bigcup \tau$, where $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and τ is the union of certain holes in $w(M_C)$ which happen to be a subset of $w(A) \bigcap w(B)$.

Proposition 5.4.[24] (Lemma 5.7) If T is a class A_k operator for some positive integer k, then $f(w(T)) = w(f(T))$ for every $f \in H(\sigma(T))$.

Proposition 5.5.[15] (Corollary 11) Suppose $A \in B(H)$ and $B \in B(K)$ are isoloids. If Weyl's theorem holds for A and B, and if $w(A) \bigcap w(B)$ has no interior points, then Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Proposition 5.6.[16] If either $SP(A)$ or $SP(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds, then for every $C \in B(K, H)$, Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Rightarrow$ Weyl's theorem holds for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

Proposition 5.7.[7] (Theorem 5) If $T \in B(H)$ then the following are equivalent:

1. $ind(T-\lambda I)ind(T-\mu I) \geq 0$ for each pair $\lambda, \mu \in C - \sigma_e(T)$.

2. $f(w(T)) = w(f(T))$ for every $f \in H(\sigma(T))$.

Proposition 5.8.[17] (Lemma) If $T \in B(H)$ is isoloid, then

$$f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)), \text{ for every f} \in \text{H}(\sigma(\text{T})).$$

Lemma 5.9. If T is a m-quasi class A_k operator for some positive integers k and m, then $f(w(T)) = w(f(T))$ for every $f \in H(\sigma(T))$.

Proof. By Theorem 3.2, if $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^m)} \oplus kerT^{*m}$, then T_1 is class A_k operator on $\overline{ran(T^m)}$ and T_3 is nilpotent. Furthermore, $\sigma(T) = \sigma(T_1) \bigcup \{0\}$. By Proposition 5.4, $f(w(T_1)) = w(f(T_1))$. Hence by Proposition 5.3,

$$w(f(T)) = w(f(T_1)) \cup w(f(T_3)) = f(w(T_1)) \cup f(w(T_3)) = f(w(T_1) \cup w(T_3)) = f(w(T)).$$

Lemma 5.10. If T is m-quasi class A_k operator for some positive integers k and m, then $ind(T - \lambda I) \leq 0$ for all complex numbers λ.

Proof. Since T is of finite ascent by theorem 4.5, by[[8] , Proposition 3.5], $ind(T - \lambda) \leq 0$ for all complex numbers λ.

By Theorem 4.1 and Proposition 5.8, we get the following result immediately.

Lemma 5.11. If T is a m-quasi class A_k operator for some positive integers k and m, then $f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T))$, for every $f \in H(\sigma(T))$.

By Lemma 5.10 and Proposition 5.7, the following result is trivial.

Lemma 5.12. If T is a m-quasi class A_k operator for some positive integers k and m, then $f(w(T)) = w(f(T))$ for every $f \in H(\sigma(T))$.

Proof of Theorem 5.1. By Theorem 3.2, if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^m)} \oplus kerT^{*m}$, then T_1 is class A_k operator on $\overline{ran(T^m)}$ and T_3 is nilpotent. Also by [24], T_1 is isoloid and weyl's theorem holds for T_1, since $0 \notin w(T_1)$. Hence by Proposition 5.5, Weyl's theorem holds for $\begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix}$. Therefore by Proposition 5.6, Weyl's theorem holds for $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$.

Proof of Theorem 5.2. By Lemma 5.11, Theorem 5.1 and Lemma 5.12, for every $f \in H(\sigma(T))$, $\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)) = f(w(T)) = w(f(T))$. Hence Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.

§6. Tensor product

Let H and K denote the Hilbert spaces. For given non-zero operators $T \in B(H)$ and $S \in B(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. The normaloid property is invariant under the tensor products [20]. $T \otimes S$ is normal if and only if T and S are normal [12,21]. There exist paranormal operators T and S such that $T \otimes S$ is not paranormal [1]. In [3], B. P. Duggal showed that for non-zero $T \in B(H)$ and $S \in B(K)$, $T \otimes S$ is p-hyponormal if and only if T and S are p-hyponormal. This result was extended to p-quasi hyponormal operators, class A operators , quasi-class A and class A_k operators in [10], [11], [13] and [24] respectively.

In this section, we prove an analogous result for quasi-class A_k operators. Tensor product of two non-zero operators satisfy:

1. $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$.
2. $|T \otimes S|^p = |T|^p \otimes |S|^p$, for any positive real number p.

In [21], J. Stochel has proved the following result.

Proposition 6.1. Let $A_1, A_2 \in B(H), B_1, B_2 \in B(K)$ be non-negative operators. If A_1 and B_1 are non-zero, then the following assertions are equivalent:

1. $A_1 \otimes B_1 \leq A_2 \otimes B_2$.
2. there exists $c > 0$ such that $A_1 \leq cA_2$ and $B_1 \leq c^{-1}B_2$.

Theorem 6.2. If $T \in B(H)$ and $S \in B(K)$ are non-zero operators. Then $T \otimes S$ is m-quasi class A_k operator if and only if one of the following holds:

1. T and S are m-quasi class A_k operators.
2. $T^{m+1} = 0$ or $S^{m+1} = 0$.

Proof. Consider

$$\begin{aligned}
& (T \otimes S)^{*m} \left(\left|(T \otimes S)^{k+1}\right|^{\frac{2}{k+l}} - |T \otimes S|^2 \right) (T \otimes S)^m \\
= \; & (T \otimes S)^{*m} \left(\left|T^{k+1} \otimes S^{k+1}\right|^{\frac{2}{k+1}} - |T|^2 \otimes |S|^2 \right) (T \otimes S)^m \\
= \; & (T \otimes S)^{*m} \left(\left(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2 \right) \otimes \left|S^{k+1}\right|^{\frac{2}{k+1}} + |T|^2 \otimes \left(\left|S^{k+1}\right|^{\frac{2}{k+1}} - |S|^2 \right) \right) (T \otimes S)^m \\
= \; & T^{*m} \left(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2 \right) T^m \otimes S^{*m} \left|S^{k+1}\right|^{\frac{2}{k+l}} S^m + T^{*m} |T|^2 T^m \otimes S^{*m} \\
& \left(\left|S^{k+1}\right|^{\frac{2}{k+1}} - |S|^2 \right) S^m,
\end{aligned}$$

Hence, if either (i) T and S are m-quasi class A_k operators or (ii) $T^{m+1} = 0$ or $S^{m+1} = 0$, then $T \otimes S$ is m- quasi class A_k operator.

Conversely, suppose that $T \otimes S$ is m-quasi class A_k operator. Then by the above equality,

$$\begin{aligned}
T^{*m}(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)T^m \otimes S^{*m} \left|S^{k+1}\right|^{\frac{2}{k+1}} S^m \\
+ T^{*m} |T|^2 T^m \otimes S^{*m}(\left|S^{k+1}\right|^{\frac{2}{k+1}} - |S|^2)S^m \geq 0.
\end{aligned}$$

Therefore for every $x \in H$ and $y \in K$,

$$\begin{aligned}
\left\langle T^{*m}(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)T^m x, x \right\rangle \left\langle S^{*m} \left|S^{k+1}\right|^{\frac{2}{k+l}} S^m y, y \right\rangle \\
+ \left\langle T^{*m} |T|^2 T^m x, x \right\rangle \left\langle S^{*m}(\left|S^{k+1}\right|^{\frac{2}{k+1}} - |S|^2)S^m y, y \right\rangle \geq 0.
\end{aligned}$$

It is sufficient to prove that either (i) or (ii) holds. Assume the contrary that, neither of T^m and S^m is the zero operator and T is not m-quasi class A_k operator.

Then there exists $x_k \in H$ such that

$$\left\langle T^{*m}(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)T^m x_k, x_k \right\rangle < 0 \; and \; \left\langle T^{*m} |T|^2 T^m x_k, x_k \right\rangle > 0.$$

Let $\alpha = \left\langle T^{*m}(\left|T^{k+1}\right|^{\frac{2}{k+1}} - |T|^2)T^m x_k, x_k \right\rangle$ and $\beta = \left\langle T^{*m}\, |T|^2\, T^m x_k, x_k \right\rangle$.

Then $\alpha \left\langle S^{*m} \left|S^{k+1}\right|^{\frac{2}{k+l}} S^m y, y \right\rangle + \beta \left\langle S^{*m}(\left|S^{k+1}\right|^{\frac{2}{k+1}} - |S|^2)S^m y, y \right\rangle \geq 0$

$\Rightarrow (\alpha+\beta) \left\langle S^{*m} \left|S^{k+1}\right|^{\frac{2}{k+l}} S^m y, y \right\rangle \geq \beta \left\langle S^{*m}\, |S|^2\, S^m y, y \right\rangle.$

Since $\alpha+\beta<\beta$, this implies that $\left\langle S^{*m}(\left|S^{k+1}\right|^{\frac{2}{k+l}} - |S|^2)S^m y, y \right\rangle \geq 0$. Hence S is m-quasi class A_k operator.

Using Holder-McCarthy inequality,

$$\begin{aligned} \left\langle S^{*m} \left|S^{k+1}\right|^{\frac{2}{k+1}} S^m y, y \right\rangle &= \left\langle (S^{*(k+1)} S^{k+1})^{\frac{1}{k+1}} S^m y, S^m y \right\rangle \\ &\leq \left\langle S^{*(k+1)} S^{k+1} S^m y, S^m y \right\rangle^{\frac{1}{k+1}} \|S^m y\|^{\frac{2k}{k+1}} \\ &= \left\|S^{k+1+m} y\right\|^{\frac{2}{k+1}} \|S^m y\|^{\frac{2k}{k+1}} \end{aligned}$$

and

$$\left\langle S^{*m}\, |S|^2\, S^m y, y \right\rangle = \langle S^{1+m} y, S^{1+m} y \rangle = \left\|S^{m+1} y\right\|^2.$$

Therefore, $(\alpha+\beta) \left\|S^{k+1+m} y\right\|^{\frac{2}{k+1}} \|S^m y\|^{\frac{2k}{k+1}} \geq \beta \left\|S^{m+1} y\right\|^2$.

Since S is m-quasi class A_k operator, S has a decomposition of the form

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & 0 \end{pmatrix} \; on\, H = \overline{ran(S^m)} \oplus ker(S^{*m}),$$

where S_1 is class A_k operator on $\overline{ran(S^m)}$.

Hence $(\alpha+\beta) \left\|S_1^{k+1+m} \xi\right\|^{\frac{2}{k+1}} \|S_1^m \xi\|^{\frac{2k}{k+1}} \geq \beta \left\|S_1^{m+1} \xi\right\|^2$ for all $\xi \in \overline{ran(S^m)}$. Since S_1 is normaloid [24],

$$(\alpha+\beta) \|S_1\|^{\frac{2(k+1+m)}{(k+1)}} \|S_1\|^{\frac{2km}{(k+1)}} \geq \beta \|S_1\|^{2(m+1)} \;\; i.e \; (\alpha+\beta) \|S_1\|^{2(m+1)} \geq \beta \|S_1\|^{2(m+1)}.$$

Hence $S_1 = 0$. Hence $S^2 y = S_1(Sy) = 0$ for all $y \in K$. This is a contradiction to that S^2 is not a zero operator. Hence T must be m-quasi class A_k operator. In a similar manner, we can show that S is m-quasi class A_k operator. Hence the result.

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Scientia Magna
Vol. 7 (2011), No. 2, 11-16

A short interval result for the function $\log \rho(n)$[1]

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Abstract let $n > 1$ be an integer, $\log \rho(n)$ denote the number of regular integers $m(\text{mod } n)$ such that$1 \leq m \leq n$. In this paper we shall establish a short interval result for the function $\log \rho(n)$.

Keywords Regular integers (mod n), convolution method, short interval.

§1. Introduction

Let $n > 1$ be an integer. Consider the integers m for which there exists an integer x such that $m^2 x \equiv m \pmod n$. Let $\text{Reg}_n = \{m : 1 \leq m \leq n,\ m$ is regular $(\text{mod } n)\}$ and let $\rho(n) = \#\text{Reg}_n$ denote the number of regular integers $m(\text{mod } n)$ such that $1 \leq m \leq n$. This function is multiplicative and $\rho(p^v) = \phi(p^v)+1 = p^v - p^{v-1}+1$ for every prime power $p^v (v \geq 1)$, where ϕ is the Euler function.

The average order of the function $\rho(n)$ was consider in [4], [2]. One has

$$\lim_{n\to\infty} \frac{1}{x^2} \sum_{n\leq x} \rho(n) = \frac{1}{2} A \approx 0.4407, \tag{1}$$

where

$$A = \prod_p \left(1 - \frac{1}{p^2(p+1)}\right) = \zeta(2) \prod_p \left(1 - \frac{1}{p^2} - \frac{1}{p^3} + \frac{1}{p^4}\right) \approx 0.8815 \tag{2}$$

is the so called quadratic class -number constant. More exactly, V. S. Joshi [2] proved

$$\sum_{n\leq x} \rho(n) = \frac{1}{2} A x^2 + R(x), \tag{3}$$

where $R(x) = O(x \log^3 x)$, This was improved into $R(x) = O(x \log^2 x)$ in [3], and into $R(x) = O(x \log x)$ in [5], using analytic methods. Also, $R(x) = \Omega_{\pm}(x\sqrt{\log\log x})$ was proved in [5].

László Tóth [1] proved the following three results:

$$\sum_{n\leq x} \frac{\rho(n)}{\phi(n)} = \frac{3}{\pi^2} x + O(\log^2 x), \tag{4}$$

[1]This work is supported by N. N. S. F. of China (Grant Nos: 10771127, 11001154) and N. S. F. of Shandong Province (Nos: BS2009SF018, ZR2010AQ009).

$$\sum_{n\leq x}\frac{\phi(n)}{\rho(n)} = Bx + O\left((\log x)^{5/3}(\log\log x)^{4/3}\right), \tag{5}$$

$$\sum_{n\leq x}\frac{1}{\rho(n)} = C_1\log x + C_2 + O\left(\frac{\log^9 x}{x}\right), \tag{6}$$

where C_1 and C_2 are constants,

$$C_1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_p\left(1-\frac{p(p-1)}{p^2-p+1}\sum_{v=1}^{\infty}\frac{1}{p^v-p^{v-1}+1}\right).$$

Recently Lixia Li [6] proved a result about the mean value of $\log\rho(n)$:

$$\sum_{n\leq x}\log\rho(n) = x\log x + Cx + O(x^{1/2}\log^{3/2} x), \tag{7}$$

where

$$C = \sum_p(1-p^{-1})\sum_{\alpha=2}^{\infty}p^{-\alpha}\log(1-p^{-1}+p^{-\alpha}). \tag{8}$$

In this paper, we shall prove the following short interval result.

Theorem. If $x^{\frac{1}{5}+2\epsilon}\leq y\leq x$, then

$$\sum_{x<n\leq x+y}\log\rho(n) = Cy + \int_x^{x+y}\log t\,dt + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3\epsilon}{2}}), \tag{9}$$

where C is given by (8).

Notations. Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas and in the section we suppose that u is a complete number such that $\Re e u\leq\varepsilon^2$.

Lemma 1. Suppose s is a complex number ($\Re e s>1$), then

$$\sum_{n=1}^{\infty}\frac{(\rho(n))^u}{n^s} = \zeta(s-u)\zeta^{-u}(2s-2u+1)G(s,u), \tag{10}$$

where the Dirichlet series $G(s,u) := \sum_{n=1}^{\infty}\frac{g(n,u)}{n^s}$ is absolutely convergent for $\Re e s > \Re e u - \frac{1}{4}$.

Proof. Here $(\rho(n))^u$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(\rho(n))^u}{n^s} &= \prod_p \left(1 + \frac{(\rho(p))^u}{p^s} + \frac{\left(\rho(p^2)\right)^u}{p^{2s}} + \frac{\left(\rho(p^3)\right)^u}{p^{3s}} + \frac{\left(\rho(p^4)\right)^u}{p^{4s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{p^u}{p^s} + \frac{p^{2u}(1 - \frac{1}{p} + \frac{1}{p^2})^u}{p^{2s}} + \frac{p^{3u}(1 - \frac{1}{p} + \frac{1}{p^3})^u}{p^{3s}} + \cdots\right) \\
&= \prod_p \Bigg(1 + \frac{1}{p^{s-u}} + \frac{1 + u(-\frac{1}{p} + \frac{1}{p^2}) + \frac{u(u-1)}{2}(-\frac{1}{p} + \frac{1}{p^2})^2 + \cdots}{p^{2s-2u}} \\
&\quad + \frac{1 + u(-\frac{1}{p} + \frac{1}{p^3}) + \frac{u(u-1)}{2}(-\frac{1}{p} + \frac{1}{p^3})^2 + \cdots}{p^{3s-3u}} + \cdots\Bigg) \\
&= \zeta(s-u) \prod_p \Bigg(1 + \frac{u(-\frac{1}{p} + \frac{1}{p^2}) + \frac{u(u-1)}{2}(-\frac{1}{p} + \frac{1}{p^2})^2 + \cdots}{p^{2s-2u}} \\
&\quad + \frac{u(-\frac{1}{p^2} + \frac{1}{p^3}) + \frac{u(u-1)}{2}[(-\frac{1}{p} + \frac{1}{p^3})^2 - (-\frac{1}{p} + \frac{1}{p^2})^2] + \cdots}{p^{3s-3u}} + \cdots\Bigg) \\
&= \frac{\zeta(s-u)}{\zeta^u(2s-2u+1)} \prod_p \left(1 + \frac{u\frac{1}{p^2} + \frac{u(u-1)}{2}(-\frac{1}{p} + \frac{1}{p^2})^2 + \cdots}{p^{2s-2u}} + \cdots\right) \\
&= \frac{\zeta(s-u)}{\zeta^u(2s-2u+1)} G(s,u).
\end{aligned}$$

So we get $G(s,u) := \sum_{n=1}^{\infty} \frac{g(n,u)}{n^s}$ and by the properties of Dirichlet series, it is absolutely convergent for $\Re es > \Re eu - \frac{1}{4}$.

Lemma 2.

$$\sum_{n \le x} n^u = \frac{1}{u+1} x^{u+1} + O(1) + O(x^{\Re eu}). \tag{11}$$

Proof. This is easily from partial summation formula.

Let $f(n,u), h(n.u)$ be arithmetic functions defined by the following Dirichlet series (for $\Re es > 1$):

$$\sum_{n=1}^{\infty} \frac{f(n,u)}{n^s} = \zeta(s-u) G(s,u), \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{h(n,u)}{n^{2s}} = \zeta^{-u}(2s - 2u + 1). \tag{13}$$

Lemma 3. Let $f(n,u)$ be an arithmetic function defined by (12), then we have

$$\sum_{n \le x} f(n,u) = \frac{x^{u+1}}{u+1} G(u+1,u) + O(1) + O(x^{\Re eu}). \tag{14}$$

Proof. From Lemma 1 the infinite series $\sum_{n=1}^{\infty} \frac{g(n,u)}{n^s}$ converges absolutely for $\sigma > \Re eu - \frac{1}{4}$, it follows that

$$\sum_{n \le x} g(n,u) \ll 1.$$

Therefore from the definition of $f(n,u)$and Lemma 2, we obtain

$$\begin{aligned}\sum_{n\le x} f(n,u) &= \sum_{n\le x} m^u g(k,u)\\ &= \sum_{k\le x} g(k,u)\sum_{m\le \frac{x}{k}} m^u\\ &= \sum_{k\le x} g(k,u)\left[\frac{1}{u+1}(\frac{x}{k})^{u+1}+O(1)+O(x^{\Re e u})\right]\\ &= \frac{1}{u+1}x^{u+1}+O(1)+O(x^{\Re e u}).\end{aligned}$$

Lemma 4. Let $k\ge 2$ be a fixed integer, $1<y\le x$ be large real numbers and

$$B(x,y;k,\epsilon):=\sum_{\substack{x\,<\,nm^k\,\le\,x\,+\,y\\ m\,>\,x^{\epsilon}}} 1.$$

Then we have

$$B(x,y;k,\epsilon)\ll yx^{-\epsilon}+x^{\frac{1}{2k+1}}\log x. \tag{15}$$

Proof. This Lemma is very important when studying the short interval distribution of l-free numbers; Using Lemma 1, see for example [7].

Next we prove our Theorem. From Lemma 4 and the definition of $f(n,u),h(n,u)$, we get

$$h(n,u)=d_{-u}(n)n^{2u-1}\ll n^{2\Re e u+\varepsilon-1}$$

and

$$\rho^u(n)=\sum_{n=km^2} f(k,u)h(m,u).$$

So we have

$$\begin{aligned}Q(x+y)-Q(x) &= \sum_{x<km^2\le x+y} f(k,u)h(m,u)\\ &=\sum_1+O(\sum_2),\end{aligned} \tag{16}$$

where

$$\begin{aligned}\sum_1 &= \sum_{m\le x^{\varepsilon}} h(m,u)\sum_{\frac{x}{m^2}<k\le\frac{x+y}{m^2}} f(k,u),\\ \sum_2 &= \sum_{\substack{x\,<\,km^2\,\le\,x\,+\,y\\ m\,>\,x^{\epsilon}}} |f(k,u)h(m,u)|.\end{aligned}$$

In view of Lemma 3,

$$\begin{aligned}\sum_1 &= \sum_{m\le x^{\varepsilon}} h(m,u)\left(\frac{G(u+1,u)}{m^{2u+2}}\int_x^{x+y} t^u dt+O(1)+O(\frac{x^{\Re e u}}{m^{2\Re e u}})\right)\\ &= \frac{G(u+1,u)}{\zeta^u(3)}\int_x^{x+y} t^u dt+O(yx^{-\frac{\varepsilon}{2}})+O(x^{\varepsilon}).\end{aligned} \tag{17}$$

By Lemma 4, we have

$$\begin{aligned}\sum_{2} &\ll x^{\epsilon^2}\sum_{\substack{x<km^2\le x+y\\ m>x^{\epsilon}}} 1\\ &\ll x^{\epsilon^2}(yx^{-\epsilon}+x^{\frac{1}{5}+\epsilon})\\ &\ll yx^{-\frac{\epsilon}{2}}+x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \qquad (18)\end{aligned}$$

Now following, we obtain

$$\begin{aligned}\sum_{n<x\le x+y}\rho^u(n) &= \frac{G(u+1,u)}{\zeta^u(3)}\int_x^{x+y} t^u dt+O(yx^{-\frac{\epsilon}{2}})+O(x^{\frac{1}{5}+\frac{3}{2}\epsilon})\\ &= H(u)\int_x^{x+y} t^u dt+O(yx^{-\frac{\epsilon}{2}})+O(x^{\frac{1}{5}+\frac{3}{2}\epsilon}). \qquad (19)\end{aligned}$$

where $H(u):=\frac{G(u+1,u)}{\zeta^u(3)}$.
By differentiating (19) term by term, we derive

$$\begin{aligned}\sum_{n<x\le x+y}\rho^u(n)\log\rho(n) = H'(u)\int_x^{x+y} t^u dt+H(u)\int_x^{x+y} t^u\log t dt\\ +O(yx^{-\frac{\epsilon}{2}})+O(x^{\frac{1}{5}+\frac{3}{2}\epsilon}). \qquad (20)\end{aligned}$$

Letting $u=0$ in (20), we get

$$\sum_{n<x\le x+y}\log\rho(n) = H'(0)y+H(0)\int_x^{x+y}\log t dt+O(yx^{-\frac{\epsilon}{2}})+O(x^{\frac{1}{5}+\frac{3}{2}\epsilon}). \qquad (21)$$

Now we evaluate $H(0)$ and $H'(0)$. From the definition of $H(u)$, we obtain

$$H(u)=\prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}+\sum_{\alpha=2}^{\infty}\frac{(1-p^{-1}+p^{-\alpha})^u}{p^{\alpha}}\right), \qquad (22)$$

which immediately that $H(0)=1$.
Taking the logarithm derivative from both sides of (22) we get

$$\frac{H'(u)}{H(u)}=\sum_p\frac{\sum_{\alpha=2}^{\infty}p^{-\alpha}(1-p^{-1}+p^{-\alpha})^u\log(1-p^{-1}+p^{-\alpha})}{1+p^{-1}+\sum_{\alpha=2}^{\infty}\frac{(1-p^{-1}+p^{-\alpha})^u}{p^{\alpha}}},$$

which together with $H(0)=1$ gives

$$H'(0)=\sum_p(1-p^{-1})\sum_{\alpha=2}^{\infty}p^{-\alpha}\log(1-p^{-1}+p^{-\alpha}). \qquad (23)$$

Now our theorem follows from (21) and (23).

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Scientia Magna
Vol. 7 (2011), No. 2, 17-21

On the mean value of the function $(\frac{\phi(n)}{\rho(n)})^r$ [1]

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Abstract An integer a is called regular (mod n) if there is an integer x such that $a^2x \equiv a$ (mod n). Let $\rho(n)$ denote the number of regular integers $a(\bmod\ n)$ such that $1 \leq a \leq n$, $\phi(n)$ is the Euler function. In this paper we investigate the mean value of the function $(\frac{\phi(n)}{\rho(n)})^r$, where $r > 1$ is a fixed integer.

Keywords Regular integer(mod n), Euler's function, Euler product, convolution method.

§1. Introduction

Let $n > 1$ be an integer. Consider the integer a for which there exist an x such that $a^2x \equiv a$ (mod n). Properties of these integer were investigated by J. Morgado [1,2] who called them regular (mod n).

Let $\mathrm{Reg_n} = \{\mathrm{a} : 1 \leq \mathrm{a} \leq \mathrm{n}, \mathrm{a} \text{ is regular (mod n)}\}$ and let $\rho(n)$ denote the number of regular integers a(mod n) such that $1 \leq a \leq n$. This function is multiplicate and $\rho(p^v) = \phi(p^v) + 1 = p^v - p^{v-1} + 1$ for every prime power $p^v (v \geq 1)$, where ϕ is the Euler function.

László Tóth [3] proved that

$$\sum_{n\leq x} \frac{\phi(n)}{\rho(n)} = Cx + O((\log x)^{5/3}(\log\log x)^{4/3}), \tag{1}$$

where C is a constant.

Let $r > 1$ be a fixed integer. The aim of the short paper is to establish the following asymptotic formula for the mean value of the function $(\frac{\phi(n)}{\rho(n)})^r$, which generalizes (1).

Theorem. Suppose $r > 1$ be a fixed integer, then

$$\sum_{n\leq x} (\frac{\phi(n)}{\rho(n)})^r = A_r x + O(\log^{2r} x), \tag{2}$$

where A_r is a constant.

[1] This work is supported by N. N. S. F. of China (Grant Nos: 10771127, 11001154) and N. S. F. of Shandong Province (Nos: BS2009SF018, ZR2010AQ009).

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas, which can be found in Ivić [4]. Form now on, suppose $\zeta(s)$ denotes the Riemann-zeta function.

Lemma 1. Suppose $t \geq 2$, then uniformly for σ we have

$$\zeta(\sigma+it) \ll \begin{cases} 1, & \text{for } \sigma \geq 2; \\ \log t, & \text{for } 1 \leq \sigma \leq 2; \\ t^{(1-\sigma)/2}\log t, & \text{for } 0 \leq \sigma \leq 1. \end{cases}$$

$$\zeta(\sigma+it)^{-1} \ll \begin{cases} 1, & \text{for } \sigma \geq 2; \\ \log t, & \text{for } 1 \leq \sigma \leq 2. \end{cases}$$

Lemma 2. There exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for $\sigma > 1 - \frac{c}{\log(|t|+2)}$.

Proof of the theorem.

Let

$$f(s) := \sum_{n=1}^{\infty} \frac{(\frac{\phi(n)}{\rho(n)})^r}{n^s}, \text{ Re}s > 1.$$

It is easy to see that $(\frac{\phi(n)}{\rho(n)})^r$ is multiplicative, so by the Euler product formula, for Re$s > 1$ we have

$$\begin{aligned} f(s) :&= \sum_{n=1}^{\infty} \frac{(\frac{\phi(n)}{\rho(n)})^r}{n^s} \\ &= \prod_p (1 + \frac{(\frac{\phi(p)}{\rho(p)})^r}{p^s} + \frac{(\frac{\phi(p^2)}{\rho(p^2)})^r}{p^{2s}} + \cdots) \\ &= \prod_p (1 + \frac{(\frac{p-1}{p})^r}{p^s} + \frac{(\frac{p^2-p}{p^2-p+1})^r}{p^{2s}} + \cdots) \\ &= \prod_p (1 + \frac{(1+\frac{1}{p-1})^{-r}}{p^s} + \frac{(1+\frac{1}{p^2-p})^{-r}}{p^{2s}} + \cdots) \\ &= \prod_p (1 + \frac{(1+\frac{1}{p}(1+\frac{1}{p}+\cdots))^{-r}}{p^s} + \frac{(1+\frac{1}{p}(1+\frac{1}{p^2}+\cdots))^{-r}}{p^{2s}} + \cdots) \\ &= \prod_p (1 + \frac{1}{p^s} - \frac{\frac{r}{p}(1+\frac{1}{p}+\cdots)}{p^s} + \cdots + \frac{1}{p^{2s}} - \frac{\frac{r}{p^2}(1+\frac{1}{p}+\cdots)}{p^{2s}} + \cdots) \\ &= \zeta(s) \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} - \frac{\frac{r}{p}(1+\frac{1}{p}+\cdots)}{p^s} + \cdots \\ &\quad + \frac{1}{p^{2s}} - \frac{\frac{r}{p^2}(1+\frac{1}{p}+\cdots)}{p^{2s}} + \cdots) \end{aligned}$$

$$
\begin{aligned}
&= \zeta(s)\prod_p\Big(1-\frac{\frac{r}{p}(1+\frac{1}{p}+\cdots)}{p^s}+\frac{\frac{r(r+1)}{2p^2}(1+\frac{1}{p}+\cdots)^2}{p^s}+\cdots\\
&\quad+\frac{\frac{r}{p}(1+\frac{1}{p}+\cdots)}{p^{2s}}-\frac{\frac{r}{p^2}(1+\frac{1}{p}+\cdots)}{p^{2s}}+\cdots\Big)\\
&= \zeta(s)\prod_p\Big(1-\frac{r}{p^{s+1}}+\frac{\frac{r(r+1)}{2}-r}{p^{s+2}}+\cdots+\frac{r}{p^{2s+1}}-\frac{\frac{r(r+1)}{2}+r}{p^{2s+2}}+\cdots\Big)\\
&= \frac{\zeta(s)}{\zeta^r(s+1)}\prod_p\Big(1+\frac{r}{p^{2s+1}}+\frac{\frac{r(r+1)}{2}-r}{p^{s+2}}-\frac{\frac{r(r+1)}{2}+r}{p^{2s+2}}+\cdots\Big)\\
&= \frac{\zeta(s)\zeta^r(2s+1)}{\zeta^r(s+1)}\prod_p\Big(1+\frac{\frac{r(r+1)}{2}-r}{p^{s+2}}-\frac{\frac{r(r+1)}{2}+r^2}{p^{2s+2}}+\cdots\Big)\\
&= \frac{\zeta(s)\zeta^r(2s+1)}{\zeta^r(s+1)}G(s,r).
\end{aligned}
$$

Write

$$G(s,r) := \prod_p\Big(1+\frac{\frac{r(r+1)}{2}-r}{p^{s+2}}-\frac{\frac{r(r+1)}{2}+r^2}{p^{2s+2}}+\cdots\Big)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}.$$

It is easy to check that Dirichlet series $\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ is absolutely convergent for Res $\geq -\frac{2}{5}$, so we have

$$\sum_{n\leq x}|g(n)| \ll 1. \tag{3}$$

Let

$$\frac{\zeta^r(2s+1)}{\zeta^r(s+1)}=\sum_{n=1}^{\infty}\frac{v_r(n)}{n^s},$$

then according to the Dirichlet convolution, we obtain

$$
\begin{aligned}
\sum_{n\leq x}(\frac{\phi(n)}{\rho(n)})^r &= \sum_{mkl\leq x} g(k)v_r(l)\\
&= \sum_{k\leq x} g(k)\sum_{l\leq \frac{x}{k}} v_r(l)\sum_{m\leq\frac{x}{kl}}1\\
&= \sum_{k\leq x} g(k)\sum_{l\leq \frac{x}{k}} v_r(l)(\frac{x}{kl}+O(1))\\
&= x\sum_{k\leq x}\frac{g(k)}{k}\sum_{l\leq\frac{x}{k}}\frac{v_r(l)}{l}+O(\sum_{k\leq x}g(k)\sum_{l\leq\frac{x}{k}}|v_r(n)|).
\end{aligned}\tag{4}
$$

So it is reduced to compute $\sum_{l\leq x} v_r(l)$ and $\sum_{l\leq x}|v_r(l)|$. Similar to the proof of the prime number theorem, with the help of lemma 1, lemma 2 and perron's formula we get

$$\sum_{l\leq x} v_r(l) = C + O(x^{-\epsilon}), \tag{5}$$

where

$$C = \mathrm{Res}_{s=0}\frac{\zeta^{\mathrm{r}}(2\mathrm{s}+1)}{\zeta^{\mathrm{r}}(\mathrm{s}+1)}\frac{\mathrm{x^s}}{\mathrm{s}}$$

is a constant, ϵ is a small positive real number. By the partial summation, we get form (4) that

$$\sum_{l>x}\frac{v_r(l)}{l}\ll x^{-1}, \tag{6}$$

$$\sum_{l\le x}\frac{v_r(l)}{l}=\sum_{l=1}^{\infty}\frac{v_r(l)}{l}-\sum_{l>x}\frac{v_r(l)}{l}=C_1+O(x^{-1}). \tag{7}$$

Now we go on to bound the sum $\sum_{l\le x}(|v_r(l)|)$. Since for Res >1,

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{v_r(l)}{l^s}&=\frac{\zeta^r(2s+1)}{\zeta^r(s+1)}\\&=\sum_{m=1}^{\infty}\frac{\mu_r(m)}{m^{s+1}}\sum_{n=1}^{\infty}\frac{d_r(n)}{n^{2s+1}}\\&=\sum_{m,n}\frac{\mu_r(m)d_r(n)}{(mn^2)^s(mn)},\end{aligned}$$

where $d_r(n)=\sum_{n=n_1\cdots n_r}1,\ \mu_r(m)=\sum_{m=m_1\cdots m_r}\mu(m_1)\cdots\mu(m_r)$, we obtain

$$v_r(l)=\sum_{l=mn^2}\frac{\mu_r(m)d_r(n)}{mn}.$$

So

$$|v_r(l)|\le\sum_{l=mn^2}\frac{d_r(m)d_r(n)}{mn},$$

which combining the well-known estimate

$$\sum_{n\le x}d_r(n)\ll x\log^{r-1}x$$

gives

$$\sum_{l\le x}|v_r(l)|\ll\log^{2r}x. \tag{8}$$

Form (4)-(8), we obtain

$$\begin{aligned}\sum_{n\le x}(\frac{\phi(n)}{\rho(n)})^r&=x\sum_{k\le x}\frac{g(k)}{k}\sum_{l\le\frac{x}{k}}\frac{v_r(l)}{l}+O(\sum_{k\le x}|g(k)|\sum_{l\le\frac{x}{k}}|v_r(l)|)\\&=x\sum_{k\le x}\frac{g(k)}{k}(C_1+O((\frac{x}{k})^{-1}))+O(\sum_{k\le x}|g(k)|\sum_{l\le\frac{x}{k}}|v_r(l)|)\\&=C_1x\sum_{k\le x}\frac{g(k)}{k}+O(\sum_{k\le x}g(k))+O(\sum_{k\le x}|g(k)|\log^{2r}\frac{x}{k})\\&=C_1x\sum_{k=1}^{\infty}\frac{g(k)}{k}+O(x\sum_{k>x}\frac{g(k)}{k}+\sum_{k\le x}g(k)+\sum_{k\le x}|g(k)|\log^{2r}\frac{x}{k})\\&=A_rx+O(\log^{2r}x),\end{aligned}$$

where

$$A_r = C_1 x \sum_{k=1}^{\infty} \frac{g(k)}{k} = \sum_{l=1}^{\infty} \frac{v_r(l)}{l} \sum_{k=1}^{\infty} \frac{g(k)}{k}$$

is a constant.

This completes the proof of the theorem.

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Scientia Magna
Vol. 7 (2011), No. 2, 22-31

Fixed points of occasionally weakly compatible mappings in fuzzy metric spaces

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Abstract In 2008, Al-Thagafi and Shahzad [7] introduced the notion of occasionally weakly compatible mappings (shortly, owc maps) which is more general than the concept of weakly compatible maps. In the present paper, we prove some common fixed point theorems for owc maps in fuzzy metric spaces without considering the completeness of the whole space or any subspace, continuity of the involved maps and containment of ranges amongst involved maps.

Keywords Triangle norm (t-norm), fuzzy metric space, weakly compatible maps, occasional -ly weakly compatible maps.

§1. Introduction

In 1965, Zadeh [40] introduced the concept of fuzzy sets. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. For example, Kramosil and Michálek [32], Erceg [21], Deng [20], Kaleva and Seikkala [30], Grabiec [24], Fang [22], George and Veeramani [23], Mishra et al.[33], Subrahmanyam [38], Gregori and Sapena [25] and Singh and Jain [37] have introduced the concept of fuzzy metric spaces in different ways. In applications of fuzzy set theory, the field of engineering has undoubtedly been a leader. All engineering disciplines such as civil engineering, electrical engineering, mechanical engineering, robotics, industrial engineering, computer engineering, nuclear engineering etc. have already been affected to various degrees by the new methodological possibilities opened by fuzzy sets.

In 1998, Jungck and Rhoades [27] introduced the notion of weakly compatible mappings in metric spaces. Singh and Jain [37] formulated the notion of weakly compatible maps in fuzzy metric spaces. This condition has further been weakened by introducing the notion of owc maps by Al-Thagafi and Shahzad [7]. While Khan and Sumitra [31] extended the notion of owc maps in fuzzy metric spaces and proved some common fixed point theorems. It is worth to mention that every pair of weak compatible self-maps is owc but the reverse is not always true. Many authors proved common fixed point theorems for owc maps on various spaces (see [1-15, 17-19, 28, 29, 31, 34, 35, 39]).

In this paper, we prove some common fixed point theorems for owc maps in fuzzy metric spaces. Our results do not require the completeness of the whole space or any subspace, continuity of the involved maps and containment of ranges amongst involved maps.

§2. Preliminaries

Definition 2.1.[36] A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ and the following conditions are satisfied:

1. $a * 1 = a$,
2. $a * b = b * a$,
3. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,
4. $(a * b) * c = a * (b * c)$.

Two typical examples of continuous t-norms are $a * b = \min\{a, b\}$ and $a * b = ab$.

Definition 2.2.[32] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X, t, s > 0$,

1. $M(x, y, t) = 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
5. $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.3.[23] Let (X, d) be a metric space. Denote $a * b = a.b$ (or $a * b = \min\{a, b\}$) for all $a, b \in [0, 1]$ and let M_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \tfrac{t}{(t+d(x,y))}.$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d.

Lemma 2.4.[16,24] For all $x, y \in X, (X, M, \cdot)$ is non-decreasing function.

Definition 2.5.[37] Let $(X, M, *)$ be a fuzzy metric space, A and B be self maps of non-empty X. A point $x \in X$ is called a coincidence point of A and B if and only if $Ax = Bx$. In this case $w = Ax = Bx$ is called a point of coincidence of A and B.

Definition 2.6.[37] Two self mappings A and B of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if they commute at their coincidence points, that is, if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Lemma 2.7. If a fuzzy metric space $(X, M, *)$ satisfies $M(x, y, t) = C$, for all $t > 0$ with fixed $x, y \in X$. Then we have $C = 1$ and $x = y$.

Lemma 2.8.[26] Let the function $\phi(t)$ satisfy the following condition $(\Phi) : \phi(t) : [0, \infty) \to [0, \infty)$ is non-decreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, when $\phi^n(t)$ denotes the n^{th} iterative function of $\phi(t)$. Then $\phi(t) < t$ for all $t > 0$.

The following concept due to Al-Thagafi and Shahzad [7−8] is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point. The counterpart of the concept of owc maps in fuzzy metric spaces is as follows:

Definition 2.9. Two self maps A and B of a fuzzy metric space $(X, M, *)$ are owc if and only if there is a point $x \in X$ which is a coincidence point of A and B at which A and B commute.

From the following example it is clear that the notion of owc maps is more general than the concept of weakly compatible maps.

Example 2.10. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, \infty)$ and

$$M(x, y, t) = \frac{t}{(t+|x-y|)}$$

for all $t > 0$ and $x, y \in X$. Define $A, B : X \to X$ by $A(x) = 3x$ and $B(x) = x^2$ for all $x \in X$. Then $A(x) = B(x)$ for $x = 0, 3$ but $AB(0) = BA(0)$ and $AB(3) \neq BA(3)$. Thus A and B are owc maps but not weakly compatible.

The following lemma is on the lines of Jungck and Rhoades [28].

Lemma 2.11. Let $(X, M, *)$ be a fuzzy metric space, A and B are owc self maps of X. If A and B have a unique point of coincidence, $w = Ax = Bx$, then w is the unique common fixed point of A and B.

Proof. Since A and B are owc, there exists a point x in X such that $Ax = Bx = w$ and $ABx = BAx$. Thus, $AAx = ABx = BAx$, which says that Ax is also a point of coincidence of A and B. Since the point of coincidence $w = Ax$ is unique by hypothesis, $BAx = AAx = Ax$, and $w = Ax$ is a common fixed point of A and B.

Moreover, if z is any common fixed point of A and B, then $z = Az = Bz = w$ by the uniqueness of the point of coincidence.

§3. Results

First, we prove a common fixed point theorem for four single-valued self maps in fuzzy metric space.

Theorem 3.1. Let A, B, S and T be self maps on fuzzy metric space $(X, M, *)$, where $*$ is a continuous t-norm with $a * a \geq a$ for all $a \in [0, 1]$. Further, let the pairs (A, S) and (B, T) are each owc satisfying:

$$M(Ax, By, \phi(t)) \geq \left\{ \begin{array}{c} M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ *M(Ax, Ty, t) * M(By, Sx, 2t) \end{array} \right\} \tag{1}$$

for all $x, y \in X$ and $t > 0$. Here, the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ). Then there exists a unique point $w \in X$ such that $Aw = Sw = w$

and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point A, B, S and T.

Proof. Since the pairs (A, S) and (B, T) are each owc, there exist points $u, v \in X$ such that $Au = Su, ASu = SAu$ and $Bv = Tv, BTv = TBv$. Now we show that $Au = Bv$. Putting $x = u$ and $y = v$ in inequality (1), then we get

$$\begin{aligned} M(Au, Bv, \phi(t)) &\geq \left\{ \begin{array}{c} M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) \\ *M(Au, Tv, \alpha t) * M(By, Su, 2t) \end{array} \right\} \\ &= \left\{ \begin{array}{c} M(Au, Bv, t) * M(Au, Au, t) * M(Bv, Bv, t) \\ *M(Au, Bv, t) * M(Bv, Au, 2t) \end{array} \right\} \\ &= \{M(Au, Bv, t) * 1 * 1 * M(Au, Bv, t) * M(Bv, Au, 2t)\}, \end{aligned}$$

then we have

$$M(Au, Bv, \phi(t)) \geq M(Au, Bv, t).$$

On the other hand, since M is non-decreasing, we get $M(Au, Bv, \phi(t)) \leq M(Au, Bv, t)$. Hence, $M(Au, Bv, t) = C$ for all $t > 0$. From Lemma 2.7, we conclude that $C = 1$, that is $Au = Bv$. Therefore, $Au = Su = Bv = Tv$. Moreover, if there is another point z such that $Az = Sz$. Then using inequality (1) it follows that $Az = Sz = Bv = Tv$, or $Au = Az$. Hence $w = Au = Su$ is the unique point of coincidence of A and S. By Lemma 2.11, w is the unique common fixed point of A and S. Similarly, there is a unique point $z \in X$ such that $z = Bz = Tz$. Suppose that $w \neq z$ and taking $x = w, y = z$ in inequality (1), then we get

$$\begin{aligned} M(Aw, Bz, \phi(t)) &\geq \left\{ \begin{array}{c} M(Sw, Tz, t) * M(Aw, Sw, t) * M(Bz, Tz, t) \\ *M(Aw, Tz, t) * M(Bz, Sw, 2t) \end{array} \right\} \\ M(w, z, \phi(t)) &\geq \left\{ \begin{array}{c} M(w, z, t) * M(w, w, t) * M(z, z, t) \\ *M(w, z, t) * M(z, w, 2t) \end{array} \right\} \\ &= \{M(w, z, t) * 1 * 1 * M(w, z, t) * M(z, w, 2t)\}, \end{aligned}$$

thus it follows that

$$M(w, z, \phi(t)) \geq M(w, z, t).$$

Since M is non-decreasing, we get $M(w, z, \phi(t)) \leq M(w, z, t)$. Hence, $M(w, z, t) = C$ for all $t > 0$. From Lemma 2.7, we conclude that $C = 1$, that is $w = z$. Hence w is the unique common fixed point of the self maps A, B, S and T in X.

Now, we give an example which illustrates Theorem 3.1.

Example 3.2. Let $X = [0, 4]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Clearly $(X, M, *)$ be a fuzzy metric space, where $*$ is a continuous t-norm with $* = \min$. Define $\phi(t) = kt$, where $k \in (0,1)$ and the self maps A, B, S and T by

$$A(x) = \begin{cases} x, & \text{if } 0 \le x \le 2; \\ 3, & \text{if } 2 < x \le 4. \end{cases} \quad S(x) = \begin{cases} 2, & \text{if } 0 \le x \le 2; \\ 0, & \text{if } 2 < x \le 4. \end{cases}$$

$$B(x) = \begin{cases} 2, & \text{if } 0 \le x \le 2; \\ 4, & \text{if } 2 < x \le 4. \end{cases} \quad T(x) = \begin{cases} 2, & \text{if } 0 \le x \le 2; \\ \frac{x}{4}, & \text{if } 2 < x \le 4. \end{cases}$$

Then A, B, S and T satisfy all the conditions of Theorem 3.1. Notice that $AS(2) = A(2) = 2 = S(2) = SA(2)$ and $BT(2) = B(2) = 2 = T(2) = TB(2)$, that is A and S as well as B and T are owc. Hence, 2 is the unique common fixed point of A, B, S and T. This example never requires any condition on containment of ranges amongst involved maps. On the other hand, it is clear to see that the self maps A, B, S and T are discontinuous at 2.

On taking $A = B$ and $S = T$ in Theorem 3.1, then we get the following result:

Corollary 3.3. Let A and S be self maps on fuzzy metric space $(X, M, *)$ where $*$ is a continuous t-norm and $a * a \ge a$ for all $a \in [0,1]$. Further, let the pair (A, S) is owc satisfying:

$$M(Ax, Ay, \phi(t)) \ge \left\{ \begin{array}{c} M(Sx, Sy, t) * M(Ax, Sx, t) * M(Ay, Sy, t) \\ *M(Ax, Sy, t) * M(Ay, Sx, 2t) \end{array} \right\} \tag{2}$$

for all $x, y \in X$ and $t > 0$. Here, the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ). Then A and S have a unique common fixed point in X.

Now, we extend Theorem 3.1 and Corollary 3.3 to any even number of self-maps in fuzzy metric space.

Theorem 3.4. Let $P_1, P_2, \ldots, P_{2n}, A$ and B be self maps on fuzzy metric space $(X, M, *)$, where $*$ is a continuous t-norm with $a * a \ge a$ for all $a \in [0,1]$. Further, let the pairs $(A, P_1P_3 \ldots P_{2n-1})$ and $(B, P_2P_4 \ldots P_{2n})$ are each owc satisfying:

$$M(Ax, By, \phi(t)) \ge \left\{ \begin{array}{c} M(P_1P_3 \ldots P_{2n-1}x, P_2P_4 \ldots P_{2n}y, t) \\ *M(Ax, P_1P_3 \ldots P_{2n-1}x, t) * M(By, P_2P_4 \ldots P_{2n}y, t) \\ *M(Ax, P_2P_4 \ldots P_{2n}y, t) * M(By, P_1P_3 \ldots P_{2n-1}x, 2t) \end{array} \right\} \tag{3}$$

for all $x, y \in X$ and $t > 0$. Here, the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ). Suppose that

$P_1(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})P_1$,

$P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3$,

$\vdots$

$P_1 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \ldots P_{2n-3}$,

$A(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})A$,

$A(P_5 \dots P_{2n-1}) = (P_5 \dots P_{2n-1})A,$

$\vdots$

$AP_{2n-1} = P_{2n-1}A,$

similarly,

$P_2(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})P_2,$

$P_2P_4(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})P_2P_4,$

$\vdots$

$P_2 \dots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \dots P_{2n-2},$

$B(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})B,$

$B(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})B,$

$\vdots$

$BP_{2n} = P_{2n}B.$

Then $P_1, P_2, \dots, P_{2n}, A$ and B have a unique common fixed point in X.

Proof. Since the pairs $(A, P_1P_3 \dots P_{2n-1})$ and $(B, P_2P_4 \dots P_{2n})$ are each owc then there exist points $u, v \in X$ such that $Au = P_1P_3 \dots P_{2n-1}u$, $A(P_1P_3 \dots P_{2n-1})u = (P_1P_3 \dots P_{2n-1})Au$ and $Bv = P_2P_4 \dots P_{2n}v$, $B(P_2P_4 \dots P_{2n})v = (P_2P_4 \dots P_{2n})Bv$. Now we show that $Au = Bv$. Taking $x = u$ and $y = v$ in inequality (3), then we get

$$\begin{aligned} M(Au, Bv, \phi(t)) &\geq \left\{ \begin{array}{c} M(P_1P_3 \dots P_{2n-1}u, P_2P_4 \dots P_{2n}v, t) \\ *M(Au, P_1P_3 \dots P_{2n-1}u, t) * M(Bv, P_2P_4 \dots P_{2n}v, t) \\ *M(Au, P_2P_4 \dots P_{2n}v, t) * M(Bv, P_1P_3 \dots P_{2n-1}u, 2t) \end{array} \right\} \\ &= \left\{ \begin{array}{c} M(Au, Bv, t) * M(Au, Au, t) * M(Bv, Bv, t) \\ *M(Au, Bv, t) * M(Bv, Au, 2t) \end{array} \right\} \\ &= \{M(Au, Bv, t) * 1 * 1 * M(Au, Bv, t) * M(Bv, Au, 2t)\}, \end{aligned}$$

then we have

$$M(Au, Bv, \phi(t)) \geq M(Au, Bv, t).$$

On the other hand, since M is non-decreasing, we get $M(Au, Bv, \phi(t)) \leq M(Au, Bv, t)$. Hence, $M(Au, Bv, t) = C$ for all $t > 0$. From Lemma 2.7, we conclude that $C = 1$, that is $Au = Bv$. Moreover, if there is another point z such that $Az = P_1P_3 \dots P_{2n-1}z$. Then using inequality (3) it follows that $Az = P_1P_3 \dots P_{2n-1}z = Bv = P_2P_4 \dots P_{2n}v$, or $Au = Az$. Hence, $w = Au = P_1P_3 \dots P_{2n-1}u$ is the unique point of coincidence of A and $P_1P_3 \dots P_{2n-1}$. From Lemma 2.11, it follows that w is the unique common fixed point of A and $P_1P_3 \dots P_{2n-1}$. By symmetry, $q = Bv = P_2P_4 \dots P_{2n}v$ is the unique common fixed point of B and $P_2P_4 \dots P_{2n}$. Since $w = q$, we obtain that w is the unique common fixed point of B and $P_2P_4 \dots P_{2n}$. Now, we show that w is the fixed point of all the component mappings. Putting $x = P_3 \dots P_{2n-1}w$, $y = w, P_1^{'} = P_1P_3 \dots P_{2n-1}$ and $P_2^{'} = P_2P_4 \dots P_{2n}$ in inequality (3), we have

$$M(AP_3 \ldots P_{2n-1}w, Bw, \phi(t)) \geq \left\{ \begin{array}{c} M(P_1^{'}P_3 \ldots P_{2n-1}w, P_2^{'}w, t) \\ *M(AP_3 \ldots P_{2n-1}w, P_1^{'}P_3 \ldots P_{2n-1}w, t) \\ *M(Bw, P_2^{'}w, t) * M(AP_3 \ldots P_{2n-1}w, P_2^{'}w, t) \\ *M(Bw, P_1^{'}P_3 \ldots P_{2n-1}w, 2t) \end{array} \right\}$$

$$M(P_3 \ldots P_{2n-1}w, w, \phi(t)) \geq \left\{ \begin{array}{c} M(P_3 \ldots P_{2n-1}w, w, t) \\ *M(P_3 \ldots P_{2n-1}w, P_3 \ldots P_{2n-1}w, t) * M(w, w, t) \\ *M(P_3 \ldots P_{2n-1}w, w, t) * M(w, P_3 \ldots P_{2n-1}w, 2t) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} M(P_3 \ldots P_{2n-1}w, w, t) * 1 * 1 \\ *M(P_3 \ldots P_{2n-1}w, w, t) * M(w, P_3 \ldots P_{2n-1}w, 2t) \end{array} \right\},$$

thus, it follows that

$$M(P_3 \ldots P_{2n-1}w, w, \phi(t)) \geq M(P_3 \ldots P_{2n-1}w, w, t).$$

Since M is non-decreasing, we get $M(P_3 \ldots P_{2n-1}w, w, \phi(t)) \leq M(P_3 \ldots P_{2n-1}w, w, t)$. Hence, $M(P_3 \ldots P_{2n-1}w, w, t) = C$ for all $t > 0$. From Lemma 2.7 we conclude that $C = 1$, that is $P_3 \ldots P_{2n-1}w = w$. Hence, $P_1 w = w$. Continuing this procedure, we have

$$Aw = P_1 w = P_3 w = \ldots = P_{2n-1}w = w.$$

So, $Bw = P_2 w = P_4 w = \ldots = P_{2n}w = w$. So, w is the unique common fixed point of $P_1, P_2, \ldots, P_{2n}, A$ and B.

The following result is a slight generalization of Theorem 3.4.

Corollary 3.5. Let $\{T_\zeta\}_{\zeta \in J}$ and $\{P_i\}_{i=1}^{2n}$ be two families of self maps on fuzzy metric space $(X, M, *)$ where $*$ is a continuous t-norm with $a * a \geq a$ for all $a \in [0, 1]$. Further, let the pairs $(T_\zeta, P_1P_3 \ldots P_{2n-1})$ and $(T_\xi, P_2P_4 \ldots P_{2n})$ are each owc satisfying: for a fixed $\xi \in J$,

$$M(T_\zeta x, T_\xi y, \phi(t)) \geq \left\{ \begin{array}{c} M(P_1P_3 \ldots P_{2n-1}x, P_2P_4 \ldots P_{2n}y, t) \\ *M(T_\zeta x, P_1P_3 \ldots P_{2n-1}x, t) * M(T_\xi y, P_2P_4 \ldots P_{2n}y, t) \\ *M(T_\zeta x, P_2P_4 \ldots P_{2n}y, t) * M(T_\xi y, P_1P_3 \ldots P_{2n-1}x, 2t) \end{array} \right\} \quad (4)$$

for all $x, y \in X$ and $t > 0$. Here, the function $\phi(t) : [0, \infty) \to [0, \infty)$ is onto, strictly increasing and satisfies condition (Φ). Suppose that

$P_1(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})P_1$,
$P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3$,
$\vdots$
$P_1 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \ldots P_{2n-3}$,
$T_\zeta(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})T_\zeta$,
$T_\zeta(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})T_\zeta$,

$\vdots$

$T_\zeta P_{2n-1} = P_{2n-1}T_\zeta$,

similarly,

$P_2(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})P_2$,

$P_2P_4(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})P_2P_4$,

$\vdots$

$P_2 \dots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \dots P_{2n-2}$,

$T_\xi(P_4 \dots P_{2n}) = (P_4 \dots P_{2n})T_\xi$,

$T_\xi(P_6 \dots P_{2n}) = (P_6 \dots P_{2n})T_\xi$,

$\vdots$

$T_\xi P_{2n} = P_{2n}T_\xi$.

Then all $\{P_i\}$ and $\{T_\zeta\}$ have a unique common fixed point in X.

Remark 3.6. The conclusions of our results remain true if we take $\phi(t) = kt$, where $k \in (0, 1)$.

Acknowledgment

The authors would like to express their sincere thanks to Professor B. E. Rhoades [28,29] for his papers.

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Scientia Magna
Vol. 7 (2011), No. 2, 32-38

Operators satisfying the condition $\left\|T^{2+k}x\right\|^{\frac{1}{1+k}} \|Tx\|^{\frac{k}{1+k}} \geq \left\|T^2x\right\|$ [1]

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Abstract In this paper, a super class of k-paranormal operators properly containing it is studied. Composition operators and weighted composition operators of this class are also characterized.
Keywords Paranormal, k-paranormal, ek-paranormal, weighted composition operators and Aluthge transformation.

§1. Introduction

Let $B(H)$ be the Banach Algebra of all bounded linear operators on a non-zero complex Hilbert space H. By an operator, we mean an element from $B(H)$. If T lies in $B(H)$, then T^* denotes the adjoint of T in $B(H)$. For $0 < p \leq 1$, an operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. If $p = 1$, T is called hyponormal. If $p = \frac{1}{2}$, T is called semi-hyponormal. An operator T is called paranormal, if $\|Tx\|^2 \leq \left\|T^2x\right\| \|x\|$, for every $x \in H$. An operator T is normaloid if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T or $\|T\|^n = \|T^n\|$ for all positive integers n.

In general, hyponormal $\Rightarrow$ p-hyponormal $\Rightarrow$paranormal $\Rightarrow$ k-paranormal.

Ando [4] has characterized paranormal operators as follows:

Theorem 1.1. An operator $T \in B(H)$ is paranormal if and only if ${T^*}^2T^2 - 2kT^*T + k^2 \geq 0$, for every $k \in R$.

Generalising this, Yuan and Gao [13] has characterised k-paranormal operators as follows:

Theorem 1.2. For each positive integer k, an operator $T \in B(H)$ is k-paranormal if and only if ${T^*}^{1+k}T^{1+k} - (1+k)\mu^kT^*T + k\mu^{1+k}I \geq 0$, for every $\mu > 0$.

In [10], Uchiyama gives a matrix representation for a paranormal operator with respect to the direct sum of an eigenspace and its orthogonal complement.

In this paper we characterize the new class of operators which properly contains k-paranorm -al operators, discuss its matrix representation and prove some more properties. We also characterize the composition operators and weighted composition operators of this class.

[1]This work is supported by the UGC, New Delhi (Grant F. No: 34-148/2008(SR)).

§2. Preliminaries

Let (X, Σ, λ) be a sigma-finite measure space. The relation of being almost everywhere, denoted by a.e, is an equivalence relation in $L^2(X, \Sigma, \lambda)$ and this equivalence relation splits $L^2(X, \Sigma, \lambda)$ into equivalence classes. Let T be a measurable transformation from X into itself. $L^2(X, \Sigma, \lambda)$ is denoted as $L^2(\lambda)$. The equation $C_T f = f \circ T$, $f \in L^2(\lambda)$ defines a composition transformation on $L^2(\lambda)$. T induces a composition operator C_T on $L^2(\lambda)$ if (i) the measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and (ii) the Radon-Nikodym derivative $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$ is essentially bounded (Nordgren). Harrington and Whitley have shown that if $C_T \in B(L^2(\lambda))$, then $C_T^* C_T f = f_0 f$ and $C_T C_T^* f = (f_0 \circ T)Pf$ for all $f \in L^2(\lambda)$ where P denotes the projection of $L^2(\lambda)$ onto $\overline{ran(C_T)}$. Thus it follows that C_T has dense range if and only if $C_T C_T^*$ is the operator of multiplication by $f_0 \circ T$, where f_0 denotes $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$. Every essentially bounded complex valued measurable function f_0 induces a bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0} f = f_0 f$, for every $f \in L^2(\lambda)$. Further $C_T^* C_T = M_{f_0}$ and ${C_T^*}^2 {C_T}^2 = M_{h_0}$. Let us denote $\frac{\mathrm{d}(\lambda T^{-1})}{\mathrm{d}\lambda}$ by h i.e f_0 by h and $\frac{\mathrm{d}(\lambda T^{-k})}{\mathrm{d}\lambda}$ by h_k, where k is a positive integer greater than or equal to one. Then $C_T^* C_T = M_h$ and ${C_T^*}^2 {C_T}^2 = M_{h_2}$. In general, ${C_T^*}^k {C_T}^k = M_{h_k}$, where M_{h_k} is the multiplication operator on $L^2(\lambda)$ induced by the complex valued measurable function h_k. Hyponormal composition operators are studied by Alan Lambert [1]. Paranormal composition operators are studied by T. Veluchamy and S. Panayappan [11].

§3. Definition and properties

Defnition 3.1. An operator T satisfying the condition $\left\|T^{2+k}x\right\|^{\frac{1}{1+k}} \|Tx\|^{\frac{k}{1+k}} \geq \left\|T^2 x\right\|$, for some integer $k \geq 1$ and for every $x \in H$ is called extended k-paranormal operator or, in short ek-paranormal operators.

If we replace x by Tx in the definition of k-paranormal operators, we get ek-paranormal operators. But the converse is not true. This is clear from the following example.

Example 3.2. Let $H = C^2$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then T is not k-paranormal for any positive integer k. But T is ek-paranormal.

We characterize ek-paranormal operators as below.

Theorem 3.3. For each positive integer k, an operator T is ek-paranormal if and only if

$$T^{*2+k} T^{2+k} - (1+k)\mu^k T^{*2} T^2 + k\mu^{1+k} T^* T \geq 0 \text{ for every } \mu > 0.$$

Example 3.4. Let H be the direct sum of a denumerable number of copies of two dimensional Hilbert space $R \times R$. Let A and B be two positive operators on $R \times R$. For any fixed positive integer n, define an operator $T = T_{A,B,n}$ on as follows:

$$T((x_1, x_2, \ldots) = (0, A(x_1), A(x_2), \ldots, A(x_n), B(x_{n+1}), \ldots).$$

Its adjoint is $T^*((x_1, x_2, \ldots)) = (A(x_2), A(x_3), \ldots, A(x_{n+1}), B(x_{n+2}), \ldots)$.

Let $n \geq k$. Then by Theorem 3.3, T is ek-paranormal if the following conditions are satisfied by A and B.

$$\begin{aligned}
A^{4+2k} - (1+k)\mu^k A^4 + k\mu^{1+k} A^2 &\geq 0 \\
B^{4+2k} - (1+k)\mu^k B^4 + k\mu^{1+k} B^2 &\geq 0 \\
A^{2+k-m} B^{2m} A^{2+k-m} - (1+k)\mu^k A^4 + k\mu^{1+k} A^2 &\geq 0 \quad \text{for } m = 1, 2, \ldots, k, \\
\text{and} \quad AB^{2+2k} A - (1+k)\mu^k AB^2 A + k\mu^{1+k} A^2 &\geq 0.
\end{aligned}$$

For $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $T_{A,B,n}$ satisfies the above conditions for every integer $k \geq 1$. Hence T is ek-paranormal, for every $k \geq 1$.

Theorem 3.5. If T is ek-paranormal for $k = 1$, then T is ek-paranormal for every positive integer k.

Proof. Let T be ek-paranormal for $k = 1$. Then

$$\left\|T^4 x\right\| \left\|Tx\right\|^2 \geq \frac{\left\|T^3 x\right\|^2}{\left\|T^2 x\right\|} \left\|Tx\right\|^2 \geq \left\|T^2 x\right\|^3.$$

Hence T is ek-paranormal for $k = 2$. Similarly we can show that if T is ek-paranormal for both $k = 1$ and $k = 2$, then T is ek-paranormal for $k = 3$, and so on for every positive integer k.

Theorem 3.6. If T is ek-paranormal and if α is a scalar , then αT is also ek-paranormal.

Proof. If $\alpha = 0$, the result is trivial. So assume that $\alpha \neq 0$.. Then for any $\mu > 0$,

$$\begin{aligned}
&(\alpha T)^{*2+k} (\alpha T)^{2+k} - (1+k)\mu^k (\alpha T)^{*2} (\alpha T)^2 + k\mu^{1+k} (\alpha T)^* (\alpha T) \\
= \; & \left(|\alpha|^2\right)^{2+k} \left(T^{*2+k} T^{2+k} - (1+k) \left(\frac{\mu}{|\alpha|^2}\right)^k T^{*2} T^2 + k \left(\frac{\mu}{|\alpha|^2}\right)^{1+k} T^* T \right) \geq 0.
\end{aligned}$$

Hence αT is ek-paranormal.

The following example shows that the ek-paranormal operators are not translation invariant.

Example 3.7. Recall that if $H = C^2$, then $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is ek-paranormal for every positive integer k. But $T + 1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is not ek-paranormal for any positive integer k.

Theorem 3.8. Let T be ek-paranormal, $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $ker(T - \lambda) \oplus ker(T - \lambda)^{\perp}$, then

$$T_2 T_3 \left(1 + \frac{T_3}{\lambda} + \left(\frac{T_3}{\lambda}\right)^2 + \cdots + \left(\frac{T_3}{\lambda}\right)^k\right) = (1+k) T_2 T_3.$$

Proof. Without loss of generality, assume that $\lambda = 1$. Since T is ek-paranormal, taking $\mu = 1$, we have

$0 \leq T^{*2+k}T^{2+k} - (1+k)T^{*2}T^2 + kT^*T =$

$$\begin{pmatrix} 0 & T_2T_3 + \cdots + T_2T_3^{1+k} - (1+k)T_2T_3 \\ T_3^*T_2^* + \cdots + T_3^{*1+k}T_2^* & (T_3^*T_2^* + \cdots + T_3^{*1+k}T_2^*)(T_2T_3 + \cdots + T_2T_3^{1+k}) + T_3^{*2+k}T_3^{2+k} \\ -(1+k)T_3^*T_2^* & -(1+k)\left((T_2^* + T_3^*T_2^*)(T_2 + T_2T_3) + T_3^{*2}T_3^2\right) + k(T_2^*T_2 + T_3^*T_3) \end{pmatrix}$$

A matrix of the form $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0 \quad \Leftrightarrow \quad X \geq 0,\ Z \geq 0$ and $Y = X^{1/2}WZ^{1/2}$, for some contraction W. Hence we get the required result.

Theorem 3.9. Let T be ek-paranormal, $0 \neq \lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T-\lambda) \oplus \ker(T-\lambda)^{\perp}$, then T_3 is also ek-paranormal.

Proof. Using Theorem 3.3,

$$0 \leq T^{*2+k}T^{2+k} - (1+k)\mu^kT^{*2}T^2 + k\mu^{1+k}T^*T = \begin{pmatrix} X(\mu) & Y(\mu) \\ Y(\mu)^* & Z(\mu) \end{pmatrix},$$

where $X(\mu) = 1-(1+k)\mu^k + k\mu^{k+1}, Y(\mu) = (1-(1+k)\mu^k + k\mu^{k+1})T_2 + (1+k)T_2T_3(1-\mu^k)$ and $Z(\mu) = (1-(1+k)\mu^k + k\mu^{k+1})T_2^*T_2 + (1+k)(T_2^*T_2T_3 + T_3^*T_2^*T_2)(1-\mu^k) + (1+k)T_3^*T_2^*T_2T_3(1+k-\mu^k) + T_3^{*2+k}{T_3}^{2+k} - (1+k)\mu^kT_3^{*2}{T_3}^2 + k\mu^{1+k}T_3^*T_3$.

Using the result $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0 \Leftrightarrow X \geq 0,\ Z \geq 0\,\mathrm{and Y = X^{1/2}WZ^{1/2}}$, for some contraction W, we get (i) $X(\mu) \geq 0$, (ii) $Z(\mu) \geq 0$ and (iii) there exists a contraction $W(\mu)$ such that $Y(\mu) = X(\mu)^{1/2}W(\mu)Z(\mu)^{1/2}$. Therefore,

$$T_3^{*2+k}{T_3}^{2+k} - (1+k)\mu^kT_3^{*2}{T_3}^2 + k\mu^{k+1}T_3^*T_3 \geq \frac{(1+k)f(\mu)}{X(\mu)}T_3^*T_2^*T_2T_3,$$

where $f(\mu) = (1+k)(1-\mu^k)X(\mu) \geq 0$ for all $\mu \geq 0$, since $f(\mu)$ has a minimum value at $\mu = 1$. Hence T_3 is ek-paranormal.

Theorem 3.10. If T is a ek-paranormal operator and $\|T^j\| = \|T\|^j$ for some $j \geq 2$, then T is a normaloid.

Proof. For any $j \geq 2$,

$$\begin{aligned} \|T^jx\|^{k+1} &\leq \|T^{k+j}x\|\,\|T^{j-1}x\|^k \\ &\leq \|T^{k+j}\|\,\|T^{j-1}\|^k\,\|x\|^{k+1} \\ \Rightarrow \qquad \|T^j\|^{k+1} &\leq \|T^{k+j}\|\,\|T^{j-1}\|^k \\ \text{Hence}\quad \|T\|^{j(k+1)} &= \|T^j\|^{k+1} \leq \|T^{k+j}\|\,\|T^{j-1}\|^k \leq \|T^{k+j}\|\,\|T\|^{(j-1)k} \\ \text{Hence}\quad \|T\|^{k+j} &\leq \|T^{k+j}\| \\ \text{and hence}\quad \|T\|^{k+j} &= \|T^{k+j}\|. \end{aligned}$$

Hence by induction, for all positive integers l, $\|T\|^{lk+2} = \|T^{lk+2}\|$. Therefore, there exists a subsequence $\{T^{n_l}\}$ of $\{T^n\}$ such that $\lim \|T^{n_l}\|^{1/n_l} \Rightarrow \|T\|$. Hence $r(T) = \|T\|$. Hence T is normaloid.

Theorem 3.11. If T is ek-paranormal, for some positive integer k, then asc(T) is finite.

Proof. By the definition of the operator, $\ker T^{k+2}$ is a subset of $\ker T^2$, which in turn is a subset of $\ker T^{k+1}$. Hence $\ker T^{k+1} = \ker T^{k+2}$ and hence the result.

Theorem 3.12. If T is ek-paranormal for some positive integer k and commutes with an isometric operator S, then ST is ek-paranormal.

Proof. Since S is an isometry, $S^*S = I$. Therefore,

$$\begin{aligned} & (ST)^{*2+k}(ST)^{2+k} - (1+k)\mu^k (ST)^{*2}(ST) + k\mu^{1+k}(ST)^*(ST) \\ = \quad & T^{*2+k}T^{2+k} - (1+k)\mu^k T^{*2}T^2 + k\mu^{1+k}T^*T \geq 0. \end{aligned}$$

Hence is ek-paranormal.

Theorem 3.13. An operator unitarily equivalent to a ek-paranormal for some positive integer k, is also a ek-paranormal operator.

Proof. Let S be unitarily equivalent to a ek-paranormal operator T, for some positive integer k. Then $S = UTU^*$ for some unitary operator U. Hence

$$\begin{aligned} \left\|S^{2+k}x\right\| \|Sx\|^k &= \left\|UT^{2+k}U^*x\right\| \|UTU^*x\| \\ &\geq \left\|U^*T^2U\right\|^{1+k} \\ &= \left\|S^2x\right\|^{1+k} \end{aligned}$$

Hence S is also ek-paranormal. Hence the result.

§4. Composition operators of ek-paranormal operators

Theorem 4.1. For each positive integer k, C_T is ek-paranormal if and only if

$$h_{2+k} - (1+k)\mu^k h_2 + k\mu^{1+k}h \geq 0 \text{ a.e., for every } \mu > 0.$$

Proof. C_T is ek-paranormal for a positive integer k if and only if

$$C_T^{*\,2+k}{C_T}^{2+k} - (1+k)\mu^k C_T^{*\,2}{C_T}^2 + k\mu^{1+k}C_T^*C_T \geq 0, \text{for every } \mu > 0,$$

if and only if for every $f \in L^2(\lambda)$,

$$\left\langle C_T^{*\,2+k}{C_T}^{2+k}f,\, f\right\rangle - (1+k)\mu^k \left\langle C_T^{*\,2}C_T^2 f,\ f\right\rangle + k\mu^{1+k}\left\langle C_T^*C_T f,\, f\right\rangle \geq 0,$$

if and only if $\langle h_{2+k}f,\, f\rangle - (1+k)\mu^k \langle h_2 f,\, f\rangle + k\mu^k \langle hf,\, f\rangle \geq 0$,
if and only if for every characteristic function χ_E of E in Σ,

$$\int_X (h_{2+k} - (1+k)\mu^k h_2 + k\mu^{1+k}h)\chi_E\overline{\chi_E}\,\mathrm{d}\lambda \geq 0,$$

if and only if $h_{2+k} - (1+k)\mu^k h_2 + k\mu^{1+k}h \geq 0$ a.e. for every $\mu > 0$.

Corollary 4.2. C_T is ek-paranormal for a positive integer k if and only if $h_2^{k+1} \leq h_{2+k}h^k$ a.e.

§5. Weighted composition operators and Aluthge transformation of k-paranormal operators

A weighted composition operator induced by T is defined as $Wf = w(f \circ T)$, is a complex valued Σ measurable function. Let w_k denote $w(w \circ T)(w \circ T^2)\cdots(w \circ T^{k-1})$. Then $W^k f = w_k(f \circ T)^k$ [9]. To examine the weighted composition operators effectively Alan Lambert [1] associated conditional expectation operator E with T as $E(\cdot/T^{-1}\Sigma) = E(\cdot)$. $E(f)$ is defined for each non-negative measurable function $f \in L^p(p \geq 1)$ and is uniquely determined by the conditions

1. $E(f)$ is $T^{-1}\Sigma$ measurable.
2. if B is any $T^{-1}\Sigma$ measurable set for which $\int_B f\mathrm{d}\lambda$ converges, we have $\int_B f\mathrm{d}\lambda = \int_B E(f)\mathrm{d}\lambda$.

As an operator on L^p, E is the projection onto the closure of range of T and E is the identity operator on L^p if and only if $T^{-1}\Sigma = \Sigma$. Detailed discussion of E is found in [6], [12] and [7].

The following proposition due to Campbell and Jamison is well-known.

Proposition 5.1.[2] For $w \geq 0$,

1. $W^*Wf = h[E(w^2)] \circ T^{-1}f$.
2. $WW^*f = w(h \circ T)E(wf)$.

Since $W^k f = w_k(f \circ T^k)$ and $W^{*k} f = h_k E(w_k f) \circ T^{-k}$, we have $W^{*k}W^k = h_k E(w_k^2) \circ T^{-k} f$, for $f \in L^2(\lambda)$. Now we are ready to characterize k-paranormal weighted composition operators.

Theorem 5.2. Let $W \in B(L^2(\lambda))$. Then W is ek-paranormal if and only if $h_{k+2}E(w_{k+2}^2) \circ T^{-(k+2)} - (1+k)\mu^k h_2 E(w_2^2) \circ T^{-2} + k\mu^{1+k} hE(w^2) \circ T^{-1} \geq 0$ a.e, for every $\mu > 0$.

Proof. Since W is ek-paranormal,

$$W^{*2+k}W^{2+k} - (1+k)\mu^k W^{*2}W^2 + k\mu^{1+k}W^*W \geq 0, \text{ for every } \mu > 0.$$

Hence

$$\int_E h_{k+2}E(w_{k+2}^2) \circ T^{-(k+2)} - (1+k)\mu^k h_2 E(w_2^2) \circ T^{-2} + k\mu^{1+k} hE(w^2) \circ T^{-1}\, \mathrm{d}\lambda \geq 0$$

for every $E \in \Sigma$ and so

$$h_{k+2}E(w_{k+2}^2) \circ T^{-(k+2)} - (1+k)\mu^k h_2 E(w_2^2) \circ T^{-2} + k\mu^{1+k} hE(w^2) \circ T^{-1} \geq 0 \text{ a.e. for every } \mu > 0.$$

Corollary 5.3. Let $T^{-1}\Sigma = \Sigma$. Then W is ek-paranormal if and only if $h_{k+2}w_{k+2}^2 \circ T^{-(k+2)} - (1+k)\mu^k h_2 w_2^2 \circ T^{-2} + k\mu^{1+k} hw^2 \circ T^{-1} \geq 0$ a.e. for every $\mu > 0$.

The Aluthge transformation of T is the operator $\widetilde{T}$ given by $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$. It was introduced by Aluthge [2]. More generally we may form the family of operators $T_r : 0 < r \leq 1$ where $T_r = |T|^r U |T|^{1-r}$ [3]. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{h}f$ and $Uf = \frac{1}{\sqrt{h \circ T}} f \circ T$. Lambert [5] has given a more general Aluthge transformation for composition operators as $C_r = |C|^r U |C|^{1-r}$ as $C_r f = \left(\frac{h}{h \circ T}\right)^{r/2} f \circ T$. i.e C_r is weighted composition with weight $\pi = \left(\frac{h}{h \circ T}\right)^{r/2}$.

Corollary 5.4. Let $C_r \in B(L^2(\lambda))$. Then C_r is of ek-paranormal if and only if

$$h_{k+2}E(\pi_{k+2}^2) \circ T^{-(k+2)} - (1+k)\mu^k h_2 E(\pi_2^2) \circ T^{-2} + k\mu^{k+1}E(\pi^2) \circ T^{-1} \geq 0$$

a.e. for every $\mu > 0$.

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Scientia Magna
Vol. 7 (2011), No. 2, 39-44

On isomorphisms of SU-algebras

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Abstract In this paper, we introduce a new algebraic structure, called SU-algebra. We presented a concept of ideal and congruences of SU-algebras and investigate its properties. Moreover, some consequences of the relations between quotient SU-algebras and isomorphisms are shown.

Keywords Homomorphism, isomorphism, ideal, congruence, SU-algebra.
2000 Mathematics Subject Classification: 06F35.

§1. Introduction

In 1966, K. Iseki introduced the notion of a BCI-algebras which is a generalization of BCK-algebras. He defined a BCI-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: (BCI 1) $((x*y)*(x*z))*(z*y) = 0$, (BCI 2) $(x*(x*y))*y = 0$, (BCI 3) $x*x = 0$, (BCI 4) $x*y = 0 = y*x$ imply $x = y$, (BCI 5) $x*0 = 0$ imply $x = 0$, for all $x, y, z \in X$. If (BCI 5) is replaced by (BCI 6) $0*x = 0$ for all $x \in X$, the algebra $(X, *, 0)$ is called BCK-algebra. In 1983, Hu and Li introduced the notion of a BCH-algebras which is a generalization of the notions of BCK-algebra and BCI-algebras. They have studied a few properties of these algebras and defined a BCH-algebra as an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: (BCH 1) $x*x = 0$, (BCH 2) $(x*y)*z = (x*z)*y$, (BCH 3) $x*y = 0 = y*x$ imply $x = y$, for all $x, y, z \in X$. In 1998, Dudek and Zhang studied ideals and congruences of BCC-algebras. They gave the concept of homomorphisms and quotient of BCC-algebras. They presented some related properties of them. In 2006 Dar and Akram studied properties of endomorphism in BCH-algebra.

In this paper, we introduce a new algebraic structure, called SU-algebra and a concept of ideal and homomorphisms in SU-algebra. We also describe connections between ideals and congruences. We investigated some related properties of them. Moreover, this paper is to derive some straightforward consequences of the relations between quotient SU-algebras and isomorphisms and also investigate some of its properties.

§2. The structure of SU-algebra

Definition 2.1. A SU-algebra is an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(1) $((x * y) * (x * z)) * (y * z) = 0$,

(2) $x * 0 = x$,

(3) if $x * y = 0$ imply $x = y$ for all $x, y, z \in X$.

From now on, X denotes a SU-algebra $(X, *, 0)$ and a binary operation will be denoted by juxtaposition.

Example 2.2. Let $X = \{0, 1, 2, 3\}$ be a set in which operation $*$ is defined by the following:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a SU-algebra.

Theorem 2.3. Let X be a SU-algebra. Then the following results hold for all $x, y, z \in X$.

(1) $xx = 0$,

(2) $xy = yx$,

(3) $0x = x$,

(4) $((xy)x)y = 0$,

(5) $((xz)(yz))(xy) = 0$,

(6) $xy = 0$ if and only if $(xz)(yz) = 0$,

(7) $xy = x$ if and only if $y = 0$.

Theorem 2.4. Let X be a SU-algebra. A relation $\leq$ on X is defined by $x \leq y$ if $xy = 0$. Then $(X, \leq)$ is a partially ordered set.

Theorem 2.5. Let X be a SU-algebra. Then the following results hold for all $x, y, z \in X$.

(1) $x \leq y$ if and only if $y \leq x$,

(2) $x \leq 0$ if and only if $x = 0$,

(3) if $x \leq y$, then $xz \leq yz$.

Theorem 2.6. Let X be a SU-algebra. Then the following results hold for all $x, y, z \in X$.

(1) $(xy)z = (xz)y$,

(2) $x(yz) = z(yx)$,

(3) $(xy)z = x(yz)$.

Theorem 2.7. Let X be a SU-algebra. If $xz = yz$, then $x = y$ for all $x, y, z \in X$.

Theorem 2.8. Let X be a SU-algebra and $a \in X$. If $ax = x$ for all $x \in X$, then $a = x$.

§3. Ideal and congruences in SU-algebra

Definition 3.1. Let X be a SU-algebra. A nonempty subset I of X is called an ideal of X if it satisfies the following properties :

(1) $0 \in I$,

(2) if $(xy)z \in I$ and $y \in I$ for all $x, y, z \in X$, then $xz \in I$.

Clearly, X and $\{0\}$ are ideals of X.

Example 3.2. Let X be a SU-algebra as defined in Example 2.2. If $A = \{0, 1\}$, then A is an ideal of X. If $B = \{0, 1, 2\}$, then B is not an ideal of X because $(1 * 1) * 2 = 0 * 2 = 2 \in B$ but $1 * 2 = 3 \notin B$.

Theorem 3.3. Let X be a SU-algebra and I be an ideal of X. Then

(1) if $xy \in I$ and $y \in I$, then $x \in I$ for all $x, y \in X$,

(2) if $xy \in I$ and $x \in I$, then $y \in I$ for all $x, y \in X$.

Proof. Let X be a SU-algebra and I be an ideal of X.

(1) Let $xy \in I$ and $y \in I$. Since $(xy)0 = xy \in I$ and $y \in I$, $x = x0 \in I$ (by Definition 3.1).

(2) It is immediately followed by (1) and Theorem 2.3 (2).

Theorem 3.4. Let X be a SU-algebra and A_i be ideal of X for $i = 1, 2, \ldots, n$. Then $\bigcap_{i=1}^{n} A_i$ is an ideal of X.

Proof. Let X be a SU-algebra and A_i be ideal of X for $i = 1, 2, \ldots, n$. Clearly, $0 \in \bigcap_{i=1}^{n} A_i$. Let $x, y, z \in X$ be such that $(xy)z \in \bigcap_{i=1}^{n} A_i$ and $y \in \bigcap_{i=1}^{n} A_i$. Then $(xy)z \in A_i$ and $y \in A_i$ for all $i = 1, 2, \ldots, n$. Since A_i is an ideal, $xz \in A_i$ for all $i = 1, 2, \ldots, n$. Thus $xz \in \bigcap_{i=1}^{n} A_i$. Hence $\bigcap_{i=1}^{n} A_i$ is a ideal of X.

Definition 3.5. Let X be a SU-algebra. A nonempty subset S of X is called a SU-subalgebra of X if $xy \in S$ for all $x, y \in S$.

Theorem 3.6. Let X be a SU-algebra and I be an ideal of X. Then I is a SU-subalgebra of X.

Proof. Let X be a SU-algebra and I be an ideal of X. Let $x, y \in I$. By Theorem 2.3 (4), $((xy)x)y = 0 \in I$. Since I is an ideal and $x \in I$, $(xy)y \in I$. Since I is an ideal and $y \in I$, $xy \in I$.

Definition 3.7. Let X be a SU-algebra and I be an ideal of X. A relation $\sim$ on X is defined by $x \sim y$ if and only if $xy \in I$.

Theorem 3.8. Let X be a SU-algebra and I be an ideal of X. Then $\sim$ is a congruence on X.

Proof. A reflexive property and a symmetric property are obvious. Let $x, y, z \in X$. Suppose that $x \sim y$ and $y \sim z$. Then $xy \in I$ and $yz \in I$. Since $((xz)(xy))(zy) = 0 \in I$ and I is an ideal, $(xz)(yz) = (xz)(zy) \in I$. By Theorem 3.3 (1), $xz \in I$. Thus $x \sim z$. Hence $\sim$ is an equivalent relation.

Next, let $x, y, u, v \in X$ be such that $x \sim u$ and $y \sim v$. Then $xu \in I$ and $yv \in I$. Thus $ux \in I$ and $vy \in I$. Since $((xy)(xv))(yv) = 0 \in I$ and Theorem 3.3 (1), $(xy)(xv) \in I$. Hence $xy \sim xv$. Since $((uv)(ux))(vx) = 0 \in I$ and Definition 3.1 (2), $(uv)(vx) \in I$. Since $vx = xv$, $(uv)(vx) = (uv)(xv)$. Hence $(uv)(xv) \in I$ and so $xv \sim uv$. Thus $xy \sim uv$. Hence $\sim$ is a congruence on X.

Let X be a SU-algebra, I be an ideal of X and $\sim$ be a congruence relation on X. For any $x \in X$, we define $[x]_I = \{y \in X | x \sim y\} = \{y \in X | xy \in I\}$. Then we say that $[x]_I$ is an equivalence class containing x.

Example 3.9. Let X be a SU-algebra as defined in Example 2.2. It is easy to show that $I = \{0, 1\}$ is an ideal of X, then $[0]_I = \{0, 1\}$, $[1]_I = \{0, 1\}$, $[2]_I = \{2, 3\}$, $[3]_I = \{2, 3\}$.

Remarks 3.10. Let X be a SU-algebra. Then

1. $x \in [x]_I$ for all $x \in X$,

2. $[0]_I = \{x \in X | 0 \sim x\}$ is an ideal of X.

Theorem 3.11. Let X be a SU-algebra, I be an ideal of X and $\sim$ be a congruence relation on X. Then $[x]_I = [y]_I$ if and only if $x \sim y$ for all $x, y \in X$.

Proof. Let $[x]_I = [y]_I$. Since $y \in [y]_I$, $y \in [x]_I$. Hence $x \sim y$. Conversely, let $x \sim y$. Then $y \sim x$. Let $z \in [x]_I$. Then $x \sim z$. Since $y \sim x$ and $x \sim z$, $y \sim z$, $z \in [y]_I$. Similarly, let $w \in [y]_I$. Then $w \in [x]_I$. Therefore $[x]_I = [y]_I$.

The family $\{[x]_I | x \in X\}$ gives a partition of X which is denoted by quotient SU-algebra X/I. For $x, y \in X$ we define $[x]_I [y]_I = [xy]_I$. The following theorem show that X/I is a SU-algebra.

Theorem 3.12. Let X be a SU-algebra and I be an ideal of X. Then X/I is a SU-algebra.

Proof. Let $[x]_I, [y]_I, [z]_I \in X/I$.

1) $(([x]_I[y]_I)([x]_I[z]_I))([y]_I[z]_I) = ([xy]_I[xz]_I)[yz]_I = [((xy)(xz))(yz)]_I = [0]_I$.

2) $[x]_I[0]_I = [x0]_I = [x]_I$.

3) Suppose that $[x]_I[y]_I = [0]_I$. Then $[xy]_I = [0]_I = [yx]_I$. Since $xy \in [xy]_I, 0 \sim xy$. Hence $xy \in [0]_I$. Since $[0]_I$ is an ideal, $x \sim y$. Hence $[x]_I = [y]_I$. Thus X/I is a SU-algebra.

§4. Isomorphism of a SU-Algebra

In this section, we defined homomorphism and isomorphism of SU-algebras, then we show some consequences of the relations between quotient SU-algebras and isomorphisms.

Definition 4.1. Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be a SU-algebra and let $f : X \to Y$. We called f is a homomorphism if and only if $f(xy) = f(x)f(y)$ for all $x, y \in X$.

The kernel of f defined to be the set $ker(f) = \{x \in X | f(x) = 0_Y\}$.

The image of f defined to be the set $im(f) = \{f(x) \in Y | x \in X\}$.

Definition 4.2. Let X and Y be a SU-algebra and let $f : X \to Y$ be a homomorphism, then:

(1) f called a monomorphism if f is injective,

(2) f called an epimorphism if f is surjective,

(3) f called an isomorphism if f is bijective.

Definition 4.3. Let X and Y be a SU-algebra, then we say that X isomorphic Y $(X \cong Y)$ if we have $f : X \to Y$ which $f$is an isomorphism.

Theorem 4.4. Let X be a SU-algebra, I be an ideal of X and $\sim$ be a congruence on X. Then $f : X \to X/I$ defined by $f(x) = [x]_I$ for all $x \in X$ is an epimorphism.

Proof. Let $f : X \to X/I$ and defined f by $f(x) = [x]_I$ for all $x \in X$. Let $x, y \in X$ and $x = y$, then $[x]_I = [y]_I$. Thus $f(x) = f(y)$. Hence f is a function. Let $[x]_I \in X/I$, then $f(x) = [x]_I$. Hence f is surjective. Since $f(xy) = [xy]_I = [x]_I[y]_I = f(x)f(y)$. Thus f is a homomorphism on X. Hence f is an epimorphism on X.

Theorem 4.5. Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be a SU-algebra and let $f : X \to Y$ be a homomorphism, then:

(1) $f(0_X) = 0_Y$,

(2) $im(f)$ is a SU-subalgebra,

(3) $ker(f) = \{0_X\}$ if and only if f is a injective,

(4) $ker(f)$is an ideal of X.

Proof. (1) Let $x \in X$, then $f(x) \in Y$. Since $0_Y f(x) = f(x) = f(0_X x) = f(0_X)f(x)$, then by Theorem 2.7 we have $0_Y = f(0_X)$.

(2) Let $a, b \in im(f)$, then there exists $x, y \in X$ such that $f(x) = a$ and $f(y) = b$. Thus $ab = f(x)f(y) = f(xy) \in im(f)$. Hence $im(f)$ is a SU-subalgebra.

(3) Suppose $ker(f) = \{0_X\}$. Let $x, y \in X$ and $f(x) = f(y)$, then $f(xy) = f(x)f(y) = 0_Y$. Thus $xy \in ker(f) = 0_X$. Hence $x = y$. Therefore f is an injective. Conversely, it is obviously.

(4) By (1) we have $f(0_X) = 0_Y$, Thus $0_X \in ker(f)$. Let $(xy)z \in ker(f)$ and $y \in ker(f)$, then $f((xy)z) = 0_Y$ and $f(y) = 0_Y$. Since f is a homomorphism, $f((xy)z) = f(xy)f(z) = (f(x)f(y))f(z)$. Since $f((xy)z) = 0_Y$ and $f(y) = 0_x$, $0_Y = (f(x)0_Y)f(z) = f(x)f(z) = f(xz)$. Thus $xz \in ker(f)$. Hence $ker(f)$ is an ideal of X.

Theorem 4.6. Let X and Y be a SU-algebra and let $f : X \to Y$ be a homomorphism, then $X/ker(f) \cong im(f)$. In particular, if f is surjective, then $X/ker(f) \cong Y$.

Proof. Consider the mapping $g : X/ker(f) \to im(f)$ given by $g([x]_{ker(f)}) = f(x)$for all $x \in X$.

1) Let $x, y \in X$ and $[x]_{ker(f)} = [y]_{ker(f)}$, then $x \sim y$. Thus $xy \in ker(f)$. Hence $f(xy) = 0$. Since $f(y)f(x) = f(x)f(y) = f(xy) = 0$, $f(x) = f(y)$. Therefore $g([x]_{ker(f)}) = f(x) = f(y) = g([y]_{ker(f)})$. Hence g is a function .

2) Let $x, y \in X$ and $g([x]_{ker(f)}) = g([y]_{ker(f)})$, then $f(x) = f(y)$. Thus $0_X = f(x)f(y) = f(xy)$. Hence $xy \in ker(f)$. Since $ker(f)$ is an ideal, $x \sim y$. Thus $[x]_{ker(f)} = [y]_{ker(f)}$. Hence g is a injective.

3) Let $f(x) \in im(f)$. Since $g([x]_{ker(f)}) = f(x)$, g is a surjective.

4) Let $x, y \in X$, then $g([x]_{ker(f)}[y]_{ker(f)}) = g([xy]_{ker(f)}) = f(xy) = f(x)f(y) = g([x]_{ker(f)})$ $g([y]_{ker(f)})$. Hence g is a homomorphism. Therefore $X/ker(f) \cong im(f)$. In particular, let f be a surjective, then $im(f) = Y$. Hence $X/ker(f) \cong Y$.

Theorem 4.7. Let H and K be an ideal of SU-algebra X and $K \subseteq H$, then $(X/K)/(H/K)$ $\cong X/H$.

Proof. Consider the mapping $g : X/K \to X/H$ given by $g([x]_K) = [x]_H$ for all $x \in X$.

1) Let $x, y \in X$ and $[x]_K = [y]_K$, then $x \sim y$. Since K is a ideal , $xy \in K$. Since $K \subseteq H$, $xy \in H$. Thus $g([x]_K) = [x]_H = [y]_H = g([y]_K)$. Hence g is a function.

2) Let $[x]_H \in X/H$. Since $g([x]_K) = [x]_H$, g is a surjective.

3) Let $x, y \in X$, then $g([x]_K[y]_K) = g([xy]_K) = [xy]_H = [x]_H[y]_H = g([x]_K)g([y]_K)$. Hence g is a homomorphism.

4) $ker(g) = \{[x]_K | g([x]_K) = [0]_H\} = \{[x]_K | [x]_H = [0]_H\} = \{[x]_K | x \sim 0\} = \{[x]_K | x = x0 \in H\} = H/K$. By Theorem 4.6 we have $(X/K)/(H/K) \cong X/H$.

Let X be a SU-algebra and A, B be a subset of X defined AB by $AB = \{xy \in X | x \in A, y \in B\}$.

Theorem 4.8. Let A and B be a subset of SU-algebra X, then AB is a SU-subalgebra of X.

Proof. Let $a \in AB$. By definition of AB we have $a = xy$ for some $x \in A, y \in B$. Since $A, B \subseteq X, x, y \in X$. Thus $a = xy \in X$. Hence $AB \subseteq X$. Let $m, n \in AB$ such that $m = a_1b_1, n = a_2b_2$ for some $a_1, a_2 \in A, b_1, b_2 \in B$, then $mn = (a_1b_1)(a_2b_2) = (a_1(a_2b_2))b_1 = (b_2(a_2a_1))b_1 = (b_2b_1)(a_2a_1) = (a_2a_1)(b_2b_1)$. Since $a_2a_1 \in A$ and $b_2b_1 \in B$, $mn \in AB$. Hence AB is a SU-subalgebra of X.

Remark 4.9. Let H and K be an ideal of SU-algebra X, then N is an ideal of HN.

Theorem 4.10. Let H and N be an ideal of SU-algebra X, then $H/(H \cap N) \cong HN/N$.

Proof. Consider the mapping $g : H \to HN/N$ given by $g(x) = [x]_N$ for all $x \in H$.

1) Let $x, y \in H$ and $x = y$, then $[x]_N = [y]_N$. Thus $g(x) = g(y)$. Hence g is a function.

2) Let $[x]_N \in HN/N$, then $g(x) = [x]_N$ Hence g is a surjective.

3) Let $x, y \in H$. Since $g(xy) = [xy]_N = [x]_N[y]_N = g(x)g(y)$. Hence g is a homomorphism on X.

4) Let $x \in ker(g)$, then $g(x) = [0]_N$. Since $g(x) = [x]_N$, $[x]_N = [0]_N$. Thus $x \sim 0$. Since N is an ideal, $x = x0 \in N$. Since $ker(g) \subseteq H$, $x \in H$. Thus $x \in H \cap N$. Hence $ker(g) \subseteq H \cap N$. Let $x \in H \cap N$, then $x \in H$ and $x \in N$. Since $g(x) = [x]_N = [0]_N, x \in ker(g)$. Thus $H \cap N \subseteq ker(g)$. Hence $ker(g) = H \cap N$. By Theorem 4.6 we have $H/(H \cap N) \cong HN/N$.

Acknowledgements

The authors are highly grateful to the referees for their valuable comments and suggestions for the paper. Moreover, this work was supported by a grant for Kasetsart University.

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Scientia Magna
Vol. 7 (2011), No. 2, 45-58

$rg\alpha-$ separation axioms

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Abstract In this paper we discuss new separation axioms using $rg\alpha-$open sets.
Keywords $rg\alpha_i$ spaces.

Mathematics Subject Classification Number: 54D10, 54D15.

§1. Introduction

Norman Levine introduced generalized closed sets, K. Balachandaran and P. Sundaram studied generalized continuous functions and generalized homeomorphism. N. Palaniappan and K. C. Rao defined regular generalized closed sets. V. K. Sharma studied generalized separation axioms. Following V. K. Sharma the author of the present paper define a new variety of generalized axioms called $rg\alpha-$separation axioms and study their basic properties and interrelation with other type of generalized separation axioms. Throughout the paper a space X means a topological space (X,τ). For any subset A of X its complement, interior, closure, $rg\alpha-$interior, $rg\alpha-$closure are denoted respectively by the symbols A^c, A^0, $\overline{A}$, $rg\alpha(A)^0$ and $rg\alpha\overline{(A)}$.

Definition 1.1. $A \subset X$ is called

(i) regularly closed if $A = \overline{(A)^o}$ and regularly open if $A = (\overline{A})^0$.

(ii) semi-closed [resp: $\nu-$closed] if $\exists$ an open [resp: regular open] set $U \ni U^o \subseteq A \subseteq U$.

(iii) g-closed if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open.

(iv) rg-closed if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is regular-open.

(v) semi-open [resp: $\nu-$open] if $\exists$ an open [resp: regular open] set $U \ni U \subseteq A \subseteq \overline{U}$.

(vi) g-open [resp: rg-open] if its complement is g-closed [resp: rg-closed].

(vii) $rg\alpha-$closed if $\alpha\overline{(A)} \subset U$ whenever $A \subset U$ and U is regular-$\alpha-$open in X.

Note 1. The class of regular open sets, open sets, g-open sets, and $rg\alpha-$open sets are denoted by $RO(X)$, $\tau(X)$, $GO(X)$ and $RG\alpha O(X)$ respectively. Clearly $RO(X) \subset \tau(X) \subset GO(X) \subset RG\alpha O(X)$.

Note 2. For any $A \subset X, A \in RG\alpha O(X, x)$ means A is a rgα-open set [neighborhood] in X containing x.

Definition 1.2. $A \subset X$ is called clopen [resp: nearly-clopen; $\nu-$clopen; semi-clopen; g-clopen; $rg\alpha-$clopen] if it is both open [resp: regular-open; $\nu-$open; semi-open; g-open; $rg\alpha-$open] and closed [resp: regular-closed; $\nu-$closed; semi-closed; g-closed; $rg\alpha-$closed].

Definition 1.3. A function f: X$\rightarrow Y$ is said to be

(i) Continuous [resp: nearly continuous, semi-continuous] if inverse image of open set is open[resp: regular-open, semi-open].

(ii) g-continuous [resp: $rg\alpha-$continuous] if inverse image of closed set is g-closed [resp: $rg\alpha-$closed].

(iii) irresolute [resp: nearly irresolute] if inverse image of semi-open [resp: regular-open] set is semi-open[resp: regular-open].

(iv) g-irresolute [resp: $rg\alpha-$irresolute] if inverse image of g-closed [resp: $rg\alpha-$closed] set is g-closed[resp: $rg\alpha-$closed].

(v) open [resp: nearly open, semi-open, g-open, $rg\alpha-$open] if the image of open set is open [resp: regular-open, semi-open, g-open, $rg\alpha-$open].

(vi) homeomorphism [resp: nearly homeomorphism, semi-homeomorphism, g-homeomorphism, $rg\alpha-$homeomorphism] if f is bijective continuous [resp: nearly-continuous, semi-continuous, g-continuous, $rg\alpha-$continuous] and f^{-1} is continuous [resp: nearly-continuous, semi-continuous, g-continuous, $rg\alpha-$continuous].

(vii) rc-homeomorphism [resp: sc-homeomorphism, gc-homeomorphism, $RG\alpha C$-homeomorphism] if f is bijective r-irresolute [resp: irresolute, g-irresolute, $rg\alpha-$irresolute] and f^{-1} is r-irresolute [resp: irresolute, g-irresolute, $rg\alpha-$irresolute].

Definition 1.4. X is said to be

(i) compact [resp: nearly compact, semi-compact, g-compact, $rg\alpha-$compact] if every open [resp: regular-open, semi-open, g-open, $rg\alpha-$open] cover has a finite subcover.

(ii) T_0 [resp: $r-T_0, s-T_0, g_0$] space if for each $x \neq y \in X \exists U \in \tau(X)$ [resp: RO(X); SO(X); GO(X)] containing either x or y.

(iii) T_1 [resp: $r-T_1, s-T_1, g_1$] space if for each $x \neq y \in X \exists U, V \in \tau(X)$ [resp: RO(X); SO(X); GO(X)] such that $x \in U - V$ and $y \in V - U$.

(iv) T_2 [resp: $r-T_2, s-T_2, g_2$] space if for each $x \neq y \in X \exists U, V \in \tau(X)$ [resp: RO(X); SO(X); GO(X)] such that $x \in U$; $y \in V$ and $U \cap V = \phi$.

(v) $T_{\frac{1}{2}}$[resp: $r-T_{\frac{1}{2}}, s-T_{\frac{1}{2}}$] if every generalized [resp: regular generalzed, semi-generalized] closed set is closed[resp: regular-closed, semi-closed].

(vi) C_0 [resp: rC_0, sC_0, gC_0] space if for each $x \neq y \in X \exists U \in \tau(X)$[resp: RO(X); SO(X); GO(X)] whose closure contains either x or y.

(vii) C_1 [resp: rC_1, sC_1, gC_1] space if for each $x \neq y \in X \exists U, V \in \tau(X)$[resp: RO(X); SO(X); GO(X)] such that $x \in \overline{(U)}$ and $y \in \overline{(V)}$.

(viii) C_2 [resp: rC_2, sC_2, gC_2] space if for each $x \neq y \in X \exists U, V \in \tau(X)$[resp: RO(X); SO(X); GO(X)] such that $x \in \overline{(U)}$; $y \in \overline{(V)}$ and $U \cap V = \phi$.

(ix) D_0 [resp: rD_0, sD_0, gD_0] space if for each $x \neq y \in X \exists U \in D(X)$[resp: RD(X); SD(X); GD(X)] containing either x or y.

(x) D_1 [resp: rD_1, sD_1, gD_1] space if for each $x \neq y \in X \exists U, V \in D\tau(X)$[resp: RD(X); SD(X); GD(X)] $\ni x \in U - V$ and $y \in V - U$.

(xi) D_2 [resp: rD_2, sD_2, gD_2]space if for each $x \neq y \in X \exists U, V \in D\tau(X)$[resp: RD(X); SD(X); GD(X)] $\ni x \in U - V; y \in V - U$ and $U \cap V = \phi$.

(xii) R_0 [resp: rR_0, sR_0, gR_0]space if for each $x \in X \exists U \in \tau(X)$ [resp: RO(X); SO(X);

GO(X)]$\ni$ $\overline{\{x\}} \subseteq U$[resp: $r\overline{\{x\}} \subseteq U; s\overline{\{x\}} \subseteq U; g\overline{\{x\}} \subseteq U$] whenever $x \in U \in \tau(X)$[resp: $x \in U \in RO(X); x \in U \in SO(X); x \in U \in GO(X)$].

(xiii) R_1 [resp: rR_1, sR_1, gR_1]space for $x, y \in X \ni \overline{\{x\}} \neq \overline{\{y\}}$ [resp:$\ni r\overline{\{x\}} \neq r\overline{\{y\}}; \ni s\overline{\{x\}} \neq s\overline{\{y\}}; \ni g\overline{\{x\}} \neq g\overline{\{y\}}$;[resp: RO(X); SO(X); GO(X)], $\exists$ disjoint $U; V \in \tau(X) \ni \overline{\{x\}} \subseteq U$ [resp: $RO(X) \ni r\overline{\{x\}} \subseteq U; SO(X) \ni s\overline{\{x\}} \subseteq U; GO(X) \ni g\overline{\{x\}} \subseteq U$] and $\overline{\{y\}} \subseteq V$[resp: $r\overline{\{y\}} \subseteq V; s\overline{\{y\}} \subseteq V; g\overline{\{y\}} \subseteq V$].

Theorem 1.1. (i) If x is a $rg\alpha-$limit point of any $A \subset X$, then every $rg\alpha-$neighborhood of x contains infinitely many distinct points.

(ii) Let $A \subseteq Y \subseteq X$ and Y is regularly open subspace of X then A is $rg\alpha-$open in X iff A is $rg\alpha-$open in $\tau_{/Y}$.

Theorem 1.2. If f is $rg\alpha-$continuous [resp: $rg\alpha-$irresolute{$rg\alpha-$homeomorphism}] and G is open [resp: $rg\alpha-$open] set in Y, then $f^{-1}(G)$ is $rg\alpha-$open [resp: $rg\alpha-$open] in X.

§2. $rg\alpha-$continuity and product spaces

Theorem 2.1. If f is nearly continuous then f is $rg\alpha-$continuous. Converse is true if X is $r - T_{\frac{1}{2}}$.

Theorem 2.2. If $f: X \to Y$ is $rg\alpha-$continuous, $g: Y \to Z$ is $rg\alpha-$continuous and Y is $r - T_{\frac{1}{2}}$, then $g \circ f$ is $rg\alpha-$continuous.

Theorem 2.3. Let Y and $\{X_\alpha : \alpha \in I\}$ be Topological Spaces. Let $f: Y \to \Pi X_\alpha$ be a function. If f is $rg\alpha-$continuous, then $\pi_\alpha \circ f: Y \to X_\alpha$ is $rg\alpha-$continuous.

Proof. Suppose f is $rg\alpha-$continuous. Since $\pi_\alpha : \Pi X_\beta \to X_\alpha$ is continuous for each $\alpha \in I$, it follows that $\pi_\alpha \circ f$ is $rg\alpha-$continuous.

Converse of the above theorem is not true in general as shown by the following example:

Example 2.1. Let $X = \{p, q, r, s\}; \tau_X = \{\phi, \{p\}, \{q\}, \{s\}, \{p, q\}, \{p, s\}, \{q, s\}, \{p, q, r\}, \{p, q, s\}, X\}, Y_1 = Y_2 = \{a, b\}; \tau_{Y_1} = \{\phi, \{a\}, Y_1\}; \tau_{Y_2} = \{\phi, \{a\}, Y_2\}; Y = Y_1 \times Y_2; = \{(a, a), (a, b), (b, a), (b, b)\}$ and $\tau_Y = \{\phi, \{(a, a)\}, \{(a, a), (a, b)\}, \{(a, a), (b, a)\}, \{(a, a), (a, b), (b, a)\}, Y_1 \times Y_2\}$.

(i) Define $f: X \to Y$ by $f(p) = (a, a), f(q) = (b, b), f(r) = (a, b), f(s) = (b, a)$. Then $\pi_1 \circ f$; $\pi_2 \circ f$ and f is $rg\alpha-$continuous.

(ii) Define $f: X \to Y$ by $f(p) = (a, a), f(q) = (a, b), f(r) = (b, b), f(s) = (b, a)$. It is easy to see that $\pi_1 \circ f$ and $\pi_2 \circ f$ are $rg\alpha-$continuous. However $\{(b, b)\}$ is closed in Y but $f^{-1}(\{(b, b)\}) = \{q\}$ is not $rg\alpha-$closed in X. Therefore f is not $rg\alpha-$continuous.

Theorem 2.4. If Y is $rT_{\frac{1}{2}}$ and $\{X_\alpha : \alpha \in I\}$ be Topological Spaces. Let $f: Y \to \Pi X_\alpha$ be a function, then f is $rg\alpha-$continuous iff $\pi_\alpha \circ f: Y \to X_\alpha$ is $rg\alpha-$continuous.

Corollary 2.5. Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let $f: \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. If f is $rg\alpha-$continuous then each f_α is $rg\alpha-$continuous.

Corollary 2.6. For each α, let X_α be $rT_{\frac{1}{2}}$ and let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let $f: \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$, then f is $rg\alpha-$continuous iff each f_α is $rg\alpha-$continuous.

§3. $rg\alpha_i$ spaces i= 0, 1, 2

Definition 3.1. X is said to be

(i) a $rg\alpha_0$ space if for each pair of distinct points x, y of X, there exists a $rg\alpha-$open set G containing either x or y.

(ii) a $rg\alpha_1$ space if for each pair of distinct points x, y of X there exists a $rg\alpha-$open set G containing x but not y and a $rg\alpha-$open set H containing y but not x.

(iii) a $rg\alpha_2$ space if for each pair of distinct points x, y of X there exists disjoint $rg\alpha-$open sets G and H such that G containing x but not y and H containing y but not x.

Note 2. (i) $rT_i \Rightarrow T_i \Rightarrow g_i \Rightarrow rg\alpha_i$, $i = 0, 1, 2$, but the converse is not true in general.

(ii) X is $rg\alpha_2 \Rightarrow X$ is $rg\alpha_1 \Rightarrow X$ is $rg\alpha_0$.

Example 3.1. Let $X = \{a, b, c\}$ and

(i) $\tau = \{\phi$, {a, c}, $X\}$ then X is $rg\alpha_i$ for $i = 0, 1, 2$.

(ii) $\tau = \{\phi$, {a}, {a, c}, $X\}$ then X is not $rg\alpha_i$ for $i = 0, 1, 2$.

Example 3.2. (i) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi$, {a}, {a, b}, {c, d}, {a, c, d}, $X\}$ then X is $rg\alpha_i$ but not g_i, r-T_0 and T_0, for $i = 0, 1, 2$.

Theorem 3.1. We have the following properties:

(i) Every [resp: regular open] open subspace of $rg\alpha_i$ space is $rg\alpha_i$ for $i = 0, 1, 2$.

(ii) The product of $rg\alpha_i$ spaces is again $rg\alpha_i$ for $i = 0, 1, 2$.

(iii) X is $rg\alpha_0$ iff $\forall x \in X, \exists U \in RG\alpha O(X)$ containing $x \ni$ the subspace U is $rg\alpha_0$.

(iv) X is $rg\alpha_0$ iff distinct points of X have disjoint $rg\alpha-$closures.

Theorem 3.2. The following are equivalent:

(i) X is $rg\alpha_1$.

(ii) Each one point set is $rg\alpha-$closed.

(iii) Each subset of X is the intersection of all $rg\alpha-$open sets containing it.

(iv) For any $x \in X$, the intersection of all $rg\alpha-$open sets containing the point is the set {x}.

Theorem 3.3. (i) If X is $rg\alpha_1$ then distinct points of X have disjoint $rg\alpha-$closures.

(ii) If x is a $rg\alpha-$limit point of a subset A of a $rg\alpha_1$ space X. Then every neighborhood of x contains infinitely many distinct points of A.

(iii) X is $rg\alpha_2$ iff the intersection of all $rg\alpha-$closed, $rg\alpha-$neighborhoods of each point of the space is reduced to that point.

(iv) If to each point $x \in X$, there exist a $rg\alpha-$closed, $rg\alpha-$open subset of X containing x which is also a $rg\alpha_2$ subspace of X, then X is $rg\alpha_2$.

(v) In $rg\alpha_2$-space, $rg\alpha-$limits of sequences, if exists, are unique.

Theorem 3.4. If X is $rg\alpha_2$ then the diagonal Δ in $X \times X$ is $rg\alpha-$closed.

Proof. Suppose $(x, y) \in X \times X - \Delta$. As $(x, y) \notin \Delta$ and $x \neq y$. Since X is $rg\alpha_2$, $\exists U, V \in RG\alpha O(X) \ni x \in U,\ y \in V$ and $U \cap V = \phi$. $U \cap V = \phi \Rightarrow (U \times V) \cap \Delta = \phi$ and therefore $(U \times V) \subset X \times X - \Delta$. Further $(x, y) \in (U \times V)$ and $(U \times V)$ is $rg\alpha-$open in $X \times X$ gives $X \times X - \Delta$ is $rg\alpha-$open. Hence Δ is $rg\alpha-$closed.

Theorem 3.5. In a $rg\alpha_2$ space, a point and disjoint $rg\alpha-$compact subspace can be separated by disjoint $rg\alpha-$open sets.

Proof. Let X be a $rg\alpha_2$ space, $x \in X$ and C a $rg\alpha-$compact subspace of X not containing x. Let $y \in C$ then for $x \neq y$ in X, there exist disjoint $rg\alpha-$open neighborhoods G_x and H_y. Allowing this for each y in C, we obtain a class $\{H_y\}$ whose union covers C; and since C is $rg\alpha-$compact, some finite subclass $\{H_i, i = 1 \text{ to } n\}$ covers C. If G_i is $rg\alpha-$neighborhood of x corresponding to H_i, we put $G = \bigcup_{i=1}^{n} G_i$ and $H = \bigcap_{i=1}^{n} H_i$, satisfying the required properties.

Theorem 3.6. (i) Every $rg\alpha-$compact subspace of a $rg\alpha_2$ space is $rg\alpha-$closed.

(ii) Every compact [resp: nearly-compact; g-compact] subspace of a T_2 [resp: rT_2; g_2] space is $rg\alpha-$closed.

Theorem 3.7. (i) If f is injective, $rg\alpha-$irresolute and Y is $rg\alpha_i$ then X is $rg\alpha_i$, $i = 0, 1, 2$.

(ii) If f is injective, $rg\alpha-$continuous and Y is T_i then X is $rg\alpha_i$, $i = 0, 1, 2$.

(iii) The property of being a space is $rg\alpha_0$ is a $rg\alpha-$topological property.

(iv) Let f is a $RG\alpha C$-homeomorphism, then X is $rg\alpha_i$ if Y is $rg\alpha_i$, $i = 0, 1, 2$.

Theorem 3.8. We have the following

(i) Let X be T_1 and f be $rg\alpha-$closed surjection. Then X is $rg\alpha_1$.

(ii) Every $rg\alpha-$irresolute map from a $rg\alpha-$compact space into a $rg\alpha_2$ space is $rg\alpha-$closed.

(iii) Any $rg\alpha-$irresolute bijection from a $rg\alpha-$compact space onto a $rg\alpha_2$ space is a $RG\alpha C$-homeomorphism.

(iv) Any $rg\alpha-$continuous bijection from a $rg\alpha-$compact space onto a $rg\alpha_2$ space is a $rg\alpha-$homeomorphism.

Theorem 3.9. The following are equivalent:

(i) X is $rg\alpha_2$.

(ii) For each pair $x \neq y \in X$, $\exists$ a $rg\alpha-$open, $rg\alpha-$closed set $V \ni x \in V$ and $y \notin V$, and

(iii) For each pair $x \neq y \in X$, $\exists$ $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$ and f is $rg\alpha-$continuous.

Theorem 3.10. If f: X$\to$ Y is $rg\alpha-$irresolute and Y is $rg\alpha_2$ then

(i) the set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is $rg\alpha-$closed in $X \times X$.

(ii) G(f), Graph of f, is $rg\alpha-$closed in $X \times Y$.

Proof. (i) Let $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow$ $\exists$ disjoint V_1 and $V_2 \in RG\alpha O(Y) \ni f(x_1) \in V_1$ and $f(x_2) \in V_2$, then by $rg\alpha-$irresoluteness of f, $f^{-1}(V_j) \in RG\alpha O(X, x_j)$ for each j. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in RG\alpha O(X \times X)$. Therefore $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A \Rightarrow X \times X - A$ is $rg\alpha-$open. Hence A is $rg\alpha-$closed.

(ii) Let $(x, y) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint $rg\alpha-$open sets V and $W \ni f(x) \in V$ and y$\in W$. Since f is $rg\alpha-$irresolute, $\exists U \in RG\alpha O(X) \ni x \in U$ and $f(U) \subset W$. Therefore we obtain $(x, y) \in U \times V \subset X \times Y - G(f)$. Hence $X \times Y - G(f)$ is $rg\alpha-$open. Hence G(f) is $rg\alpha-$closed in $X \times Y$.

Theorem 3.11. If f: $X \to Y$ is $rg\alpha-$open and the set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is closed in $X \times X$. Then Y is $rg\alpha_2$.

Theorem 3.12. Let X be an arbitrary space, R an equivalence relation in X and $p : X \to X/R$ the identification map. If $R \subset X \times X$ is $rg\alpha-$closed in $X \times X$ and p is $rg\alpha-$open map, then X/R is $rg\alpha_2$.

Proof. Let p(x), p(y) be distinct members of X/R. Since x and y are not related, $R \subset X \times X$ is $rg\alpha-$closed in $X \times X$. There are $rg\alpha-$open sets U and V such that $x \in U, y \in V$

and $U \times V \subset R^c$. Thus *p(U), p(V)* are disjoint and also $rg\alpha-$open in X/R since p is $rg\alpha-$open.

Theorem 3.13. The following four properties are equivalent:

(i) X is $rg\alpha_2$.

(ii) Let $x \in X$. For each $y \neq x, \exists U \in RG\alpha O(X) \ni x \in U$ and $y \notin rg\alpha\overline{(U)}$.

(iii) For each $x \in X, \cap\{rg\alpha\overline{(U)}/U \in RG\alpha O(X)$ and $x \in U\} = \{x\}$.

(iv) The diagonal $\Delta = \{(x,x)/x \in X\}$ is $rg\alpha-$closed in $X \times X$.

Proof. $(i) \Rightarrow (ii)$: Let $x \in X$ and $y \neq x$. Then there are disjoint $rg\alpha-$open sets U and V such that $x \in U$ and $y \in V$. Clearly V^c is $rg\alpha-$closed, $rg\alpha\overline{(U)} \subset V^c, y \notin V^c$ and therefore $y \notin rg\alpha\overline{(U)}$.

$(ii) \Rightarrow (iii)$: If $y \neq x$, then $\exists U \in RG\alpha O(X) \ni x \in U$ and $y \notin rg\alpha\overline{(U)}$. So $y \notin \cap\{rg\alpha\overline{(U)}/U \in RG\alpha O(X)$ and $x \in U\}$.

$(iii) \Rightarrow (iv)$: We prove Δ^c is $rg\alpha-$open. Let $(x,y) \notin \Delta$. Then $y \neq x$ and $\cap\{rg\alpha\overline{(U)}/U \in RG\alpha O(X)$ and $x \in U\} = \{x\}$ there is some $U \in RG\alpha O(X)$ with $x \in U$ and $y \notin rg\alpha\overline{(U)}$. Since $U \cap (rg\alpha\overline{(U)})^c = \phi$, $U \times (rg\alpha\overline{(U)})^c$ is a $rg\alpha-$open set such that $(x,y) \in U \times (rg\alpha\overline{(U)})^c \subset \Delta^c$.

$(iv) \Rightarrow (i)$: $y \neq x$, then $(x,y) \notin \Delta$ and thus there exist $rg\alpha-$open sets U and V such that $(x,y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for the $rg\alpha-$open sets U and V we have: $x \in U, y \in V$ and $U \cap V = \phi$.

§4. $rg\alpha - R_i$ spaces; i = 0,1

Definition 4.1. Let $x \in X$. Then

(i) $rg\alpha-$kernel of x is defined and denoted by $Ker_{rg\alpha}\{x\} = \cap\{U : U \in RG\alpha O(X)$ and $x \in U\}$.

(ii) $Ker_{rg\alpha}F = \cap\{U : U \in RG\alpha O(X)$ and $F \subset U\}$.

Lemma 4.1. Let $A \subset X$, then $Ker_{rg\alpha}\{A\} = \{x \in X : rg\alpha\overline{\{x\}} \cap A \neq \phi\}$.

Lemma 4.2. Let $x \in X$. Then $y \in Ker_{rg\alpha}\{x\}$ iff $x \in rg\alpha\overline{\{y\}}$.

Proof. Suppose that $y \notin Ker_{rg\alpha}\{x\}$. Then $\exists V \in RG\alpha O(X)$ containing $x \ni y \notin V$. Therefore we have $x \notin rg\alpha\overline{\{y\}}$. The proof of converse part can be done similarly.

Lemma 4.3. For any points $x \neq y \in X$, the following are equivalent:

(1) $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\}$;

(2) $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$.

Proof. (1)$\Rightarrow$ (2): Let $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\}$, then $\exists z \in X \ni z \in Ker_{rg\alpha}\{x\}$ and $z \notin Ker_{rg\alpha}\{y\}$. From $z \in Ker_{rg\alpha}\{x\}$ it follows that $x \cap rg\alpha\overline{\{z\}} \neq \phi \Rightarrow x \in rg\alpha\overline{\{z\}}$. By $z \notin Ker_{rg\alpha}\{y\}$, we have $\{y\} \cap rg\alpha\overline{\{z\}} = \phi$. Since $x \in rg\alpha\overline{\{z\}}, rg\alpha\overline{\{x\}} \subset rg\alpha\overline{\{z\}}$ and $\{y\} \cap rg\alpha\overline{\{x\}} = \phi$. Therefore $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. Now $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\} \Rightarrow rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$.

(2)$\Rightarrow$(1): If $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. Then $\exists z \in X \ni z \in rg\alpha\overline{\{x\}}$ and $z \notin rg\alpha\overline{\{y\}}$. Then $\exists$ a $rg\alpha-$open set containing z and therefore containing x but not y, namely, $y \notin Ker_{rg\alpha}\{x\}$. Hence $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\}$.

Definition 4.2. X is said to be

(i) $rg\alpha - R_0$ iff $rg\alpha\overline{\{x\}} \subseteq G$ whenever $x \in G \in RG\alpha O(X)$.

(ii) weakly $rg\alpha - R_0$ iff $\cap rg\alpha\overline{\{x\}} = \phi$.

(iii) $rg\alpha - R_1$ iff for $x, y \in X \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}} \exists$ disjoint $U; V \in RG\alpha O(X) \ni rg\alpha\overline{\{x\}} \subseteq U$ and $rg\alpha\overline{\{y\}} \subseteq V$.

Example 4.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$, then X is weakly $rg\alpha R_0$ and $rg\alpha R_i$, $i = 0, 1$.

Remark 4.1. (i) $r - R_i \Rightarrow R_i \Rightarrow gR_i \Rightarrow rg\alpha R_i$, $i = 0, 1$.

(ii) Every weakly-R_0 space is weakly $rg\alpha R_0$.

Lemma 4.4. Every $rg\alpha R_0$ space is weakly $rg\alpha R_0$.

Converse of the above theorem is not true in general by the following examples.

Example 4.2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Clearly, X is weakly $rg\alpha R_0$, since $\cap rg\alpha\overline{\{x\}} = \phi$. But it is not $rg\alpha R_0$, for $\{a\} \subset X$ is $rg\alpha-$open and $rg\alpha\overline{\{a\}} = \{a, b\} \not\subset \{a\}$.

Theorem 4.1. Every $rg\alpha-$regular space X is rT_2 and $rg\alpha - R_0$.

Proof. Let X be $rg\alpha-$regular and let $x \neq y \in X$. By Lemma 4.1, {x} is either $r-$open or $r-$closed. If {x} is $r-$open, {x} is $rg\alpha-$open and hence $r-$clopen. Thus {x} and $X - \{x\}$ are separating $r-$open sets. Similar argument, for {x} is $rg\alpha-$closed gives {x} and $X - \{x\}$ are separating $r-$closed sets. Thus X is rT_2 and $rg\alpha - R_0$.

Theorem 4.2. X is $rg\alpha - R_0$ iff given $x \neq y \in X; rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$.

Proof. Let X be $rg\alpha - R_0$ and let let $x \neq y \in X$. Suppose U is a $rg\alpha-$open set containing x but not y, then $y \in rg\alpha\overline{\{y\}} \subset X - U$ and so $x \notin rg\alpha\overline{\{y\}}$. Hence $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$.

Conversely, let $x \neq y \in X \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}} \Rightarrow rg\alpha\overline{\{x\}} \subset X - rg\alpha\overline{\{y\}} = U$(say) a $rg\alpha-$open set in X. This is true for every $rg\alpha\overline{\{x\}}$. Thus $\cap rg\alpha\overline{\{x\}} \subseteq U$ where $x \in rg\alpha\overline{\{x\}} \subseteq U \in RG\alpha O(X)$, which in turn implies $\cap rg\alpha\overline{\{x\}} \subseteq U$ where $x \in U \in RG\alpha O(X)$. Hence X is $rg\alpha R_0$.

Theorem 4.3. X is weakly $rg\alpha R_0$ iff $Ker_{rg\alpha}\{x\} \neq$ X for any $x \in X$.

Proof. Let $x_0 \in X \ni ker_{rg\alpha}\{x_0\} = X$. This means that x_0 is not contained in any proper $rg\alpha-$open subset of X. Thus x_0 belongs to the $rg\alpha-$closure of every singleton set. Hence $x_0 \in \cap rg\alpha\overline{\{x\}}$, a contradiction.

Conversely assume $Ker_{rg\alpha}\{x\} \neq X$ for any x$\in X$. If there is an $x_0 \in X \ni x_0 \in \cap\{rg\alpha\overline{\{x\}}\}$, then every $rg\alpha-$open set containing x_0 must contain every point of X. Therefore, the unique $rg\alpha-$open set containing x_0 is X. Hence $Ker_{rg\alpha}\{x_0\} = X$, which is a contradiction. Thus X is weakly $rg\alpha - R_0$.

Theorem 4.4. The following statements are equivalent:

(i) X is $rg\alpha - R_0$ space.

(ii) For each x$\in X, rg\alpha\overline{\{x\}} \subset Ker_{rg\alpha}\{x\}$.

(iii) For any $rg\alpha-$closed set F and a point x$\notin F, \exists U \in RG\alpha O(X) \ni x \notin U$ and $F \subset U$.

(iv) Each $rg\alpha-$closed set F can be expressed as $F = \cap\{G : G$ is $rg\alpha-$open and F$\subset G\}$.

(v) Each $rg\alpha-$open set G can be expressed as $G = \cup\{A : A$ is $rg\alpha-$closed and A$\subset G\}$.

(vi) For each $rg\alpha-$closed set F, x$\notin F$ implies $rg\{x\}^- \cap F = \phi$.

Proof. (i)$\Rightarrow$(ii): For any $x \in X$, we have $Ker_{rg\alpha}\{x\} = \cap\{U : U \in RG$O(X) and $x \in U\}$. Since X is $rg\alpha - R_0$, each $rg\alpha-$open set containing x contains $rg\alpha\overline{\{x\}}$. Hence $rg\alpha\overline{\{x\}} \subset Ker_{rg\alpha}\{x\}$.

(ii)$\Rightarrow$(iii): Let $x \notin F \in RG\alpha C(X)$. Then for any $y \in F$, $rg\alpha\overline{\{y\}} \subset F$ and so $x \notin rg\alpha\overline{\{y\}} \Rightarrow y \notin rg\alpha\overline{\{x\}}$ that is $\exists U_y \in RG\alpha O(X) \ni y \in U_y$ and $x \notin U_y$, $\forall y \in F$. Let $U = \cup\{U_y : U_y$ is; $rg\alpha-$open, $y \in U_y$ and $x \notin U_y\}$. Then U is $rg\alpha-$open such that $x \notin U$ and $F \subset U$.

(iii)$\Rightarrow$(iv): Let F be any $rg\alpha-$closed set and $N = \cap\{G : G$ is $rg\alpha-$open and $F \subset G\}$. Then $F \subset N \longrightarrow (1)$. Let $x \notin F$, then by (iii) $\exists G \in RG\alpha O(X) \ni x \notin G$ and $F \subset G$, hence $x \notin N$ which implies $x \in N \Rightarrow x \in F$. Hence $N \subset F \longrightarrow (2)$.

Therefore from (1)&(2), each $rg\alpha-$closed set $F = \cap\{G : G$ is $rg\alpha-$open and $F \subset G\}$.

(iv)$\Rightarrow$(v): obvious.

(v)$\Rightarrow$(vi): Let $x \notin F \in RG\alpha C(X)$. Then $X - F = G$ is a $rg\alpha-$open set containing x. Then by (v), G can be expressed as the union of $rg\alpha-$closed sets A contained in G, and so there is an $M \in RG\alpha C(X) \ni x \in M \subset G$; and hence $rg\alpha\overline{\{x\}} \subset G$ which implies $rg\alpha\overline{\{x\}} \cap F = \phi$.

(vi)$\Rightarrow$(i): Let $x \in G \in RG\alpha O(X)$. Then $x \notin (X - G)$, which is a $rg\alpha-$closed set. Therefore by (vi) $rg\alpha\overline{\{x\}} \cap (X - G) = \phi$, which implies that $rg\alpha\overline{\{x\}} \subseteq G$. Thus X is $rg\alpha R_0$ space.

Theorem 4.5. Let f be a $rg\alpha-$closed one-one function. If X is weakly $rg\alpha - R_0$, then so is Y.

Theorem 4.6. If X is weakly $rg\alpha - R_0$, then for every space $Y, X \times Y$ is weakly $rg\alpha - R_0$.

Proof. $\cap rg\alpha\overline{\{(x, y)\}} \subseteq \cap\{rg\alpha\overline{\{x\}} \times rg\alpha\overline{\{y\}}\} = \cap[rg\alpha\overline{\{x\}}] \times [rg\alpha\overline{\{y\}}] \subseteq \phi \times Y = \phi$. Hence $X \times Y$ is $rg\alpha R_0$.

Corollary 4.1. (i) If X and Y are weakly $rg\alpha R_0$, then $X \times Y$ is weakly $rg\alpha R_0$.

(ii) If X and Y are (weakly-)R_0, then $X \times Y$ is weakly $rg\alpha R_0$.

(iii) If $X$and Y are $rg\alpha R_0$, then $X \times Y$ is weakly $rg\alpha R_0$.

(iv) If X is $rg\alpha R_0$ and Y are weakly R_0, then $X \times Y$ is weakly $rg\alpha R_0$.

Theorem 4.7. X is $rg\alpha R_0$ iff for any $x, y \in X, rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}} \Rightarrow rg\alpha\overline{\{x\}} \cap rg\alpha\overline{\{y\}} = \phi$.

Proof. Let X is $rg\alpha R_0$ and $x, y \in X \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. Then, $\exists z \in rg\alpha\overline{\{x\}} \ni z \notin rg\alpha\overline{\{y\}}(or z \in rg\alpha\overline{\{y\}}) \ni z \notin rg\alpha\overline{\{x\}}$. There exists $V \in RG\alpha O(X) \ni y \notin V$ and $z \in V$; hence $x \in V$. Therefore, $x \notin rg\alpha\overline{\{y\}}$. Thus $x \in [G\overline{y}]^c \in RG\alpha O(X)$, which implies $rg\alpha\overline{\{x\}} \subset [rg\alpha\overline{\{y\}}]^c$ and $rg\alpha\overline{\{x\}} \cap rg\alpha\overline{\{y\}} = \phi$. the proof for otherwise is similar.

Sufficiency: Let $x \in V \in RG\alpha O(X)$. We show that $rg\alpha\overline{\{x\}} \subset V$. Let $y \notin V$, i.e., $y \in [V]^c$. Then $x \neq y$ and $x \notin rg\alpha\overline{\{y\}}$. Hence $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. By assumption, $rg\alpha\overline{\{x\}} \cap rg\alpha\overline{\{y\}} = \phi$. Hence $y \notin rg\alpha\overline{\{x\}}$. Therefore $rg\alpha\overline{\{x\}} \subset V$.

Theorem 4.8. X is $rg\alpha R_0$ iff for any points $x, y \in X, Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\} \Rightarrow Ker_{rg\alpha}\{x\} \cap Ker_{rg\alpha}\{y\} = \phi$.

Proof. Suppose X is $rg\alpha R_0$. Thus by Lemma 4.3, for any $x, y \in X$ if $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\}$ then $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. Assume that $z \in Ker_{rg\alpha}\{x\} \cap Ker_{rg\alpha}\{y\}$. By $z \in Ker_{rg\alpha}\{x\}$ and Lemma 4.2, it follows that $x \in rg\alpha\overline{\{z\}}$. Since $x \in rg\alpha\overline{\{z\}}, rg\alpha\overline{\{x\}} = rg\alpha\overline{\{z\}}$. Similarly, we have $rg\alpha\overline{\{y\}} = rg\alpha\overline{\{z\}} = rg\alpha\overline{\{x\}}$. This is a contradiction. Therefore, we have $Ker_{rg\alpha}\{x\} \cap Ker_{rg\alpha}\{y\} = \phi$.

Conversely, let $x, y \in X, \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$, then by Lemma 4.3, $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\}$. Hence by hypothesis $Ker_{rg\alpha}\{x\} \cap Ker_{rg\alpha}\{y\} = \phi$, which implies $rg\alpha\overline{\{x\}} \cap rg\alpha\overline{\{y\}} = \phi$. Because $z \in rg\alpha\overline{\{x\}}$ implies that $x \in Ker_{rg\alpha}\{z\}$ and therefore $Ker_{rg\alpha}\{x\} \cap Ker_{rg\alpha}\{z\} \neq \phi$. Therefore by Theorem 4.7 X is a $rg\alpha R_0$ space.

Theorem 4.9. The following properties are equivalent:

(1) X is a $rg\alpha - R_0$ space.

(2) For any $A \neq \phi$ and $G \in RG\alpha O(X) \ni A \cap G \neq \phi \exists F \in RG\alpha C(X) \ni A \cap F \neq \phi$ and $F \subset G$.

Proof. (1)⇒(2): Let $A \neq \phi$ and $G \in RG\alpha O(X) \ni A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in RG\alpha O(X), rg\alpha\overline{\{x\}} \subset G$. Set $F = rg\alpha\overline{\{x\}}$, then $F \in RG\alpha C(X)$, $F \subset G$ and $A \cap F \neq \phi$.

(2)⇒(1): Let $G \in RG\alpha O(X)$ and $x \in G$. By (2) $rg\alpha\overline{\{x\}} \subset G$. Hence X is $rg\alpha - R_0$.

Theorem 4.10. The following properties are equivalent:

(1) X is a $rg\alpha - R_0$ space;

(2) $x \in rg\alpha\overline{\{y\}}$ iff $y \in rg\alpha\overline{\{x\}}$, for any points x and y in X.

Proof. (1)⇒(2): Assume X is $rg\alpha R_0$. Let $x \in rg\alpha\overline{\{y\}}$ and D be any $rg\alpha-$open set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every $rg\alpha-$open set which contain y contains x. Hence $y \in rg\alpha\overline{\{x\}}$.

(2)⇒(1): Let U be a $rg\alpha-$open set and $x \in U$. If $y \notin U$, then $x \notin rg\alpha\overline{\{y\}}$ and hence $y \notin rg\alpha\overline{\{x\}}$. This implies that $rg\alpha\overline{\{x\}} \subset U$. Hence X is $rg\alpha R_0$.

Theorem 4.11. The following properties are equivalent:

(1) X is a $rg\alpha R_0$ space;

(2) If F is $rg\alpha-$closed, then $F = Ker_{rg\alpha}(F)$;

(3) If F is $rg\alpha-$closed and $x \in F$, then $Ker_{rg\alpha}\{x\} \subseteq F$;

(4) If $x \in X$, then $Ker_{rg\alpha}\{x\} \subset rg\alpha\overline{\{x\}}$.

Proof. (1)⇒(2): Let $x \notin F \in RG\alpha C(X) \Rightarrow (X - F) \in RG\alpha O(X)$ and contains x. For X is $rg\alpha R_0, rg\alpha\overline{(\{x\})} \subset (X - F)$. Thus $rg\alpha\overline{(\{x\})} \cap F = \phi$ and $x \notin Ker_{rg\alpha}(F)$. Hence $Ker_{rg\alpha}(F) = F$.

(2)⇒(3): $A \subset B \Rightarrow Ker_{rg\alpha}(A) \subset Ker_{rg\alpha}(B)$. Therefore, by (2), $Ker_{rg\alpha}\{x\} \subset Ker_{rg\alpha}(F) = F$.

(3)⇒(4): Since $x \in rg\alpha\overline{\{x\}}$ and $rg\alpha\overline{\{x\}}$ is $rg\alpha-$closed, by (3), $Ker_{rg\alpha}\{x\} \subset rg\alpha\overline{\{x\}}$.

(4)⇒(1): Let $x \in rg\alpha\overline{\{y\}}$. Then by Lemma 4.2, $y \in Ker_{rg\alpha}\{x\}$. Since $x \in rg\alpha\overline{\{x\}}$ and $rg\alpha\overline{\{x\}}$ is $rg\alpha-$closed, by (4), we obtain $y \in Ker_{rg\alpha}\{x\} \subseteq rg\alpha\overline{\{x\}}$. Therefore $x \in rg\alpha\overline{\{y\}}$ implies $y \in rg\alpha\overline{\{x\}}$.

The converse is obvious and X is $rg\alpha R_0$.

Corollary 4.2. The following properties are equivalent:

(1) X is $rg\alpha R_0$.

(2) $rg\alpha\overline{\{x\}} = Ker_{rg\alpha}\{x\} \forall x \in X$.

Proof. Straight forward from Theorem 4.4 and 4.11.

Recall that a filterbase F is called $rg\alpha-$convergent to a point x in X, if for any $rg\alpha-$open set U of X containing x, there exists $B \in F$ such that $B \subset U$.

Lemma 4.5. Let x and y be any two points in X such that every net in X $rg\alpha-$converging to y $rg\alpha-$converges to x. Then $x \in rg\alpha\overline{\{y\}}$.

Theorem 4.12. The following statements are equivalent:

(1) X is a $rg\alpha R_0$ space;

(2) If $x, y \in X$, then $y \in rg\alpha\overline{\{x\}}$ iff every net in X $rg\alpha-$converging to y $rg\alpha-$converges to x.

Proof. (1)⇒(2): Let $x, y \in X \ni y \in rg\alpha\overline{\{x\}}$. Suppose that $\{x_\alpha\}_{\alpha\in\Lambda}$ is a net in $X \ni \{x_\alpha\}_{\alpha\in\Lambda} rg\alpha-$converges to y. Since $y \in rg\alpha\overline{\{x\}}$, by Theorem 4.7, we have $rg\alpha\overline{\{x\}} = rg\alpha\overline{\{y\}}$. Therefore $x \in rg\alpha\overline{\{y\}}$. This means that $\{x_\alpha\}_{\alpha\in\Lambda} rg\alpha-$converges to x.

Conversely, let x, $y \in X$ such that every net in X $rg\alpha-$converging to y $rg\alpha-$converges to x. Then $x \in rg\{y\}^-$ by Theorem 4.4. By Theorem 4.7, we have $rg\alpha\overline{\{x\}} = rg\alpha\overline{\{y\}}$. Therefore $y \in rg\alpha\overline{\{x\}}$.

(2)⇒(1): Let $x, y \in X \ni rg\alpha\overline{\{x\}} \cap rg\alpha\overline{\{y\}} \neq \phi$. Let $z \in rg\alpha\overline{\{x\}} \cap rg\alpha\overline{\{y\}}$. So ∃ a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $rg\alpha\overline{\{x\}} \ni \{x_\alpha\}_{\alpha\in\Lambda} rg\alpha-$converges to z. Since $z \in rg\alpha\overline{\{y\}}$, then $\{x_\alpha\}_{\alpha\in\Lambda} rg\alpha-$conve -rges to y. It follows that $y \in rg\alpha\overline{\{x\}}$. Similarly we obtain $x \in rg\alpha\overline{\{y\}}$. Therefore $rg\alpha\overline{\{x\}} = rg\alpha\overline{\{y\}}$. Hence, X is $rg\alpha R_0$.

Theorem 4.13. (i) Every subspace of $rg\alpha R_1$ space is again $rg\alpha R_1$.

(ii) Product of any two $rg\alpha R_1$ spaces is again $rg\alpha R_1$.

Theorem 4.14. X is $rg\alpha R_1$ iff given $x \neq y \in X, rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$.

Theorem 4.15. Every $rg\alpha_2$ space is $rg\alpha R_1$.

The converse is not true. However, we have the following result.

Theorem 4.16. Every $rg\alpha_1$, $rg\alpha R_1$ space is $rg\alpha_2$.

Proof. Let $x \neq y \in X$. Since X is $rg\alpha_1$, $\{x\}$ and $\{y\}$ are $rg\alpha-$closed sets $\ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. Since X is $rg\alpha R_1$, ∃ disjoint $U; V \in RG\alpha O(X) \ni x \in U,\ y \in V$. Hence X is $rg\alpha_2$.

Corollary 4.3. X is $rg\alpha_2$ iff it is $rg\alpha R_1$ and $rg\alpha_1$.

Theorem 4.17. The following are equivalent

(i) X is $rg\alpha - R_1$.

(ii) $\cap rg\alpha\overline{\{x\}} = \{x\}$.

(iii) For any $x \in X$, intersection of all $rg\alpha-$neighborhoods of x is $\{x\}$.

Proof. (i)⇒(ii): Let $y \neq x \in X \ni y \in rg\alpha\overline{\{x\}}$. Since X is $rg\alpha R_1$, $\exists U \in RG\alpha O(X) \ni y \in U, x \notin U$ or x∈U, $y \notin U$. In either case $y \notin rg\alpha\overline{\{x\}}$. Hence $\cap rg\alpha\overline{\{x\}} = \{x\}$.

(ii)⇒(iii): If $y \neq x \in X$, then x$\notin \cap rg\alpha\overline{\{y\}}$, so there is a $rg\alpha-$open set containing x but not y. Therefore y does not belong to the intersection of all $rg\alpha-$neighborhoods of x. Hence intersection of all $rg\alpha-$neighborhoods of x is $\{x\}$.

(iii)⇒(i): Let $x \neq y \in X$. By hypothesis, y does not belong to the intersection of all $rg\alpha-$neighborhoods of x and x does not belong to the intersection of all $rg\alpha-$neighborhoods of y, which implies $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$. Hence X is $rg\alpha - R_1$.

Theorem 4.18. The following are equivalent:

(i) X is $rg\alpha - R_1$.

(ii) For each pair $x, y \in X \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$, ∃ a $rg\alpha-$open, $rg\alpha-$closed set $V \ni x \in V$ and $y \notin V$, and

(iii) For each pair $x, y \in X \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}, \exists f : X \to [0, 1] \ni f(x) = 0$ *and* $f(C) = 1$ and f is $rg\alpha-$continuous.

Proof. (i)⇒(ii): Let $x, y \in X \ni rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$, ∃ disjoint $U; W \in RG\alpha O(X) \ni rg\alpha\overline{\{x\}} \subset U$ and $rg\alpha\overline{\{y\}} \subset W$and $V = rg\alpha\overline{(U)}$ is $rg\alpha-$open and $rg\alpha-$closed such that $x \in V$ and $y \notin V$.

(ii)⇒(iii): Let $x, y \in X$ such that $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$, and let V be $rg\alpha-$open and $rg\alpha-$closed such that $x \in V$ and $y \notin V$. Then $f : X \to [0, 1]$ defined by $f(z) = 0$ if $z \in V$

and $f(z) = 1$ if $z \notin V$ satisfied the desired properties.

(iii)$\Rightarrow$(i): Let $x, y \in X$ such that $rg\alpha\overline{\{x\}} \neq rg\alpha\overline{\{y\}}$, let $f\colon X \to [0,1]$ be $rg\alpha-$continuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$ are disjoint $rg\alpha-$open and $rg\alpha-$closed sets in X, such that $rg\alpha\overline{\{x\}} \subset U$ and $rg\alpha\overline{\{y\}} \subset V$.

Theorem 4.19: (i) If X is $rg\alpha - R_1$, then X is $rg\alpha - R_0$.

(ii) X is $rg\alpha - R_1$ iff for $x, y \in X$, $Ker_{rg\alpha}\{x\} \neq Ker_{rg\alpha}\{y\}$, $\exists$ disjoint $U; V \in RG\alpha O(X) \ni rg\alpha\overline{\{x\}} \subset U$ and $rg\alpha\overline{\{y\}} \subset V$.

§5. $rg\alpha - C_i$ and $rg\alpha - D_i$ spaces, i = 0,1,2

Definition 5.1. X is said to be a

(i) $rg\alpha - C_0$ space if for each pair of distinct points x, y of X there exists an $rg\alpha-$open set G whose closure contains either x or y.

(ii) $rg\alpha - C_1$ space if for each pair of distinct points x, y of X there exists $rg\alpha-$open sets G and H such that $\overline{G}$ containing x but not y and $\overline{H}$ containing y but not x.

(iii) $rg\alpha - C_2$ space if for each pair of distinct points x, y of X there exists disjoint $rg\alpha-$open sets G and H such that $\overline{G}$ containing x but not y and $\overline{H}$ containing y but not x.

Note. $rg\alpha - C_2 \Rightarrow rg\alpha - C_1 \Rightarrow rg\alpha - C_0$ but converse need not be true in general as shown by the following example.

Example 5.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then X is $rg\alpha - C_i$, $i = 1, 2$.

Theorem 5.1. The following statements are true:

(i) Every subspace of $rg\alpha - C_i$ space is $rg\alpha - C_i$.

(ii) Every $rg\alpha_i$ spaces is $rg\alpha - C_i$.

(iii) Product of $rg\alpha - C_i$ spaces are $rg\alpha - C_i$.

(iv) If $f\colon X \to Y$ is $rg\alpha-$continuous and Y is C_i then X is $rg\alpha - C_i$.

(v) If $f\colon X \to Y$ is $rg\alpha-$irresolute and Y is $rg\alpha - C_i$ then X is $rg\alpha - C_i$.

(vi) Let (X, τ) be any $rg\alpha - C_i$ space and $A \subset X$ then A is $rg\alpha - C_i$ iff $(A, \tau_{/A})$ is $rg\alpha - C_i$.

(vii) If X is $rg\alpha - C_1$ then each singleton set is $rg\alpha-$closed.

(viii) In an $rg\alpha - C_1$ space disjoint points of X has disjoint $rg\alpha-$ closures.

Definition 5.2. $A \subset X$ is called a $rg\alpha$Difference (Shortly $rg\alpha$D-set) set if there are two U, $V \in RG\alpha O(X, \tau)$ such that $U \neq X$ and $A = U - V$.

Clearly every $rg\alpha-$open set U different from X is a $rg\alpha$D-set if $A = U$ and $V = \phi$.

Definition 5.3. X is said to be a

(i) $rg\alpha - D_0$ if for any pair of distinct points x and y of X there exist a $rg\alpha$D-set in X containing x but not y or a $rg\alpha$D-set in X containing y but not x.

(ii) $rg\alpha - D_1$ if for any pair of distinct points x and y in X there exist a $rg\alpha$D-set of X containing x but not y and a $rg\alpha$D-set in X containing y but not x.

(iii) $rg\alpha - D_2$ if for any pair of distinct points x and y of X there exists disjoint $rg\alpha$D-sets G and H in X containing x and y, respectively.

Example 5.2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then X is $rg\alpha - D_i$, $i = 0, 1, 2$.

Remark 5.2. (i) If X is $r - T_i$, then it is $rg\alpha_i$, $i = 0, 1, 2$ and converse is false.

(ii) If X is $rg\alpha_i$, then it is $rg\alpha_{i-1}$, $i = 1, 2$.

(iii) If X is $rg\alpha_i$, then it is $rg\alpha - D_i$, $i = 0, 1, 2$.

(iv) If X is $rg\alpha - D_i$, then it is $rg\alpha - D_{i-1}$, $i = 1, 2$.

Theorem 5.2. The following statements are true:

(i) X is $rg\alpha - D_0$ iff it is $rg\alpha_0$.

(ii) X is $rg\alpha - D_1$ iff it is $rg\alpha - D_2$.

Proof. (i) The sufficiency is stated in Remark 5.1(iii).

To prove necessity, let X be $rg\alpha - D_0$. Then for each distinct pair of points x, $y \in X$, at least one of x, y, say x, belongs to a $rg\alpha$D-set G but $y \notin G$. Let $G = U_1 - U_2$ where $U_1 \neq X$ and $U_1, U_2 \in RG\alpha O(X)$. Then $x \in U_1$ and for $y \notin G$ we have two cases:

(a) $y \notin U_1$; (b) $y \in U_1$ and $y \notin U_2$.

In case (a), $x \in U_1$ but $y \notin U_1$;

In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is $rg\alpha_0$.

(ii) Sufficiency. Remark 5.1(iv).

Necessity. Suppose X is $rg\alpha - D_1$. Then for each $x \neq y \in X$, we have $rg\alpha$D-sets $G_1, G_2 \ni x \in G_1; y \notin G_1; y \in G_2, x \notin G_2$. Let $G_1 = U_1 - U_2, G_2 = U_3 - U_4$. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1 - U_2$, it follows that $x \in U_1 - (U_2 \cup U_3)$ and by $y \in U_3 - U_4$, we have $y \in U_3 - (U_1 \cup U_4)$. Therefore $(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4) = \phi$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2, y \in U_2 . (U_1 - U_2) \cap U_2 = \phi$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4, x \in U_4 . (U_3 - U_4) \cap U_4 = \phi$.

Therefore X is $rg\alpha - D_2$.

Corollary 5.1. If X is $rg\alpha - D_1$, then it is $rg\alpha_0$.

Proof. Remark 5.1(iv) and Theorem 5.1(vii).

Definition 5.4. A point $x \in X$ which has X as the unique $rg\alpha-$neighborhood is called rgc.c point.

Theorem 5.3. For an $rg\alpha_0$ space X the following are equivalent:

(1) X is $rg\alpha - D_1$;

(2) X has no rgc.c point.

Proof. (1)⇒(2): Since X is $rg\alpha - D_1$, then each point x of X is contained in a $rg\alpha$D-set $O = U - V$ and thus in U. By definition $U \neq X$. This implies that x is not a rgc.c point.

(2)⇒(1): If X is $rg\alpha_0$, then for each $x \neq y \in X$, at least one of them, x (say) has a $rg\alpha-$neighborhood U containing x and not y. Thus $U \neq X$ is a $rg\alpha$D-set. If X has no rgc.c point, then y is not a rgc.c point. This means $\exists$ $rg\alpha-$neighborhood V of y) $\ni V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is a $rg\alpha$D-set. Hence X is $rg\alpha - D_1$.

Remark 5.2. It is clear that an $rg\alpha_0$ space X is not $rg\alpha - D_1$ iff there is a unique $rg\alpha-$c.c point in X. It is unique because if x and y are both rgc.c point in X, then at least one of them say x has a $rg\alpha-$neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

Definition 5.5. X is $rg\alpha-$symmetric if for x and y in X, $x \in rg\alpha\overline{\{y\}}$ implies $y \in rg\alpha\overline{\{x\}}$.

Theorem 5.4. X is $rg\alpha-$symmetric iff {x} is $rg\alpha$-closed for each $x \in X$.

Proof. Assume that $x \in rg\alpha\overline{\{y\}}$ but $y \notin rg\alpha\overline{\{x\}}$. This means that $[rg\alpha\overline{\{x\}}]^c$ contains y. This implies that $rg\alpha\overline{\{y\}}$ is a subset of $[rg\alpha\overline{\{x\}}]^c$. Now $[rg\alpha\overline{\{x\}}]^c$ contains x which is a contradiction. Conversely, suppose that $\{x\} \subset E \in RG\alpha O(X)$ but $rg\alpha\overline{\{x\}}$ is not a subset of E. This means that $rg\alpha\overline{\{x\}}$ and E^c are not disjoint. Let y belongs to their intersection. Now we have $x \in rg\alpha\overline{\{y\}}$ which is a subset of E^c and $x \notin E$. But this is a contradiction.

Corollary 5.2. If X is a $rg\alpha_1$, then it is $rg\alpha-$symmetric.

Proof. In a $rg\alpha_1$ space, singleton sets are $rg\alpha-$closed (Theorem 3.2(ii)) and therefore $rg\alpha-$closed (Remark 5.3). By Theorem 5.4, the space is $rg\alpha-$symmetric.

Corollary 5.3. The following are equivalent:

(1) X is $rg\alpha-$symmetric and $rg\alpha_0$;

(2) X is $rg\alpha_1$.

Proof. By Corollary 5.2 and Remark 5.1 it suffices to prove only (1) $\rightarrow$ (2). Let $x \neq y$ and by $rg\alpha_0$, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in RG\alpha O(X)$. Then $x \notin rg\{y\}^-$ and hence $y \notin rg\{x\}^-$. There exists $G_2 \in RG\alpha O(X) \ni y \in G_2 \subset \{x\}^c$ and X is a $rg\alpha_1$ space.

Theorem 5.5. For an $rg\alpha-$symmetric space X the following are equivalent: (1) X is $rg\alpha_0$; (2) X is $rg\alpha - D_1$; (3) X is $rg\alpha_1$.

Proof. (1)$\Rightarrow$(3): Corollary 5.3 and (3) $\Rightarrow$ (2) $\Rightarrow$ (1): Remark 5.1.

Theorem 5.6. If *f* is a $rg\alpha-$irresolute surjection and E is $rg\alpha$D-set in Y, then $f^{-1}(E)$ is $rg\alpha$D-set in X.

Proof. Let E be a $rg\alpha$D-set in Y. Then there are $rg\alpha-$open sets U_1 and U_2 in Y such that $E = U_1 - U_2$ and $U_1 \neq Y$. By the $rg\alpha-$irresoluteness of $f, f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $rg\alpha-$open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$ Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a $rg\alpha-$ D-set.

Theorem 5.7. If Y is $rg\alpha - D_1$ and *f* is bijective, $rg\alpha-$irresolute, then X is $rg\alpha - D_1$.

Proof. Suppose that Y is a $rg\alpha - D_1$ space. Let x and y be any pair of distinct points in X. Since f is injective and Y is $rg\alpha - D_1$, there exist $rg\alpha-$D-sets G_z and G_y of Y containing f(x) and f(y) respectively, such that $f(y) \notin G_z$ and $f(x) \notin G_y$. By Theorem 5.6, $f^{-1}(G_z)$ and $f^{-1}(G_y)$ are $rg\alpha-$D-sets in X containing x and y, respectively. Therefore X is a $rg\alpha - D_1$ space.

Theorem 5.8. X is $rg\alpha - D_1$ iff for each pair $x \neq y \in X$, $\exists$ a $rg\alpha-$irresolute surjective function *f*, where Y is an $rg\alpha - D_1$ space $\ni f(x) \neq f(y)$.

Proof. Necessity. For every $x \neq y \in X$, it suffices to take the identity function on X. Sufficiency. Let $x \neq y \in X$. By hypothesis, $\exists$ a $rg\alpha-$irresolute, surjective function f from X onto a $rg\alpha - D_1$ space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint $rg\alpha-$D-sets $G_x; G_y \subset Y \ni f(x) \in G_x$ and $f(y) \in G_y$. Since f is $rg\alpha-$irresolute and surjective, by Theorem 5.6, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $rg\alpha-$D-sets in X containing x and y respectively. Therefore X is $rg\alpha - D_1$ space.

Corollary 5.4. Let $\{X_\alpha/\alpha \in I\}$ be any family of topological spaces. If X_α is $rg\alpha - D_1$ for each $\alpha \in I$, then the product ΠX_α is $rg\alpha - D_1$.

Proof. Let (x_α) and (y_α) be any pair of distinct points in ΠX_α. Then there exists an index $\beta \in I$ such that $x_\beta \neq y_\beta$. The natural projection $P_\beta : \Pi X_\alpha \rightarrow X_\beta$ is almost continuous and almost open and $P_\beta((x_\alpha)) = P_\beta((y_\alpha))$. Since X_β is $rg\alpha - D_1$, ΠX_α is $rg\alpha - D_1$.

Conclusion. In this paper we defined new separation axioms using $rg\alpha-$open sets and studied their interrelations with other separation axioms.

Acknowledgment

Author is thankful to the referees for their critical comments to the development of the paper.

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Scientia Magna
Vol. 7 (2011), No. 2, 59-67

Intuitionistic fuzzy resolvable and intuitionistic fuzzy irresolvable spaces

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Abstract In this paper the concepts of intuitionistic fuzzy resolvable, intuitionistic fuzzy irresolvable, intuitionistic fuzzy open hereditarily irresolvable spaces and maximally intuitionistic fuzzy irresolvable spaces are introduced. Also we study several interest properties of the intuitionistic fuzzy open hereditarily irresolvable spaces besides giving characterization of these spaces by means of somewhat intuitionistic fuzzy continuous functions and somewhat intuitionistic fuzzy open functions.

Keywords Intuitionistic fuzzy resolvable, intuitionistic fuzzy irresolvable, intuitionistic fuzzy submaximal, intuitionistic fuzzy open hereditarily irresolvable space, somewhat intuitionistic fuzzy continuous and somewhat intuitionistic fuzzy open functions.
2000 Mathematics Subject Classification: 54A40, 03E72.

§1. Introduction

The fuzzy concept has invaded almost all branches of mathematics ever since the introduction of fuzzy sets by L. A. Zadeh [14]. The theory of fuzzy topological space was introduced and developed by C. L. Chang [7] and since then various notions in classical topology have been extended to fuzzy topological space. The idea of "intuitionistic fuzzy set" was first published by Atanassov [1] and many works by the same author and his colleagues appeared in the literature [2−4]. Later, this concept was generalized to "intuitionistic L - fuzzy sets" by Atanassov and Stoeva [5]. In classical topology the class of somewhat continuous functions was introduced by Karl R. Gentry and Hunghes B. Hoyle III in [11]. We have extended this concepts to fuzzy topological space and in this connection, we have introduced the concept of somewhat fuzzy continuous functions and somewhat fuzzy open hereditarily irresolvable by G. Thangaraj and *G*. Balasubramanian in [12]. The concepts of resolvability and irresolvability in topological spaces was introduced by E. Hewit in [10]. The concept of open hereditarily irresolvable spaces in the classical topology was introduced by A. Geli'kin in [9]. The concept on fuzzy resolvable and fuzzy irresolvable spaces was introduced by G. Thangaraj and G. Balasubramanian in [13]. In this paper the concept of intuitionistic fuzzy resolvable, intuitionistic fuzzy irresolvable, intuitionistic fuzzy open hereditarily irresolvable spaces and maximally intuitionistic fuzzy irresolvable space are introduced. Also we discuss and study several interest properties of the intuitionistic fuzzy open hereditarily irresolvable spaces besides giving characterization of these spaces by means of somewhat intuitionistic fuzzy continuous functions and somewhat intuitionistic fuzzy open functions. Some interesting properties and related examples are given.

§2. Preliminaries

Definition 2.1.[3] Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS for short) A is an object having the form $A = \{\langle x, \mu_A(x), \delta_A(x)\rangle : x \in X\}$ where the function $\mu_A : X \to I$ and $\delta_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership($\delta_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \delta_A(x) \leq 1$ for each $x \in X$.

Definition 2.2.[3] Let X be a nonempty set and the intuitionistic fuzzy sets A and B in the form $A = \{\langle x, \mu_A(x), \delta_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \delta_B(x)\rangle : x \in X\}$. Then

(a) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \delta_A(x) \vee \delta_B(x)\rangle : x \in X\}$;

(b) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \delta_A(x) \wedge \delta_B(x)\rangle : x \in X\}$.

Now we shall define the image and preimage of intuitionistic fuzzy sets. Let X and Y be two nonempty sets and $f : X \to Y$ be a function.

Definition 2.3.[3] **(a)** If $B = \{\langle y, \mu_B(y), \delta_B(y)\rangle : y \in Y\}$ is an intuitionistic fuzzy set in Y, then the preimage of B under f, denoted by $f^{-1}(B)$, is the intuitionistic fuzzy set in X defined by $f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\delta_B)(x)\rangle : x \in X\}$.

(b) If $A = \{\langle x, \lambda_A(x), \vartheta_A(x)\rangle : x \in X\}$ is an intuitionistic fuzzy set in X, then the image of A under f, denoted by $f(A)$, is the intuitionistic fuzzy set in Y defined by $f(A) = \{\langle y, f(\lambda_A)(y), (1 - f(1 - \vartheta_A))(y)\rangle : y \in Y\}$.

Where

$$f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$(1 - f(1 - \vartheta_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \vartheta_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

For the sake of simplicity, let us use the symbol $f_-(\vartheta_A)$ for $1 - f(1 - \vartheta_A)$.

Definition 2.4.[8] Let A be an intuitionistic fuzzy set in intuitionistic fuzzy topological space (X, T). Then

$IFint(A) = \bigcup\{G \mid G$ is an intuitionistic fuzzy open in X and $G \subseteq A\}$ is called an intuitionistic fuzzy interior of A;

$IFcl(A) = \bigcap\{G \mid$G is an intuitionistic fuzzy closed in X and $G \supseteq A\}$ is called an intuitionistic fuzzy closure of A.

Proposition 2.1.[8] Let (X,T) be any intuitionistic fuzzy topological space. Let A be an intuitionistic fuzzy sets in (X,T). Then the intuitionistic fuzzy closure operator satisfy the following properties:

(i) $1 - IFcl(A) = IFint(1 - A)$;

(ii) $1 - IFint(A) = IFcl(1 - A)$.

Definition 2.5.[12] A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy dense if there exists no fuzzy closed set μ in (X,T) such that $\lambda < \mu < 1$.

Definition 2.6.[13] Let (X,T) be a fuzzy topological space. (X,T) is called fuzzy resolvable if there exists a fuzzy dense set λ in (X,T) such that $cl(1-\lambda) = 1$. Otherwise (X,T) is called fuzzy irresolvable.

Definition 2.7.[6] A fuzzy topological space (X,T) is called a fuzzy submaximal space if for each fuzzy set λ in (X,T) such that $cl(\lambda) = 1$, then $\lambda \in T$.

Definition 2.8.[13] Let (X,T) be a fuzzy topological space. (X,T) is called fuzzy open hereditarily irresolvable if $intcl(\lambda) \neq 0$ then $int\lambda \neq 0$ for any fuzzy set λ in (X,T).

Definition 2.9.[12] Let (X,T) and (Y,S) be any two fuzzy topological spaces. A function $f : (X,T) \to (Y,S)$ is called somewhat fuzzy continuous if $\lambda \in S$ and $f^{-}(\lambda) \neq 0 \implies$ there exists $\mu \in T$ such that $\mu \neq 0$ and $\mu \leq f^{-}(\lambda)$.

Definition 2.10.[12] Let (X,T) and (Y,S) be any two fuzzy topological spaces. A function $f : (X,T) \to (Y,S)$ is called somewhat fuzzy open if $\lambda \in T$ and $\lambda \neq 0 \implies$ there exists $\mu \in S$ such that $\mu \neq 0$ and $\mu \leq f(\lambda)$.

§3. Intuitionistic fuzzy resolvable and intuitionistic fuzzy irresolvable

Definition 3.1. An intuitionistic fuzzy set A in intuitionistic fuzzy topological space (X,T) is called intuitionistic fuzzy dense if there exists no intuitionistic fuzzy closed set B in (X,T) such that $A \subset B \subset 1_{\sim}$.

Definition 3.2. Let (X,T) be an intuitionistic fuzzy topological space. (X,T) is called intuitionistic fuzzy resolvable if there exists a intuitionistic fuzzy dense set A in (X,T) such that $IFcl(1-A) = 1_{\sim}$. Otherwise, (X,T) is called intuitionistic fuzzy irresolvable.

Example 3.1. Let $X = \{a,b,c\}$. Define the intuitionistic fuzzy sets A, B and C as follows,

$$A = \langle x, (\tfrac{a}{0.6}, \tfrac{b}{0.6}, \tfrac{c}{0.5}), (\tfrac{a}{0.3}, \tfrac{b}{0.3}, \tfrac{c}{0.5}) \rangle,$$

$$B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.5}), (\tfrac{a}{0.5}, \tfrac{b}{0.5}, \tfrac{c}{0.4}) \rangle,$$

and

$$C = \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.3}, \tfrac{c}{0.4}), (\tfrac{a}{0.7}, \tfrac{b}{0.7}, \tfrac{c}{0.6}) \rangle.$$

Clearly $T = \{0_{\sim}, 1_{\sim}, A\}$ is an intuitionistic fuzzy topology on X. Thus (X,T) is an intuitionistic fuzzy topological space. Now $IFint(B) = 0_{\sim}$, $IFint(C) = 0_{\sim}$, $IFint(1-B) = 0_{\sim}$, $IFint(1-C) = A$, $IFcl(B) = 1_{\sim}$, $IFcl(C) = 1_{\sim}$, $IFcl(1-B) = 1_{\sim}$ and $IFcl(1-C) = 1-$

A. Hence there exists a intuitionistic fuzzy dense set B in (X,T), such that $IFint(1-B) = 1_\sim$. Hence the intuitionistic fuzzy topological space (X,T) is called a intuitionistic fuzzy resolvable.

Definition 3.2. Let $X = \{a,b,c\}$. Define the intuitionistic fuzzy sets A, B and C as follows,

$$A = \langle x, (\tfrac{a}{0.6}, \tfrac{b}{0.5}, \tfrac{c}{0.5}), (\tfrac{a}{0.4}, \tfrac{b}{0.5}, \tfrac{c}{0.5})\rangle,$$

$$B = \langle x, (\tfrac{a}{0.7}, \tfrac{b}{0.8}, \tfrac{c}{0.6}), (\tfrac{a}{0.3}, \tfrac{b}{0.1}, \tfrac{c}{0.3})\rangle,$$

and

$$C = \langle x, (\tfrac{a}{0.6}, \tfrac{b}{0.5}, \tfrac{c}{0.5}), (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.4})\rangle.$$

Clearly $T = \{0_\sim, 1_\sim, A\}$ is an intuitionistic fuzzy topology on X. Thus (X,T) is an intuitionistic fuzzy topological space. Now $IFint(B) = A$, $IFint(C) = A$, $IFcl(B) = 1_\sim$, $IFcl(C) = 1_\sim$ and $IFcl(B) = 1_\sim$. Thus B and C are intuitionistic fuzzy dense set in (X,T), such that $IFcl(1-B) = 1-A$ and $IFcl(1-C) = 1-A$. Hence the intuitionistic fuzzy topological space (X,T) is called a intuitionistic fuzzy irresolvable.

Proposition 3.1. Let (X,T) be an intuitionistic fuzzy topological space. (X,T) is an intuitionistic fuzzy resolvable space iff (X,T) has a pair of intuitionistic fuzzy dense set A_1 and A_2 such that $A_1 \subseteq 1 - A_2$.

Proof. Let (X,T) be an intuitionistic fuzzy topological space and (X,T) is an intuitionistic fuzzy resolvable space. Suppose that for all intuitionistic fuzzy dense sets A_i and A_j, we have $A_i \nsubseteq 1 - A_j$. Then $A_i \supset 1 - A_j$. Then $IFcl(A_i) \supset IFcl(1 - A_j)$ which implies that $1_\sim \supset IFcl(1 - A_j)$ then $IFcl(1 - A_j) \neq 1_\sim$. Also $A_j \supset 1 - A_i$ then $IFcl(A_j) \supset IFcl(1 - A_i)$ which implies that $1_\sim \supset IFcl(1 - A_i)$. Then $IFcl(1 - A_i) \neq 1_\sim$. Hence $IFcl(A_i) = 1_\sim$, but $IFcl(1 - A_i) \neq 1_\sim$ for all intuitionistic fuzzy set A_i in (X,T). Which is a contradiction. Hence (X,T) has a pair of intuitionistic fuzzy dense set A_1 and A_2 such that $A_1 \subseteq 1 - A_2$.

Converse, suppose that the intuitionistic fuzzy topological space (X,T) has a pair of intuitionistic fuzzy dense set A_1 and A_2, such that $A_1 \subseteq 1 - A_2$. Suppose that (X,T) is a intuitionisic fuzzy irresolvable space. Then for all intuitionistic fuzzy dense set A_1 and A_2 in (X,T), we have $IFcl(1 - A_1) \neq 1_\sim$. Then $IFcl(1 - A_2) \neq 1_\sim$ implies that there exists a intuitionistic fuzzy closed set B in (X,T), such that $1 - A_2 \subset B \subset 1_\sim$. Then $A_1 \subseteq 1 - A_2 \subset B \subset 1_\sim$ implies that $A_1 \subset B \subset 1_\sim$. Which is a contradiction. Hence (X,T) is a intuitionistic fuzzy resolvable space.

Proposition 3.2. If (X,T) is intuitionistic fuzzy irresolvable iff $IFint(A) \neq 0_\sim$ for all intuitionistic dense set A in (X,T).

Proof. Since (X,T) is an intuitionistic fuzzy irresolvable space, for all intuitionistic fuzzy dense set A in (X,T), $IFcl(1 - A) \neq 1_\sim$. Then $1 - IFint(A) \neq 1_\sim$, which implies $IFint(A) \neq 0_\sim$.

Conversely $IFint(A) \neq 0_\sim$, for all intuitionistic fuzzy dense set A in (X,T). Suppose that (X,T) is intuitionistic fuzzy resolvable. Then there exists a intuitionistic fuzzy dense set A in (X,T), such that $IFcl(1 - A) = 1_\sim$ implies that $1 - IFint(A) = 1_\sim$, implies $IFint(A) = 0_\sim$. Which is a contradiction. Hence (X,T) is intuitionistic fuzzy irresolvable space.

Definition 3.3. An intuitionistic fuzzy topological space (X, T) is called a intuitionistic fuzzy submaximal space if each intuitionistic fuzzy set A in (X, T) such that $IFcl(A) = 1_\sim$, then $A \in T$.

Proposition 3.3. If the intuitionistic fuzzy topological space (X, T) is intuitionistic fuzzy submaximal, then (X, T) is intuitionistic fuzzy irresolvable.

Proof. Let (X, T) be a intuitionistic fuzzy submaximal space. Assume that (X, T) is a intuitionistic fuzzy resolvable space. Let A be a intuitionistic fuzzy dense set in (X, T). Then $IFcl(1 - A) = 1_\sim$. Hence $1 - IFint(A) = 1_\sim$, which implies that $IFint(A) = 0_\sim$. Then $A \notin T$. Which is a contradiction to intuitionistic fuzzy submaximal space of (X, T). Hence (X, T) is intuitionistic fuzzy irresolvable space.

The converse Proposition 3.3 is not true. See Example 3.2.

Definition 3.4. An intuitionistic fuzzy topological space (X, T) is called a maximal intuitionistic fuzzy irresolvable space if (X, T) is intuitionistic fuzzy irresolvable and every intuitionistic fuzzy dense set A of (X, T) is intuitionistic fuzzy open.

Example 3.3. Let $X = \{a, b, c\}$. Define the intuitionistic fuzzy sets A, B, $A \cap B$ and $A \cup B$ as follows,

$$A = \langle x, (\tfrac{a}{0.5}, \tfrac{b}{0.4}, \tfrac{c}{0.5}), (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.4}) \rangle,$$

$$B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.5}, \tfrac{c}{0.5}), (\tfrac{a}{0.5}, \tfrac{b}{0.5}, \tfrac{c}{0.5}) \rangle,$$

$$A \cap B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.5}), (\tfrac{a}{0.5}, \tfrac{b}{0.5}, \tfrac{c}{0.5}) \rangle,$$

and

$$A \cup B = \langle x, (\tfrac{a}{0.5}, \tfrac{b}{0.5}, \tfrac{c}{0.5}), (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.4}) \rangle.$$

Clearly $T = \{0_\sim, 1_\sim, A, B, A \cap B, A \cup B\}$ is an intuitionistic fuzzy topology on X. Thus (X, T) is an intuitionistic fuzzy topological space. Now $IFint(1 - A) = 0_\sim$, $IFint(1 - B) = \vee\{0_\sim, B, A \cap B\} = B$, $IFint(1 - A \cup B) = 0_\sim$, $IFint(1 - A \cap B) = \vee\{0_\sim, B, A \cap B\} = B$ and $IFcl(A) = 1_\sim$, $IFcl(B) = 1 - B$, $IFcl(A \cup B) = 1_\sim$, $IFcl(A \cap B) = 1 - B$, $IFcl(1 - A \cup B) = \wedge\{1_\sim, 1 - A \cup B, 1 - B, 1 - A \cap B\} = 1 - A \cup B$, $IFcl(1 - A) = \wedge\{1_\sim, 1 - A, 1 - A \cap B\} = 1 - A$, $IFcl(0_\sim) \neq 1_\sim$. Thus intuitionistic fuzzy dense set in (X, T) are $A, A \cup B, 1_\sim$ are intuitionistic fuzzy open in (X, T). Hence (X, T) is an intuitionistic fuzzy irresolvable and every intuitionistic fuzzy dense set of (X, T) is intuitionistic fuzzy open. Therefore (X, T) is a maximally intuitionistic fuzzy irresolvable space.

§4. Intuitionistic fuzzy open hereditarily irresolvable

Definition 4.1. (X, T) is said to be intuitionistic fuzzy open hereditarily irresolvable if $IFint(IFcl(A)) \neq 0_\sim$ then $IFint(A) \neq 0_\sim$ for any intuitionistic fuzzy set A in (X, T).

Example 4.1. Let $X = \{a, b, c\}$. Define the intuitionistic fuzzy sets A_1, A_2 and A_3 as follows,

$$A_1 = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.5}, \tfrac{c}{0.5})\rangle,$$

$$A_2 = \langle x, (\tfrac{a}{0.6}, \tfrac{b}{0.5}, \tfrac{c}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.5}, \tfrac{c}{0.4})\rangle,$$

and

$$A_3 = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.5}), (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.5})\rangle.$$

Clearly $T = \{0_\sim, 1_\sim, A_1, A_2\}$ is an intuitionistic fuzzy topology on X. Thus (X, T) is an intuitionistic fuzzy topological space. Now $IFcl(A_1) = 1 - A_1; IFcl(A_2) = 1_\sim$ and $IFint(A_3) = A_1$. Also $IFint(IFcl(A_1)) = IFint(1 - A_1) = 1 - A_1 \neq 0_\sim$ and $IFint(A_1) = A_1 \neq 0_\sim$, $IFint(IFcl(A_2)) = IFint(1_\sim) = 1_\sim \neq 0_\sim$ and $IFint(A_2) = A_2 \neq 0_\sim$, $IFint(IFcl(A_3)) = IFint(1 - A_1) = 1 - A_1 \neq 0_\sim$ and $IFint(A_3) = A_1 \neq 0_\sim$ and $IFint(IFcl(1 - A_3)) = IFint(1 - A_1) = 1 - A_1 \neq 0_\sim$ and $IFint(1 - A_3) = A_1 \neq 0_\sim$. Hence if $IFint(IFcl(A)) \neq 0_\sim$ then $IFint(A) \neq 0_\sim$ for any non zero intuitionistic fuzzy set A in (X, T). Thus (X, T) is a intuitionistic fuzzy open hereditarily irresolvable space.

Proposition 4.1. Let (X, T) be an intuitionistic fuzzy topological space. If (X, T) is intuitionistic fuzzy open hereditarily irresolvable then (X, T) is intuitionistic fuzzy irresolvable.

Proof. Let A be an intuitionistic fuzzy dense set in (X, T). Then $IFcl(A) = 1_\sim$, which implies that $IFint(IFcl(A)) = 1_\sim \neq 0_\sim$. Since (X, T) is intuitionistic fuzzy open hereditarily irresolvable, we have $IFint(A) \neq 0_\sim$. Therefore by Proposition 3.2, $IFint(A) \neq 0_\sim$ for all intuitionistic fuzzy dense set in (X, T), implies that (X, T) is intuitionistic fuzzy irresolvable. The converse is not true (See Example 4.2).

Example 4.2. Let $X = \{a, b, c\}$. Define the intuitionistic fuzzy sets A, B and C as follows,

$$A = \langle x, (\tfrac{a}{0.3}, \tfrac{b}{0.3}, \tfrac{c}{0.4}), (\tfrac{a}{0.5}, \tfrac{b}{0.5}, \tfrac{c}{0.5})\rangle,$$

$$B = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.5}, \tfrac{c}{0.4}), (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.4})\rangle,$$

and

$$C = \langle x, (\tfrac{a}{0.4}, \tfrac{b}{0.4}, \tfrac{c}{0.4}), (\tfrac{a}{0.3}, \tfrac{b}{0.3}, \tfrac{c}{0.3})\rangle.$$

Clearly $T = \{0_\sim, 1_\sim, A, B\}$ is an intuitionistic fuzzy topology on X. Thus (X, T) is an intuitionistic fuzzy topological space. Now C and $1_\sim$ are intuitionistic fuzzy dense sets in (X, T). Then $IFint(C) = A \neq 0_\sim$ and $IFint(1_\sim) \neq 0_\sim$. Hence (X, T) i an intuitionistic fuzzy irresolvable. But $IFint(IFcl(1 - C)) = IFint(1 - A) = A \neq 0_\sim$ and $IFint(1 - C) = 0_\sim$. Therefore (X, T) is not a intuitionistic fuzzy open hereditarily irresolvable space.

Proposition 4.2. Let (X, T) be an intuitionistic fuzzy open hereditarily irresolvable. Then $IFint(A) \not\subseteq 1 - IFint(B)$ for any two intuitionistic fuzzy dense sets A and B in (X, T).

Proof. Let A and B be any two intuitionistic fuzzy dense sets in (X, T). Then $IFcl(A) = 1_\sim$ and $IFcl(B) = 1_\sim$ implies that $IFint(IFcl(A)) \neq 0_\sim$ and $IFint(IFcl(B)) \neq 0_\sim$. Since (X, T) is intuitionistic fuzzy open hereditarily irresolvable, $IFint(A) \neq 0_\sim$ and $IFint(B) \neq 0_\sim$. Hence by Proposition 3.1, $A \not\subseteq 1 - B$. Therefore $IFint(A) \subseteq A \not\subseteq 1 - B \subseteq 1 - IFint(B)$. Hence we have $IFint(A) \subseteq 1 - IFint(B)$ for any two intuitionistic fuzzy dense sets A and B in (X, T).

Proposition 4.3. Let (X, T) be an intuitionistic fuzzy topological space. If (X, T) is intuitionistic fuzzy open hereditarily irresolvable then $IFint(A) = 0_{\sim}$ for any nonzero intuitionistic fuzzy dense set A in (X, T) implies that $IFint(IFcl(A)) = 0_{\sim}$.

Proof. Let A be an intuitionistic fuzzy set in (X, T), such that $IFint(A) = 0_{\sim}$. We claim that $IFint(IFcl(A)) = 0_{\sim}$. Suppose that $IFint(IFcl(A)) = 0_{\sim}$. Since (X, T) is intuitionistic fuzzy open hereditarily irresolvable, we have $IFint(A) \neq 0_{\sim}$. Which is a contradiction to $IFint(A) = 0_{\sim}$. Hence $IFint(IFcl(A)) = 0_{\sim}$.

Proposition 4.4. Let (X, T) be an intuitionistic fuzzy topological space. If (X, T) is intuitionistic fuzzy open hereditarily irresolvable then $IFcl(A) = 1_{\sim}$ for any nonzero intuitionistic fuzzy dense set A in (X, T) implies that $IFcl(IFint(A)) = 0_{\sim}$.

Proof. Let A be an intuitionistic fuzzy set in (X, T), such that $IFcl(A) = 1_{\sim}$. Then we have $1 - IFcl(A) = 0_{\sim}$, which implies that $IFint(1 - A) = 0_{\sim}$. Since (X, T) is intuitionistic fuzzy open hereditarily irresolvable by Proposition 4.3. We have that $IFint(IFcl(1-A)) = 0_{\sim}$. Therefore $1 - IFcl(IFint(A)) = 0_{\sim}$ implies that $IFcl(IFint(A)) = 1_{\sim}$.

§5. Somewhat intuitionistic fuzzy continuous and somewhat intuitionistic fuzzy open

Definition 5.1. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called somewhat intuitionistic fuzzy continuous if $A \in S$ and $f^{-1}(A) \neq 0_{\sim}$, then there exists a $B \in T$, such that $B \neq 0_{\sim}$ and $B \subseteq f^{-1}(A)$.

Definition 5.2. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. A function $f : (X, T) \to (Y, S)$ is called somewhat intuitionistic fuzzy open if $A \in T$ and $A \neq 0_{\sim}$, then there exists a $B \in S$, such that $B \neq 0_{\sim}$ and $B \subseteq f(A)$.

Proposition 5.1. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. If the function $f : (X, T) \to (Y, S)$ is somewhat intuitionistic fuzzy continuous and 1-1 and if $IFint(A) = 0_{\sim}$ for any nonzero intuitionistic fuzzy set A in (X, T) then $IFint(f(A)) = 0_{\sim}$ in (Y, S).

Proof. Let A be a nonzero intuitionistic fuzzy set in (X, T), such that $IFint(A) = 0_{\sim}$. To prove that $IFint(f(A)) = 0_{\sim}$. Suppose that $IFint(f(A)) \neq 0_{\sim}$ in (Y, S). Then there exists an nonzero intuitionistic fuzzy set B in (Y, S), such that $B \subseteq f(A)$. Then $f^{-1}(B) \subseteq f^{-1}(f(A))$. Since f is somewhat intuitionistic fuzzy continuous, there exists a $C \in T$, such that $C \neq 0_{\sim}$ and $C \subseteq f^{-1}(B)$. Hence $C \subseteq f^{-1}(B) \subseteq A$, which implies that $IFint(A) \neq 0_{\sim}$. Which is a contradiction. Hence $IFint(f(A)) = 0_{\sim}$ in (Y, S).

Proposition 5.2. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. If the function $f : (X, T) \to (Y, S)$ is somewhat intuitionistic fuzzy continuous and 1-1 and if $IFint(IFcl(A)) = 0_{\sim}$ for any nonzero intuitionistic fuzzy set A in (X, T) then $IFint(IFcl(f(A))) = 0_{\sim}$ in (Y, S).

Proof. Let A be a nonzero intuitionistic fuzzy set in (X, T), such that $IFint(IFcl(A)) = 0_{\sim}$. We claim that $IFint(IFcl(f(A))) = 0_{\sim}$ in (Y, S). Suppose that $IFint(IFcl(f(A))) \neq 0_{\sim}$ in (Y, S). Then $IFcl(f(A)) \neq 0_{\sim}$. Then $1 - IFcl(f(A)) \neq 0_{\sim}$. Now $1 - IFcl(f(A)) \neq 0_{\sim} \in S$ and since f is somewhat intuitionistic fuzzy continuous, there exists a $B \in T$, such that

$B \neq 0_\sim$ and $B \subseteq f^{-1}(1 - IFcl(f(A)))$. That is $B \subseteq 1 - f^{-1}(IFcl(f(A)))$, which implies that $f^{-1}(IFcl(f(A))) \subseteq 1 - B$. Since f is 1-1, thus $A \subseteq f^{-1}(f(A) \subseteq f^{-1}(IFcl(f(A))) \subseteq 1 - B$, implies that $A \subseteq 1 - B$. Therefore $B \subseteq 1 - A$ implies that $IFint(1 - A) \neq 0_\sim$. Let $IFint(1 - A) = C \neq 0_\sim$. Then we have $IFcl(IFint(1 - A)) = IFcl(C) \neq 1_\sim$, implies that $IFint(IFcl(A)) \neq 0_\sim$. Which is a contradiction. Hence $IFint(Ifcl(f(A))) = 0_\sim$ in (Y, S).

Proposition 5.3. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. If the function $f : (X, T) \to (Y, S)$ is somewhat intuitionistic fuzzy open and if $IFint(A) = 0_\sim$ for any nonzero intuitionistic fuzzy set A in (Y, S) then $IFint(f^{-1}(A)) = 0_\sim$ in (X, T).

Proof. Let A be a nonzero intuitionistic fuzzy set in (Y, S), such that $IFint(A) = 0_\sim$. We claim that $IFint(f^{-1}(A)) = 0_\sim$ in (X, T). Suppose that $IFint(f^{-1}(A)) \neq 0_\sim$ in (X, T). Then there exists a nonzero intuitionistic fuzzy open set B in (X, T), such that $B \subseteq f^{-1}(A)$. Then we have $f(B) \subseteq f(f^{-1}(A)) \subseteq A$. Which implies that $f(B) \subseteq A$. Since f is somewhat intuitionistic fuzzy open, there exists a $C \in S$, such that $C \neq 0_\sim$ and $C \subseteq f(B)$. Hence $C \subseteq f(B) \subseteq A$, which implies that $C \subseteq A$. Hence $IFint(A) \neq 0_\sim$. Which is a contradiction. Hence $IFint(f^{-1}(A)) = 0_\sim$ in (X, T).

Proposition 5.4. Let (X, T) and (Y, S) be any two intuitionistic fuzzy topological spaces. Let (X, T) be an intuitionistic fuzzy open hereditarily irresolvable space. If $f : (X, T) \to (Y, S)$ is somewhat intuitionistic fuzzy open and somewhat intuitionistic fuzzy continuous, 1-1 and onto function then (Y, S) is intuitionistic fuzzy open hereditarily space.

Proof. Let A be a nonzero intuitionistic fuzzy set in (Y, S), such that $IFint(A) = 0_\sim$. Now $IFint(A) = 0_\sim$ and f is somewhat intuitionistic fuzzy open implies that by Proposition 5.3, $IFint(f^{-1}(A)) = 0_\sim$ in (X, T). Since (X, T) is intuitionistic fuzzy open hereditarily irresolvable space, we have $IFint(IFcl(f^{-1}(A))) = 0_\sim$ in (X, T), by Proposition 4.3. Since $IFint(IFcl(f^{-1}(A))) = 0_\sim$ and f is somewhat intuitionistic fuzzy continuous by Proposition 5.2, we have $IFint(IFcl(f(f^{-1}(A)))) = 0_\sim$. Since f is onto, thus $IFintIFcl(A) = 0_\sim$. Hence by Proposition 4.3. (Y, S) is an intuitionistic fuzzy open hereditarily irresolvable space.

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Scientia Magna
Vol. 7 (2011), No. 2, 68-72

Super Weyl transform and some of its properties

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Abstract In this paper, we define the Wigner transform and the corresponding Weyl transform associated with the supersymbols that extend classical Weyl transform theory. This is motivated by acting with a net of symbols that generalize the class of S^m. Then we give some relations between super Weyl transform and generalized Wigner transform.

Keywords Microlocal analysis, Weyl transform, Wigner transform, supersingular pseudodif -ferential operator, supersymbol.

2000 Mathematics Subject Classification: Primary 35S30; Secondary 81S30.

§1. Introduction

The classical Weyl transform was first introduced in [6] by Hermann Weyl arising in quantum mechanics. The theory of Weyl transform is a vast subject of remarkable interest both in mathematical analysis and physics. In the theory of partial differential equations Weyl operators have been studied as a particular type of pseudo-differential operators. They have proved to be a useful technique in a quantity of problems like elliptic theory, spectral asymptotics, regularity problems, etc [7].

There is so many operators in theoretical mathematics that have powerful analytic methods for achieving the relationships and proposed which are require. But in the Theoretical physics (or applied mathematics!) we deal with functions or equations that do not abbey ordinary laws.

As operators acting on $L^2(\mathbb{R}^n)$, Weyl operators have been deeply investigated mainly in the case where the symbol is a smooth function belonging to some special symbol classes [1−2].

In the microlocal analysis we deal with the space of symbols which are infinitely differentiable functions and make it into Fréchet space by means of seminorms [5]. But in the Physics observations we often deal with functions which vary in respect of time and thus make nets of functions. In [4] introduced a class of symbols that vary in respect of time and are integrable with respect to an arbitrary measure. By means of this class of symbols we can generalize the classical theory with supersymbols and supersigular pseudodifferential operators.

§2. Preliminaries

Suppose N is a fixed natural number. The pseudodifferential operator (abb. ψDO) generates from S^m symbols as follow [4]

$$OP\psi = \int\int e^{i(x-y)\xi}\psi(x,\xi)f(y)dy\ d\xi,$$

in which $\psi(x,\xi) \in S^m$, such that for all $(\alpha,\beta,n) \in \mathbb{Z}_+^N \times \mathbb{Z}_+^N \times \mathbb{Z}_+$,

$$\sup_{|x|\leq n}\sup_{\xi\in\mathbb{R}^N}|\partial_\xi^\alpha\partial_x^\beta\psi(x,\xi)|(1+|\xi|)^{-m+|\alpha|} < \infty.$$

One of the advantage operators which is used in the quantum meachanics is Wigner transform [7]. Let f and g be in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. Then the $W(f,g)$ on the $\mathbb{R}^{2N}$, is defined by

$$W(f,g)(x,\xi) = (2\pi)^{-\frac{N}{2}}\int e^{-i\xi y}f\left(x+\frac{y}{2}\right)\overline{g\left(x-\frac{y}{2}\right)}dy,$$

which is called the Wigner transform of f and g.

In addition to usefulness of the Wigner transform, the other application of that, is its beautiful relationship with one of the most important operator in the quantum machanics, i. e. Weyl transform,

Definition 2.1. Suppose that ψ lies in S^m. Then the linear operator W_ψ defined by

$$(W_\psi f)(x) = \int\int e^{i(x-y)\cdot\xi}\psi\left(\frac{x+y}{2},\xi\right)f(y)dy\ d\xi,$$

is the Weyl transform of the function $f \in S(\mathbb{R}^N)$.

Now we want to define a class of ψDOs such that be more general and applicable in physics phenomena.

Definition 2.2. Given an arbitrary measure σ on $\mathbb{R}^N$. If $\psi : \mathbb{R}^N \to S^m$ $(m \in \mathbb{R})$, ψ is said to be supersymbol if for all $(\alpha,\beta,n) \in \mathbb{Z}_+^N \times \mathbb{Z}_+^N \times \mathbb{Z}_+$,

$$\int\sup_{|x|<n}\sup_{\xi\in\mathbb{R}}|\partial_\xi^\alpha\partial_x^\beta\psi(x,\xi)|(1+|\xi|)^{-m+|\alpha|}d\sigma(t) < \infty.$$

The class of such ψ is denoted by SS^m.

Each supersymbol, regarded together with the measure generates a supersingular pseudodifferential operator (abb. sψDO) as follows:

$$T(\psi,\sigma)(f) = \int\int\int e^{ix\xi}\psi(t)(x,\xi)e^{-iy\theta}f(y-t)dy\ d\theta\ d\sigma(t).$$

As the trivial case when σ is the unit measure $\delta(t)$ supported at origin, $T(\psi,\sigma)$ is the pseudodofferential operator OP$\psi(0)$. As usual, notations OP$SS^m(\sigma)$ and OP$SS^{-\infty}$ etc. Stand for the space of operators generated by the corresponding space of supersymbols, i.e. of $SS^m(\sigma)$, $SS^{-\infty} \equiv \cap_m SS^m$. It is easy to check that for $T(\psi,\ \sigma)$ we can rewrite it as

$$T(\psi,\sigma)(f) = \int\int\int e^{i(x-y-t)\cdot\xi}\psi(t)(x,\ \theta)f(y)dy\ d\theta\ d\sigma(t).$$

This technique is very useful for generalization of the differential operators to operators for an arbitrary measure space. So we can define a general Wigner transform that is integrated with an arbitrary measure σ on $\mathbb{R}^N$ as follows

$$SW(f,g)(x,t,\xi) = (2\pi)^{-\frac{N}{2}} \int e^{-i\xi\cdot y} f\left(x - t + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)} dy,$$

where $x, t, \xi \in \mathbb{R}^N$.

Now we are ready for introducing super Weyl transform that is more general and that will extend the classical theory and make a framework of operators that will be useful in theoretical physics and applied mathematics.

§3. Super Weyl transform

Definition 3.1. Let $\psi(t)$ be a super symbol and σ be an arbitrary measure on $\mathbb{R}^N$. For both functions f and g in the Schwartz space, the integral

$$(SW_{\psi(t)}f)(x) = \int\int\int e^{i(x-y-t)\cdot\xi} \psi(t)\left(\frac{x+y+t}{2}, \xi\right) f(y)dy\, d\xi\, d\sigma(t),$$

is called the super Weyl transform of f and g.

In the next theorem we will illuminate the relationship between the super Weyl transform and the generalized Wigner transform. The following lemma is needed.

Lemma 3.2. If θ is in $C_0^\infty(\mathbb{R}^N)$, such that $\theta(0) = 1$, then

$$\lim_{\epsilon\to 0+} (2\pi)^{-N} \int\int\int \theta(\epsilon\xi) e^{i(x-y-t)\cdot\xi} \psi(t)\left(\frac{x+y+t}{2}, \xi\right) f(y)dy\, d\xi\, d\sigma(t),$$

exists and is independent of the choice of the function θ. Moreover, the convergence is uniform with respect to x on $\mathbb{R}^N$.

Hint: Note that for any positive integer L,

$$(1 - \Delta_y)^L \{e^{i(x-y-t)\cdot\xi}\} = (1 + |\xi|^2)^L e^{i(x-y-t)\cdot\xi}.$$

Theorem 3.3. Let $\psi(t) \in SS^m$, $m \in \mathbb{R}$, and σ be an finite measure on $\mathbb{R}^N$. Then

$$\langle SW_{\psi(t)}f,\ g\rangle = (2\pi)^{-\frac{N}{2}} \int\int\int \psi(t)(x,\xi) SW(f,g)(x,t,\xi) dx\, d\xi\, d\sigma(t), \quad f,\ g \in \mathcal{S}(\mathbb{R}^N).$$

Proof. Let θ be any function in $C_0^\infty(\mathbb{R}^N)$ such that $\theta(0) = 1$. Then, by the Lemma, Lebesgue dominated convergence theorem, and Fubini's theorem,

$$
\begin{aligned}
&\int\int\int \psi(t)(x,\xi)SW(f,g)(x,t,\xi)dx\ d\xi\ d\sigma(t)\\
&= \lim_{\epsilon\to 0}\int\int\int \theta(\epsilon\xi)\psi(t)(x,\xi)SW(f,g)(x,t,\xi)dx\ d\xi\ d\sigma(t)\\
&= \lim_{\epsilon\to 0}(2\pi)^{-\frac{N}{2}}\int\int \theta(\epsilon\xi)\psi(t)(x,\xi)\\
&\quad \times\left\{\int\int e^{-i\xi\cdot y}f\left(x+\frac{y}{2}\right)\overline{g\left(x-t-\frac{y}{2}\right)}dy\right\}dx\ d\xi\ d\sigma(t)\\
&= \lim_{\epsilon\to 0}(2\pi)^{-\frac{N}{2}}\int\int \theta(\epsilon\xi)\\
&\quad \times\left\{\int\int \psi(t)(x,\xi)e^{-i\xi\cdot y}f\left(x+\frac{y}{2}\right)\overline{g\left(x-t-\frac{y}{2}\right)}dy\ dx\right\}d\xi\ d\sigma(t).
\end{aligned}
$$

Let $u = x - t + \frac{y}{2}$ and $v = x - \frac{y}{2}$ in the last term, by Lemma, Fubini's Theorem and the Lebesgue dominated convergence theorem,

$$
\begin{aligned}
&\int\int\int \psi(t)(x,\xi)SW(f,g)(x,t,\xi)dx\ d\xi\ d\sigma(t)\\
&= \lim_{\epsilon\to 0}(2\pi)^{-\frac{N}{2}}\int\int \theta(\epsilon\xi)\\
&\quad \times\left\{\int\int \psi(t)\left(\frac{u+v+t}{2},\xi\right)e^{i(v-u-t)\xi}f(u)\overline{g(v)}du\ dv\right\}d\xi d\sigma(t)\\
&= \lim_{\epsilon\to 0}(2\pi)^{-\frac{N}{2}}\int \overline{g(v)}\\
&\quad \times\left\{\int\int\int \theta(\epsilon\xi)\psi(t)\left(\frac{u+v+t}{2},\xi\right)e^{i(v-u-t)\xi}f(u)du\ d\xi\ d\sigma(t)\right\}dv\\
&= (2\pi)^{-\frac{N}{2}}\int \overline{g(v)}(SW_{\psi(t)}f)(v)dv\ = (2\pi)^{-\frac{N}{2}}\langle SW_{\psi(t)}f,\ g\rangle
\end{aligned}
$$

In classical mechanics, the phase space used to describe the motion of a particle moving in $\mathbb{R}^N$ is given by

$$\mathbb{R}^{2N} = \{(x,\xi);\quad x,\xi\in\mathbb{R}^N\},$$

where the variables x and ξ are used to denote the position and momentum of the particle, respectively. The observables of the motion are given by real-valued tempered distributions on $\mathbb{R}^{2N}$. The rules of quantization, with Planck's constant adjusted to 1, say that a quantum-mechanical model of the motion can be set up using the Hilbert space $L^2(\mathbb{R}^N)$ for the phase space, the multiplication operator on $L^2(\mathbb{R}^N)$ by the function x_j for the position variable x_j, and the differential operator D_j for the momentum variable ξ_j.

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Scientia Magna
Vol. 7 (2011), No. 2, 73-77

A short interval result for the e-squarefree e-divisor function[1]

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Abstract Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n. The aim of this paper is to establish a short interval result for the function $t^{(e)}(n)$. This enriches the properties of the e-squarefree e-divisor function.

Keywords The e-squarefree e-divisor function, arithmetic function, the generalized divisor function short interval.

§1. Introduction and preliminaries

Let $n > 1$ be an integer of canonical from $n = \prod_{i=1}^{s} p_i^{a_i}$. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of n if $b_i | a_i$ for every $i \in$ {1,2,. . . ,s}, notation: $d|_e n$. By convention $1|_e 1$.

The integer $n > 1$ is called e-squarefree, if all exponents $a_1, \ldots, a_s$ are squarefree. The integer 1 is also considered to be e-squarefree. Consider now the exponential squarefree exponential divisor (e-squarefree e-divisor) of n. Here $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an e-squarefree e-divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$, if $b_1|a_1, \cdots b_s|a_s, b_1, \ldots, b_s$ are squarefree. Note that the integer 1 is e-squarefree but is not an e-divisor of $n > 1$.

Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n. The function $t^{(e)}(n)$ is called the e-squarefree e-divisor function, which is a multiplicative and if $n = \prod_{i=1}^{s} p_i^{\alpha_i} > 1$, then (see [1])

$$t^{(e)}(n) = 2^{\omega(\alpha_1)} \cdots 2^{\omega(\alpha_s)},$$

where $\omega(\alpha) = s$ denotes the number of distinct prime factors of α. The properties of the function $t^{(e)}(n)$ were investigated by many authors, see example [4]. Let

$$A(x) := \sum_{n \le x} t^{(e)}(n),$$

Recently László Tóth proved that the estimate

$$\sum_{n \le x} t^{(e)}(n) = c_1 x + c_2 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\epsilon})$$

[1]This work is supported by N. N. S. F. of China (Grant Nos: 10771127, 11001154) and N. S. F. of Shandong Province (Nos: BS2009SF018, ZR2010AQ009).

holds for every $\varepsilon > 0$, where

$$c_1 := \prod_p (1 + \sum_{\alpha=6}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)}}{p^{\alpha}}), \qquad (*)$$

$$c_2 := \zeta(\frac{1}{2}) \prod_p (1 + \sum_{\alpha=4}^{\infty} \frac{2^{\omega(\alpha)} - 2^{\omega(\alpha-1)} - 2^{\omega(\alpha-2)} + 2^{\omega(\alpha-4)}}{p^{\frac{\alpha}{2}}}).$$

Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant. We assume that $1 \leq a \leq b$ are fixed integers, and we denote by $d(a, b; k)$ the number of representations of k as $k = n_1^a n_2^b$, where n_1, n_2 are natural numbers, that is,

$$d(a, b; k) = \sum_{k = n_1^a n_2^b} 1,$$

and $d(a, b; k) \ll n^{\epsilon^2}$ will be used freely.

The aim of this short text is to study the short interval case and prove the following.

Theorem. If $x^{\frac{1}{5}+2\varepsilon} < y \leq x$, then

$$\sum_{x < n \leq x+y} t^{(e)}(n) = c_1 y + O(y x^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}),$$

where c_1 is given by $(*)$.

§2. Proof of the theorem

In order to prove our theorem, we need the following lemmas.

Lemma 1. Suppose s is a complex number ($\Re es > 1$), then

$$F(s) := \sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \frac{\zeta(s)\zeta(2s)}{\zeta(4s)} G(s),$$

where the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re es > 1/6$.

Proof. Here $t^{(e)}(n)$ is multiplicative and by Euler product formula we have for $\sigma > 1$ that,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^e(n)}{n^s} &= \prod_p (1 + \frac{t^e}{p^s} + \frac{t^e(p^2)}{p^{2s}} + \frac{t^e(p^3)}{p^{3s}} + \cdots) \qquad (1)\\
&= \prod_p (1 + \frac{1}{p^s} + \frac{2}{+} \frac{2}{p^{3s}} + \frac{2}{p^{4s}} + \cdots)\\
&= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \cdots)\\
&= \zeta(s)\zeta(2s) \prod_p (1 - \frac{1}{p^{2s}})(1 + \frac{1}{p^{2s}} + \frac{1}{p^{6s}} + \cdots)\\
&= \frac{\zeta(s)\zeta(2s)}{\zeta(4s)} G(s).
\end{aligned}$$

So we get $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. It is easily seen the Dirichlet series is absolutely convergent for $\Re es > 1/6$.

Lemma 2. Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and

$$B(x, y; k, \epsilon) := \sum_{\substack{x < nm^k \leq x+y \\ m > x^{\epsilon}}} 1.$$

Then we have

$$B(x, y; k, \epsilon) \ll yx^{-\epsilon} + x^{\frac{1}{2k+1}} \log x. \tag{2}$$

Proof. This lemma is very important when studying the short interval distribution of 1-free number, see example [3].

Lemma 3. Let $a(n)$ be an arithmetic function defined by (2), then we have

$$\sum_{n \leq x} a(n) = Cx + O(x^{\frac{1}{6}+\epsilon}), \tag{3}$$

where $C = \Re es_{s=1}\zeta(s)G(s)$.

Proof. Using Lemma 1, it is easy to see that

$$\sum_{n \leq x} |g(n)| \ll x^{\frac{1}{6}+\epsilon}.$$

Therefore from the definition of $g(n)$ and (2), it follows that

$$\begin{aligned} \sum_{n \leq x} a(n) &= \sum_{mn \leq x} g(n) \\ &= \sum_{n \leq x} g(n) \sum_{m \leq \frac{x}{n}} 1 \\ &= \sum_{n \leq x} g(n)(\frac{x}{n} + O(1)) \\ &= Cx + O(x^{\frac{1}{6}+\epsilon}), \end{aligned}$$

and $C = \Re es_{s=1}\zeta(s)G(s)$.

Next we prove our theorem. From Lemma 3 and the definition of $a(n)$, we get

$$t^{(e)}(n) = \sum_{n = n_1 n_2^2 n_3^4} a(n_1)\mu(n_3),$$

and

$$a(n) \ll n^{\epsilon^2}, |\mu(n)| \ll 1. \tag{4}$$

So we have

$$\begin{aligned} A(x+y) - A(x) &= \sum_{x < n_1 n_2^2 n_3^4 \leq x+y} a(n_1)\mu(n_3) \\ &= \sum_1 + O(\sum_2 + \sum_3). \end{aligned} \tag{5}$$

where

$$\begin{aligned}
\sum_1 &= \sum_{\substack{n_2 \le x^{\epsilon} \\ n_3 \le x^{\epsilon}}} \mu(n_3) \sum_{\frac{x}{n_2^2 n_3^4} < n_1 \le \frac{x+y}{n_2^2 n_3^4}} a(n_1), \\
\sum_2 &= \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^{\epsilon}}} |a(n_1)\mu(n_3)|, \\
\sum_3 &= \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_3 > x^{\epsilon}}} |a(n_1)\mu(n_3)|.
\end{aligned} \tag{6}$$

In view of Lemma 3,

$$\begin{aligned}
\sum_1 &= \sum_{\substack{n_2 \le x^{\epsilon} \\ n_3 \le x^{\epsilon}}} \mu(n_3)(\frac{Cy}{n_2^2 n_3^4} + O((\frac{x}{n_2^2 n_3^4})^{\frac{1}{6}+\epsilon})) \\
&= c_1 y + O(yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{6}+\frac{3}{2}\epsilon}),
\end{aligned} \tag{7}$$

where $c_1 = \Re es_{s=1} F(s)$.

$$\begin{aligned}
\sum_2 &\ll \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^{\epsilon}}} (n_1)^{\epsilon^2} \\
&\ll x^{\epsilon^2} \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^{\epsilon}}} 1 \\
&= x^{2\epsilon^2} \sum_{\substack{x < n_1 n_2^2 n_3^4 \le x+y \\ n_2 > x^{\epsilon}}} d(1,4;m) \\
&\ll x^{\epsilon^2} B(x,y;2,\epsilon) \\
&\ll x^{2\epsilon^2}(yx^{-\epsilon} + x^{\frac{1}{5}+\epsilon}) \\
&\ll yx^{2\epsilon^2-\epsilon} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}\log x \\
&\ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}.
\end{aligned} \tag{8}$$

Similarly we have

$$\sum_3 \ll yx^{-\frac{\epsilon}{2}} + x^{\frac{1}{5}+\frac{3}{2}\epsilon}. \tag{9}$$

Now our theorem follows from (5)-(9).

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Scientia Magna
Vol. 7 (2011), No. 2, 78-81

Log convexity and concavity of some double sequences

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Abstract In this paper, we obtained some results on log convexity and log concavity of double sequences.

Keywords Double sequences, convexity, monotonicity, means.

§1. Introduction

In literature, the well known means respectively called Arithmetic mean, Geometric mean and Harmonic mean are as follows;

For $a, b > 0$, then

$$A(a,b) = \frac{a+b}{2}, \qquad G(a,b) = \sqrt{ab} \qquad \text{and} \qquad H(a,b) = \frac{2ab}{a+b}.$$

Several researchers introduced and studied some interesting results on double sequences in the form of above said means. Also, proved the convergence properties and obtained there limit values. As an application to estimate the best accurate value of π, the authors considered the following double sequences [4−8];

$$a_{n+1} = H(a_n, b_n) \qquad \text{and} \qquad b_{n+1} = G(a_{n+1}, b_n), \tag{1}$$

where H and G stands for harmonic mean and geometric mean respectively.

In finding the roots of an equation one of the famous iteration method is called the Heron's method of extracting of square root is achieved by the following double sequences;

$$a_{n+1} = H(a_n, b_n) \qquad \text{and} \qquad b_{n+1} = A(a_n, b_n), \tag{2}$$

where H and A stands for Harmonic mean and Arithmetic mean respectively.

Also, several researchers introduced and studied the double sequences for other applications as follows;

$$a_{n+1} = A(a_n, b_n) \qquad \text{and} \qquad b_{n+1} = G(a_n, b_n), \tag{3}$$

where A and G stands for Arithmetic mean and Geometric mean respectively.

In [1, 3], contains many results on convergence and monotonicity. The double sequences were generalized as Archimedean double sequences and Gauss double sequences.

The sequence c_n is said to be *log-convex*, if $c_n^2 \leq c_{n+1}c_{n-1}$ and the sequence c_n is said to be *log-concave*, if $c_n^2 \geq c_{n+1}c_{n-1}$.

In this paper, the logarithmic convexity and logarithmic concavity of the double sequences are presented (1)-(3).

§2. Results

In this section, some results on log convexity and concavity of double sequences are proved.

Theorem 2.1. For $n \geq 0$, $a_0 < b_0$, then the sequences $a_{n+1} = H(a_n, b_n)$ and $b_{n+1} = G(a_{n+1}, b_n)$ are respectively Log-concave and Log-convex.

Proof. From definitions of harmonic mean and geometric mean, consider

$$a_{n+1} = H(a_n, b_n) = \frac{2a_nb_n}{a_n + b_n} \qquad \text{and} \qquad b_{n+1} = G(a_{n+1}, b_n) = \sqrt{a_{n+1}b_n}. \tag{4}$$

It is proved that for $a_0 < b_0$,

$$a_0 < a_1 < a_2 < ... < a_n < a_{n+1} < ... < b_{n+1} < b_n < ... < b_2 < b_1 < b_0 \tag{5}$$

from (4), $b_n^2 = a_nb_{n-1}$ and $a_n < b_{n+1}$ from (5), implies that $b_n^2 = a_nb_{n-1} < b_{n+1}b_{n-1}$, which is equivalently

$$b_n^2 < b_{n+1}b_{n-1}, \tag{6}$$

consider

$$\begin{aligned} \frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n} &= \frac{a_n + b_n}{2b_n} - \frac{a_{n-1} + b_{n-1}}{2b_{n-1}} \\ &= \frac{1}{2b_nb_{n-1}}[a_nb_{n-1} - a_{n-1}b_n] \end{aligned} \tag{7}$$

form (5), $b_1 < b_0, -b_1 > -b_0, -a_0b_1 > -a_1b_0$, this leads to

$$= \frac{1}{2b_nb_{n-1}}[a_nb_{n-1} - a_{n-1}b_n] = \frac{1}{2b_nb_{n-1}}[a_nb_{n-1} - a_{n-1}b_{n-1}] = \frac{1}{2b_n}[a_n - a_{n-1}] > 0.$$

This proves that

$$a_n^2 > a_{n+1}a_{n-1}. \tag{8}$$

Thus the (6) and (8) satisfies the conditions of log concave and log convex for sequence. That is the sequence a_n is log concave and the sequence b_n is log convex.

Theorem 2.2. For $n \geq 0$, $a_0 < b_0$, then the sequences $a_{n+1} = H(a_n, b_n)$ and $b_{n+1} = A(a_n, b_n)$ are respectively log-concave and log-convex.

Proof. Consider

$$a_n^2 - a_{n-1}a_{n+1} = \left(\frac{2a_{n-1}b_{n-1}}{a_{n-1} + b_{n-1}}\right)^2 - a_{n-1}\left(\frac{2a_nb_n}{a_n + b_n}\right)$$

on rearranging the above expression leads to

$$\frac{(4a_{n-1}^2 b_{n-1})(a_{n-1}^2 + 3a_{n-1}b_{n-1})}{(a_{n-1}+b_{n-1})^2(a_{n-1}^2 + b_{n-1}^2 + 6a_{n-1}b_{n-1})}(b_{n-1} - a_{n-1}) > 0,$$

this proves that

$$a_n^2 > a_{n+1}a_{n-1}, \tag{9}$$

again consider

$$b_n^2 - b_{n-1}b_{n+1} = \left(\frac{a_{n-1}+b_{n-1}}{2}\right)^2 - b_{n-1}\left(\frac{a_n + b_n}{2}\right)$$

on rearranging the above expression leads to

$$\frac{(a_{n-1}^2 + 3a_{n-1}b_{n-1})}{4(a_{n-1}+b_{n-1}))}(a_{n-1} - b_{n-1}) < 0,$$

this proves that

$$b_n^2 < b_{n+1}b_{n-1}. \tag{10}$$

Thus the (9) and (10) satisfies the conditions of log concave and log convex. That is the sequence a_n is log concave and the sequence b_n is log convex.

Theorem 2.3. For $n \geq 0$, $a_0 < b_0$, then the sequences $a_{n+1} = A(a_n, b_n)$ and $b_{n+1} = G(a_n, b_n)$ are respectively log concave and log convex.

Proof. Consider

$$a_n^2 - a_{n-1}a_{n+1} = \left(\frac{a_{n-1}+b_{n-1}}{2}\right)^2 - a_{n-1}\left(\frac{a_n + b_n}{2}\right)$$

on substituting and using the fact $A(a,b) > G(a,b)$, from (5) the above expression leads to

$$\frac{1}{2}\left(b_{n-1}\left(\frac{a_{n-1}+b_{n-1}}{2}\right) - a_{n-1}\sqrt{a_{n-1}b_{n-1}}\right) < 0,$$

this proves that

$$a_n^2 > a_{n+1}a_{n-1}, \tag{11}$$

again consider

$$b_n^2 - b_{n-1}b_{n+1} = a_{n-1}b_{n-1} - b_{n-1}\sqrt{a_n b_n}$$

on substituting and using the fact $A(a,b) > a_{n-1}, G(a,b) > a_{n-1}$, the above expression leads to

$$b_{n-1}\left(a_{n-1} - \sqrt{\frac{a_{n-1}+b_{n-1}}{2}\sqrt{a_{n-1}b_{n-1}}}\right) < 0,$$

this proves that

$$b_n^2 < b_{n+1}b_{n-1}. \tag{12}$$

Thus the (11) and (12) satisfies the conditions of log concave and log convex. That is the sequence a_n is log concave and the sequence b_n is log convex.

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Scientia Magna
Vol. 7 (2011), No. 2, 82-94

Almost contra $\nu-$continuity

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Abstract The object of the paper is to study basic properties and characterizations of almost contra $\nu-$continuous functions.
Keywords $\nu-$continuity, $\nu-$irresolute, contra $\nu-$continuity.
AMS-classification Numbers: 54C10, 54C08, 54C05.

§1. Introduction

In 1980, Joseph and Kwack introduced the notion of (θ, s)-continuous functions. In 1982, Jankovic introduced the notion of almost weakly continuous functions. In 1996, Dontchev introduced contra-continuous functions. C. W. Baker defined Subcontra-continuous functions in 1998 and contra almost $\beta-$continuous functions in 2006. Dontchev, Ganster and Reilly introduced regular set-connected functions and Dontchev and T. Noiri introduced Contra-semicontinuous functions in 1999. S. Jafari and T. Noiri introduced and studied Contra-super-continuous functions and Jafari introduced the notion of (p, s)-continuous functions in 1999; Contra-$\alpha-$continuous functions in 2001 and contra-precontinuous functions in 2002. M. Caldas and S. Jafari studied Some Properties of Contra-$\beta-$Continuous Functions in 2001. T. Noiri and V. Popa studied unified theory of contra-continuity in 2002, Some properties of almost contra-precontinuity in 2005 and unified theory of almost contra-continuity in 2008. E. Ekici introduced Almost contra-precontinuous functions in 2004 and recently have been investigated further by Noiri and Popa. A. A. Nasef studied some properties of contra-$\gamma-$continuous functions in 2005. M. K. R. S. Veera Kumar introduced Contra-Pre-Semi-Continuous Functions in 2005. Ekici E., introduced Almost contra-precontinuous functions and studied another form of contra-continuity in 2006. During 2007, N. Rajesh studied total $\omega-$Continuity, Strong $\omega-$Continuity and contra $\omega-$Continuity. Recently, Ahmad Al-Omari and Mohd. Salmi Md. Noorani studied Some Properties of Contra-b-Continuous and Almost Contra-b-Continuous Functions in 2009 and Jamal M. Mustafa introduced Contra Semi-I-Continuous functions in 2010. Inspired with these developments, we introduce a new class of functions called contra $\nu-$continuous functions. Moreover, we obtain basic properties, preservation theorem of contra $\nu-$continuous function and relationship with other types of functions are verified.

§2. Preliminaries

Definition 2.1. $A \subset X$ is said to be

(i) regular open [pre-open; semi-open; α-open; β-open] if $A = (\overline{A})^0[A \subseteq (\overline{A})^o$; $A \subseteq \overline{(A^o)}$; $A \subseteq (\overline{(A^o)})^o$; $A \subseteq \overline{((\overline{A})^o)}]$ and regular closed [pre-closed; semi-closed; α-closed; β-closed] if $A = \overline{A^0}[\overline{(A^o)} \subseteq A$; $(\overline{A})^o \subseteq A$; $\overline{((\overline{A})^o)} \subseteq A$; $(\overline{(A^o)})^o \subseteq A]$.

(ii) $\nu-$open [r$\alpha-$open] if there exists a regular open set O such that $O \subset A \subset \overline{O}[O \subset A \subset \alpha\overline{(O)}]$.

(iii) θ-semi-closed if $A = sCl_\theta(A) = \{x \in X : \overline{V} \cap A \neq \phi;\ \forall\ V \in SO(X, x)\}$; $sCl_\theta(A)$ is θ-semi-closure of A. The complement of a θ-semi-closed set is said to be θ-semi-open.

(iv) ν-dense in X if $\nu\overline{(A)} = X$.

(v) θ-closed if $A = Cl_\theta(A)$. The complement of a θ-closed set is said to be θ-open.

Remark 1. We have the following implication diagrams for closed sets.

$\nearrow$ $r\alpha-$open $\rightarrow$ $\nu-$**open** $\searrow$

Regular open $\rightarrow$ open $\rightarrow$ $\alpha-$open $\rightarrow$ semi open $\rightarrow$ $\beta-$open

$\searrow$

pre-open

Definition 2.2. A cover $\Sigma = \{U_\alpha : \alpha \in I\}$ of subsets of X is called a $\nu-$cover if U_α is $\nu-$open for each $\alpha \in I$.

Definition 2.3. A filter base Λ is said to be $\nu-$convergent (resp. rc-convergent) to a point x in X if for any $U \in \nu O(X, x)$(resp. $U \in RC(X, x)$), there exists a $B \in \Lambda$ such that $B \subset U$.

Definition 2.4. A function $f\colon X \rightarrow Y$ is called

(i) almost-[resp: almost-semi-; almost-pre-;almost-$r\alpha-$; almost-$\alpha-$; almost-$\beta-$; almost-$\omega-$; almost-pre-semi-; almost-$\lambda-$]continuous if $f^{-1}(V)$ is open [resp: semi-open; pre-open;$r\alpha-$open; $\alpha-$open; $\beta-$open; $\omega-$open; pre-semi-open; $\lambda-$open] in X $\forall V \in RO(Y)$.

(ii) contra-[resp: contra-semi-; contra-pre-;contra-r-;contra-$r\alpha-$; contra-$\alpha-$; contra-$\beta-$; contra-$\omega-$; contra-pre-semi-; contra-$\nu-$]continuuos if $f^{-1}(V)$ is closed [resp: semi-closed; pre-closed; regular-closed; $r\alpha-$closed; $\alpha-$closed; $\beta-$closed; $\omega-$closed; pre-semi-closed; $\nu-$closed] in X $\forall V \in \sigma$.

(iii) regular set-connected if inverse image of every regular open set V in Y is clopen in X.

(iv) perfectly continuous if inverse image of every open set V in Y is clopen in X.

(v) almost s-continuous if for each $x \in X$ and each $V \in SO(Y)$ with $f(x) \in V$, there exists an open set U in X containing x such that $f(U) \subset scl(V)$.

(vi) M-$\nu-$open if image of each $\nu-$open set is $\nu-$open.

(vii) (ν, s)-continuous if for each $x \in X$ and each $V \in SO(Y, f(x))$, $\exists\ U \in \nu O(X, x) \ni f(U) \subset \overline{V}$.

(viii) weakly continuous if for each $x \in X$ and each $V \in (\sigma(Y), f(x))$, $\exists\ U \in (\tau(X), x) \ni f(U) \subset \overline{V}$.

(ix) (θ, s)-continuous iff for each θ-semi-open set V of Y , $f^{-1}(V)$ is open in X.

(x) faintly $\nu-$continuous if for each $x \in X$ and each θ-open set V of Y containing f(x), $\exists U \in \nu O(X, x)] \ni f(U) \subset V$.

Definition 2.5. A space X is said to be

(i) strongly compact [resp:strongly Lindelof] if every preopen cover of X has a finite [resp: countable] subcover and P-closed [resp: P-Lindelof] if every preclosed cover of X has a finite [resp:countable] subcover.

(ii) strongly countably compact if every countable cover of X by preopen sets has a finite subcover and countably S-closed [resp:countably P-closed] if every countable cover of X by regular closed [resp: preclosed] sets has a finite subcover.

(iii) mildly compact (mildly countably compact, mildly Lindelof) if every clopen cover (respectively, clopen countable cover, clopen cover) of X has a finite (respectively, a finite, a countable) subcover.

(iv) S-closed [resp: S-Lindelof] if every regular closed cover of X has a finite [resp:countable] subcover and nearly compact [resp:nearly Lindelof] if every regular open cover of X has a finite [resp: countable] subcover.

(v) $\nu-$connected provided that X is not the union of two disjoint nonempty $\nu-$open sets.

(vi) $\nu-$ultra-connected if every two non-void $\nu-$closed subsets of X intersect.

(vii) hyperconnected if every open set is dense.

(viii) weakly Hausdorff if each element of X is an intersection of regular closed sets.

(ix) $\nu-T_0$ if for each pair of distinct points in X, there exists a $\nu-$open set of X containing one point but not the other.

(x) $\nu - T_1$ [resp: $\nu - T_2$] if for each pair of distinct points x and y of X, there exist [resp: disjoint] $\nu-$open sets U and V containing x and y respectively such that $y \notin U$ and $x \notin V$.

(xi) almost regular if for each regular closed set F of X and each $x \in X - F$, there exists disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

(xii) extremally disconnected (briefly E.D.) if the closure of every open set of X is open in X.

Lemma 2.1. If V is an open [r-open] set, then

(i) sCl(V) = Int(Cl(V)).

(ii) $sCl_\theta(V) = sCl(V)$.

(iii) If $B \subseteq A \subseteq X$ and $A \in RO(X)$, then $\nu_A\overline{(B)} \subseteq \nu\overline{B}$.

Lemma 2.2. For $V \subset Y$, the following properties hold:

(i) $\alpha\overline{V} = \overline{V}$ for every $V \in \beta(Y)$.

(ii) $\nu\overline{V} = \overline{V}$ for every $V \in SO(Y)$.

(iii) $s\overline{V} = (\overline{V})^o$ for every $V \in RO(Y)$.

Lemma 2.3. For $f: X \to Y$, the following properties are equivalent:

(i) f is faintly-$\nu-$continuous.

(ii) $f^{-1}(V) \in \nu O(X)$ for every θ-open set V of Y.

(iii) $f^{-1}(K)$ is $\nu-$closed in X for every θ-closed set K of Y.

Lemma 2.4. f is al.ν.c. iff $\forall\, x \in X$ and each $V \in RO(Y, f(x))$, $\exists\, U \in \nu O(X, x) \ni f(U) \subset V$.

Definition 2.6. For a function $f: X \to Y$,

(i) The subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

(ii) A graph $G(f)$ of a function f is said to be $\nu-$regular if for each $(x, y) \in (X \times Y) - G(f)$, $\exists\, U \in \nu C(X, x)$ and $V \in RO(Y, y) \ni (U \times V) \cap G(f) = \phi$.

Lemma 2.5. The following properties are equivalent for a graph $G(f)$ of a function:

(i) $G(f)$ is $\nu-$regular;

(ii) for each point $(x, y) \in (X \times Y) - G(f)$, $\exists\, U \in \nu C(X, x)$ and $V \in RO(Y, y) \ni f(U) \cap V = \phi$.

§3. Almost contra ν-continuous maps

Definition 3.1. A function $f\colon X \to Y$ is said to be

(i) almost contra $\nu-$continuous at x if for each regular closed set F in Y containing f(x), there exists a $\nu-$open set U in X containing x such that $f(U) \subset F$.

(ii) almost contra $\nu-$continuous if the inverse image of every regular-open set is $\nu-$closed.

Note 1. Here after we call almost contra $\nu-$continuous function as al.$c.\nu.c$ function shortly.

Example 1. (i) $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$. Let f be identity function, then f is al.$c.\nu.c.$

(ii) f: on $\Re$ defined by f(x) = [x], where [x] is the Gaussian symbol; is al.c.ν.c; al.c.s.c. but not al.c.c; r-irreslute and c.r.c.

Example 2. (i) $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. Let f be identity function, then f is al.c.rα.c. but not al.$c.\nu.c.$

(ii) The identity function on $\Re$ with usual topology is not al.c.ν.c and al.c.s.c. but it is al.c.c;c.c; c.r.c. and r-irresolute.

Theorem 3.1. (i) f is al.$c.\nu.c$ iff $f^{-1}(U) \in \nu O(X)$ whenever $U \in RC(Y)$.

(ii) If f is c.ν.c., then f is al.c.ν.c. Converse is true if X is discrete space.

Theorem 3.2. (i) f is al.c.ν.c. iff for each $x \in X$ and each $U_Y \in \nu O(Y, f(x))$, $\exists A \in \nu$O(X) $\ni x \in A$ and $f(A) \subset U_Y$.

(ii)f is al.c.ν.c. iff for each $x \in X$ and each $V \in RO(Y, f(x))$, $\exists\, U \in \nu O(X, x) \ni f(U) \subset V$.

Proof. Let $U_Y \in RC(Y)$ and let $x \in f^{-1}(U_Y)$. Then $f(x) \in U_Y$ and thus $\exists\, A_x \in \nu$O(X) $\ni x \in A_x$ and $f(A_x) \subset U_Y$. Then $x \in A_x \subset f^{-1}(U_Y)$ and $f^{-1}(U_Y) = \cup A_x$. Hence $f^{-1}(U_Y) \in \nu$O(X).

Remark 2. We have the following implication diagram for a function $f\colon (X, \tau) \to (Y, \sigma)$

$\nearrow$ al.c.rαc. $\to$ **al.**$c.\nu.c$ $\searrow$
r-irresolute $\to$ al.c.c. $\to$ al.c.α.c. $\to$ al.c.s.c. $\to$ al.c.β.c.
$\searrow$
al.c.p.c.

Example 3. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. The identity function f is not al.c.c., al.c.s.c., al.c.p.c., al.c.α.c., al.c.rα.c., al.c.β.c., al.c.ν.c., contra r-irresolute and r-irresolute and f defined as f(a) = f(b) = a; f(c) = c is al.c.c., al.c.s.c., al.c.p.c., al.c.α.c., al.c.β.c., but not al.c.ν.c., al.c.rα.c., contra r-irresolute and r-irresolute.

Example 4. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$. The identity function f is al.c.s.c., al.c.α.c., al.c.rα.c., al.c.β.c., al.c.ν.c., but not al.c.c.., al.c.p.c., contra r-irresolute and r-irresolute. Under usual topology on $\Re$ both al.c.c. and r-irresolute are same as well both al.c.s.c. and al.$c.\nu.c$ are same.

Theorem 3.3. If f is $\nu-$open and al.$c.\nu.c$, then $f^{-1}(U)$ is $\nu-$closed if U is $\nu-$open in Y.

Proof. Let U be $\nu-$open in Y. Then $\exists V \in RO(Y) \ni V \subseteq U \subseteq \overline{V}$. $V \in RO(Y) \Rightarrow V \in \nu O(Y) \Rightarrow f^{-1}(V) \in \nu C(X)$ and $f^{-1}(V) \subseteq f^{-1}(U) \subseteq \overline{f^{-1}(V)} \Rightarrow f^{-1}(U)$ is $\nu-$closed.

Theorem 3.4. Let f be al.$c.\nu.c$ and r-open, then

(i) $f^{-1}(A) \in SO(X)$ $[f^{-1}(A) \in SC(X)]$ for each $A \in SC(Y)$ $[A \in SO(Y)]$.

(ii) $f^{-1}(A) \in RO(X)$ $[f^{-1}(A) \in RC(X)]$ for each $A \in RC(Y)$ $[A \in RO(Y)]$.

(iii) If f is r-open and r-irresolute, then $f^{-1}(U) \in \nu C(X)$ for each $U \in \nu O(Y)$.

Theorem 3.5. Let $f_i : X_i \to Y_i$ be al.$c.\nu.c$ for $i = 1, 2$. Let $f : X_1 \times X_2 \to Y_1 \times Y_2$ be defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f : X_1 \times X_2 \to Y_1 \times Y_2$ is al.$c.\nu.c$.

Proof. Let $U_1 \times U_2 \subset Y_1 \times Y_2$ where U_i be regular open in Y_i for $i = 1, 2$. Then $f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \times f_2^{-1}(U_2)$. But $f_1^{-1}(U_1)$ and $f_2^{-1}(U_2)$ are $\nu-$closed in X_1 and X_2 respectively and thus $f_1^{-1}(U_1) \times f_2^{-1}(U_2)$ is $\nu-$closed in $X_1 \times X_2$. Therefore f is al.c.ν.c.

Theorem 3.6. Let $h : X \to X_1 \times X_2$ be al.$c.\nu.c$, where $h(x) = (h_1(x), h_2(x))$. Then $h_i : X \to X_i$ is al.$c.\nu.c$ for $i = 1, 2$.

Proof. Let U_1 is regular open in X_1. Then $U_1 \times X_2$ is regular open in $X_1 \times X_2$, and $h^{-1}(U_1 \times X_2)$ is $\nu-$closed in X. But $h_1^{-1}(U_1) = h^{-1}(U_1 \times X_2)$, therefore $h_1 : X \to X_1$ is al.$c.\nu.c$ Similar argument gives $h_2 : X \to X_2$ is al.$c.\nu.c$ and thus $h_i : X \to X_i$ is al.$c.\nu.c$ for $i = 1, 2$.

In general we have the following extenstion of Theorem 3.5 and 3.6:

Theorem 3.7. (i) If $f : X \to \Pi Y_\lambda$ is al.$c.\nu.c$, then $P_\lambda \circ f : X \to Y_\lambda$ is al.$c.\nu.c$ for each $\lambda \in \Lambda$, where P_λ is the projection of ΠY_λ onto Y_λ.

(ii) $f : \Pi X_\lambda \to \Pi Y_\lambda$ is al.$c.\nu.c$, iff $f_\lambda : X_\lambda \to Y_\lambda$ is al.$c.\nu.c$ for each $\lambda \in \Lambda$.

Note 2. Converse of Theorem 3.7 is not true in general, as shown by the following example.

Example 5. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1 : X \to X_1$ be defined as follows: $f_1(x) = 1$ if $0 \leq x \leq \frac{1}{2}$ and $f_1(x) = 0$ if $\frac{1}{2} < x \leq 1$. Let $f_2 : X \to X_2$ be defined as follows: $f_2(x) = 1$ if $0 \leq x < \frac{1}{2}$ and $f_2(x) = 0$ if $\frac{1}{2} < x < 1$. Then $f_i : X \to X_i$ is clearly al.$c.\nu.c$ for $i = 1, 2$, but $h(x) = (f_1(x_1), f_2(x_2)) : X \to X_1 \times X_2$ is not al.$c.\nu.c$, for $S_{\frac{1}{2}}(1, 0)$ is regular open in $X_1 \times X_2$, but $h^{-1}(S_{\frac{1}{2}}(1, 0)) = \{\frac{1}{2}\}$ which is not $\nu-$closed in X.

Remark 3. In general,

(i) The algebraic sum and product of two al.$c.\nu.c$ functions is not al.$c.\nu.c$ However the scalar multiple of a al.$c.\nu.c$ function is al.$c.\nu.c$.

(ii) The pointwise limit of a sequence of al.$c.\nu.c$ functions is not al.$c.\nu.c$ as shown by.

Example 6. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1 : X \to X_1$ and $f_2 : X \to X_2$ are defined as follows: $f_1(x) = x$ if $0 < x < \frac{1}{2}$ and $f_1(x) = 0$ if $\frac{1}{2} < x < 1; f_2(x) = 0$ if $0 < x < \frac{1}{2}$ and $f_2(x) = 1$ if $\frac{1}{2} < x < 1$.

Example 7. Let $X = Y = [0, 1]$. Let f_n is defined as follows: $f_n(x) = x_n$ for $n \geq 1$ then f is the limit of the sequence where $f(x) = 0$ if $0 \leq x < 1$ and $f(x) = 1$ if $x = 1$. Therefore f is not al.$c.\nu.c$ For $(\frac{1}{2}, 1]$ is open in Y, $f^{-1}((\frac{1}{2}, 1]) = (1)$ is not $\nu-$closed in X.

However we can prove the following theorem.

Theorem 3.8. Uniform Limit of sequence of al.c.ν.c. functions is al.c.ν.c.

Problem. (i) Are sup$\{f,\ g\}$ and inf$\{f,\ g\}$ are al.$c.\nu.c$ if f, g are al.$c.\nu.c$.

(ii) Is $C_{al.c.\nu.c}(X, R)$, the set of all al.$c.\nu.c$ functions.

(1) a Group.

(2) a Ring.

(3) a Vector space.

(4) a Lattice.

(iii) Suppose $f_i : X \to X_i (i = 1, 2)$ are al.$c.\nu.c$ If $f : X \to X_1 \times X_2$ defined by $f(x) = (f_1(x), f_2(x))$, then f is al.$c.\nu.c.$

Solution. No.

Note 3. In general al.c.c., al.c.α.c. and al.c.p.c. are independent of al.$c.\nu.c$ as shown by Example 1 and 3.

Theorem 3.9. (i) If f is $\nu-$irresolute and g is al.$c.\nu.c$, then $g \circ f$ is al.$c.\nu.c.$

(ii) If f is al.$c.\nu.c$ and g is continuous [resp: r-continuous] then $g \circ f$ is al.$c.\nu.c.$

(iii) If f and g are r-irresolute then $g \circ f$ is $\nu-$continuous.

(iv) If f is al.$c.\nu.c$ and g is r-irresolute, then $g \circ f$ is al.$c.\nu.c$; al.c.s.c and al.$c.\beta.c.$

(v) If f is al.$c.\nu.c$[contra r-irresolute] g is al.g.c.[al.rg.c] and GO(Y) = RGO(Y) = RO(Y), then $g \circ f$ is al.$c.\nu.c.$

Theorem 3.10. If f is $\nu-$irresolute, $\nu-$open and $\nu O(X) = \tau$ and g be any function, then $g \circ f : X \to Z$ is c.ν.c iff g is al.c.ν.c.

Proof. If part: Theorem 3.9 only if part: Let $A \in RC(Z)$. Then $(g \circ f)^{-1}(A)$ is a $\nu-$open and hence open in X [by assumption]. Since f is $\nu-$open $f(g \circ f)^{-1}(A) = g^{-1}(A)$ is $\nu-$open in Y. Thus $g : Y \to Z$ is al.c.ν.c.

Corollary 3.1: (i) If f is a surjective M-$\nu-$open [resp: M-$\nu-$closed] and g is a function such that $g \circ f$ is al.c.ν.c., then g is al.c.ν.c.

(ii) If f is $\nu-$irresolute, M-$\nu-$open and bijective, g is a function. Then g is al.c.ν.c. iff $g \circ f$ is al.c.ν.c.

Theorem 3.11. If $g : X \to X \times Y$, defined by $g(x) = (x, f(x)) \forall x \in X$ be the graph function of f. Then g is al.$c.\nu.c$ iff f is al.$c.\nu.c.$

Proof. Let $V \in RC(Y)$, then $X \times V = X \times \overline{V^0} = \overline{X^0} \times \overline{V^0} = \overline{(X \times V)^0} \in RC(X \times Y)$. Since g is al.c.$\nu$.c., then $f^{-1}(V) = g^{-1}(X \times V) \in \nu O(X)$. Thus, f is al.c.$\nu$.c.

Conversely, let $x \in X$ and $F \in RC(X \times Y)$ containing g(x). Then $F \cap (\{x\} \times Y)$ is r-closed in $\{x\} \times Y$ containing g(x). Also $\{x\} \times Y$ is homeomorphic to Y. Hence $\{y \in Y : (x, y) \in F\}$ is r-closed subset of Y. Since f is al.c.ν.c. $\bigcup\{f^{-1}(y) : (x, y) \in F\}$ is $\nu-$open in X. Further $x \in \bigcup\{f^{-1}(y) : (x, y) \in F\} \subseteq g^{-1}(F)$. Hence $g^{-1}(F)$ is $\nu-$open. Thus g is al.c.ν.c.

Remark 4. In general, composition of two al.$c.\nu.c$ functions is not al.$c.\nu.c.$ However we have the following example:

Example 8. Let $X = Y = Z = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$; $\sigma = \{\phi, \{a\}, \{b, c\}, Y\}$, and $\eta = \{\phi, \{a\}, \{b\}, \{a, b\}, Z\}$. Let f be identity map and g be be defined as g(a) = a = g(b); g(c) = c; are al.$c.\nu.c$ and $g \circ f$ is also al.c.ν.c.

Theorem 3.12. Let X, Y, Z be spaces and every $\nu-$closed set be r-open in Y, then the composition of two al.$c.\nu.c$ maps is al.$c.\nu.c.$

Corollary 3.2. If f is al.$c.\nu.c$ [r-irresolute],

(i) g is al.c [r-continuous], then $g \circ f$ is al.c.s.c. and hence al.c.β.c.

(ii)g is al.g.c.{al.rg.c.} and Y is $r - T_{\frac{1}{2}}$, then $g \circ f$ is al.c.s.c. and hence al.c.β.c.

Theorem 3.13. (i) If RαC(X) = RC(X) then f is al.c.rα.c. iff f is contra r-irresolute.

(ii) If RαC(X) = νC(X) then f is al.c.rα.c. iff f is al.$c.\nu.c.$

(iii) If νC(X) = RC(X) then f is r-irresolute iff f is al.$c.\nu.c.$

(iv) If νC(X) = αC(X) then f is al.c.α.c. iff f is al.$c.\nu.c.$

(v) If νC(X) = SC(X) then f is al.c.s.c. iff f is al.$c.\nu.c.$

(vi) If νC(X) = βC(X) then f is al.c.β.c. iff f is al.$c.\nu.c.$

Note 4. Pasting Lemma is not true with respect to al.$c.\nu.c$ functions. However we have the following weaker versions.

Theorem 3.14. Let X and Y be such that $X = A \cup B$. Let $f_{/A} : A \to Y$ and $g_{/B} : B \to Y$ are r-irresolute functions such that $f(x) = g(x) \forall x \in A \cap B$. Suppose A and B are r-closed sets in X and RC(X) is closed under finite unions, then the combination $\alpha : X \to Y$ is al.$c.\nu.c.$

Theorem 3.15. Pasting Lemma. Let X and Y be such that $X = A \cup B$. Let $f_{/A} : A \to Y$ and $g_{/B} : B \to Y$ are al.$c.\nu.c$ maps such that f(x) = g(x) $\forall x \in A \cap B$. Suppose A, B are r-closed sets in X and νC(X) is closed under finite unions, then the combination $\alpha : X \to Y$ is al.$c.\nu.c.$

Proof. Let F be r-open set in Y, then $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$ where $f^{-1}(F)$ is $\nu-$closed in A and $g^{-1}(F)$ is $\nu-$closed in $B \Rightarrow f^{-1}(F)$ and $g^{-1}(F)$ are $\nu-$closed in $X \Rightarrow f^{-1}(F) \cup g^{-1}(F)$ is $\nu-$closed in X[by assumption] $\Rightarrow \alpha^{-1}(F)$ is $\nu-$closed in X. Hence α is al.$c.\nu.c.$

Theorem 3.16. The following statements are equivalent for a function f:

(i) f is al.c.ν.c.;

(ii) $f^{-1}(F) \in \nu O(X)$ for every $F \in RC(Y)$;

(iii) for each $x \in X$ and each regular closed set F in Y containing f(x), there exists a $\nu-$open set U in X containing x such that $f(U) \subset F$;

(iv) for each $x \in X$ and each regular open set V in Y non-containing f(x), there exists a $\nu-$closed set K in X non-containing x such that $f^{-1}(V) \subset K$;

(v) $f^{-1}((\overline{G})^o) \in \nu C(X)$ for every open subset G of Y;

(vi) $f^{-1}(\overline{F^o}) \in \nu O(X)$ for every closed subset F of Y.

Proof. (i) $\Leftrightarrow$ (ii): Let $F \in RC(Y)$. Then $Y - F \in RO(Y)$. By (i), $f^{-1}(Y - F) = X - f^{-1}(F) \in \nu C(X)$. We have $f^{-1}(F) \in \nu O(X)$. Reverse can be obtained similarly.

(ii)$\Rightarrow$(iii): Let $F \in RC(Y, f(x))$. By (ii), $x \in f^{-1}(F) \in \nu O(X)$. Take $U = f^{-1}(F)$. Then $f(U) \subset F$.

(iii)$\Rightarrow$(ii): Let $F \in RC(Y)$ and $x \in f^{-1}(F)$. From (iii), $\exists\ U_x \in \nu O(X, x) \ni U \subset f^{-1}(F)$. We have $f^{-1}(F) = \bigcup_{x \in f^{-1}(F)} U_x$. Thus $f^{-1}(F)$ is $\nu-$open.

(iii)$\Leftrightarrow$(iv): Let $V \in RO(Y)$ not containing f(x). Then, $Y - V \in RC(Y, f(x))$. By (3), $\exists$ $U \in \nu O(X, x) \ni f(U) \subset Y - V$. Hence, $U \subset f^{-1}(Y - V) \subset X - f^{-1}(V)$ and then $f^{-1}(V) \subset X - U$. Take $H = X - U$, then H is $\nu-$closed in X non-containing x. The converse can be shown easily.

(i)$\Leftrightarrow$(v): Let $G \in \sigma(Y)$. Since $(\overline{G})^o \in RO(Y)$, by (i), $f^{-1}((\overline{G})^o) \subset \nu C(X)$. The converse can be shown easily.

(ii)$\Leftrightarrow$(vi): It can be obtained smilar as (i)$\Leftrightarrow$(v).

Example 9. Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} = \sigma$. Then the identity function $f : X \to X$ is al.c.ν.c. But it is not regular set-connected.

Example 10. Let $X = \{a, b, c\}, \tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, X\}$. Then the identity function f on X is al.c.ν.c. but not c.ν.c. and ν.c.

Theorem 3.17. Let f be al.$c.\nu.c$ and $A \in RC(X)$, then $f_{/A} : A \to Y$ is al.$c.\nu.c.$

Proof. Let $V \in RO(Y) \Rightarrow f_{/A}^{-1}(V) = f^{-1}(V) \cap A$ is $\nu-$closed in A. Hence $f_{/A}$ is al.$c.\nu.c.$

Remark 5. Every restriction of an al.c.ν.c. function is not necessarily al.c.ν.c.

Example 11. Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b, c, d\}, X\}$. The identity function $f \colon X \to X$ is al.c.ν.c., but, if $A = \{a, c, d\}$ is not regular-open in (X, σ) and σ_A is the relative topology on A induced by σ, then $f_{|A} : (A, \sigma_A) \to (X, \tau)$ is not al.c.ν.c.

Note that {b, c, d} is regular closed in (X, τ), but that $f_{|A}^{-1}(\{b, c, d\}) = \{c, d\}$ is not $\nu-$open in (A, σ_A).

Theorem 3.18. Let f be a function and $\Sigma = \{U_\alpha : \alpha \in I\}$ be a $\nu-$cover of X. If for each $\alpha \in I$, $f_{|U_\alpha}$ is al.c.ν.c., then f is an al.c.ν.c. function.

Proof. Let $F \in RC(Y)$. $f_{|U_\alpha}$ is al.c.ν.c. for each $\alpha \in I$, $f_{|U_\alpha}^{-1}(F) \in \nu O_{|U_\alpha}$. Since $U_\alpha \in \nu O(X)$, by the previous lemma, $f_{|U_\alpha}^{-1}(F) \in \nu O(X)$ for each $\alpha \in I$. Then $f^{-1}(F) = \bigcup_{\alpha \in I} f_{|U_\alpha}^{-1}(F) \in \nu O(X)$. This gives f is an al.c.ν.c.

Theorem 3.19. Let f be a function and $x \in X$. If there exists $U \in \nu O(X) \ni x \in U$ and the restriction of $f_{|U}$ is al.c.ν.c. at x, then f is al.c.ν.c. at x.

Proof. Suppose that $F \in RC(Y)$ containing f(x). Since $f_{|U}$ is al.c.ν.c. at x, $\exists\ V \in \nu O(U, x) \ni f(V) = (f_{|U})(V) \subset F$. Since $U \in \nu O(X, x)$; then $V \in \nu O(X, x)$. Thus f is al.c.ν.c. at x.

Theorem 3.20. For f and g, the following properties hold:

(i) If f is al.c.ν.c. and g is regular set-connected, then $g \circ f$ is al.c.ν.c. and al.ν.c.

(ii) If f is al.c.ν.c. and g is perfectly continuous, then $g \circ f$ is $\nu-$continuous and c.ν.c.

Proof. (i) Let$V \in RO(Z)$. Since g is regular set-connected, $g^{-1}(V)$ is clopen. Since f is al.c.ν.c., $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\nu-$clopen. Therefore, $g \circ f$ is al.c.ν.c. and al.ν.c.

(ii) can be obtained similarly.

Theorem 3.21. If f is r-irresolute and al.c.c., then f is regular set-connected.

Theorem 3.22. If f is al.c.ν.c., then for each point $x \in X$ and each filter base Λ in X $\nu-$converging to x, the filter base $f(\Lambda)$ is rc-convergent to f(x).

Theorem 3.23. For f, the following properties are equivalent:

(i) f is (ν, s)-continuous;

(ii) f is al.c.ν.c.;

(iii) $f^{-1}(V)$ is $\nu-$open in X for each θ-semi-open set V of Y;

(iv) $f^{-1}(F)$ is $\nu-$closed in X for each θ-semi-closed set F of Y.

Proof. (i)$\Rightarrow$(ii): Let $F \in RC(Y)$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and F is semi-open. Since f is (ν, s)-continuous, $\exists U \in \nu O(X, x) \ni f(U) \subset \overline{F} = F$. Hence $x \in U \subset f^{-1}(F)$ which implies that $x \in \nu(f^{-1}(F))^0$. Therefore, $f^{-1}(F) \subset \nu(f^{-1}(F))^0$ and hence $f^{-1}(F) = \nu(f^{-1}(F))^0$. This shows that $f^{-1}(F) \in \nu O(X)$. Hence f is al.c.ν.c.

(ii) $\Rightarrow$(iii): Follows from the fact that every θ-semi-open set is the union of regular closed sets.

(iii) $\Leftrightarrow$(iv): This is obvious.

(iv) $\Rightarrow$ (i): Let $x \in X$ and $V \in SO(Y, f(x))$. Since $\overline{V}$ is regular closed, it is θ-semi-open. Now, put $U = f^{-1}(\overline{V})$. Then $U \in \nu O(X, x)$ and $f(U) \subset \overline{V}$. This shows that f is ($\nu$, s)-continuous.

Theorem 3.24. For f, the following properties are equivalent:

(i) f is al.c.ν.c.;

(ii) $f(\nu(\overline{A})) \subset sCl_\theta(f(A))$ for every subset A of X;

(iii) $\nu\overline{(f^{-1}(B))} \subset f^{-1}(sCl_\theta(B))$ for every subset B of Y.

Proof. (i)⇒ (ii): Let A be any subset of X. Suppose that $x \in \nu\overline{(A)}$ and $G \in SO(Y, f(x))$. Since f is al.c.ν.c., by Theorem 3.23, $\exists U \in \nu O(X,x) \ni f(U) \subset \overline{G}$. Since $x \in \nu\overline{(A)}$, $U \cap A \neq \phi$; and hence $\phi \neq f(U) \cap f(A) \subset \overline{G} \cap f(A)$. Therefore, $f(x) \in \overline{sCl_\theta(f(A))}$ and hence $f(\nu\overline{(A)}) \subset sCl_\theta(f(A))$.

(ii) ⇒(iii): Let B be any subset of Y. Then $f(\nu\overline{(f^{-1}(B))}) \subset sCl_\theta(\overline{f(f^{-1}(B))}) \subset sCl_\theta(B)$ and hence $\nu\overline{(f^{-1}(B))} \subset f^{-1}(sCl_\theta(B))$.

(iii) ⇒(i): Let $V \in SO(Y, f(x))$. Since $\overline{V} \cap (Y - \overline{V}) = \phi$, we have $f(x) \notin sCl_\theta(Y - \overline{V})$ and hence $x \notin f^{-1}(sCl_\theta(Y - \overline{V}))$. By (3), $x \notin \nu\overline{(f^{-1}(Y - \overline{V}))}$, then $\exists\ U \in \nu O(X,x) \ni U \cap f^{-1}(Y - \overline{V}) = \phi$; hence $f(U) \cap (Y - \overline{V}) = \phi$. This shows that $f(U) \subset \overline{V}$. Therefore f is al.c.ν.c.

Theorem 3.25. For f, the following properties are equivalent:

(i) f is al.c.ν.c.;

(ii) $f^{-1}(\overline{V})$ is $\nu-$open in X for every $V \in \beta(Y)$;

(iii) $f^{-1}(\overline{V})$ is $\nu-$open in X for every $V \in SO(Y)$;

(iv) $f^{-1}((\overline{V})^o)$ is $\nu-$closed in X for every $V \in RO(Y)$.

Proof. (i) ⇒ (ii): Let V be any β-open set of Y. It follows from Theorem 2.4 of [2] that $\overline{V}$ is regular closed. Then by Theorem 3.16, $f^{-1}(\overline{V}) \in \nu O(X)$.

(ii) ⇒ (iii): This is obvious since $SO(Y) \subset \beta(Y)$.

(iii) ⇒ (iv): Let $V \in RO(Y)$. Then $Y - (\overline{V})^o$ is regular closed and hence semi-open. Then, $X - f^{-1}((\overline{V})^o) = f^{-1}(Y - (\overline{V})^o) = f^{-1}(\overline{(Y - (\overline{V})^o)}) \in \nu O(X)$. Hence $f^{-1}((\overline{V})^o) \in \nu C(X)$.

(iv) ⇒ (i): Let $V \in RO(Y)$. Then $V \in \nu O(Y)$ and hence $f^{-1}(V) = f^{-1}((\overline{V})^o) \in \nu C(X)$.

Corollary 3.3. For f, the following properties are equivalent:

(i) f is al.c.ν.c.;

(ii) $f^{-1}(\alpha\overline{V})$ is $\nu-$open in X for every $V \in \beta(Y)$;

(iii) $f^{-1}(\nu\overline{V})$ is $\nu-$open in X for every $V \in SO(Y)$;

(iv) $f^{-1}(s\overline{V})$ is $\nu-$closed in X for every $V \in RO(Y)$.

Proof. This is an immediate consequence of Theorem 3.25 and Lemma 2.2.

The $\nu-$frontier of $A \subset X$; is defined by $\nu Fr(A) = \nu\overline{(A)} - \nu\overline{(X - A)} = \nu\overline{(A)} - \nu(A)^0$.

Theorem 3.26. For f, the following conditions are equivalent:

(i) f is al.c.ν.c.;

(ii) $\nu\overline{(f^{-1}(V))} \subseteq f^{-1}(sCl_\theta(V))$ for every open subset V of Y;

(iii) $\nu\overline{(f^{-1}(V))} \subseteq f^{-1}(s\overline{(V)})$ for every open subset V of Y;

(iv) $\nu\overline{(f^{-1}(V))} \subseteq f^{-1}((\overline{V})^o)$ for every open subset V of Y;

(v) $\overline{(f^{-1}(V))^o} \subseteq f^{-1}((\overline{V})^o)$ for every open subset V of Y.

Proof. (i)⇒(ii) follows from Theorem 3.24(c).

(ii) ⇒(iii) follows from Lemma 2.1(ii).

(iii) ⇒(iv) follows from Lemma 2.1(iii).

(iv) ⇒(v). Since $\nu\overline{(f^{-1}(V))} = f^{-1}(V) \cup \overline{(f^{-1}(V))^o)}$, it follows from (iv) that $\overline{(f^{-1}(V))^o)} \subseteq f^{-1}((\overline{V})^o)$.

(v) ⇒(i). Let $V \in RO(Y)$. Then by (v), $\overline{(f^{-1}(V))^o)} \subseteq f^{-1}((\overline{V})^o) = f^{-1}(V)$. Therefore $f^{-1}(V)$ is $\nu-$closed, which proves that f is al.c.ν.c.

The next result is an immediate consequence of Theorems 3.24 and 3.26.

Theorem 3.27. Let f be a function and let S be any collection of subsets of Y containing the open sets. Then f is al.c.ν.c. iff $\nu\overline{(f^{-1}(S))} \subseteq f^{-1}(sCl_\theta(S))$ for every $S \in S$.

Corollary 3.4. For f, the following properties are equivalent:

(i) f is al.c.ν.c.;

(ii) $\nu\overline{(f^{-1}(V))} \subseteq f^{-1}(sCl_\theta(V))$ for every $V \in SO(Y)$;

(iii) $\nu\overline{(f^{-1}(V))} \subseteq f^{-1}(sCl_\theta(V))$ for every $V \in PO(Y)$;

(iv) $\nu\overline{(f^{-1}(V))} \subseteq f^{-1}(sCl_\theta(V))$ for every $V \in \beta O(Y)$.

Theorem 3.28. $\{x \in X : f : X \to Y$ is not al.c.$\nu.c.\}$ is identical with the union of the $\nu-$frontier of the inverse images of regular closed sets of Y containing $f(x)$.

Proof. Suppose that f is not al.c.ν.c. at $x \in X$. By Theorem 3.16, $\exists\ F \in RC(Y, f(x) \ni f(U) \cap (Y - F) \neq \phi$ for every $U \in \nu O(X, x)$. Then, $x \in \nu\overline{(f^{-1}(Y - F))} = \nu\overline{(X - f^{-1}(F))}$. On the other hand, we get $x \in f^{-1}(F) \subset \nu\overline{(f^{-1}(F))}$ and hence $x \in \nu Fr(f^{-1}(F))$.

Conversely, suppose that f is al.c.ν.c. at x and let $F \in RC(Y, f(x))$. By Theorem 3.16, there exists $U \in \nu O(X, x) \ni x \in U \subset f^{-1}(F)$. Therefore, $x \in \nu(f^{-1}(F))^o$. This contradicts that $x \in \nu Fr(f^{-1}(F))$. Thus f is not al.c.ν.c.

Theorem 3.29. If f is al.c.ν.c. and Y is T_2, then $G(f)$ is $\nu-$regular graph in $X \times Y$.

Proof. Assume Y is T_2. Let $(x, y) \in (X \times Y) - G(f)$. It follows that $f(x) \neq y$. Since Y is T_2, there exist disjoint open sets V and W containing f(x) and y, respectively. We have $((\overline{V})^o) \cap ((\overline{W})^o) = \phi$. Since f is al.c.$\nu$.c., $f^{-1}((\overline{V})^o)$ is $\nu-$closed in X containing x. Take $U = f^{-1}((\overline{V})^o)$. Then $f(U) \subset ((\overline{V})^o)$. Therefore, $f(U) \cap ((\overline{W})^o) = \phi$ and $G(f)$ is $\nu-$regular in $X \times Y$.

Remark 6. al.ν.c. and al.c.ν.c. are independent to each other.

It is shown that $Cl_\theta(V) = \overline{V}$ for every open set V and $Cl_\theta(S)$ is closed for every subset S of X.

Theorem 3.30. Let (Y, σ) be E.D. Then, a function f is al.c.ν.c. iff it is al.ν.c.

Proof. Necessity. Let $x \in X$ and $V \in RO(Y, f(x))$. Since Y is E. D., by Lemma 5.6 of [26] V is clopen and hence V is regular closed. By Theorem 3.16, there exists $U \in \nu O(X, x) \ni f(U) \subset V$. By Lemma 2.4, f is al.ν.c.

Sufficiency. Let F be any regular closed set of Y . Since (Y, σ) is E. D., F is also regular open and $f^{-1}(F)$ is $\nu-$open in X. This shows that f is al.c.ν.c.

§4. The preservation theorems and some other properties

Theorem 4.1. If f is al.c.ν.c.[resp: al.c.r.c] surjection and X is $\nu-$compact, then Y is nearly closed compact.

Proof. Let $\{G_i : i \in I\}$ be any regular-closed cover for Y. Since f is al.c.ν.c., $\{f^{-1}(G_i)\}$ forms a $\nu-$open cover for X and hence have a finite subcover, since X is $\nu-$compact. Since f is surjection, $Y = f(X) = \bigcup_{i=1}^{n} G_i$. Therefore Y is nearly closed compact.

Corollary 4.1. If f is al.c.ν.c.[r-irresolute], surjection, then the following statements hold:

(i) If X is locally $\nu-$compact, then Y is locally nearly closed compact; locally mildly compact.

(ii) If X is $\nu-$Lindeloff [locally $\nu-$lindeloff], then Y is nearly closed Lindeloff [resp: locally nearly closed Lindeloff; locally mildly lindeloff].

(iii)If X is $\nu-$compact [countably $\nu-$compact], then Y is S-closed [countably S-closed].

(iv) If X is $\nu-$Lindelof, then Y is S-Lindelof [nearly Lindelof].

(v) If X is $\nu-$closed [countably $\nu-$closed], then Y is nearly compact [nearly countably compact].

Theorem 4.2. If f is contra r-iresolute and al.c., surjection and X is mildly compact (resp. mildly countably compact, mildly Lindelof), then Y is nearly compact (resp. nearly countably compact, nearly Lindelof) and S-closed (resp. countably S-closed, S-Lindelof).

Proof. Since f is contra r-iresolute and al.c., for $\{V_\alpha : \alpha \in I\}$ be any regular closed (resp: regular open) cover of Y, $\{f^{-1}(V_\alpha : \alpha \in I\}$ is a clopen cover of X and since X is mildly compact, $\exists$ a finite subset I_0 of I $\ni X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$. Hence Y is S-closed (resp: nearly compact). The other proofs can be obtained similarly.

Theorem 4.3. If f is al.c.ν.c., surjection and

(i) X is $\nu-$compact [$\nu-$lindeloff] then Y is mildly closed compact [mildly closed lindeloff].

(ii) X is s-closed then Y is mildly compact [mildly lindeloff].

Theorem 4.4. If X is $\nu-$ultra-connected and f is al.c.ν.c. and surjective, then Y is hyperconnected.

Proof. If Y is not hyperconnected. Then $\exists\ V \in \sigma \ni \overline{V} \neq Y$. Then $\exists$ disjoint non-empty regular open subsets B_1 and B_2 in Y. Since f is al.c.ν.c. and onto, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint non-empty $\nu-$closed subsets of X. By assumption, the $\nu-$ultra-connectedness of X implies that A_1 and A_2 must intersect, which is a contradiction. Therefore Y is hyperconnected.

Theorem 4.5. If f is al.c.ν.c.[contra $\nu-$irreolute] surjection and X is $\nu-$connected, then Y is connected [$\nu-$connected].

Proof. If Y is disconnected. Then $Y = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$. Since f is al.c.ν.c., $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $\nu-$open sets in X and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$, which is a contradiction for $\nu-$connectedness of X. Hence Y is connected.

Corollary 4.2. The inverse image of a disconnected [$\nu-$disconnected] space under a al.c.ν.c.[contra $\nu-$irreolute] surjection is $\nu-$disconnected.

Theorem 4.6. If f is al.c.ν.c., injection and

(i) Y is UT_i, then X is $\nu - T_i$ [hence semi-T_i and $\beta - T_i$], $i = 0, 1, 2$.

(ii) Y is UR_i, then X is $\nu - R_i$ [hence semi-R_i and $\beta - R_i$], $i = 0, 1$.

(iii) If f is closed and Y is UT_i, then X is $\nu - T_i$ [hence semi-T_i and $\beta - T_i$], $i = 3, 4$.

(iv) Y is UC_i[resp : $\mathrm{UD_i}$] then X is $\nu - T_i$[resp: $\nu - D_i$], hence X is semi-T_i [resp: semi-D_i] and $\beta - T_i$ [resp: $\beta - D_i$], $i = 0, 1, 2$.

Theorem 4.7. If f is al.c.ν.c.[resp: al.c.r.c] and Y is UT_2, then the graph G(f) of f is $\nu-$closed in the product space $X \times Y$.

Proof. Let $(x_1, x_2) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint clopen sets V and $W \ni f(x) \in V$ and y$\in W$. Since f is al.c.ν.c., $\exists U \in \nu O(X) \ni x \in U$ and $f(U) \subset W$. Therefore $(x, y) \in U \times V \subset X \times Y - G(f)$. Hence G($f$) is $\nu-$closed in $X \times Y$.

Corollary 4.3. If f is al.c.ν.c. and Y is UT_2, then the graph G(f) of f is semi-closed [resp: $\beta-$closed and semi-θ-closed] in the product space $X \times Y$.

Theorem 4.8. If f is al.c.ν.c.[al.c.r.c] and Y is UT_2, then $A = \{(x_1, x_2)|f(x_1) = f(x_2)\}$ is $\nu-$closed[and hence semi-closed and $\beta-$closed] in the product space $X \times X$.

Proof. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow \exists$ disjoint $V_j \in CO(\sigma) \ni f(x_j) \in V_j$, and since f is al.c.ν.c., $f^{-1}(V_j) \in \nu O(X, x_j)$ for each $j = 1, 2$. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in \nu O(X \times X)$ and $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A$. Hence A is $\nu-$closed.

Theorem 4.9. If f is r-irresolute{al.c.c.}; $g : X \to Y$ is c.ν.c; and Y is UT_2, then $E = \{x \in X : f(x) = g(x)\}$ is $\nu-$closed [and hence semi-closed and $\beta-$closed] in X.

Theorem 4.10. If f is an al.c.ν.c. injection and Y is weakly Hausdorff, then X is $\nu - T_1$.

Proof. Suppose that Y is weakly Hausdorff. For any $x \neq y \in X, \exists V, W \in RC(Y) \ni f(x) \in V, f(y) \notin V, f(x) \notin W$ and $f(y) \in W$. Since f is al.c.ν.c., $f^{-1}(V)$ and $f^{-1}(W)$ are $\nu-$open subsets of X such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is $\nu - T_1$.

Theorem 4.11. If for each pair $x_1 \neq x_2 \in X$ there exists a function f of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ and f is al.c.ν.c., at x_1 and x_2, then X is $\nu - T_2$.

Proof. Let $x_1 \neq x_2$. Then by the hypothesis $\exists$ a function f which satisfies the condition of this theorem. Since Y is Urysohn and $f(x_1) \neq f(x_2)$, $\exists$ open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$, respectively, such that $\overline{V_1} \cap \overline{V_2} = \phi$. Since f is al.$\nu$.c., at x_i, $\exists U_i \in \nu O(X, x_i) \ni f(U_i) \subset \overline{V_i}$ for $i = 1, 2$. Hence, we obtain $U_1 \cap U_2 = \phi$. Therefore, X is $\nu - T_2$.

Corollary 4.4. If f is r-irresolute injection and Y is Urysohn, then X is $\nu - T_2$.

Definition 4.1. A function f is said to have a strongly contra-$\nu-$closed graph if for each $(x, y) \in (X \times Y) - G(f)$ there exists $U \in \nu O(X, x)$ and a regular closed set V of Y containing y such that $(U \times V) \cap G(f) = \phi$.

Lemma 4.1. f has a strongly contra-$\nu-$closed graph iff for each $(x, y) \in (X \times Y) - G(f)$ $\exists U \in \nu O(X, x)$ and $V \in RC(Y, y) \ni f(U) \cap V = \phi$.

Theorem 4.12. If f is injective al.c.ν.c. with the strongly contra-$\nu-$closed graph, then X is $\nu - T_2$.

Proof. Let $x \neq y \in X$. Since f is injective, we have $f(x) \neq f(y)$ and $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly contra-$\nu-$closed, by Lemma 5.1, $\exists$ $U \in \nu O(X, x)$ and a $V \in RC(Y, f(y)) \ni f(U) \cap V = \phi$. Since f is al.c.ν.c., by Theorem 3.16, $\exists$ $G \in \nu O(X, y) \ni f(G) \subset V$. Therefore, we have $f(U) \cap f(G) = \phi$; hence $U \cap G = \phi$. Hence X is $\nu - T_2$.

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Scientia Magna
Vol. 7 (2011), No. 2, 95-98

Counterexamples to a theorem concerning solution of certain quadratic Diophantine equation

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Abstract By considering the Diophantine equation of the form $x^2 - Dy^2 = 4$ where D is a positive integer that is not perfect square and fundamental solution of $x^2 - Dy^2 = 4$ with central norm equal to 4 associated with a principal norm of 8 we reformulate a theorem which is given by R. A. Mollin in [2]. In this paper we find some counterexamples to the theorem, thus disproves it and it is verified by using C computer programme that it is satisfied.

Keywords Quadratic Diophantine equations, central norms.

2000 Mathematics Subject Classifacition: 11D09, 11R11, 11A55.

§1. Introduction

In [2], R. A. Mollin considered the problem of giving necessary and sufficent conditions for the solvability of the Diophantine equation $x^2 - Dy^2 = 4$ and posed several theorems concerning the relations between fundamental solution of the Diophantine equation $x^2 - Dy^2 = 4$ with $\gcd(x, y) = 1$ and central norm equals to 4 associated with a principal norm of 8, which is an analogue of the generalized Lagrange result.

The following theorem was given by R. A. Mollin in [2].

Theorem 1. If $D = 4c$, c is odd, $\ell(\sqrt{D}) = \ell$ is even with $Q_{\ell/2} = 4$, and $Q_j = 8$ for some j, then the following hold:

(1) $c \equiv 3, 7 \pmod{16}$ if and only if j is even.

(2) $c \equiv 11, 15 \pmod{16}$ if and only if j is odd.

But the theorem is incorrect. In this paper we revise Theorem 1. Also, we describe an algorithm for calculate j with value of c and give some counterexamples to Theorem 1.

§2. Notation and preliminaries

We will be concerned with the simple continued fraction expansions of $\sqrt{D}$ where D is an integer that is not perfect square. We denote this expansion

$$\sqrt{D} = \langle q_0; \overline{q_1, q_2, \ldots, q_{\ell-1}, 2q_0} \rangle,$$

where $\ell(\sqrt{D}) = \ell$ is the period length, $q_0 = \lfloor \sqrt{D} \rfloor$ (the floor of $\sqrt{D}$) and $q_1, q_2, \ldots, q_{\ell-1}$ is a palindrome. The jth convergent of $\sqrt{D}$ for $j \geq 0$ is given by,

$$\frac{A_j}{B_j} = \langle q_0; q_1, q_2, \ldots, q_j, \rangle,$$

where

$$A_j = q_j A_{j-1} + A_{j-2}, \quad B_j = q_j B_{j-1} + B_{j-2},$$

with $A_{-2} = 0, A_{-1} = 1, B_{-2} = 1, B_{-1} = 0$. The complete quotients are given by, $(P_j + \sqrt{D})/Q_j$, where $P_0 = 0, Q_0 = 0$ and for $j \geq 1$,

$$P_{j+1} = q_j Q_j - P_j, \quad q_j = \lfloor (P_j + \sqrt{D})/Q_j \rfloor, \quad D = P_{j+1}^2 + Q_j Q_{j+1}.$$

We will also need the following facts:

$$A_j B_{j-1} - A_{j-1} B_j = (-1)^{j-1}, \quad A_{j-1}^2 - B_{j-1}^2 D = (-1)^j Q_j.$$

When ℓ is even, $P_{\ell/2} = P_{\ell/2+1}$ and $Q_{\ell/2} | 2P_{l/2}$, where $Q_{\ell/2} | 2D$ and $Q_{\ell/2}$ is called the central norm. In general, the values Q_j are called the principal norms, since they are the norms of principal reduced ideals in order $Z[\sqrt{D}]$. (Also, see [1] for a more advanced exposition)

We will be considering Diophantine equations $x^2 - Dy^2 = 4$. The fundamental solution of such an equation means the (unique) least positive integers $(x, y) = (x_0, y_0)$ satisfying it.

§3. Revision of Theorem 1

Firstly, we reformulate Theorem 1 as the following:

Theorem 2. If $D = 4c,\ c$ is old, $\ell(\sqrt{D}) = \ell$ is even with $Q_{\ell/2} = 4$ and $Q_j = 8$ for some j, then the following hold:

(i) $c \equiv 7, 15 (\bmod 16)$ if and only if j is even.

(ii) $c \equiv 3, 11 (\bmod 16)$ if and only if j is odd.

Proof. Theorem 1 is correct for $c \equiv 7 (\bmod 16)$ and for $c \equiv 11 (\bmod 16)$ in part (1) and in part (2) of Theorem 1, respectively. Therefore, we only prove for $c \equiv 15 (\bmod 16)$ and $c \equiv 3 (\bmod 16)$.

Let $c \equiv 15 (\bmod 16)$. The solution $A_{j-1}^2 - B_{j-1}^2 D = (-1)^j 8$ exists if and only if

$$1 = (A_{j-1}^2 / c) = ((-1)^j 8 / c) = ((-1)^j / c)(2/c) = (-1)^{(4j(c-1)+c^2-1)/8}.$$

Since $D = 4c$ and c is odd, there exists an integer k such that $c = 16k + 15$. If we calculate $(4j(c-1) + c^2 - 1)/8$ with respec to the values of c except for j, then we have

$$(4j(c-1) + c^2 - 1)/8 = j(8k + 7) + (32k^2 + 60k + 28).$$

Now we assume j is an odd integer. Then $(4j(c-1)+c^2-1)/8$ is odd. Therefore, from equation (1), we have

$$(-1)^{(4j(c-1)+c^2-1)/8} = -1 \neq 1.$$

This is a contradiction and so j is an even integer for $c \equiv 15 \pmod{16}$.

Conversely, suppose that j is an even integer. Then there exists an integer m such that $j = 2m$. From equation (1), we get for integers k

$$1 = (-1)^{(4j(c-1)+c^2-1)/8} = (-1)^{(8m(c-1)+c^2-1)/8} = (-1)^{2k}$$

and so $(8m(c-1)+c^2-1)/8 = 2k$. Thus, we have $2c^2 - 2 \equiv 0 \pmod{16}$ and so $c \equiv 15 \pmod{16}$.

Proof of (ii) is the analogue of (i).

Now we describe a procedure to calculate j with value of c.

```
P[0] = 0 : Q[0] = 1 : A[0] = 0 : A[1] = 1 : B[0] = 1 : B[1] = 0
For k = 0, ··· , max d
c ← 16 * k + 15
D ← 4 * c
q[0] ← int(sqrt(D))
For n = 1, ··· , max d
P[n] ← q[n − 1] * Q[n − 1] − P[n − 1]
Q[n] ← (D − P[n] * P[n])/Q[n − 1]
q[n] ← int(P[n]+sqrt(float D)/Q[n]
For j = 1, ··· , n
if q[n] equal 2*q[0] and q[j] equal q[j + n]
period ← period +1
End of For
For i = 1, ··· , n
if (Q⌊n/2⌋ equal 4) compute Q[i]
if (Q[i] equal 8) Display i
End of For
End of For
End of For
```

Using above algorithms, we seek some values of j on computer and get many counterexamples to Theorem 1.

Example 1. Let we take $D = 204 = 4.51$ where $c = 51 \equiv 3 + 3.16 \equiv 3 (\mathrm{mod}\, 16)$. Then we have, $\ell(\sqrt{D}) = \ell = 8$, $Q_{\ell/2} = 4$ and $Q_j = 8$. But, this case holds only for odd numbers $j = 1$ and $j = 7$.

Example 2. Now let we take $D = 508 = 4.127$ where $c = 127 \equiv 15 + 7.16 \equiv 15 (\mathrm{mod}\, 16)$. Then we have, $\ell(\sqrt{D}) = \ell = 32$, $Q_{\ell/2} = 4$ and $Q_j = 8$. But, this case holds only for even numbers $j = 6$ and $j = 26$.

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Scientia Magna
Vol. 7 (2011), No. 2, 99-105

Neighborhood symmetric n-sigraphs

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Abstract An n-tuple $(a_1, a_2, ..., a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. The neighborhood graph of a graph $G = (V, E)$, denoted by $N(G)$, is a graph on the same vertex set V, where two vertices in $N(G)$ are adjacent if, and only if, they have a common neighbor. Analogously, one can define the neighborhood symmetric n-sigraph $N(S_n)$ of a symmetric n-sigraph $S_n = (G, \sigma)$ as a symmetric n-sigraph, $N(S_n) = (N(G), \sigma')$, where $N(G)$ is the underlying graph of $N(S_n)$, and for any edge $e = uv$ in $N(S_n)$, $\sigma'(e) = \mu(u)\mu(v)$, where for any $v \in V$, $\mu(v) = \prod_{u \in N(v)} \sigma(uv)$. In this paper, we characterize symmetric n-sigraphs S_n for which $S_n \sim N(S_n)$, $S_n^c \sim N(S_n)$ and $N(S_n) \sim J(S_n)$, where $J(S_n)$ and S_n^c denotes jump symmetric n-sigraph and complement of symmetric n-sigraph of S_n respectively.

Keywords Symmetric n-sigraphs, symmetric n-marked graphs, balance, switching, neighborhood symmetric n-sigraphs, line symmetric n-sigraphs, jump symmetric n-sigraphs.

§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [6]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An n-tuple $(a_1, a_2, ..., a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function.

In this paper by an n-tuple/n-sigraph/n-marked graph we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the identity n-tuple, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a non-identity n-tuple. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an identity edge, otherwise it is a non-identity edge.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the n-tuple $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [11], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. S. K. Reddy [8]):

Definition. Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is identity balanced (or i-balanced), if product of n-tuples on each cycle of S_n is the identity n-tuple, and

(ii) S_n is balanced, if every cycle in S_n contains an even number of non-identity edges.

Note. An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [11].

Proposition 1.1.[11] An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. Complement of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1.[13]

In [11], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See [7,9,10,13-16]).

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be isomorphic, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, switching S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the μ-switched n-sigraph or just switched n-sigraph.

Further, an n-sigraph S_n switches to n-sigraph S'_n (or that they are switching equivalent to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be cycle isomorphic, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n.

We make use of the following known result.

Proposition 1.2.[11] Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

§2. Neighborhood n-sigraphs

For any graph G, neighborhood graph $N(G)$ of G is a graph on the same vertex set $V(G)$, with two vertices are adjacent if, and only if, they have a common neighbor. Neighborhood graphs are also known as 2-path graphs (See [1]). Further, a graph G is said to be neighborhood graph if $G \cong N(H)$. The neighborhood of a vertex v is the set of all vertices adjacent to v. Clearly, $N(G)$ is the intersection graph of neighborhoods of G. Neighborhood graphs was first introduced by C. R. Cook [4] as H_2-graph of a graph. B. D. Acharya [2] introduced the notion as open neighborhood graph of a given graph as intersection graph of neighbors of vertices of G. Later F. Escalante et al.[5] introduced the notion of n-path graphs as follows: For any integer n, the n-path graph $(G)_n$ of a graph G, as a graph on the same vertex set and two vertices are adjacent if, and only if, there exists a path of length n in G. Thus 2-path graphs are nothing but neighborhood graph.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of neighborhood graphs to n-sigraphs as follows: The neighborhood n-sigraph $N(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $N(G)$ and the n-tuple of any edge uv in $N(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called neighborhood n-sigraph, if $S_n \cong N(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion of neighborhood n-sigraph as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to neighborhood n-sigraphs.

Proposition 2.1. For any n-sigraph $S_n = (G, \sigma)$, its neighborhood n-sigraph $N(S_n)$ is i-balanced.

Proof. Since the n-tuple of any edge uv in $N(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Proposition 1.1, $N(S_n)$ is i-balanced.

The following result is due to B. D. Acharya and M. N. Vartak [2] which gives a characterization of neighborhood graphs:

Proposition 2.2. A graph $G = (V, E)$, where $V = \{v_1, v_2, ..., v_p\}$ is a neighborhood graph if, and only if edges of G can be included in p complete subgraphs $H_1, H_2, ..., H_p$, where the subgraphs can be indexed so that

(i) $v_i \notin H_i$ and;

(ii) $v_i \in H_j$ if, and only if $v_j \in H_i$.

Proposition 2.3. Suppose an n-sigraph $S_n = (G, \sigma)$ is a neighborhood n-sigraph. Then S_n is i-balanced and G is a neighborhood graph.

Proof. Suppose that S_n is a neighborhood n-sigraph. That is there exists an n-sigraph $S'_n = (G', \sigma')$ such that $N(S'_n) = S_n$ and hence $N(G') \cong G$. That is G is a neighborhood graph. Also, by Proposition 2.1, the neighborhood n-sigraph of any n-sigraph is i-balanced, it follows that $N(S'_n) = S_n$ is i-balanced.

Problem 2.4. Characterize neighborhood n-sigraphs.

Proposition 2.5. For any two n-sigraphs S_n and S'_n with the same underlying graph, their neighborhood n-sigraphs are switching equivalent.

The following results are due to R. C. Brigham and R. D. Dutton [3] which gives characterization of graphs for which $N(G) \cong G$ and $N(G) \cong \overline{G}$.

Proposition 2.6. For a connected graph G, $N(G) \cong G$ if, and only if, G is either a complete graph or an odd cycle of order ≥ 3.

Proposition 2.7. For a graph $G = (V, E)$, the following are equivalent:

(i) $N(G) \cong \overline{G}$;

(ii) There is a permutation f of the vertex set V such that uv is an edge in G if, and only if, $f(u)$ and $f(v)$ have no common neighbor.

For any positive integer k, the iterated neighborhood graph of G is defined as follows:

$$N^0(G) = G, N^k(G) = N(N^{k-1}(G)).$$

Proposition 2.8. For any graph G and any integer $k \geq 1$, the k^{th}-iterated neighborhood graph $N^k(G) \cong G$ if, and only if, $N(G) \cong G$.

The following result characterizes the family of n-sigraphs satisfies $S_n \sim N(S_n)$.

Proposition 2.9. A connected n-sigraph $S_n = (G, \sigma)$ satisfies $S_n \sim N(S_n)$ if, and only if, S_n is i-balanced and G is either an odd cycle or a complete graph.

Proof. Suppose $S_n \sim N(S_n)$. This implies, $G \cong N(G)$ and hence by Proposition 2.6, we see that the graph G is either an odd cycle or a complete graph. Now, if S_n is any n-sigraph with underlying graph is complete or is an odd cycle, Proposition 3 implies that $N(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its neighborhood n-sigraph $N(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n i-balanced n-sigraph on a complete graph or an odd cycle. Then, since $N(S_n)$ is i-balanced as per Proposition 2.1 and since $G \cong N(G)$ by Proposition 2.6, the result follows from Proposition 2.1 again.

Proposition 2.10. For an n-sigraph $S_n = (G, \sigma)$, the following are equivalent:

(i) $N(S_n) \sim S_n^c$;

(ii) There is a permutation f of the vertex set V such that uv is an edge in G if, and only if, $f(u)$ and $f(v)$ have no common neighbor.

Proof. Suppose that $N(S_n) \sim S_n^c$. Then clearly we have $N(G) \cong \bar{G}$. Hence by Proposition 2.7, there is a permutation f of the vertex set V such that uv is an edge in G if, and only if, $f(u)$ and $f(v)$ have no common neighbor.

Conversely, suppose that there is a permutation f of the vertex set V such that uv is an edge in G if, and only if, $f(u)$ and $f(v)$ have no common neighbor. Then again by Proposition 2.7, $N(G) \cong \bar{G}$. Since both $N(S_n)$ and S_n^c are balanced for any n-sigraph S_n, the result follows by Proposition 1.2 again.

§3. Switching equivalence of neighborhood n-sigraphs and line n-sigraphs

The line graph $L(G)$ of graph G has the edges of G as the vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent. The line n-sigraph of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph $L(S_n) = (L(G), \sigma')$, where for any edge ee' in $L(S_n)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by E. Sampatkumar et al.[12]

Proposition 3.1. (**E. Sampathkumar et al.**[12]) For any n-sigraph $S_n = (G, \sigma)$, its line n-sigraph $L(S_n)$ is i-balanced.

For any positive integer k, the k^{th} iterated line n-sigraph, $L^k(S_n)$ of S_n is defined as follows:

$$L^0(S_n) = S_n,\ L^k(S_n) = L(L^{k-1}(S_n)).$$

Corollary 3.2. For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $L^k(S_n)$ is i-balanced.

The following result due to B. D. Acharya [1] gives a characterization of graphs for which $L(G) \cong N(G)$.

Proposition 3.3. (**B. D. Acharya** [1]) For a connected graph $G = (V, E)$, $L(G) \cong N(G)$ if and only if G satisfies the following conditions:

(i) G is unicyclic and the cycle C of G is of odd length m=2n+1, $n \geq 1$.

(ii) If G contains a vertex v not on the cycle then, $d(v, C) \leq 2$.

(iii) If there exists at least one vertex v not on the cycle C, with $d(v, C) = 2$, then $C = C_3$. Further, all such vertices not on the cycle C and at a distance 2 from C have degree 1 and are adjacent to a unique point, say v, which is adjacent to exactly one vertex of C.

(iv) If degrees of all the vertices are distinct, then $C = C_3$ and any vertex not on the cycle C, is at a distance 1 from C.

(v) If the cycle C is of length more than 3 say $C = C_m$ with m=2n+1 then, there exists vertices v_i and v_j of C (not necessarily distinct) such that at least one of the two systems S_1 and S_2 given below, among the degrees d_k of vertices $v_k \in C$ holds:

S_1: $d_i = d_j$, $d_{i+r} = d_{j+r}$, $1 \leq r \leq n-i$, $d_{n+r} = d_{j+2(n-i+r)-1}$, $1 \leq r \leq n$, $d_r = d_{j+2(n-i+r)+1}$, $1 \leq r \leq n-i$.

S_2: $d_i = d_j$, $d_{i+r} = d_{j+2(n-r)+1}, 1 \leq r\ n-i+1$, $d_{n+r} = d_{j+2(i+n-r)-2}, 2 \leq r \leq n+1$, $d_r = d_{j+2(i+n-r)-1}, 1 \leq r \leq i-1$.

The following result gives a characterization of those n-sigraphs whose neighborhood n-sgraphs are switching equivalent to their line n-sigraphs.

Proposition 3.4. For any n-sigraph $S_n = (G, \sigma)$, $N(S_n) \sim L(S_n)$ if, and only if, S_n is an i-balanced n-sigraph and satisfies conditions (i) to (iv) of Proposition 3.3.

§4. Switching equivalence of neighborhood n-sigraphs and jump n-sigraphs

The jump graph $J(G)$ of a graph $G = (V, E)$ is $\overline{L(G)}$, the complement of the line graph $L(G)$ of G (see [6]).

We now give a characterization of graphs for which $N(G) \cong J(G)$.

Proposition 4.1. The jump graph $J(G)$ of a connected graph G is isomorphic to $N(G)$, the neighborhood graph of G if, and only if, G is C_5.

Proof. Suppose G is a connected graph such that $N(G) \cong J(G)$. Hence number of vertices and number of edges are equal and so G must be unicyclic. Since $J(C_n)$ is a cycle if, and only if, $n = 5$ and $N(C_n)$ is either C_n or two copies of $C_{n/2}$ according as n is odd or even, it follows that the cycle in G is C_5. Now suppose that there exists a vertex in C_5 of degree ≥ 3, then the edge not on the cycle is adjacent to 3 vertices in $J(G)$, where as the vertex in $N(G)$ is adjacent two vertices of the cycle. Hence $G \cong C_5$. The converse part is obvious.

The jump n-sigraph of an n-sigraph $S_n = (G, \sigma)$ is an signed graph $J(S) = (J(G), \sigma')$, where for any edge ee' in $J(S_n)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by E. Sampathkumar et al.[11]

Proposition 4.2. (**E. Sampathkumar et al.**[11]) For any n-sigraph $S_n = (G, \sigma)$, its jump n-sigraph $J(S_n)$ is i-balanced.

For any positive integer k, the k^{th} iterated jump n-sigraph, $J^k(S_n)$ of S_n is defined as follows:

$$J^0(S_n) = S_n,\ J^k(S_n) = J(J^{k-1}(S_n)).$$

Corollary 4.3. For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $J^k(S_n)$ is i-balanced.

We now give a characterization of n-sigraphs whose jump n-sigraphs are switching equivalent to their neighborhood n-sigraphs.

Proposition 4.4. A connected n-sigraph $S_n = (G, \sigma)$ satisfies $N(S_n) \sim J(S_n)$ if and only if S_n is an n-sigraph on C_5, cycle on 5 vertices.

Proof. Suppose that $N(S_n) \sim J(S_n)$. Then clearly $N(G) \cong J(G)$. Hence by Proposition 4.1, G must be C_5.

Conversely, suppose that S_n is an n-sigraph on C_5. Then by Proposition 4.1, $N(G) \cong J(G)$. Since for any n-sigraph S_n, both $N(S_n)$ and $J(S_n)$ are balanced, the result follows by Proposition 1.2.

Acknowledgement

The authors would like to acknowledge and extend their heartfelt gratitude to Sri. B. Premnath Reddy, Chairman, Acharya Institutes, for his vital encouragement and support.

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Scientia Magna
Vol. 7 (2011), No. 2, 106-113

The influence of $SCAP$ and S-supplemented subgroups on the p-nilpotency of finite groups[1]

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Abstract In this paper, we investigate the influence of some subgroups of Sylow subgroups with semi cover-avoiding property and S-supplementation on the structure of finite groups. Some conditions of p-nilpotency are obtained and some recent results are generalized.

Keywords $SCAP$, S-supplemented, p-nilpotent, subgroup.

§1. Introduction

Throughout the paper, all groups are finite. We use conventional notions and notation, as in Huppert [1]. G always denotes a group, $|G|$ is the order of G, $O_p(G)$ is the maximal normal p-subgroup of G, $O^p(G) =< g \in G \mid p \nmid o(g) >$ and $\Phi(G)$ is the Frattini subgroup of G.

Let L/K be a normal factor of a group G. A subgroup H of G is said to cover L/K if $HL = HK$, and H is said to avoid L/K if $H \cap L = H \cap K$. If H covers or avoids every chief factor of G, then H is said to have the cover-avoiding property in G, i.e., H is a CAP-subgroup of G. This conception was first studied by Gaschütz (see [2]) to study the solvable groups, later by Gillam (see [3]) and Tomkinson (see [4]). In Ezquerro (see [5]) gave some characterizations for a group G to be p-supersolvable and supersolvable under the assumption that all maximal subgroups of some Sylow subgroups of G have the cover-avoiding property in G. For example, Ezquerrohas proved: Let G be a group with a normal subgroup H such that G/H is supersolvable. Then G is supersolvable if one of following holds: (1) all maximal subgroups of the Sylow subgroups of H are CAP-subgroups of G; (2) H is solvable and all maximal subgroups of the Sylow subgroups of $F(H)$ are CAP-subgroups of G. Asaad (see [6]) said that it is possible to extend Ezquerro's results with formation theory. Recently, Guo and Shum pushed further this approach and obtained some charaterizations for a solvable group and a p-solvable group based on the assumption that some of its subgroups are CAP-subgroups

[1]The project is supported by the N. S. F. of China (No: 11071229) and the N. S. F. of the Jiangsu Higher Education Institutions (No: 10KJD110004).

(see [7]). More recently, in Fan et al. (see [8]) introduced the semi cover-avoiding property, which is the generalization not only of the cover-avoiding property but also of c-normality (see [9]). A subgroup H of a group G is said to have the semi cover-avoiding property in G, i.e., H is an $SCAP$-subgroup of G, if there exists a chief series of G such that H either covers or avoids every G-chief factor of this series. The results in Guo and Shum (see [7]) and Wang (see [9]) were extended with the requirement that the certain subgroups of G are $SCAP$-subgroups (see [10, 11]). More recently, many authors invest presented some conditions for a group to be p-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup are $SCAP$-subgroups (see [12, 13, 14]).

A subgroup H of a group G is said to be S-quasinormal (or π-quasinormal) in G if H permutes with all Sylow subgroups of G, i.e., $HS = SH$ for any Sylow subgroup S of G. This concept was introduced by Kegel in [15]. As another generalization of S-quasinormal subgroups, A. N. Skiba (see [16]) introduced the following concept: A subgroup H of a group G is called weakly S-supplemented (or S-supplemented) in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-quasinormal in G. In fact, this concept is also a generalization of c-supplemented subgroups given in [17]. By using S-supplemented subgroups, many interesting results in finite groups were obtained (see [18, 19, 20]). For example, Skiba proved: Let E be a normal subgroup of a finite group G. Suppose that for every non-cyclic Sylow subgroup P of E, either all maximal subgroups of P or all cyclic subgroups of P of prime order and order 4 are S-supplemented in G. Then each G-chief factor below E is cyclic.

There are examples to show that semi cover-avoiding property and S-supplementation can not imply from one to the other one. In this paper, we will try an attempt to unify the two concepts and establish the structure of groups under the assumption that all maximal subgroups or all minimal subgroups of a Sylow subgroup or are $SCAP$ or S-supplemented subgroups. Our theorems generalize and unify some known results, such as in [11, 13, 26, 27, 28].

§2. Preliminaries

In this section, we list some lemmas which will be useful for the proofs of our main results.

Lemma 2.1.[11] (Lemma 2.5 and 2.6) Let H be an $SCAP$ subgroup of a group G.

(1) If $H \leq L \leq G$, then H is an $SCAP$ subgroup of L.

(2) If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is an $SCAP$ subgroup of G/N.

(3) If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is an $SCAP$ subgroup of G/N.

Lemma 2.2.[16] (Lemma 2.10) Let H be an S-supplemented subgroup of a group G.

(1) If $H \leq L \leq G$, then H is S-supplemented in L.

(2) If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is S-supplemented in G/N.

(3) If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is S-supplemented in G/N.

Lemma 2.3.[16] (Lemma 3.1) Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let P be a p-Sylow subgroup of G. If there is a maximal subgroup P_1 of P

such that P_1 has the semi cover-avoiding property in G, then G is p-solvable.

Lemma 2.4.[22] (Lemma 2.8) Let M be a maximal subgroup of G and P a normal p-subgroup of G such that $G = PM$, where p is a prime. Then $P \cap M$ is a normal subgroup of G.

Lemma 2.5.[23] (Lemma 2.7) Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1) If N is normal in G of order p, then $N \leq Z(G)$.

(2) If G has cyclic Sylow p-subgroup, then G is p-nilpotent.

(3) If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.

Lemma 2.6.[25] (Lemma 2.6) If P is a S-quasinormal p-subgroup of a group G for some prime p, then $O^p(G) \leq N_G(P)$.

Lemma 2.7.[24] (Main Theorem) Suppose that G has a Hall π-subgroup where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.

Lemma 2.8. ([1], IV, 5.4) Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.9. ([1], III, 5.2) Suppose G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then

(a) G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(b) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(c) If P is non-abelian and $p > 2$, then the exponent of P is p; If P is non-abelian and $p = 2$, then the exponent of P is 4.

(d) If P is abelian, then the exponent of P is p.

(e) $Z(G) = \Phi(P) \times \Phi(Q)$.

§3. P-nilpotentcy

Theorem 3.1. Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either an $SCAP$ or an S-supplemented subgroup of G, then G is p-nilpotent.

Proof. We distinguish two cases:

Case I. $H = G$.

Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Suppose that $M/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then there exists a maximal subgroup P_1 of P such that $M = P_1O_{p'}(G)$. By the hypothesis, P_1 is either an $SCAP$ or an S-supplemented subgroup of G, then $M/O_{p'}(G) = P_1O_{p'}(G)/O_{p'}(G)$ is either an $SCAP$ or an S-supplemented subgroup of $G/O_{p'}(G)$ by Lemma 2.1 and 2.2. It is clear that

$(|G/O_{p'}(G)|, p-1) = 1$. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. Therefore, we have $O_{p'}(G) = 1$.

(2) $O_p(G) \neq 1$.

If all maximal subgroups of P are S-supplemented in G, then G is p-nilpotent by [21, Lemma 3.1]. Therefore we may assume that there is a maximal subgroup P_1 of P which is an $SCAP$ subgroup of G. By Lemma 2.3, G is p-solvable. Since $O_{p'}(G) = 1$ by Step (1), we have $O_p(G) \neq 1$.

(3) G is solvable.

If $p \neq 2$, then G is odd from the assumption that $(|G|, p-1) = 1$. By the famous Odd Order Theorem, G is solvable. If $p = 2$, then $O_2(G) \neq 1$ by Step (2). Suppose that $M/O_2(G)$ is a maximal subgroup of $P/O_2(G)$. Then M is a maximal subgroup of P. By the hypothesis, M is either an $SCAP$ or an S-supplemented subgroup of G, then $M/O_2(G)$ is either an $SCAP$ or an S-supplemented subgroup of $G/O_2(G)$ by Lemma 2.1 and 2.2. Therefore $G/O_2(G)$ satisfies the hypothesis of the theorem. The minimal choice of G implies that $G/O_2(G)$ is 2-nilpotent, and so $G/O_2(G)$ is solvable. It follows that G is solvable.

(4) $O_p(G)$ is the unique minimal normal subgroup of G.

Let N be a minimal normal subgroup of G. By Step (3), N is an elementary abelian subgroup. Since $O_{p'}(G) = 1$, we have N is p-subgroup and so $N \leq O_p(G)$. It is easy to see that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Choose M to be a maximal subgroup of G such that $G = NM$. Obviously, $G = O_p(G)M$ and so $O_p(G) \cap M$ is normal in G by Lemma 2.4. The uniqueness of N yields $N = O_p(G)$.

(5) The final contradiction.

By the proof in Step (4), G has a maximal subgroup M such that $G = MO_p(G)$ and $G/O_p(G) \cong M$ is p-nilpotent. Clearly, $P = O_p(G)(P \cap M)$. Furthermore, $P \cap M < P$. Thus, there exists a maximal subgroup V of P such that $P \cap M \leq V$. Hence, $P = O_p(G)V$. By the hypothesis, V is either an $SCAP$ or a S-supplemented subgroup of G. First, we assume that V is an $SCAP$ of G. Since $O_p(G)$ is the unique minimal normal subgroup of G, V covers or avoids $O_p(G)/1$. If V covers $O_p(G)/1$, then $VO_p(G) = V$, i.e., $O_p(G) \leq V$. It follows that $P = O_p(G)V = V$, a contradiction. If V avoids $O_p(G)/1$, then $V \cap O_p(G) = 1$. Since $V \cap O_p(G)$ is a maximal subgroup of $O_p(G)$, we have that $O_p(G)$ is of order p and so $O_p(G)$ lies in $Z(G)$ by Lemma 2.5. By the proof in Step (4), we have $G/O_p(G)$ is p-nilpotent. Then $G/Z(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction. Now, we may assume that V is an S-supplemented subgroup of G. Then there is a subgroup T of G such that $G = VT$ and $V \cap T \leq V_{sG}$. From Lemma 2.6, we have $O^p(G) \leq N_G(V_{sG})$. Since V_{sG} is subnormal in G, we have $V \cap T \leq V_{sG} \leq O_p(G)$. Thus, $V_{sG} \leq V \cap O_p(G)$ and

$$V_{sG} \leq (V_{sG})^G = (V_{sG})^{O^p(G)P} = (V_{sG})^P \leq (V \cap O_p(G))^P = V \cap O_p(G) \leq O_p(G).$$

It follows that $(V_{sG})^G = 1$ or $(V_{sG})^G = V \cap O_p(G) = O_p(G)$. If $(V_{sG})^G = V \cap O_p(G) = O_p(G)$, then $O_p(G) \leq V$ and $P = O_p(G)V = V$, a contradiction. If $(V_{sG})^G = 1$, then $V \cap T = 1$ and so $|T|_p = p$. Hence, T is p-nilpotent by Lemma 2.5. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \trianglelefteq G$ and $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence, we may assume $M = N_G(M_{p'})$. By applying Lemma 2.7 and Feit-Thompson's theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence, $T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of V. Thus, $G = VT^g = VM$ and $P = V(P \cap M) = V$, a contradiction.

Case II. $H < G$.

By Lemma 2.1 and 2.2, every maximal subgroup of P is an $SCAP$ or S-supplemented subgroup of H. By Case I, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \trianglelefteq G$. Assume $H_{p'} \neq 1$ and consider $G/H_{p'}$. Applying Lemma 2.1 and 2.2, it is easy to see that $G/H_{p'}$ satisfies the hypotheses for the normal subgroup $H/H_{p'}$. Therefore, by induction $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence, we may assume $H_{p'} = 1$ and so $H = P$ is a p-group. Since G/H is p-nilpotent, we can let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. A new application of Case II yields K is p-nilpotent and so $K = H \times K_{p'}$. Hence, $K_{p'}$ is a normal p-complement of G, i.e., G is p-nilpotent.

Corollary 3.2. Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P is either an $SCAP$ or an S-supplemented subgroup of G, then G is p-nilpotent.

Proof. It is clear that $(|G|, p-1) = 1$ if p is the smallest prime dividing the order of G and so Corollary 3.2 follows immediately from Theorem 3.1.

Corollary 3.3. Suppose that every maximal subgroup of any Sylow subgroup of a group G is either an $SCAP$ or an S-supplemented subgroup of G, then G is a Sylow tower group of supersolvable type.

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By Corollary 3.2, G is p-nilpotent. Let U be the normal p-complement of G. By Lemma 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of U is either an $SCAP$ or an S-supplemented subgroup of U. Thus U satisfies the hypothesis of the Corollary. It follows by induction that U, and hence G is a Sylow tower group of supersolvable type.

Corollary 3.4.[26] (Theorem 3.1) Let G be a group, p a prime dividing the order of G, and P a Sylow p-subgroup of G. If $(|G|, p-1) = 1$ and every maximal subgroup of P is an $SCAP$ subgroup of G, then G is p-nilpotent.

Corollary 3.5.[11] (Theorem 3.2) Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divisor of $|G|$. If P is cyclic or every maximal subgroup of P is an $SCAP$ subgroup of G, then G is p-nilpotent.

Proof. If P is cyclic, by Lemma 2.5, we have G is p-nilpotent. Thus we may assume that every maximal subgroup of P is an $SCAP$ subgroup of G. By Corollary 3.2, G is p-nilpotent.

Theorem 3.6. Suppose N is a normal subgroup of a group G such that G/N is p-nilpotent, where p is a fixed prime number. Suppose every subgroup of order p of N is contained in the hypercenter $Z_\infty(G)$ of G. If $p = 2$, in addition, suppose every cyclic subgroup of order 4 of N is either an $SCAP$ or an S-supplemented subgroup of G, then G is p-nilpotent.

Proof. Suppose that the theorem is false, and let G be a counterexample of minimal order.

(1) The hypotheses are inherited by all proper subgroups, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

In fact, $\forall K < G$, since G/N is p-nilpotent, $K/K \cap N \cong KN/N$ is also p-nilpotent. The cyclic subgroup of order p of $K \cap N$ is contained in $Z_\infty(G) \cap K \leq Z_\infty(K)$, the cyclic subgroup of order 4 of $K \cap N$ is either an $SCAP$ or an S-supplemented subgroup of G, then is either an $SCAP$ or an S-supplemented subgroup of K by Lemma 2.1 and 2.2. Thus $K, K \cap N$ satisfy the hypotheses of the theorem in any case, so K is p-nilpotent, therefore G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. By Lemma 2.8 and 2.9, $G = PQ$, $P \trianglelefteq G$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(2) $G/P \cap N$ is p-nilpotent.

Since $G/P \cong Q$ is nilpotent, G/N is p-nilpotent and $G/P \cap N \lesssim G/P \times G/N$, therefore $G/P \cap N$ is p-nilpotent.

(3) $P \leq N$.

If $P \nleq N$, then $P \cap N < P$. So $Q(P \cap N) < QP = G$. Thus $Q(P \cap N)$ is nilpotent by (1), $Q(P \cap N) = Q \times (P \cap N)$. Since $G/P \cap N = P/P \cap N \cdot Q(P \cap N)/P \cap N$, it follows that $Q(P \cap N)/P \cap N \trianglelefteq G/P \cap N$ by Step (2). So Q char $Q(P \cap N) \trianglelefteq G$. Therefore, $G = P \times Q$, a contradiction.

(4) $p = 2$.

If $p > 2$, then $\exp(P) = p$ by (a) and Lemma 2.9. Thus $P = P \cap N \leq Z_\infty(G)$. It follows that $G/Z_\infty(G)$ is nilpotent, and so G is nilpotent, a contradiction.

(5) For every $x \in P \backslash \Phi(P)$, we have $\circ(x) = 4$.

If not, there exists $x \in P \backslash \Phi(P)$ and $\circ(x) = 2$. Denote $M =< x^G >\leq P$. Then $M\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, we have that $P = M\Phi(P) = M \leq Z_\infty(G)$ as $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ by Lemma 2.9, a contradiction.

(6) For every $x \in P \backslash \Phi(P)$, $< x >$ is supplemented in G.

Let $x \in P \backslash \Phi(P)$. Then x either an $SCAP$ or an S-supplemented subgroup of G by Step (5) and the hypothesis. We assume that x has the semi cover-avoiding property in G. In this case, there exists a chief series of G

$$1 = G_0 < G_1 < \cdots < G_t = G,$$

such that x covers or avoids every G_j/G_{j-1}. Since $x \in G$, for some k, $x \notin G_k$ but $x \in G_{k+1}$. It follows from $G_k \cap < x > \neq G_{k+1} \cap < x >$ that $G_k < x >= G_{k+1} < x >= G_{k+1}$. Hence G_{k+1}/G_k is a cyclic group of order 4. The normality of $P \cap G_k$ implies that $(P \cap G_k)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we see that $(P \cap G_k)\Phi(P) = \Phi(P)$ or P. If $(P \cap G_k)\Phi(P) = P$, then $P \cap G_k = P$, contradicting $x \notin P \cap G_k$. Thus $P \cap G_k \leq \Phi(P)$. Since $x \notin \Phi(P)$ but $x \in P \cap G_{k+1}$, $P \cap G_{k+1} = P$, i.e., $P \leq G_{k+1}$.Therefore, $P = P \cap G_k < x >=< x > (P \cap G_k) = \Phi(P) < x >=< x >$. By Lemma 2.9, P is an elementary abelian group and so P does not have an element of order 4, a contradiction.

(7) Final contradiction.

For any $x \in P \backslash \Phi(P)$, we may assume that x is supplemented in G by Step (6). Then there is a subgroup T of G such that $G =< x > T$ and $< x > \cap T \leq < x >_{sG}$. It follows that $P = P \cap G = P\cap < x > T =< x > (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $< x >= P \trianglelefteq G$, a contraction. If $P = (P\cap T)\Phi(P) = P\cap T$, then $T = G$ and so $< x >=< x >_{sG}$ is s-permutable in G. We have $< x > Q$ is a proper subgroup of G and so $< x > Q =< x > \times Q$, i.e., $< x >\leq N_G(Q)$. By Lemma 2.9, $\Phi(P) \subseteq Z(G)$. Therefore we have $P \leq N_G(Q)$ and so $Q \trianglelefteq G$, a contradiction.

Corollary 3.7. Suppose N is a normal subgroup of a group G such that G/N is nilpotent and every minimal subgroup of N is contained in the hypercenter $Z_\infty(G)$ of G. If $p = 2$, in addition, suppose every cyclic subgroup of order 4 of N is either an $SCAP$ or an S-supplemented subgroup of G, then G is nilpotent.

Corollary 3.8.[27] (Theorem 4.3) Suppose N is a normal subgroup of a group G such that G/N is nilpotent and every minimal subgroup of N is contained in the hypercenter $Z_\infty(G)$ of G. If $p = 2$, in addition, suppose every cyclic subgroup of order 4 of N is c-supplemented subgroup of G, then G is nilpotent.

Corollary 3.9.[28] (Theorem 2.5) Suppose that p is a prime and $K = G^{\mathcal{N}}$ be the nilpotent residual of G. Then G is p-nilpotent if every minimal subgroup of K is contained in $Z_\infty(G)$ and every cyclic $< x >$ of K with order 4 is c-supplemented in G.

Corollary 3.10.[29] (Theorem 2.4) Let G be a finite group and $K = G^{\mathcal{N}}$ be the nilpotent residual of G. Then G is nilpotent if and only if every minimal subgroup $< x >$ of K lies in the hypercenter $Z_\infty(G)$ of G and every cyclic element of P with order 4 is c-normal in G.

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Scientia Magna
Vol. 7 (2011), No. 2, 114-116

The Smarandache adjacent number sequences and its asymptotic property

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Abstract The main purpose of this paper is using the elementary method to study the Smarand -ache adjacent number sequences, and give several interesting asymptotic formula for it.

Keywords Smarandache adjacent number sequences, elementary method, asymptotic formula.

§1. Introduction

For any positive integer n, the famous Smarandache adjacent number sequences $\{a(n,m)\}$ are defined as the number of such set, making the number of each set can be divided into several same parts, where m represent the bits of n. For example, Smarandache $a(1,1) = 1$, $a(2,1) = 22$, $a(3,1) = 333$, $a(4,1) = 4444$, $a(5,1) = 55555$, $a(6,1) = 666666$, $a(7,1) = 7777777$, $a(8,1) = 88888888$, $a(9,1) = 999999999$, $a(10,2) = 10101010101010101010,\ldots$, $a(100,3) = \underbrace{100\cdots 100}_{100},\ldots$, and so on.

In the reference [1], Professor F. Smarandache asked us to study the properties of this sequence. About this problem, it seems that none had studied them before, at least we couldn't find any reference about it.

The problem of this sequence's first n items summation is meaningful. After a simple deduction and calculation, we can get a complex formula, but it's not ideal. So we consider the asymptotic problem of the average $\ln a(n,1) + \ln a(n,2) + \cdots + \ln a(N,M)$. We use the elementary method and the property of integral nature of the carrying to prove the following conclusion:

Theorem. If m is the bits of n, for any positive integer N,we have the asymptotic formula:

$$\sum_{n\le N} \ln a(n,m) = N\cdot \ln N + O\left(N\right).$$

But the two asymptotic formulas is very rough, we will continue to study the precise asymptotic formulas.

§2. Proof of the theorem

In this section, we shall use the elementary methods to prove our theorems directly. First, we give one simple lemma which is necessary in the proof of our theorem. The proof of this lemma can be found in the reference [8].

Lemma 1. If f has a continuous derivative f' on the interval $[x, y]$, where $0 < y < x$,

$$\sum_{y<k\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

Then, we consider the structure of $\{a(n,m)\}$. We will get the following equations:
$a(1,1) = 1$,
$a(2,1) = 2\cdot 10^1 + 2\cdot 10^0$,
$a(3,1) = 3\cdot 10^2 + 3\cdot 10^1 + 3\cdot 10^0$,
$a(4,1) = 4\cdot 10^3 + 4\cdot 10^2 + 4\cdot 10^1 + 4\cdot 10^0$,
$\cdots$,
$a(9,1) = 9\cdot 10^8 + 9\cdot 10^7 + \cdots + 9\cdot 10^2 + 9\cdot 10^1 + 9\cdot 10^0$,
$a(10,2) = 10\cdot 10^{18} + 10\cdot 10^{16} + \cdots + 10\cdot 10^2 + 10\cdot 10^0$
$\cdots$,
$a(100,3) = 100\cdot 10^{297} + 100\cdot 10^{294} + \cdots + 100\cdot 10^3 + 100\cdot 10^0$
$\cdots$,
$a(n,m) = n\cdot n^{297} + n\cdot 10^{294} + \cdots + n\cdot 10^m + n\cdot 10^0$.
If we analysis the above equations, we can get :

$$\begin{aligned}\prod_{1\leq n\leq N} a(n,M) &= \left(\prod_{n=1}^{9} a(1,n)\right)\cdots\left(\prod_{n=10^{M-1}-1}^{10^{(10^{M-1}-1)\cdot(M-1)}} a(n,M-1)\right)\cdot\left(\prod_{n=10^{M-1}}^{N} a(N,M)\right)\\ &= N!\frac{(10-1)\cdot(10^2-1)\cdots(10^{(10^{M-1}-1)\cdot(M-1)}-1)\cdot(10^{(10^M-1)\cdot M}-1)}{(10-1)^9\cdot(10^2-1)^{90}\cdots(10^{M-1}-1)^{9\cdot 10^{M-2}}\cdot(10^M-1)^{9\cdot 10^{M-1}}}. \quad (1)\end{aligned}$$

When $x \to 0$, we note that the estimation $\ln(1+x) = x + O\left(x^2\right)$, so we have

$$\begin{aligned}\sum_{k=1}^{M}\ln\left(10^k+1\right)^{9\cdot 10^{k-1}} &= 9\cdot\sum_{k=1}^{M}10^{k-1}\cdot\left(k\cdot\ln 10 + \frac{1}{10^k} + O\left(\frac{1}{10^{2k}}\right)\right)\\ &= \sum_{k=1}^{M} k\cdot 10^{k-1}\cdot 9\ln 10 + \frac{9}{10}M + O(1)\\ &= M\cdot 10^M\cdot\ln 10 + \frac{1}{9}(1-10^M)\cdot\ln 10 + \frac{9}{10}M + O(1)\\ &= M\cdot 10^M\cdot\ln 10 + O(N). \quad (2)\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{M}\ln(10^{(10^k-1)\cdot k}-1)\\
=&\sum_{k=1}^{M}\left(10^k-1\right)\cdot k\ln 10-\sum_{k=1}^{M}\frac{1}{10^{(10^k-1)\cdot k}}+O\left(\frac{1}{10^{10^{2\cdot(10^k-1)\cdot k}}}\right)\\
=&\frac{M\cdot 10^{M+1}\cdot\ln 10}{9}+M\cdot 10^M\cdot\ln 10+O(N). \qquad (3)
\end{aligned}$$

Applying the Lemma 1, we obtain

$$\ln(N!)=\sum_{1\leq n\leq N}\ln n=N\cdot\ln N-N+O(1). \qquad (4)$$

Combining the equation (1), asymptotic formulas (2), (3) and (4), we obtain the asymptotic formula

$$\begin{aligned}
\sum_{1\leq n\leq N}\ln a(n,m) &= \sum_{1\leq n\leq N}\ln n+\sum_{k=1}^{M}\ln\left(10^k+1\right)^{9\cdot 10^{k-1}}-\sum_{k=1}^{M}\ln(10^{(10^k-1)\cdot k}-1)\\
&= N\cdot\ln N+O(N).
\end{aligned}$$

Thus, we have accomplished the proof of the theorem.

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Scientia Magna
Vol. 7 (2011), No. 2, 117-122

Strongly almost λ-convergence and statistically almost λ-convergence of interval numbers

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Abstract In this paper we introduce and study the concepts of strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence for interval numbers and prove some inclusion relations.

Keywords $\lambda-$sequence, interval numbers, almost convergence.
2000 Mathematics Subject Classification: 40C05, 40J05, 46A45.

§1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concept of fuzzy set was introduced by Zadeh in 1965 [5]. Later on many research workers were motivated by the introduced notion of fuzzy sets. It has been applied for the studies in almost all branches of sciences, where mathematics has been applied. Workers on sequence spaces have been also applied the notion of fuzzy real numbers and have introduced sequences of fuzzy real numbers and have studied their different properties. Interval arithmetic was first suggested by Dwyer [7] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [10] in 1959 and Moore and Yang [11] 1962. Furthermore, Moore and others [7−9] and [12] have developed applications to differential equations.

Chiao in [4] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Recently Şengönül and Eryilmaz in [6] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space.

The idea of statistical convergence for single sequences was introduced by Fast [2] in 1951. Schoenberg [3] studied statistical convergence as a summability method and listed some of elamantary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

§2. Preliminaries

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $\mathrm{I\!R}$. Any elements of $\mathrm{I\!R}$ is called closed interval and denoted by $\overline{x}$. That is $\overline{x} = \{x \in \mathbb{R} : \ a \leq x \leq b\}$. An interval number $\overline{x}$ is a closed subset of real numbers [4]. Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x_1}, \overline{x_2} \in \mathrm{I\!R}$, we have $\overline{x_1} = \overline{x_2} \Leftrightarrow x_{1_l}{=}x_{2_l}, x_{1_r}{=}x_{2_r}$. $\overline{x_1} + \overline{x_2} = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \ \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \ \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$,

$$\overline{x_1}.\overline{x_2} = \left\{ \begin{array}{c} x \in \mathbb{R} : \min\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \leq x \\ \leq \max\{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\} \end{array} \right\}.$$

The set of all interval numbers $\mathrm{I\!R}$ is a complete metric space defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\} \text{ [10]}.$$

In the special case $\overline{x_1} = [a, a]$ and $\overline{x_2} = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}$, $\overline{x} = (\overline{x_k})$. Then $\overline{x} = (\overline{x_k})$ is called sequence of interval numbers. The $\overline{x_k}$ is called k^{th} term of sequence $\overline{x} = (\overline{x_k})$. w^i denotes the set of all interval numbers with real terms and the algebric properties of w^i can be found in [13].

Now we give the definition of convergence of interval numbers:

Definition 2.1.[4] A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$, there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$.

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

§3. Main results

In this paper, we introduce and study the concepts of strongly $\lambda-$convergence and statistically $\lambda-$convergence for interval numbers.

Definition 3.1. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. The sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be strongly $\lambda-$summable if there is an interval number $\overline{x}_o$ such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\overline{x}_{k+m}, \overline{x}_o\right) = 0, \text{ uniformly in m.}$$

In which case we say that the sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be strongly almost $\lambda-$summable to interval number $\overline{x}_o$. If $\lambda_n = n$, then strongly almost $\lambda-$summable reduces to strongly almost Cesaro summable defined as follows:

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} d\left(\overline{x}_{k+m}, \overline{x}_o\right) = 0, \text{uniformly in m.}$$

In special case $m = 0$, we obtain strongly $\lambda-$summable, which was defined Esi in [1].

Definition 3.2. A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be statistically almost $\lambda-$convergent to interval number $\overline{x}_o$ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\overline{x}_{k+m}, \overline{x}_o\right) \geq \varepsilon\right\}\right| = 0, \text{ uniformly in m.}$$

In this case we write $\widehat{\overline{s}}_\lambda - \lim \overline{x}_k = \overline{x}_o$. If $\lambda_n = n$, then statistically almost $\lambda-$convergence reduces to statistically almost convergence as follows:

$$\lim_n \frac{1}{n} \left|\left\{k \leq n : \ d\left(\overline{x}_{k+m}, \overline{x}_o\right) \geq \varepsilon\right\}\right| = 0, \text{ uniformly in m.}$$

In this case we write $\widehat{\overline{s}} - \lim \overline{x}_k = \overline{x}_o$. In special case $m = 0$, we obtain statistically $\lambda-$convergence, which was defined Esi in [1].

Theorem 3.1. Let $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ be sequences of interval numbers.

(i) If $\widehat{\overline{s}}_\lambda - \lim \overline{x}_k = \overline{x}_o$ and $\alpha \in \mathbb{R}$, then $\widehat{\overline{s}}_\lambda - \lim \alpha\overline{x}_k = \alpha\overline{x}_o$.

(ii) If $\widehat{\overline{s}}_\lambda - \lim \overline{x}_k = \overline{x}_o$ and $\widehat{\overline{s}}_\lambda - \lim \overline{y}_k = \overline{y}_o$, then $\widehat{\overline{s}}_\lambda - \lim\left(\overline{x}_k + \overline{y}_k\right) = \overline{x}_o + \overline{y}_o$.

Proof. **(i)** Let $\alpha \in \mathbb{R}$. We have $d\left(\alpha\overline{x}_k, \alpha\overline{x}_o\right) = |\alpha|\, d\left(\overline{x}_k, \overline{x}_o\right)$. For a given $\varepsilon > 0$ and all m,

$$\frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\alpha\overline{x}_{k+m}, \alpha\overline{x}_o\right) \geq \varepsilon\right\}\right| \leq \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\overline{x}_{k+m}, \overline{x}_o\right) \geq \frac{\varepsilon}{|\alpha|}\right\}\right|.$$

Hence $\widehat{\overline{s}}_\lambda - \lim \alpha\overline{x}_k = \alpha\overline{x}_o$.

(ii) Suppose that $\widehat{\overline{s}}_\lambda - \lim \overline{x}_k = \overline{x}_o$ and $\widehat{\overline{s}}_\lambda - \lim \overline{y}_k = \overline{y}_o$. We have

$$d\left(\overline{x}_{k+m} + \overline{y}_{k+m}, \overline{x}_o + \overline{y}_o\right) \leq d\left(\overline{x}_{k+m}, \overline{x}_o\right) + d\left(\overline{y}_{k+m}, \overline{y}_o\right).$$

Therefore given $\varepsilon > 0$ and all m, we have

$$\begin{aligned}
& \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(\overline{x}_{k+m} + \overline{y}_{k+m}, \overline{x}_o + \overline{y}_o\right) \geq \varepsilon\right\}\right| \\
\leq \; & \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(\overline{x}_{k+m}, \overline{x}_o\right) + d\left(\overline{y}_{k+m}, \overline{y}_o\right) \geq \varepsilon\right\}\right| \\
< \; & \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\overline{x}_{k+m}, \overline{x}_o\right) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{1}{\lambda_n} \left|\left\{k \in I_n : \ d\left(\overline{y}_{k+m}, \overline{y}_o\right) \geq \frac{\varepsilon}{2}\right\}\right|.
\end{aligned}$$

Thus, $\widehat{\overline{s}}_\lambda - \lim\left(\overline{x}_k + \overline{y}_k\right) = \overline{x}_o + \overline{y}_o$.

In the following theorems, we exhibit some connections between strongly almost $\lambda-$summable and statistically almost $\lambda-$convergence of sequences of interval numbers.

Theorem 3.2. If an interval sequence $\overline{x}=(\overline{x}_k)$ is strongly almost $\lambda-$summable to interval number $\overline{x}_o$, then it is statistically almost $\lambda-$convergent to interval number $\overline{x}_o$.

Proof. Let $\varepsilon>0$. Since

$$\sum_{k\in I_n} d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq \sum_{\substack{k\in I_n\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}} d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\left|\left\{k\in I_n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}\right|\varepsilon,$$

if $\overline{x}=(\overline{x}_k)$ is strongly almost $\lambda-$summable to $\overline{x}_o$, then it is statistically almost $\lambda-$convergent to $\overline{x}_o$.

Theorem 3.3. If $\overline{x}=(\overline{x}_k)\in\widehat{\overline{m}}$ and $\overline{x}=(\overline{x}_k)$ is statistically $\lambda-$convergent to interval number $\overline{x}_o$, then it is strongly almost $\lambda-$summable to $\overline{x}_o$ and hence $\overline{x}=(\overline{x}_k)$ is strongly almost Cesaro summable to $\overline{x}_o$, where $\widehat{\overline{m}}=\left\{\overline{x}=(\overline{x}_k):\ \sup_{k,m} d\left(\overline{x}_{k+m},\overline{x}_o\right)<\infty\right\}$.

Proof. Suppose that $\overline{x}=(\overline{x}_k)\in\widehat{\overline{m}}$ and statistically almost $\lambda-$convergent to interval number $\overline{x}_o$. Since $\overline{x}=(\overline{x}_k)\in\widehat{\overline{m}}$, we write $d\left(\overline{x}_{k+m},\overline{x}_o\right)\leq A$ for all $k,m\in\mathbb{N}$. Given $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\overline{x}_{k+m},\overline{x}_o\right) &= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\overline{x}_k,\overline{x}_o)\geq\varepsilon}} d\left(\overline{x}_{k+m},\overline{x}_o\right)+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d(\overline{x}_k,\overline{x}_o)<\varepsilon}} d\left(\overline{x}_{k+m},\overline{x}_o\right)\\ &< \frac{A}{\lambda_n}\left|\left\{k\in I_n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}\right|+\varepsilon,\end{aligned}$$

which implies that $\overline{x}=(\overline{x}_k)$ is strongly almost $\lambda-$summable to $\overline{x}_o$. Further we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} d\left(\overline{x}_{k+m},\overline{x}_o\right) &= \frac{1}{n}\sum_{k=1}^{n-\lambda_n} d\left(\overline{x}_{k+m},\overline{x}_o\right)+\frac{1}{n}\sum_{k\in I_n} d\left(\overline{x}_{k+m},\overline{x}_o\right)\\ &< \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} d\left(\overline{x}_{k+m},\overline{x}_o\right)+\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\overline{x}_{k+m},\overline{x}_o\right)\\ &< \frac{2}{\lambda_n}\sum_{k\in I_n} d\left(\overline{x}_{k+m},\overline{x}_o\right).\end{aligned}$$

Hence $\overline{x}=(\overline{x}_k)$ is strongly almost Cesaro summable to $\overline{x}_o$.

Theorem 3.4. If a interval sequence $\overline{x}=(\overline{x}_k)$ is statistically almost convergent to interval number $\overline{x}_o$ and $\liminf_n \frac{\lambda_n}{n}>0$, then it is statistically almost $\lambda-$convergent to $\overline{x}_o$.

Proof. For given $\varepsilon>0$ and all m, we have

$$\left\{k\leq n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}\supset\left\{k\in I_n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}.$$

Therefore

$$\begin{aligned}\frac{1}{n}\left|\left\{k\leq n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}\right| &> \frac{1}{n}\left|\left\{k\in I_n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}\right|\\ &\geq \frac{\lambda_n}{n}\cdot\frac{1}{\lambda_n}\left|\left\{k\in I_n:\ d\left(\overline{x}_{k+m},\overline{x}_o\right)\geq\varepsilon\right\}\right|.\end{aligned}$$

Taking limit as $n \to \infty$, uniformly in m and using $\liminf_n \frac{\lambda_n}{n} > 0$, we get that $\overline{x} = (\overline{x}_k)$ is statistically almost $\lambda-$convergent to $\overline{x}_o$.

Finally we conclude this paper by stating a definition which generalizes Definition 3.1 of Section 3 and two theorems related to this definition.

Definition 3.3. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$ and $p \in (0, \infty)$. The sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be strongly almost $\lambda p-$summable if there is an interval number $\overline{x}_o$ such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(\overline{x}_{k+m}, \overline{x}_o\right)\right]^p = 0, \text{ uniformly in m.}$$

In which case we say that the sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be strongly almost $\lambda p-$summable to interval number $\overline{x}_o$. If $\lambda_n = n$, then strongly almost $\lambda p-$summable reduces to strongly almost p-Cesaro summable defined as follows:

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left[d\left(\overline{x}_{k+m}, \overline{x}_o\right)\right]^p = 0, \text{ uniformly in m.}$$

The following theorems is similar to that of Theorem 3.2 and Theorem 3.3, so the proofs omitted.

Theorem 3.5. If an interval sequence $\overline{x} = (\overline{x}_k)$ is strongly almost $\lambda p-$summable to interval number $\overline{x}_o$, then it is statistically almost $\lambda-$convergent to interval number $\overline{x}_o$.

Theorem 3.6. If $\overline{x} = (\overline{x}_k) \in \widehat{\overline{m}}$ and $\overline{x} = (\overline{x}_k)$ is statistically almost $\lambda-$convergent to interval number $\overline{x}_o$, then it is strongly almost $\lambda p-$summable to $\overline{x}_o$ and hence $\overline{x} = (\overline{x}_k)$ is strongly almost p-Cesaro summable to $\overline{x}_o$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

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ISSN 1556- 6706