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On α-T-open sets and α-T-closed sets in ditopological ideal texture spaces

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Abstract In this paper the two different concepts namely Ditopological texture spaces and ideals are combined together to define a new dimension of topology namely ditopological ideal texture spaces. Here we analyse the properties of α-T-open sets and α-T-closed sets in the ditopological ideal texture spaces.

Keywords Texture spaces, ditopology, ditopological texture spaces, ditopological ideal texture spaces.
2000 AMS Subject Classification: 54C08, 54A20.

§1. Introduction

Textures were introduced by L. M. Brown [2] as a point-set for the study of fuzzy sets in 1998. On the other hand, textures offers a convenient setting for the investigation of complement-free concepts in general, so much of the recent work has proceeded independently of the fuzzy setting. And also several authors [1,12,13] have studied ideal topological spaces. In 1992, Jankovic and Hamlett introduced the notion of I-open sets in topological spaces which received more and more attention because of their good properties. El-Monsef [1], investigated I-open sets and I-continuous functions. In 1996, Dontchev [12] introduced the notion of pre-I-open sets and obtained a decomposition of I-continuity. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions:

(i) If $A \in I$ and $B \subset A$, then $B \in I$.

(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

An ideal topological space denoted by $(X,\ \tau,\ I)$ is a topological space $(X,\ \tau)$ with an ideal I on X. For a subset A of X, $A^*(I)$=$\{x \in X :\ U \cap A \notin I$ for each neighbourhood U of $x\}$ is called the local function [6] of A with respect to I and τ. Additionally, $cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

Ditopological Texture Spaces: Let S be a set, a texturing T of S is a subset of $P(S)$. If

(i) $(T,\ \subseteq)$ is a complete lattice containing S and ϕ, and the meet and join operations in $(T,\ \subseteq)$ are related with the intersection and union operations in $(P(S),\ \subseteq)$ by the equalities

$\wedge_{i\in I} A_i = \cap_{i\in I} A_i,\ A_i \in T,\ i \in I$, for all index sets I, while

$\vee_{i\in I} A_i = \cup_{i\in I} A_i,\ A_i \in T,\ i \in I$, for all finite sets I.

(ii) T is completely distributive.

(iii) T separates the points of S. That is, given $s_1 \neq s_2$ in S we have $A \in T$ with $s_1 \in A,\ s_2 \notin A$, or $A \in T$ with $s_2 \in A,\ s_1 \notin A$.

If S is textured by T we call $(S,\ T)$ a texture space or simply a texture.

For a texture $(S;\ T)$, most properties are conveniently defined in terms of the p-sets $P_s = \cap\{A \in T \backslash s \in A\}$ and the q-sets $Q_s = \vee\{A \in T/s \notin A\}$. The following are some basic examples of textures.

Examples 1.1. Some examples of texture spaces,

(i) If X is a set and $P(X)$ the powerset of X, then $(X;\ P(X))$ is the discrete texture on X. For $x \in X,\ P_x = \{x\}$ and $Q_x = X \backslash \{x\}$.

(ii) Setting $I = [0;\ 1]$, $T = \{[0;\ r);\ [0;\ r]/r \in I\}$ gives the unit interval texture $(I;\ T)$. For $r \in I,\ P_r = [0;\ r]$ and $Q_r = [0;\ r)$.

(iii) The texture $(L;\ T)$ is defined by $L = (0;\ 1]$, $T = \{(0;\ r]/r \in I\}$. For $r \in L,\ P_r = (0;\ r] = Q_r$.

(iv) $T = \{\phi,\ \{a, b\},\ \{b\},\ \{c\},\ \{b, c\},\ S\}$ is a simple textureing of $S = \{a,\ b,\ c\}$ clearly $P_a = \{a,\ b\},\ P_b = \{b\}$ and $P_c = \{c\}$.

Since a texturing T need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of dichotomous topology or ditopology, namely a pair $(\tau,\ \kappa)$ of subsets of T, where the set of open sets τ satisfies:

(i) $S, \phi \in \tau$,

(ii) $G_1;\ G_2 \in \tau$ then $G_1 \cap G_2 \in \tau$ and

(iii) $G_i \in \tau$, $i \in I$ then $\vee_i G_i \in \tau$,

and the set of closed sets κ satisfies:

(i) $S, \phi \in \kappa$,

(ii) $K_1;\ K_2 \in \kappa$ then $K_1 \cup K_2 \in \kappa$ and

(iii) $K_i \in \kappa,\ i \in I$ then $\cap K_i \in \kappa$.

Elements of closed set topology is set such that each set in κ is $U_x \bigcap A \neq \varphi$, where A in T and U_x in τ, x in S.

For $A \in T$ we define the closure $[A]$ or $cl(A)$ and the interior $]A[$ or $int(A)$ under $(\tau,\ \kappa)$ by the equalities $[A] = \cap\{K \in \kappa/A \subset K\}$ and $]A[= \vee\{G \in \tau/G \subset A\}$:

An mapping $\sigma :\ T \to T$ is said to be complementation on $(S,\ T)$ if $\kappa = \sigma(\tau)$, then $(S,\ T,\ \sigma,\ \tau,\ \kappa)$ is said to be a complemented ditopological texture space. The ditopology $(u,\ u^c)$ is clearly complemented for the complementation $\pi_X :\ P(X) \to P(X)$ given by $\phi_X(Y) = X \backslash Y$.

We denote by $O(S;\ T;\ \tau,\ \kappa)$, or when there can be no confusion by $O(S)$, the set of open sets in S. Likewise, $C(S;\ T;\ \tau,\ \kappa)$, $C(S)$ will denote the set of closed sets. One of the most useful notions of ditopological texture spaces is that of difunction. A difunction is a special type of direlation [4]. For a difunction $(f;\ F) :\ (S_1;\ T_1) \to (S_2;\ T_2)$ we will have cause to use the inverse image $f^{\leftarrow}B$ and inverse co-image $F^{\rightarrow}B,\ B \in T$, which are equal; and the image $f^{\rightarrow}A$ and co-image $F^{\rightarrow}A,\ A \in S$, which are usually not. Now we consider complemented textures $(S_j;\ T_j;\ \sigma_j)$, $j = 1;\ 2$ and the difunction $(f;\ F) :\ (S_1;\ T_1) \to (S_2;\ T_2)$. The complement of the difunction $(f;\ F)$ is denoted by $(f;\ F)'$ [4]. If $(f;\ F) = (f;\ F)'$ then the difunction $(f;\ F)$ is called complemented.

The difunction $(f;\ F) :\ (S_1;\ T_1;\ \tau_{S_1};\ \kappa_{S_1}) \to (S_2;\ T_2;\ \tau_{S_2};\ \kappa_{S_2})$ is called continuous if

$B \in \tau_{S_2} \Rightarrow F^{\rightarrow}B \in \tau_{S_1}$, cocontinuous if $B \in \kappa_S \Rightarrow f^{\leftarrow}B \in \kappa_S$ and bicontinuous if it is both. For complemented difunctions these two properties are equivalent. Finally, we also recall from [8, 9, 11] the classes of ditopological texture spaces and difunctions.

Definition 1.1. For a ditopological texture space $(S;\ T;\ \tau,\ \kappa)$:

(i) $A \in T$ is called pre-open (resp. semi-open, β-open) if $A \subseteq intclA$ (resp. A $\subseteq$ $clintA$; $A \subseteq clintclA$). $B \in S$ is called pre-closed (resp. semi-closed, β-closed) if $clintB \subseteq B$ (resp. $intclB \subseteq B$; $intclintB \subseteq B$).

(ii) A difunction $(f;\ F) : (S;\ T;\ \tau_S,\ \kappa_S) \to (T;\ T;\ \tau_S,\ \kappa_S)$ is called pre-continuous (resp. semi-continuous, β-continuous) if $F^{\leftarrow}(G) \in PO(S)$ (resp. $F^{\leftarrow}(G) \in SO(S)$; $F^{\leftarrow}(G) \in \beta O(S)$) for every $G \in O(T)$. It is called pre-cocontinuous (resp. semi-cocontinuous, β-cocontinuous) if $F^{\leftarrow}(K) \in PC(S)$ (resp. $F^{\leftarrow}(K) \in SC(S)$; $F^{\leftarrow}(K) \in \beta C(S)$) for every $K \in C(T)$.

We denote by $PO(S;\ T;\ \tau,\ \kappa)$ ($\beta O(S;\ T;\ \tau,\ \kappa)$), more simply by $PO(S)$ ($\beta O(S)$), the set of pre-open sets (β-open sets) in S. Likewise, $PC(S;\ T;\ \tau,\ \kappa)$ ($\beta C(S;\ T;\ \tau,\ \kappa)$), $PC(S)$ ($\beta C(S)$) will denote the set of pre-closed (β-closed sets) sets.

§2. α-T-open and α-T-closed sets

A texture ideal is defined as a nonempty collection I of subsets of texture space satisfying the following two conditions:

(i) If $A \in I$ and $B \subset A$ and $B \in T$, then $B \in I$.

(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

A Ditopological ideal texture space denoted by $(S,\ T,\ \tau,\ \kappa,\ I)$ is a Ditopological texture space $(S,\ T,\ \tau,\ \kappa)$ with an texture ideal I on X. For a subset A of X, $A^*(I) = \{x \in T : U \cap A \notin I$ for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ . Here $cl^{*T}(A) = A \cup A^*$ and additionally, $int^{*T}(A) = \{x \in S/G \subset A \bigcup J,$ for some $J \in I\}$.

Definition 2.1. Let $(S;\ T;\ \tau,\ \kappa,\ I)$ be ditopological ideal texture space and $A \in T$.

(i) If $A \subseteq int(cl^*(int(A)))$ then A is α-T-open.

(ii) If $cl(int^*(cl(A))) \subseteq A$ then A is α-T-closed.

We denote by $O\alpha^*(S;\ T;\ \tau,\ \kappa,\ I)$, or when there can be no confusion by $O\alpha^*(S)$, the set of α-T-open sets in S. Likewise, $C\alpha^*(S;\ T;\ \tau,\ \kappa,\ I)$, or $C\alpha^*(S)$ will denote the set of α-T-closed sets.

Proposition 2.1. For a given ditopological ideal texture space $(S;\ T;\ \tau,\ \kappa,\ I)$:

(i) $O(S) \subseteq O\alpha^*(S)$ and $C(S) \subseteq C\alpha^*(S)$.

(ii) Arbitrary join of α-T-open sets is α-T-open.

(iii) Arbitrary intersection of α-T-closed sets is α-T-closed.

Proof. (i) Let $G \in O(S)$. Since $intG = G$ we have $G \subseteq int(cl^*(int(G)))$. Thus $G \in O\alpha^*$(S). Secondly, let $K \in C(S)$. Since $clK = K$ we have $cl(int^*(cl(K))) \subseteq K$ and so $K \in C\alpha(S)$.

(ii) Let $\{A_j\}_{j\in J}$ be a family of α-T-open sets. Then for each $j \in J$, $A_j \subseteq int(cl^*(int(A_j)))$. Now, $\vee A_j \subseteq \vee\ int(cl^*(int(A_j))) \subseteq int \vee cl^*(int(A_j)) = int(cl^* \vee int(A_j)) = int(cl^*(int \vee A_j)$. Hence $\vee A_j$ is a α-T-open set. The result (iii) is dual of (ii).

Generally there is no relation between the α-T-open and α-T-closed sets, but for a complemented ditopological space we have the following result.

Examples 2.1. The following are some examples of complemented ditopological texture space. If $(X;\ T)$ is a texture space then $(X;\ P(X);\ \pi_X,\ \tau;\ \tau_c)$ is a complemented ditopological texture space. Here $\pi_X(Y) = X\backslash Y$ for $Y \subseteq X$ is the usual complementation on $(X;\ P(X))$ and $\tau_c = \{\pi_X(G)\hat{}G \in \tau\}$, $I = \{P(A)/A \subset X\}$. Clearly the α-T-open, α-T-closed sets in $(X;\ \tau)$ correspond precisely to the α-T-open, α-T-closed respectively, in $(X;\ P(X);\ I,\ \pi_X,\ \tau,\ \tau_c)$.

Definition 2.2. Let $(S;\ T;\ \tau,\ \kappa,\ I)$ be a ditopological ideal texture space. For $A \in T$, we define:

(i) The α-T-closure $cl\alpha^*(A)$ of A under $(\tau,\ \kappa)$ by the equality $cl_{\alpha^*}(A) = \cap\{B\backslash B \in C_{\alpha^*}(S)$ and $A \subseteq B\}$.

(ii) The α-T-interior $int\alpha^* A$ of A under $(\tau,\ \kappa)$ by the equality $int\alpha^*(A) = \vee\{B\backslash B \in O\alpha(S)$ and $B \subseteq A\}$.

From the Proposition 2.1, it is obtained, $int_{\alpha^*}(A) \in O\alpha^*(S)$, $cl\alpha(A) \in C\alpha(S)$.

Proposition 2.2. Let $(S;\ T;\ \tau,\ \kappa,\ I)$ be a ditopological ideal texture space. Then the following are true:

(i) $cl_{\alpha^*}(\phi) = \phi$.

(ii) $cl_{\alpha^*}(A)$ is α-T-closed, for all $A \in T$.

(iii) If $A \subseteq B$ then $cl_{\alpha^*}(A) \subseteq cl_{\alpha^*}(B)$, for every $A;\ B \in T$.

(iv) $cl_{\alpha^*}(cl_{\alpha^*}(A)) = cl_{\alpha^*}(A)$.

Definition 2.3. For a ditopological ideal texture space $(S;\ T;\ \tau,\ \kappa,\ I)$:

(i) $A \in T$ is called pre-T-open (resp. semi-T-open, β-T-open) if $A \subseteq int(cl^*(A))$ (resp. $A \subseteq cl^* int A$; $A \subseteq cl^*(int(cl^*(A)))$.

(ii) $B \in T$ is called pre-T-closed (resp. semi-T-closed, β-T-closed) if $cl(int^*(B)) \subseteq B$ (resp. $int^*(cl(B)) \subseteq B$; $int^*(cl(int^*(B))) \subseteq B$).

Theorem 2.1. For a ditopological texture space $(S;\ T;\ \tau,\ \kappa,\ I)$:

(i) Every α-T-open is pre-T-open.

(ii) Every α-T-open is semi-T-open.

(iii) Every pre-T-open is β-T-open.

(iv) Every semi-T-open is β-T-open.

Proof. The proof is obivious.

Remark 2.1. For a ditopological ideal texture space $(S;\ T;\ \tau,\ \kappa,\ I)$ the converse of the above results need not be always true.

Example 2.2. For a ditopological ideal texture space $(S;\ T;\ \tau,\ \kappa,\ I)$. Let $S = \{a,\ b,\ c,\ d\}$ and $T = P(S)$. $\tau = \{\phi,\ S,\ \{b\},\ \{c,d\},\ \{b,c,d\}\}$, $\kappa = \{\phi,\ S,\ \{a\},\ \{b\},\ \{a,b\}\}$ and $I = \{\phi,\ \{d\}\}$. Let $A = \{a,\ b,\ c\}$, then A is pre-T-open but not α-T-open.

Remark 2.2. We know that every α-T-open set is α open in ideal topological space, but this is not always true in the case of ditopological ideal texture space, here they are independent sets. It is given in the following example:

Example 2.3. In a ditopological ideal texture space $(S;\ T;\ \tau,\ \kappa,\ I)$. Let $S = \{a,\ b,\ c\}$ and $T = P(S)$, $\tau = \{\phi,\ S,\ \{a\}\}$, $\kappa = \{\phi,\ S,\ \{a\}\}$ and $I = \{\phi,\ \{b\}\}$. Let $A = \{a,\ c\}$, then A is α-T-open but not α-open.

Example 2.4. In a ditopological ideal texture space $(S;\ T;\ \tau,\ \kappa,\ I)$. Let $S = \{a,\ b,\ c,\ d\}$ and $T = P(S)$, $\tau = \{\phi,\ S, \{a\},\ \{b\},\ \{a, b\}\}$, $\kappa = \{\phi,\ S,\ \{a\},\ \{b, c\},\ \{a, b, c\}\}$ and $I = \{\phi,\ \{a\},\ \{b\},\ \{a, b\}\}$. Let $A = \{a,\ b,\ c\}$, then A is α open but not α-T-open.

Remark 2.3. From above, the results is shown in the following figure:

$$\begin{tikzcd}
 & \boxed{\alpha\text{-open}} \arrow[d, shift left, "/" marking] & \boxed{\text{open}} \arrow[l] \arrow[d] \\
\boxed{\text{semi-}T\text{-open}} & \boxed{\alpha\text{-}T\text{-open}} \arrow[l] \arrow[r] \arrow[u, shift left, "/" marking] \arrow[d] & \boxed{\text{pre-}T\text{-open}} \arrow[dl] \\
 & \boxed{\beta\text{-}T\text{-open}} &
\end{tikzcd}$$

§3. α-T-continuous functions and α-T-cocontinuous functions

Definition 3.1. Let $(S_j;\ T_j;\ \tau_j,\ \kappa_j,\ I)$, $j = 1, 2$ be ditopological texture spaces and $(f;\ F) :\ (S_1;\ T_1) \to (S_2;\ T_2)$ a difunction.

(i) It is called α-T-continuous, if $F^{\leftarrow}(G)$ is α-T-open, for every $G \in O(S_2)$.

(ii) It is called α-T-cocontinuous, if $f^{\leftarrow}(K)$ is α-T-closed in S_1, for every $K \in C(S_2)$.

(iii) It is called α-T-bicontinuous, if it is α-T-continuous and α-T-cocontinuous.

Definition 3.2. A difunction $(f;\ F) :\ (S_1;\ T_1;\ \tau_S,\ \kappa_S) \to (S_2;\ T_2;\ \tau_S,\ \kappa_S)$ is called

(i) pre-T-continuous (resp. semi-T-continuous, β-T-continuous) if $F^{\leftarrow}(G) \in PTO(S)$ (resp. $F^{\leftarrow}(G) \in STO(S)$; $F^{\leftarrow}(G) \in \beta TO(S)$) for every $G \in O(T)$.

(ii) It is called pre-T-cocontinuous (resp. semi-T-cocontinuous, β-T-cocontinuous) if $f^{\leftarrow}(K) \in PIC(S)$ (resp. $f^{\leftarrow}(K) \in SIC(S)$; $f^{\leftarrow}(K) \in \beta IC(S)$) for every $K \in C(T)$.

We denote by $PTO(S;\ T;\ \tau,\ \kappa,\ I)$ ($\beta TO(S;\ T;\ \tau,\ \kappa,\ I)$, more simply by $PTO(S)$ ($\beta TO(S)$), the set of pre-T-open sets (β-T-open sets) in S. Likewise, $PTC(S;\ T;\ \tau,\ \kappa)$ ($\beta TC(S;\ T;\ \tau,\ \kappa)$), $PTC(S)$ ($\beta TC(S)$) will denote the set of pre-T-closed (β-T-closed sets) sets.

Proposition 3.1. Let $(f;\ F) :\ (S_1;\ T_1;\ \tau_1;\ \kappa_1) \to (S_2;\ T_2;\ \tau_2;\ \kappa_2)$ be a difunction.

(1) The following are equivalent:

(i) $(f;\ F)$ is α-T-continuous.

(ii) $int(F^{\rightarrow}A) \subseteq F^{\rightarrow}(int_{\alpha^*}A)$, for all $A \in T_1$.

(iii) $f^{\leftarrow}(intB) \subseteq int_{\alpha^*}(f^{\leftarrow}B)$, for all $B \in T_2$.

(2) The following are equivalent:

(i) $(f;\ F)$ is α-T-cocontinuous.

(ii) $f^{\rightarrow}(cl_{\alpha^*}A) \subseteq cl(f^{\rightarrow}A)$, for all $A \in T_1$.

(iii) $cl_{\alpha^*}(F^{\leftarrow}B) \subseteq F^{\leftarrow}(clB)$, for all $B \in T_1$.

Proof. (i)$\Rightarrow$(ii) Take $A \in T_1$. From the definition of interior, $f^{\leftarrow}int(F^{\leftarrow}A) \subseteq f^{\leftarrow}(F^{\rightarrow}A) \subseteq A$. Since inverse image and coimage under a difunction is equal, $f^{\leftarrow}int(F^{\rightarrow}A) = F^{\leftarrow}int(F^{\rightarrow}A)$. Thus, $f^{\leftarrow}int(F^{\rightarrow}A) \in O_{\alpha^*}(S_1)$, by α-T-continuity. Hence $f^{\leftarrow}int(F^{\rightarrow}A) \subseteq int_{\alpha^*}(A)$ and applying (ii) gives $int(F^{\rightarrow}A) \subseteq F^{\rightarrow}\ f^{\leftarrow}int(F^{\rightarrow}A) \subseteq F^{\rightarrow}(int_{\alpha^*}A)$, which is the required inclusion.

(ii)$\Rightarrow$(iii) Let $B \in T_2$. Applying inclusion (ii) to $A = f^{\leftarrow}B$ and using [4, Theorem 2.4 (2b)] gives $int(B) \subset int(F^{\rightarrow}(f^{\leftarrow}B)) \subset F^{\rightarrow}(int_{\alpha^*}(f^{\leftarrow}B))$. Hence, we have $f^{\leftarrow}(intB) \subset f^{\leftarrow}F^{\rightarrow}int_{\alpha^*}(f^{\leftarrow}B) \subset int_{\alpha^*}(f^{\leftarrow}B)$ by [4, Theorem 2.24 (2a)].

(iii)$\Rightarrow$(i) Applying (iii) for $B \in O(S_2)$ gives $f^{\leftarrow}B = f^{\leftarrow}(int(B)) \subset int_{\alpha^*}(f^{\leftarrow}B)$, so $F^{\leftarrow}B = f^{\leftarrow}B = int_{\alpha^*}(f^{\leftarrow}B) \in \alpha TO(S_1)$. Hence, $(f;\ F)$ is α-T-continuous.

Corollary 3.1. Let $(f;\ F) : (S_1;\ T_1;\ \tau_1;\ \kappa_1) \to (S_2;\ T_2;\ \tau_2;\ \sigma_2)$ be a difunction.

(1) If $(f;\ F)$ is α-T-continuous then:

(i) $f^{\rightarrow}(clA) \subset cl(f^{\rightarrow}(A))$, for every $A \in PO(T_1)$.

(ii) $cl(f^{\leftarrow}(B)) \subset f^{\leftarrow}(clB)$, for every $B \in O(T_2)$.

(2) If $(f;\ F)$ is α-T-cocontinuous then:

(i) $int(F^{\rightarrow}(A)) \subset F^{\rightarrow}(intA)$, for every $A \in PC(T_1)$.

(ii) $F^{\leftarrow}(intB) \subset int(F^{\leftarrow}B)$, for every $B \in C(T_2)$.

Proof. (i) Let $A \in PO(T_1)$. Then $clA \subset clintclA$ and so $f^{\rightarrow}(clA) \subset f \rightarrow (clintclA)$. Then, we have, $f^{\rightarrow}(clA) \subset cl(f^{\rightarrow}(A))$.

(ii) Let $B \in O(S_2)$. From the assumption, $f^{\leftarrow}(B)$ is α-T-open and by Remark 2.3, $f^{\leftarrow}(B) \in PO(T_1)$. Hence, $f^{\leftarrow}(B) \subset intcl(f^{\leftarrow}(B))$ and so $cl(f^{\leftarrow}(B)) \subset clintcl(f^{\leftarrow}(B))$. Then, we have $cl(f^{\leftarrow}(B)) \subset f^{\leftarrow}(clB)$.

Theorem 3.1. For a ditopological texture space $(S;\ T;\ \tau,\ \kappa)$:

(i) Every α-T-continuous is pre-T-continuous.

(ii) Every α-T-continuous is semi-T-continuous.

(iii) Every pre-T-continuous is β-T-continuous.

(iv) Every semi-T-continuous is β-T-continuous.

Proof. The proof is obivious from Theorem 2.1.

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A new class of generalized semiclosed sets using grills

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Abstract The aim of this paper is to apply the notion of semiclosed sets, to obtain a new class of generalized semi closed sets using Grills. Also we investigate the properties of the above mentioned sets. Further the concept is extended to derive some applications of generalized semi closed sets via Grills.

Keywords Grill, topology τ_G, operator Φ, G-gs-closed.
2000 AMS Subject Classification: 54D35, 54D95.

§1. Introduction

The idea of grill on a topological space was first introduced by Choquet in 1947. From subsequent investigations concept of grills has shown to be a powerful supporting and useful tool like nets and filters, further we get a deeper insight into studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds. In this paper, we explore the concept of semi-closed sets to define a new class of generalized semi closed sets via Grills.

§2. Preliminaries

Definition 2.1.[4] A collection G of non empty subsets of a space X is called a grill on X if

(i) $A \in G$ and $A \subseteq B \subseteq X \Longrightarrow B \in G$ and

(ii) $A,\ B \subseteq X$ and $A \cup B \in G \Longrightarrow A \in G$ or $B \in G$.

Definition 2.2.[4] Let (X, τ) be a topological space and G be a grill on X. We define a mapping $\Phi : P(X) \longrightarrow P(X)$ denoted by $\Phi_G(A, \tau)$ (for $A \in P(X)$) or $\Phi_G(A)$ or simply $\Phi(A)$, called the operator associated with the grill G and the topology τ, and is defined by $\Phi_G(A) = \{x \in X : A \cap U \in G,\ \forall U \in \tau(x)\}$. For any point x of a topological space (X, τ), we shall let $\tau(x)$ to stand for the collection of all open neighbourhood of x.

Definition 2.3.[4] Let G be grill on a space X. We define a map $\Psi : P(X) \longrightarrow P(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in P(X)$.

Definition 2.4.[11] Corresponding to a grill G on a topological space (X, τ) there exist a unique topology τ_G (say) on X given by $\tau_G = \{U \subseteq X : \Psi(X/U) = X/U\}$ where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_G - cl(A)$.

Definition 2.5.[10] Let (X, τ) be a topological space and G be a grill on X. Then for any $A,\ B \subseteq X$ the following hold:

(i) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$.

(ii) $\Phi(\Phi(A)) \subseteq \Phi(A) = cl(\Phi(A)) \subseteq cl(A)$, and hence $\Phi(A)$ is closed in (X, τ), for all $A \subseteq X$.

(iii) $A \subseteq B \Longrightarrow \Phi(A) \subseteq \Phi(B)$.

Definition 2.6.[15] A subset A of a topological space X is said to be θ-closed if $A = \theta cl(A))$ where $\theta cl(A)$ is defined as $\theta cl(A) = \{x \in X/cl(U) \cap A \neq \phi\}$ for every $U \in \tau$ and $x \in U$.

Definition 2.7.[15] A subset A of X is said to be θ-open if X/A is θ-closed.

Definition 2.8.[15] A subset A of a topological space X is said to be δ-closed if $A = \delta cl(A))$ where $\delta cl(A)$ is defined as $\delta cl(A) = \{x \in X/intcl(U) \cap A \neq \phi\}$ for every $U \in \tau$ and $x \in U$.

Definition 2.9.[15] A subset A of X is said to be δ-open if X/A is δ-closed.

Definition 2.10.[1] A subset A of a topological space X is said to be θg-closed if $\theta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Definition 2.11.[1] A subset A of a topological space X is said to be $g\theta$-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is θ-open.

Definition 2.12.[1] A subset A of X is said to be $g\theta$-open (θg-open) if X/A is $g\theta$-closed (θg-closed).

§3. Generalized semiclosed sets with respect to a grill

Definition 3.1. A subset A of a topological space X is said to be gs-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open.

Definition 3.2. Let (X, τ) be a topological space and G be a grill on X. Then a subset A of X is said to be gs-closed with respect to the grill G (G-gs-closed, for short) if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in X.

Definition 3.3. A subset A of X is said to be G-gs-open if X/A is G-gs-closed.

Proposition 3.1. For a topological space (X, τ) and a grill G on X,

(i) Every closed set in X is G-gs-closed.

(ii) For any subset A in X, $\Phi(A)$ is G-gs-closed.

(iii) Every τ_G-closed set is G-gs-closed .

(iv) Any non member of G is G-gs-closed.

(v) Every G-gs-closed set is G-g-closed.

(vi) Every gs-closed set is G-gs-closed.

(vii) Every θ-closed set in X is G-gs-closed.

(viii) Every δ-closed set in X is G-gs-closed.

Proof. (i) Let A be a closed set then $cl(A) = A$. Let U be a semi open set in $X \ni A \subseteq U$. Then, $\Phi(A) \subseteq cl(A) = A \subseteq U \Longrightarrow \Phi(A) \subseteq U \Longrightarrow A$ is G-gs-closed.

(ii) Let A be a subset in X. Then $\Phi(\Phi(A)) \subseteq \Phi(A) \subseteq U \Longrightarrow \Phi(A)$ is G-gs-closed.

(iii) Let A be a τ_G-closed set then $\tau_G - cl(A) = A \Longrightarrow A \cup \Phi(A) = A \Longrightarrow \Phi(A) \subseteq A$. Therefore, $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X. This implies A is G-gs-closed.

(iv) Let $A \notin G$ then $\Phi(A) = \phi \Longrightarrow A$ is G-gs-closed.

(v) Let A be a G-gs-closed and $A \subseteq U$ and U is open in X, we get $\Phi(A) \subseteq U \Longrightarrow A$ is G-g-closed. Therefore, every G-gs-closed set is G-g-closed.

(vi) Let A be a gs-closed set and U be a semiopen set in X, such that $A \subseteq U$, then $cl(A) \subseteq U$, consider $\Phi(A) \subseteq cl(A) \subseteq U \Longrightarrow$ A is G-gs-closed. Thus every gs-closed set is G-gs-closed.

(vii) Let A be θ-closed then $A = \theta cl(A)$. Let U be a semi open set in X such that $A \subseteq U$, then $\Phi(A) \subseteq cl(A) \subseteq \theta cl(A) = A \subseteq U$. Thus A is G-gs-closed.

(viii) Let A be δ-closed then $A = \delta cl(A)$. Let U be a semi open set in X such that $A \subseteq U$, then $\Phi(A) \subseteq cl(A) \subseteq \delta cl(A) = A \subseteq U$. Thus A is G-gs-closed.

Remark 3.1. $g\theta$-closed and G-gs-closed are independent from each other. Similarly θg-closed and G-gs-closed are independent from each other.

Remark 3.2. Every gs-closed set is G-gs-closed but the converse is not true as shown by the following example:

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$, $G = \{X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}\}$, then (X, τ) is a topological space and G is a grill on X. Let $A = \{b\}$ then $\Phi(A) = \phi$. Therefore, A is G-gs-closed. But $A \subseteq \{b, c\}$ and $cl(A) = X$ does not a subset of $\{b, c\}$. Therefore, A is not gs-closed.

Definition 3.4. Let X be a space and $(\phi \neq) A \subseteq X$. Then $[A] = \{B \subseteq X : A \cap B \neq \phi\}$ is a grill on X, called the principal grill generated by A.

Proposition 3.2. In the case of $[X]$ principal grill generated by X, it is known that $\tau = \tau_{[X]}$ so that any $[X]$-gs-closed set becomes simply a gs-closed set and vice-versa.

Theorem 3.1. Let (X, τ) be a topological space and G be a grill on X. If a subset A of X is G-gs-closed then $\tau_G - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open.

Proof. Let A be a G-gs-closed set and U be a semi open in X such that $A \subseteq U$ then $\Phi(A) \subseteq U \Longrightarrow A \cup \Phi(A) \subseteq U \Longrightarrow \tau_G - cl(A) \subseteq U$. Thus $\tau_G - cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen.

Theorem 3.2. Let (X, τ) be a topological space and G be a grill on X. If a subset A of X is G-gs-closed then for all $x \in \tau_G - cl(A)$, $cl(\{x\}) \cap A \neq \phi$.

Proof. Let $x \in \tau_G - cl(A)$. If $cl(\{x\}) \cap A = \phi \Longrightarrow A \subseteq X/cl(\{x\})$ then by Theorem 3.1, $\tau_G - cl(A) \subseteq X/cl(\{x\})$ which is a contradiction to our assumption that $x \in \tau_G - cl(A)$. Therefore, $cl(\{x\}) \cap A \neq \phi$.

Theorem 3.3. Let (X, τ) be a topological space and G be a grill on X. If a subset A of X is G-gs-closed then $\tau_G - cl(A)/A$ contains no non-empty closed set of (X, τ). Moreover $\Phi(A)/A$ contains no non-empty closed set of (X, τ).

Proof. Let F be a closed set contained in $\tau_G - cl(A)/A$ and let $x \in F$, since $F \cap A = \phi$ we get $cl(\{x\}) \cap A = \phi$ Which is a contradiction to the fact that $cl(\{x\}) \cap A \neq \phi$. $\tau_G - cl(A)/A$ contains no non-empty closed set of (X, τ). Since $\Phi(A)/A = \tau_G - cl(A)/A, \Phi(A)/A$ contains no non-empty closed set of (X, τ).

Corollary 3.1. Let (X, τ) be a T_1-space and G be a grill on X. Then every G-gs-closed set is τ_G-closed.

Proof. Let A be a G-gs-closed set and $x \in \Phi(A)$ then $x \in \tau_G - cl(A)$. By Theorem 3.2, $cl(\{x\}) \cap A \neq \phi$, $\{x\} \cap A \neq \phi$, $x \in A$. Therefore, $\Phi(A) \subseteq A$. Thus A is τ_G-closed.

Corollary 3.2. Let (X, τ) be a T_1-space and G be a grill on X. Then A $(\subseteq X)$ is G-gs-closed set iff A is τ_G-closed.

Proposition 3.3. Let G be a grill on a space (X, τ) and A be a G-gs-closed set. Then the following are equivalent:

(i) A is τ_G-closed.

(ii) $\tau_G - cl(A)/A$ is closed in (X, τ).

(iii) $\Phi(A)/A$ is closed in (X, τ).

Proof. (i)$\Longrightarrow$(ii) Let A be τ_G-closed then $\tau_G - cl(A)/A = \phi$ so $\tau_G - cl(A)/A$ is a closed set.

(ii)$\Longrightarrow$(iii) Since $\tau_G - cl(A)/A = \Phi(A)/A$.

(iii)$\Longrightarrow$(i) Let $\phi(A)/A$ be closed in (X, τ). Since A is G-gs-closed by Theorem 3.3, $\Phi(A)/A = \phi$. So A is τ_G-closed.

Lemma 3.1. Let (X, τ) be a space and G be a grill on X. If $A(\subseteq X)$ is τ_G-dense in itself, then $\Phi(A) = cl(\Phi(A)) = \tau_G - cl(A) = cl(A)$.

Proof. A is τ_G-dense in itself $\Longrightarrow A \subseteq \Phi(A) \Longrightarrow cl(A) \subseteq cl(\Phi(A)) = \Phi(A) \subseteq cl(A) \Longrightarrow cl(A) = \Phi(A) = cl(\Phi(A))$ now by definition $\tau_G - cl(A) = A \cup \Phi(A) = A \cup cl(A) = cl(A)$. Therefore, $\Phi(A) = cl(\Phi(A)) = \tau_G - cl(A) = cl(A)$.

Theorem 3.4. Let G be a grill on a space (X, τ). If $A(\subseteq X)$ is τ_G-dense in itself and G-gs-closed, then A is gs-closed.

Proof. Follows from Lemma 3.1.

Corollary 3.3. For a grill G on a space (X, τ). Let $A(\subseteq X)$ be τ_G-dense in itself. Then A is G-gs-closed iff it is A is gs-closed.

Proof. Follows from Proposition 3.1(vi) and Theorem 3.4.

Theorem 3.5. For any grill G on a space (X, τ) the following are equivalent:

(i) Every subset of X is G-gs-closed.

(ii) Every semiopen subset of (X, τ) is τ_G-closed.

Proof. (i)$\Longrightarrow$(ii) Let A be semiopen in (X, τ) then by (i), A is G-gs-closed so that $\Phi(A) \subseteq A \Longrightarrow A$ is τ_G-closed.

(ii)$\Longrightarrow$(i) Let $A \subseteq X$ and U be semi open in (X, τ) such that $A \subseteq U$. Since U is semiopen by (ii), $\Phi(U) \subseteq U$. Now $A \subseteq U \Longrightarrow \Phi(A) \subseteq \Phi(U) \subseteq U \Longrightarrow A$ is G-gs-closed.

Theorem 3.6. For any subset A of a space (X, τ) and a grill G on X. If A is G-gs-closed then $A \cup (X/\Phi(A))$ is G-gs-closed.

Proof. Let $A \cup (X/\Phi(A)) \subseteq U$, where U is semi open in X. Then $X/U \subseteq X/(A \cup (X/\Phi(A))) = \Phi(A)/A$. Since A is G-gs-closed, by Theorem 3.2, we have $X/U = \phi$, i.e., $X = U$. Since X is the only semi open set containing $A \cup (X/\Phi(A))$, $A \cup (X/\Phi(A))$ is G-gs-closed.

Proposition 3.4. For any subset A of a space (X, τ) and a grill G on X, the following are equivalent:

(i) $A \cup (X/\Phi(A))$ is G-gs-closed.

(ii) $\Phi(A)$ A is G-gs-open.

Proof. Follows from the fact that $X/(\Phi(A)/A) = A \cup (X/\Phi(A))$.

Theorem 3.7. Let (X, τ) be a space, G be a grill on X and A, B be subsets of X such that $A \subseteq B \subseteq \tau_G - cl(A)$. If A is G-gs-closed, then B is G-gs-closed.

Proof. Let $B \subseteq U$, where U is semi open in X. Since A is G-gs-closed, $\Phi(A) \subseteq U \Longrightarrow \tau_G - cl(A) \subseteq U$. Now, $A \subseteq B \subseteq \tau_G - cl(A) \Longrightarrow \tau_G - cl(A) \subseteq \tau_G - cl(B) \subseteq \tau_G - cl(A)$. Thus $\tau_G - cl(B) \subseteq U$ and hence B is G-gs-closed.

Corollary 3.4. τ_G-closure of every G-gs-closed set is G-gs-closed.

Theorem 3.8. Let G be a grill on a space (X, τ) and A, B be subsets of X such that $A \subseteq B \subseteq \Phi(A)$. If A is G-gs-closed. Then A and B are gs-closed.

Proof. $A \subseteq B \subseteq \Phi(A) \Longrightarrow A \subseteq B \subseteq \tau_G - cl(A)$ and hence by Theorem 3.7, B is G-gs-closed. Again, $A \subseteq B \subseteq \Phi(A) \Longrightarrow \Phi(A) \subseteq \Phi(B) \subseteq \Phi(\Phi(A)) \subseteq \Phi(A) \Longrightarrow \Phi(A) = \Phi(B)$. Thus A and B are τ_G-dense in itself and hence by Theorem 3.4, A and B are gs-closed.

Theorem 3.9. Let G be a grill on a space (X, τ). Then a subset A of X is G-gs-open iff $F \subseteq \tau_G - int(A)$ whenever $F \subseteq A$ and F is closed.

Proof. Let A be G-gs-open and $F \subseteq A$, where F is closed in (X, τ). Then $X/A \subseteq X/F \Longrightarrow \Phi(X/A) \subseteq X/F \Longrightarrow \tau_G - cl(X/A) \subseteq X/F \Longrightarrow F \subseteq \tau_G - int(A)$. Conversely, $X/A \subseteq U$ where U is open in $(X, \tau) \Longrightarrow X/U \subseteq \tau_G - int(A) \Longrightarrow \tau_G - cl(X/A) \subseteq U$. Thus (X/A) is G-gs-closed and hence A is G-gs-open.

§4. Some characterizations of regular and normal spaces

Theorem 4.1. Let X be a normal space and G be a grill on X then for each pair of disjoint closed sets F and K , there exist disjoint G-gs-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Proof. It is obvious, since every open set is G-gs-open.

Theorem 4.2. Let X be a normal space and G be a grill on X then for each closed set F and any open set V containing F, there exist a G-gs-open set U such that $F \subseteq U \subseteq \tau_G - cl(U) \subseteq V$.

Proof. Let F be a closed set and V an open set in (X, τ) such that $F \subseteq V$. Then F and X/V are disjoint closed sets. By Theorem 4.1, there exist disjoint G-gs-open sets U and W such that $F \subseteq U$ and $X/V \subseteq W$. Since W is G-gs-open and $X/V \subseteq W$ where X/V is closed, $X/V \subseteq \tau_G - int(W)$. So $X/\tau_G - int(W) \subseteq V$. Again, $U \cap W = \phi \Longrightarrow U \cap \tau_G - int(W) = \phi$. Hence $\tau_G - cl(U) \subseteq X/\tau_G - int(W) \subseteq V$. Thus $F \subseteq U \subseteq \tau_G - cl(U) \subseteq V$, where U is a G-gs-open set.

The following theorems gives characterizations of a normal space in terms of gs-open sets which are the consequence of Theorems 4.1, 4.2 and Proposition 3.2 if one takes $G = [X]$.

Theorem 4.3. Let X be a normal space and G be a grill on X then for each pair of disjoint closed sets F and K, there exist disjoint gs-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Theorem 4.4. Let X be a normal space and G be a grill on X then for each closed set F and any open set V containing F, there exist a gs-open set U such that $F \subseteq U \subseteq \tau_G - cl(U) \subseteq V$.

Theorem 4.5. Let X is regular and G be a grill on a space (X, τ). Then for each closed set F and each $x \in X/F$, there exist disjoint G-gs-open sets U and V such that $x \in U$ and $F \subseteq V$.

Proof. The proof is obvious.

Theorem 4.6. Let X be a regular space and G be a grill on a space (X, τ). Then for each open set V of (X, τ) and each point $x \in V$ there exist a G-gs-open set U such that $x \in U \subseteq \tau_G - cl(U) \subseteq V$.

Proof. Let V be any semi-open set in (X, τ) containing a point x of X. Then by Theorem 4.5, there exist disjoint G-gs-open sets U and W such that $x \in U$ and $X/V \subseteq W$. Now, $U \cap W = \phi$ implies $\tau_G - cl(U) \subseteq X/W \subseteq V$. Thus $x \in U \subseteq \tau_G - cl(U) \subseteq V$.

The following theorems gives characterizations of a regular space in terms of gs-open sets which are the consequence of Theorems 4.5, 4.6 and Proposition 3.2 if one takes $G = [X]$.

Theorem 4.7. Let X be a regular and G be a grill on a space (X, τ). Then for each closed set F and each $x \in X/F$, there exist disjoint gs-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 4.8. Let X be a regular space and G be a grill on a space (X, τ). Then for each open set V of (X, τ) and each point $x \in V$ there exist a gs-open set U such that $x \in U \subseteq \tau_G - cl(U) \subseteq V$.

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On Smarandache friendly numbers

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Abstract The Smarandache friendly numbers have been defined by Murthy [1]. This paper finds the Smarandache friendly numbers by solving the associated Pell's equation.

Keywords Smarandache friendly numbers, pell's equation, recurrence relation.

§1. Introduction

Murthy [1] defines the Smarandache friendly numbers as follows :

Definition 1.1. A pair of positive integers $(m,\ n)$ (with $n > m$) is called the Smarandache friendly numbers if

$$m + (m + 1) + \cdots + n = mn.$$

For example, $(3, 6)$ is a Smarandache friendly pair, since

$$3 + 4 + 5 + 6 = 18 = 3 \times 6.$$

Recently, Khainar, Vyawahare and Salunke [2] have treated the problem of finding Smarandache friendly pairs and Smarandache friendly primes.

In this paper, we show that the problem of finding the Smarandache friendly pairs can be reduced to solving a particular type of Pell's equation, which can then be used to find the sequence of all Smarandache friendly pairs.

In section 2, we give some preliminary results that would be necessary in the next section which gives the main results of this paper. It is conjectured that, if (m, n) is a Smarandache friendly pair of numbers, then $(m + 2n, 2m + 5n - 1)$ is also a friendly pair. We also prove this conjecture in the affirmative.

§2. Some preliminary results

The following result is well-known (see, for example, Hardy and Wright [3]).

Lemma 2.1. The general solution of the Diophantine equation $x^2 - 2y^2 = -1$ is

$$x + \sqrt{2}y = (1 + \sqrt{2})^{2\nu+1}; \quad \nu \geq 0. \tag{1}$$

Note that, the Diophantine equation

$$x^2 - 2y^2 = -1 \tag{2}$$

is a particular type of Pell's equation.

Lemma 2.2. Denoting by $(x_\nu,\ y_\nu)$ the ν-th solution of the Diophantine equation (2), $(x_\nu,\ y_\nu)$ satisfies the following recurrence relation:

$$x_{\nu+1} = 3x_\nu + 4y_\nu, \quad y_{\nu+1} = 2x_\nu + 3y_\nu; \quad \nu \geq 1, \tag{3}$$

with

$$x_1 = 7, \quad y_1 = 9. \tag{4}$$

Proof. Since

$$\begin{aligned} x_{\nu+1} + \sqrt{2}\, y_{\nu+1} &= (1+\sqrt{2})^{2\nu+3} \\ &= (x_\nu + \sqrt{2}\, y_\nu)(1+\sqrt{2})^2 \\ &= (x_\nu + \sqrt{2}\, y_\nu)(3+2\sqrt{2}) \\ &= (3x_\nu + 4\, y_\nu) + \sqrt{2}(2x_\nu + 3\, y_\nu), \end{aligned}$$

the result follows.

Lemma 2.2 enables us to calculate the solutions of the Diophantine equation (2) recursively, starting with $x_1 = 7,\ y_1 = 9$.

§3. Main results

We now consider the problem of finding the pair of integers $(m,\ n)$, with $n > m > 0$, such that

$$m + (m+1) + \cdots + n = mn. \tag{5}$$

Writing

$$n = m + k \quad \text{for some integer} \quad k > 0, \tag{6}$$

(5) takes the form

$$m + (m+1) + \cdots + (m+k) = m(m+k),$$

which, after some simple algebraic manipulations, gives

$$k(k+1) = 2m(m-1). \tag{7}$$

In Eq.(7), we substitute

$$k = K + \frac{1}{2}, \quad m = M + \frac{1}{2},$$

to get

$$K^2 - \frac{1}{4} = 2(M^2 - \frac{1}{4}), \tag{8}$$

that is,

$$4K^2 - 8M^2 = -1,$$

that is,

$$x^2 - 2y^2 = -1, \tag{9}$$

where

$$x = 2K, \quad y = 2M. \tag{10}$$

Note that, though K and M are not integers, each of x and y is a positive integer.

Lemma 3.1. The sequence of Smarandache friendly pair of numbers, $\{(m_\nu,\ n_\nu\}_{\nu=1}^{\infty}$, is given by

$$\begin{aligned} m_\nu &= M_\nu + \frac{1}{2} = \frac{1}{2}(y_\nu + 1), \\ k_\nu &= K_\nu - \frac{1}{2} = \frac{1}{2}(x_\nu - 1), \\ n_\nu &= m_\nu + k_\nu = \frac{1}{2}(x_\nu + y_\nu), \end{aligned} \tag{11}$$

where

$$x_\nu + \sqrt{2}\, y_\nu = (1 + \sqrt{2})^{2\nu+1};\ \nu \geq 1. \tag{12}$$

Proof. Since x and y satisfy the Diophantine equation (9), with solutions given by (12), the result follows.

Lemma 3.2. The sequence of Smarandache friendly pair of numbers $\{(m_\nu,\ n_\nu\}_{\nu=1}^{\infty}$ satisfies the following recurrence relation:

$$m_{\nu+1} = m_\nu + 2n_\nu,\ n_{\nu+1} = 2m_\nu + 5n_\nu - 1;\ \nu \geq 1,$$

with

$$m_1 = 3,\ n_1 = 6.$$

Proof. By Lemma 3.1, together with Lemma 2.2,

$$m_{\nu+1} = \frac{1}{2}(y_{\nu+1} + 1) = \frac{1}{2}(2x_\nu + 3y_\nu + 1) = x_\nu + y_\nu + \frac{1}{2}(y_\nu + 1) = 2n_\nu + m_\nu,$$

$$\begin{aligned} n_{\nu+1} &= \frac{1}{2}(x_{\nu+1} + y_{\nu+1}) = \frac{1}{2}[(3x_\nu + 4y_\nu) + (2x_\nu + 3y_\nu)] \\ &= \frac{5}{2}(x_\nu + y_\nu) + y_\nu = 5n_\nu + (2m_\nu - 1), \end{aligned}$$

and we get the desired results.

Lemma 3.2 shows that, if $(x_\nu,\ y_\nu)$ is a Smarandache friendly pair of numbers, so is the pair $(m_\nu + 2n_\nu,\ 2m_\nu + 5n_\nu - 1)$, which is the result conjectured in [2]; moreover, it is the next pair in the sequence. Thus, starting with the smallest friendly pair (3, 6), the other pairs can be obtained recursively, using Lemma 3.2.

§4. Open problems

A pair of primes $(p,\ q)$ with $q > p \geq 2$ is called a pair of Smarandache friendly primes if the sum of the primes from p through q is equal to pq.

Open problem 1. Find all the pairs of Smarandache friendly primes.

Open problem 2. Is the sequence of pairs of Smarandache primes finite?

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The double growth rate of Orlicz sequence spaces

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Abstract In this paper we introduce double entire rate sequence spaces defined by a modulus function. We study their different properties and obtain some inclusion relations involving these double entire rate sequence spaces.

Keywords Double sequence space, entire sequence, analytic sequence, rate spaces, duals, growth sequence.
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§1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m,\ n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the Coordinatewise addition and scalar multiplication.

Some initial work on double sequence spaces were found in Bromwich [4]. Later on, they were investigated by Hardy [13], Moricz [19], Moricz and Rhoades [20], Basarir and Solankan [3], Tripathy [37], Tripathy and Dutta ([41],[42]), Tripathy and Sarma ([43],[44],[45],[46]), Tripathy and Sen [48], Turkmenoglu [49], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : P\text{-}\lim_{m,n\to\infty} |x_{mn} - L|^{t_{mn}} = 0 \text{ for some } L \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : P\text{-}\lim_{m,n\to\infty} |x_{mn} - L|^{t_{mn}} = 0\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of positive reals for all $m,\ n \in \mathbb{N}$ and $P\text{-}\lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m,\ n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u,\ \mathcal{C}_p,\ \mathcal{C}_{0p},\ \mathcal{L}_u,\ \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [11,12] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and obtained the α-, β-, γ-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [50] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] and Tripathy [35] have recently introduced the statistical convergence and Cauchy for double sequences independently and given the relation between statistical convergent and strongly Cesàro summable double sequences. Later, Mursaleen [22] and Mursaleen and Edely [23] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [1] have defined the spaces $\mathcal{BS},\ \mathcal{BS}(t)$, $\mathcal{CS}_p,\ \mathcal{CS}_{bp},\ \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u,\ \mathcal{M}_u(t)$, $\mathcal{C}_p,\ \mathcal{C}_{bp},\ \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and have also examined some properties of those sequence spaces and determined the α-duals of the spaces $\mathcal{BS},\ \mathcal{BV},\ \mathcal{CS}_{bp}$ and the $\beta(\vartheta)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [5] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra ([29],[30],[33]) have studied the space $\chi^2_M(p,q,u)$ and the generalized gai of double sequences and have proved some inclusion relations.

We need the following inequality in the sequel of the paper. For $a,\ b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m,n \in \mathbb{N})$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m,n \to \infty$. The double entire sequences will be denoted by Γ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let ϕ denote the set of all finite sequences.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m,\ n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non-zero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,\ j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})\ \ (m,n \in \mathbb{N})$ are also continuous.

Orlicz [25] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [16] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\left(1 \leq p < \infty\right)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [26], Mursaleen et al. [23], Tripathy et al. [36], Rao and Subramanian [6], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [25].

Recalling [14] and [25], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [24] and further discussed by Ruckle [27] and Maddox [18], Tripathy and Chandra [40] and many others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2-condition is equivalent to $M(Lu) \leq KLM(u)$, for all values of u and for $L > 1$.

Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) $X^{'}=$ the continuous dual of X.

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\}$.

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X\right\}$.

(iv) $X^\gamma = \left\{a = (a_{mn}) : sup_{m,n} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$.

(v) let X be an FK-space $\supset \phi$, then $X^f = \left\{f(\Im_{mn}) : f \in X^{'}\right\}$.

(vi) $X^\delta = \left\{a = (a_{mn}) : sup_{m,n} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\right\}$.

X^α, X^β, X^γ and X^δ are called α- (or Köthe-Toeplitz) dual of X, β- (or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ-dual of X respectively. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [15] as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c,\ c_o$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here $w,\ c,\ c_o$ and ℓ_∞ denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1} |\Delta x_k|.$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2,\ \Gamma^2$ and χ^2 respectively. $\Delta x_{mn} = (x_{m,n} - x_{m,n+1}) - (x_{m+1,n} - x_{m+1,n+1}) = x_{m,n} - x_{m,n+1} - x_{m+1,n} + x_{m+1,n+1}$ for all $m,\ n \in \mathbb{N}$.

Let $r \in \mathbb{N}$ be fixed, then

$$Z(\Delta^r) = \{(x_{mn}) : (\Delta^r x_{mn}) \in Z\} \text{ for } Z = \chi^2,\ \Gamma^2 \text{ and } \Lambda^2,$$

where $\Delta^r x_{mn} = \Delta^{r-1} x_{mn} - \Delta^{r-1} x_{m,n+1} - \Delta^{r-1} x_{m+1,n} + \Delta^{r-1} x_{m+1,n+1}$.

Now we introduced a generalized difference double operator as follows:

Let $r,\ \mu \in \mathbb{N}$ be fixed, then

$$Z\left(\Delta^r_\eta\right) = \left\{(x_{mn}) : \left(\Delta^r_\eta x_{mn}\right) \in Z\right\} \text{ for } Z = \chi^2,\ \mu^2 \text{ and } \Lambda^2,$$

where $\Delta^r_\eta x_{mn} = \Delta^{r-1}_\eta x_{mn} - \Delta^{r-1}_\eta x_{m,n+1} - \Delta^{r-1}_\eta x_{m+1,n} + \Delta^{r-1}_\eta x_{m+1,n+1}$ and $\Delta^0_\eta x_{mn} = x_{mn}$ for all $m,\ n \in \mathbb{N}$.

The notion of a modulus function was introduced by Nakano [24]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that

(i) $f(x) = 0$ if and only if $x = 0$.

(ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0,\ y \geq 0$.

(iii) f is increasing.

(iv) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0,\infty)$.

It is immediate from (ii) and (iv) that f is continuous on $[0,\infty)$. Also from condition (ii), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ and $n^{-1}f(x) \leq f\left(xn^{-1}\right)$, for all $n \in \mathbb{N}$.

§2. Definitions and preliminaries

Let w^2 denote the set of all complex double sequences. A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all prime sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called prime sense double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m,\ n \to \infty$. The double entire sequences will be denoted by Γ^2. The space Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = \sup_{mn}\left\{|x_{mn} - y_{mn}|^{1/m+n} : m,n : 1,2,3,\cdots\right\} \tag{2}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.

Let $\pi = \{\pi_{mn}\}$ be a sequence of positive numbers. If X is a sequence space, we write $X_\pi = \left\{x \in X : \left\{\frac{x_{mn}}{\pi_{mn}}\right\} \in X\right\}$, where $X = \Gamma^2,\ \Lambda^2$.

A sequence $x = (x_{mn})$ is called prime sense double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m,\ n \to \infty$. The double gai sequences will be denoted by χ^2. The space χ^2 is a metric space with the metric

$$\widetilde{d}(x,y) = \sup_{mn}\left\{((m+n)!\,|x_{mn}-y_{mn}|)^{1/m+n} : m,n : 1,2,3,\cdots\right\} \tag{3}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in χ^2.

Definition 2.1. A sequence t is called a double analytic growth sequence for a set Λ^2 of sequences if $x_{mn} = O(t_{mn})$ for all $x \in \Lambda^2 \Leftrightarrow \left|\frac{x_{mn}}{t_{mn}}\right|^{1/m+n} \leq M,\ \forall m,\ n.$

§3. Main results

Theorem 3.1. If Λ^2 has a growth sequence, then Λ^2_π has a growth sequence.

Proof. Let be a growth sequence for Λ^2. Then $|x_{mn}|^{1/m+n} \leq M\,|t_{mn}|$ for some $M > 0$. Let $x \in \Lambda^2_\pi$. Then $\left\{\frac{x_{mn}}{\pi_{mn}}\right\}^{1/m+n} \in \Lambda^2$. We have $\left|\frac{x_{mn}}{\pi_{mn}}\right|^{1/m+n} \leq |x_{mn}|^{1/m+n} \leq M\,|t_{mn}|^{1/m+n}$ which means that $|x_{mn}|^{1/m+n} \leq M\,|t_{mn}\pi_{mn}|^{1/m+n}$. Thus $\{\pi_{mn}t_{mn}\}$ is a growth sequence of Λ^2_{mn}. In other words, Λ^2_π has the growth sequence πt.

Theorem 3.2. Let Λ^2 be a BK-space. Then the rate space Λ^2_π has a growth sequence.

Proof. Let $x \in \Lambda^2_\pi$. Then $\left\{\frac{x_{mn}}{\pi_{mn}}\right\}^{1/m+n} \in \Lambda^2$. Put $P_{mn}(x) = \left(\frac{x_{mn}}{\pi_{mn}}\right)^{1/m+n},\ \forall x \in \Lambda^2_\pi$. Then P_{mn} is a continuous functional on Λ^2_π. Hence $|||P_{mn}|||^{1/m+n} < \infty$. Also for every positive integer $m,\ n$, we have $|x_{mn}|^{1/m+n} = |P_{mn}(x)\,\pi_{mn}|^{1/m+n} = |P_{mn}(x\pi)|^{1/m+n} \leq |||P_{mn}|||^{1/m+n}\,|||x\pi|||^{1/m+n} = |||P_{mn}\pi_{mn}|||^{1/m+n}\,|||x_{mn}|||^{1/m+n}$. Hence $x_{mn} = O(P_{mn}\pi_{mn})$. Thus $\{P_{mn}\pi_{mn}\}$ is a growth sequence for Λ^2_π.

Theorem 3.3. $\left(\Gamma^2_\pi\right)^\alpha = \Lambda^2_{1/\pi}$.

Proof. Let $x \in \Lambda^2_{1/\pi}$. Then there exists $M > 0$ with $|\pi_{mn}x_{mn}| \leq M^{m+n},\ \forall m,\ n \geq 1$. Choose $\epsilon > 0$ such that $\epsilon M < 1$.

If $y \in \Gamma^2_\pi$, we have $\left|\frac{y_{mn}}{\pi_{mn}}\right| \leq \epsilon^{m+n},\ \forall m,\ n \geq m_0n_0$ depending on ϵ.

Therefore $\sum|x_{mn}y_{mn}| \leq \sum(M\epsilon)^{m+n} < \infty$, Hence

$$\Lambda^2_{1/\pi} \subset \left(\Gamma^2_\pi\right)^\alpha. \tag{4}$$

On the other hand, let $x \in \left(\Gamma^2_\pi\right)^\alpha$. Assume that $x \notin \Lambda^2_{1/\pi}$. Then there exists an increasing sequence $\{p_{mn}q_{mn}\}$ of positive integers such that $|\pi_{p_{mn}q_{mn}}x_{p_{mn}q_{mn}}| > (m+n)^{2(p_{mn}+q_{mn})}$, $\forall m$, $n > m_0n_0$. Take $y = \{y_{mn}\}$ by

$$y_{mn} = \begin{cases} \frac{\pi_{mn}}{(m+n)^{(p_{mn}+q_{mn})}}, & \text{for } (p,q) = (p_m,q_n)\,, \\ 0, & \text{for } (p,q) \neq (p_m,q_n)\,. \end{cases} \tag{5}$$

Then $\{y_{mn}\} \in \Gamma^2_\pi$, but $\sum|x_{mn}y_{mn}| = \infty$, a contradiction. This contradiction shows that

$$\left(\Gamma^2_\pi\right)^\alpha \subset \Lambda^2_{1/\pi}. \tag{6}$$

From (4) and (6) it follows that $\left(\Gamma^2_\pi\right)^\alpha = \Lambda^2_{1/\pi}$.

Theorem 3.4. $\left[\Lambda^2_{M\pi}\right]^{\beta} = \left[\Lambda^2_{M\pi}\right]^{\alpha} = \left[\Lambda^2_{M\pi}\right]^{\gamma} = \eta^2_{M\pi}$, where $\eta^2_M = \bigcap_{N\in N-\{1\}}\left\{x = x_{mn} : \sum_{m,n}\left(M\left(\frac{\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|N^{m+n}}{\rho}\right)\right) < \infty\right\}$.

Proof. (i) First we show that $\eta^2_{M\pi} \subset \left[\Lambda^2_{M\pi}\right]^{\beta}$. Let $x \in \eta^2_{M\pi}$ and $y \in \Lambda^2_{M\pi}$. Then we can find a positive integer N such that $\left(|y_{mn}|^{1/m+n}\right) < \max\left(1, \sup_{m,n\geq 1}\left(|y_{mn}|^{1/m+n}\right)\right) < N$, for all $m,\ n$.

Hence we may write

$$\left|\sum_{m,n} x_{mn}y_{mn}\right| \leq \sum_{m,n}|x_{mn}y_{mn}| \leq \sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right) \leq \sum_{m,n}\left(M\left(\frac{\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|N^{m+n}}{\rho}\right)\right).$$

Since $x \in \eta^2_{M\pi}$, the series on the right side of the above inequality is convergent, whence $x \in \left[\Lambda^2_{M\pi}\right]^{\beta}$. Hence $\eta^2_{M\pi} \subset \left[\Lambda^2_{M\pi}\right]^{\beta}$.

Now we show that $\left[\Lambda^2_{M\pi}\right]^{\beta} \subset \eta^2_{M\pi}$.

For this, let $x \in \left[\Lambda^2_{M\pi}\right]^{\beta}$, and suppose that $x \notin \Lambda^2_{M\pi}$. Then there exists a positive integer $N > 1$ such that $\sum_{m,n}\left(M\left(\frac{\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|N^{m+n}}{\rho}\right)\right) = \infty$.

If we define $y_{mn} = (N^{m+n}/\pi_{mn}t_{mn})\, Sgn\,(x_{mn})\, m, n = 1, 2, \cdots$, then $y \in \Lambda^2_{M\pi}$. But, since $\left|\sum_{m,n} x_{mn}y_{mn}\right| = \sum_{mn}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right) = \sum_{m,n}\left(M\left(\frac{|x_{mn}|\left(N^{m+n}/\pi_{mn}t_{mn}\right)}{\rho}\right)\right) = \infty$, we get $x \notin \left[\Lambda^2_{M\pi}\right]^{\beta}$, which contradicts to the assumption $x \in \left[\Lambda^2_{M\pi}\right]^{\beta}$. Therefore $x \in \eta^2_{M\pi}$. Therefore $\left[\Lambda^2_{M\pi}\right]^{\beta} = \eta^2_{M\pi}$.

(ii) and (iii) can be shown in a similar way of (i). Therefore we omit it.

Theorem 3.5. Let M be an Orlicz function or modulus function which satisfies the Δ_2-condition and if $\Gamma^2_{M\pi}$ is a growth sequence then $\Gamma^2_{\pi} \subset \Gamma^2_{M\pi}$.

Proof. Let

$$x \in \Gamma^2_{\pi}. \tag{7}$$

Then $\left((|x_{mn}/\pi_{mn}t_{mn}|)^{1/m+n}\right) \leq \epsilon$ for sufficiently large $m,\ n$ and every $\epsilon > 0$. But then by taking $\rho \geq 1/2$,

$$\begin{aligned}
\left(M\left(\frac{(|x_{mn}/\pi_{mn}t_{mn}|)^{1/m+n}}{\rho}\right)\right) &\leq \left(M\left(\frac{\epsilon}{\rho}\right)\right) \text{ (because } M \text{ is non-decreasing)}\\
&\leq (M(2\epsilon)).\\
\Rightarrow \left(M\left(\frac{(|x_{mn}/\pi_{mn}t_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} &\leq KM(\epsilon) \text{ (by the } \Delta_2\text{-condition, for some } k > 0)\\
&\leq \epsilon \text{ (by defining } M(\epsilon) < \epsilon/K).
\end{aligned}$$

$$\left(M\left(\frac{(|x_{mn}/\pi_{mn}t_{mn}|)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \to 0 \text{ as } m,\ n \to \infty. \tag{8}$$

Hence

$$x \in \Gamma^2_{M\pi}. \tag{9}$$

From (7) and (9) we get $\Gamma^2_{\pi} \subset \Gamma^2_{M\pi}$. This completes the proof.

Theorem 3.6. If $\Gamma^2_{M\pi}$ is a growth sequence then $\eta^2_{M\pi} \subset \left[\Gamma^2_{M\pi}\right]^\beta \subsetneqq \Lambda^2_\pi$.

Proof. (i) First we show that $\eta^2_{M\pi} \subset \left[\Gamma^2_{M\pi}\right]^\beta$. We know that $\Gamma^2_{M\pi} \subset \Lambda^2_{M\pi}$, $\left[\Lambda^2_{M\pi}\right]^\beta \subset \left[\Gamma_{M\pi}\right]^\beta$. But $\left[\Lambda^2_{M\pi}\right]^\beta = \eta^2_{M\pi}$, by Theorem 3.4, therefore

$$\eta^2_{M\pi} \subset \Gamma^2_{M\pi}. \tag{10}$$

(ii) Now we show that $\left[\Gamma^2_{M\pi}\right]^\beta \subsetneqq \Lambda^2_\pi$. Let $y = \{y_{mn}\}$ be an arbitrary point in $\left(\Gamma^2_{M\pi}\right)^\beta$. If y is not in Λ^2_π, then for each natural number q, we can find an index $m_q n_q$ such that

$$\left(M\left(\frac{\left(|y_{m_q n_q}/\pi_{m_q n_q} t_{m_q n_q}|\right)^{1/m_q+n_q}}{\rho}\right)\right) > q, \ (1,2,3,\cdots).$$

Define $x = \{x_{mn}\}$ by $\left(M\left(\frac{(x_{mn}\pi_{mn}t_{mn}}{\rho}\right)\right) = \frac{1}{q^{m+n}}$ for $(m,n) = (m_q, n_q)$ for some $q \in \mathbb{N}$, and $\left(M\left(\frac{x_{mn}\pi_{mn}t_{mn}}{\rho}\right)\right) = 0$ otherwise.

Then x is in $\Gamma^2_{M\pi}$, but for infinitely mn,

$$\left(M\left(\frac{|y_{mn}x_{mn}|}{\rho}\right)\right)^{p_{mn}} > 1. \tag{11}$$

Consider the sequence $z = \{z_{mn}\}$, where $\left(M\left(\frac{z_{11}/\pi_{11}t_{11}}{\rho}\right)\right) = \left(M\left(\frac{x_{11}/\pi_{11}t_{11}}{\rho}\right)\right) - s$ with $s = \sum\left(M\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)$, and $\left(M\left(\frac{z_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) = \left(M\left(\frac{x_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) (m,n = 1,2,3,\cdots)$. Then z is a point of $\Gamma^2_{M\pi}$. Also $\sum\left(M\left(\frac{z_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) = 0$. Hence z is in $\Gamma^2_{M\pi}$. But, by the equation (11), $\sum\left(M\left(\frac{z_{mn}y_{mn}}{\rho}\right)\right)$ does not converge $\Rightarrow \sum x_{mn}y_{mn}$ diverges.

Thus the sequence y would not be in $\left(\Gamma^2_{M\pi}\right)^\beta$. This contradiction proves that

$$\left(\Gamma^2_{M\pi}\right)^\beta \subset \Lambda^2_\pi. \tag{12}$$

If we now choose $M = id$, where id is the identity and $y_{1n}/\pi_{1n}t_{1n} = x_{1n}/\pi_{1n}t_{1n} = 1$ and $y_{mn}/\pi_{mn}t_{mn} = x_{mn}/\pi_{mn}t_{mn} = 0\,(m > 1)$ for all n, then obviously $x \in \Gamma^2_{M\pi}$ and $y \in \Lambda^2_\pi$, but $\sum_{m,n=1}^\infty x_{mn}y_{mn} = \infty$, hence

$$y \notin \left(\Gamma^2_{M\pi}\right)^\beta. \tag{13}$$

From (12) and (13) we are granted

$$\left(\Gamma^2_{M\pi}\right)^\beta \subsetneqq \Lambda^2_\pi. \tag{14}$$

Hence (10) and (14)we are granted $\eta^2_{M\pi} \subset \left[\Gamma^2_{M\pi}\right]^\beta \subsetneqq \Lambda^2_\pi$. This completes the proof.

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The constrained solution of a general Sylvester matrix equation[1]

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Abstract The aim of this paper is to solve the general Sylvester matrix equation $A_1X_1B_1 + A_2X_2B_2 + \cdots + A_5X_5B_5 = C$ with unknown matrix X_1, symmetric X_2, generalized centro-symmetric X_3, generalized bisymmetric X_4 and (R, S)-symmetric X_5. Motivated by the idea of the conjugate gradient method, an iterative method is proposed to find the solution of the above matrix equation. Meantime, the optimal approximation problem is also considered. Numerical examples illustrate the efficiency of this method.

Keywords Generalized Sylvester matrix equation, iterative method, symmetric matrices, generalized centro-symmetric matrices, generalized bisymmetric matrices, (R, S)-symmetric matrices, optimal approximation.

§1. Introduction

In this work, we will use the following notations: Let $R^{m\times n}$ be the set of all $m \times n$ real matrices, and $SR^{n\times n}$ be the set of all $n \times n$ real symmetric matrices, and $SOR^{n\times n}$ be the set of all $n \times n$ symmetric orthogonal matrices. For matrix $A \in R^{m\times n}$, A^T, $tr(A)$, $\mathbf{R}(A)$ denotes the transpose, trace, column space of A, respectively. The symbol $vec(A)$ stands for the stretching function that is defined by $vec(A) = (a_1^T\, a_2^T\, \cdots\, a_n^T)^T$, where a_i is the i^{th} column of A. $A \otimes B$ represents the Kronecker product of matrices $A = (a_{ij})_{m\times n}$ and B. Moreover, $tr(B^TA)$ denotes the inner product of matrices A and B, which generates the Frobenius norm denoted by $\|A\| = \sqrt{tr(A^TA)}$. If $tr(B^TA) = 0$, we say that A, B are orthogonal each other.

Definition 1.1. Let $R \in SOR^{m\times m}$, $S \in SOR^{n\times n}$, i.e., $R^T = R = R^{-1}$, $S^T = S = S^{-1}$. A matrix $X \in R^{m\times m}$ is called generalized centro-symmetric (generalized bisymmetric) with respect to the matrix R if $RXR = X$ ($X^T = X = RXR$). More, we say that matrix $Y \in R^{m\times n}$ is (R, S)-symmetric if $RYS = Y$.

The set of all $n \times n$ generalized centro-symmetric matrices, generalized bisymmetric matrices, and $m \times n$ (R, S)-symmetric matrices are denoted by $GCSR^{n\times n}$, $GBSR^{n\times n}$, $RSR^{n\times n}$, respectively.

These matrices play important roles in information theory, linear system theory, linear

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estimate theory and numerical analysis [1-4], and have been widely studied (see, e.g., [5-11]) by generalized inverse, the singular value decomposition (SVD), the generalized SVD [12], or the canonical correlation decomposition (CCD) [13]. Meanwhile, the iterative method was also involved (see, e.g., [14]).

The researching for matrices always goes with matrix equation, the latter is one of the topics of very active research in scientific computing. There have been a number of papers to discuss the solvability of matrix equation(s) over kinds of matrix spaces (see, e.g., [15-18]). The well-known matrix Sylvester equation and Lyapunov equation are very important in control theory and many other branches of engineering [19,20], which are the particular cases of matrix equations

$$AX - YB = C, \quad AXB + CYD = F$$

with unknown matrices X and Y (see, e.g., [21-25]).

Naturally, the constrained matrix equation problems with more general forms, for instance

$$\sum_{j=1}^{q} A_j X_j B_j = C \tag{1}$$

have taken many authors' attentions, where X_i fulfill some particular structures. Feng Ding, Guan-ren Duan and their collaborators have made much work on this equation or its special cases [26−29]. Motivated by the idea of conjugate gradient method and [30], in this paper, we will solve the following generalized Sylvester matrix equation

$$A_1X_1B_1 + A_2X_2B_2 + \cdots + A_5X_5B_5 = C, \tag{2}$$

where A_i, B_1, B_5, B_j, C are known, and X_l $(l = 1, 2, \cdots, 5)$ to be determined have different structures, which generalizes the results of [30].

For given $R_3 \in SOR^{p_3\times p_3}$, $R_4 \in SOR^{p_4\times p_4}$, $R_5 \in SOR^{p_5\times p_5}$ and $S_5 \in SOR^{q_5\times q_5}$, let

$$\Phi = \left\{ (X_1, X_2, X_3, X_4, X_5) \;\middle|\; \begin{array}{c} X_1 \in R^{p_1\times q_1},\ X_2 \in SR^{p_2\times p_2},\ X_3 \in GCSR^{p_3\times p_3}, \\ X_4 \in GBSR^{p_4\times p_4},\ X_5 \in RSR^{p_5\times q_5} \end{array} \right\}.$$

Then the problems to be discussed in present paper can be stated as follows:

Problem 1.1. Given $A_i \in R^{m\times p_i}$, $B_1 \in R^{q_1\times n}$, $B_5 \in R^{q_5\times n}$, $B_j \in R^{p_j\times n}$, and $C \in R^{m\times n}$, $i = 1, \cdots, 5$, $j = 2, 3, 4$. Find $(X_1, X_2, \cdots, X_5) \in \Phi$ satisfies (1).

Problem 1.2. If the Problem 1.1 is solvable, then, for given matrices $\overline{X}_1 \in R^{p_1\times q_1}$, $\overline{X}_2 \in R^{p_2\times p_2}$, $\overline{X}_3 \in R^{p_3\times p_3}$, $\overline{X}_4 \in R^{p_4\times p_4}$, $\overline{X}_5 \in R^{p_5\times q_5}$, find $(\widehat{X}_1, \widehat{X}_2, \cdots, \widehat{X}_5) \in S_\Phi$ such that

$$\sum_{l=1}^{5} ||\widehat{X}_l - \overline{X}_l||^2 = \min_{(X_1,X_2,\cdots,X_5)\in S_\Phi} \sum_{l=1}^{5} ||X_l - \overline{X}_l||^2,$$

where S_Φ is the solution set of Problem 1.1.

Problem 1.2 is to find the optimal approximation solution for given matrices $\overline{X}_l$ in the solution set of Problem 1.1. This problem is so-called the optimal approximation problem with respect to matrix equation (2) (see e.g., [35,8,11-17,24]).

§2. The iterative method for matrix equation (2)

In this section, we will establish an algorithm to obtain the solution of matrix equation (2).

The following Lemma is necessary.

Lemma 2.1. For given symmetric orthogonal matrices $R_3 \in R^{p_3\times p_3}$, $R_4 \in R^{p_4\times p_4}$, $R_5 \in R^{p_5\times p_5}$ and $S_5 \in R^{q_5\times q_5}$, then matrix equation (2) is solvable if and only if the following matrix equations are consistent, namely

$$\begin{cases} A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 + A_5X_5B_5 = C, \\ A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4R_4X_4R_4B_4 + A_5X_5B_5 = C, \\ A_1X_1B_1 + A_2X_2^TB_2 + A_3R_3X_3R_3B_3 + A_4X_4^TB_4 + A_5R_5X_5S_5B_5 = C, \\ A_1X_1B_1 + A_2X_2^TB_2 + A_3R_3X_3R_3B_3 + A_4R_4X_4^TR_4B_4 + A_5R_5X_5S_5B_5 = C. \end{cases} \tag{3}$$

In particular, if $X_i \in \Phi$, then the two matrix equations are equivalent.

Proof. If matrix equation (2) is consistent, we can easily verify that matrix equation (3) is also consistent.

Conversely, suppose that $(X_1, X_2, \cdots, X_5)$ is a solution group of matrix equation (3), let

$$\widetilde{X}_1 = X_1, \quad \widetilde{X}_2 = \frac{X_2 + X_2^T}{2}, \quad \widetilde{X}_3 = \frac{X_3 + R_3X_3R_3}{2},$$
$$\widetilde{X}_4 = \frac{X_4 + X_4^T + R_4(X_4 + X_4^T)R_4}{4}, \quad \widetilde{X}_5 = \frac{X_5 + R_5X_5S_5}{2}.$$

Obviously, $(\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_5) \in \Phi$. Then

$$\begin{aligned} & \sum_{l=1}^{5} A_l\widetilde{X}_lB_l \\ = \;& A_1\widetilde{X}_1B_1 + A_2\frac{X_2 + X_2^T}{2}B_2 + A_3\frac{X_3 + R_3X_3R_3}{2}B_3 \\ & + A_4\frac{X_4 + X_4^T + R_4(X_4 + X_4^T)R_4}{4}B_4 + A_5\frac{X_5 + R_5X_5S_5}{2}B_5 \\ = \;& \frac{1}{4}[A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4X_4B_4 + A_5X_5B_5] \\ & + \frac{1}{4}[A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 + A_4R_4X_4R_4B_4 + A_5X_5B_5] \\ & + \frac{1}{4}[A_1X_1B_1 + A_2X_2^TB_2 + A_3R_3X_3R_3B_3 + A_4X_4^TB_4 + A_5R_5X_5S_5B_5] \\ & + \frac{1}{4}[A_1X_1B_1 + A_2X_2^TB_2 + A_3R_3X_3R_3B_3 + A_4R_4X_4^TR_4B_4 + A_5R_5X_5S_5B_5] \\ = \;& \frac{1}{4}C \times 4 = C, \end{aligned}$$

which implies that $(\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_5) \in \Phi$ is a solution group of matrix equation (3), it is consistent.

Now, the iterative algorithm for solving Problem 1.1 can be stated as follows:

Algorithm 2.1. *Step* 1: Given matrices $A_i \in R^{m\times p_i}$, $B_1 \in R^{q_1\times n}$, $B_5 \in R^{q_5\times n}$, $B_j \in R^{p_j\times n}$, $i = 1, 2, 3, 4, 5$, $j = 2, 3, 4$. Input arbitrary $(X_1(0), X_2(0), X_3(0), X_4(0), X_5(0)) \in \Phi$.

Step 2: Calculate

$$\mathcal{R}(0) = C - \textstyle\sum_{l=1}^{5} A_l X_l(0) B_l,$$

$$P_l(0) = A_l^T \mathcal{R}(0) B_l^T,\ l = 1, 2, \cdots, 5,$$

$$Q_1(0) = P_1(0),$$

$$Q_2(0) = \frac{1}{2}(P_2(0) + P_2(0)^T),$$

$$Q_3(0) = \frac{1}{2}(P_3(0) + R_3 P_3(0) R_3),$$

$$Q_4(0) = \frac{1}{4}(P_4(0) + P_4(0)^T + R_4[P_4(0) + P_4(0)^T]R_4),$$

$$Q_5(0) = \frac{1}{2}(P_5(0) + R_5 P_5(0) S_5),$$

$$k := 0.$$

Step 3 : Calculate

$$X_l(k+1) = X_l(k) + \alpha_k Q_l(k),\ l = 1, 2, \cdots, 5,\ \alpha_k = \tfrac{\|\mathcal{R}(k)\|^2}{\sum_{l=1}^{5}\|Q_l(k)\|^2}.$$

Step 4 : Calculate

$$\mathcal{R}(k+1) = C - \textstyle\sum_{l=1}^{5} A_l X_l(k+1) B_l = \mathcal{R}(k) - \alpha_k \sum_{l=1}^{5} A_l Q_l(k) B_l,$$

$$P_l(k+1) = A_l^T \mathcal{R}(k+1) B_l^T, l = 1, 2, \cdots, 5,$$

$$Q_1(k+1) = P_1(k+1) + \beta_k Q_1(k),$$

$$Q_2(k+1) = \frac{1}{2}[P_2(k+1) + P_2(k+1)^T] + \beta_k Q_2(k),$$

$$Q_3(k+1) = \frac{1}{2}[P_3(k+1) + R_3 P_3(k+1) R_3] + \beta_k Q_3(k),$$

$$Q_4(k+1) = \frac{1}{4}[P_4(k+1) + P_4(k+1)^T + R_4(P_4(k+1) + P_4(k+1)^T)R_4] + \beta_k Q_4(k),$$

$$Q_5(k+1) = \frac{1}{2}[P_5(k+1) + R_5 P_5(k+1) S_5] + \beta_k Q_5(k),$$

$$\beta_k = \frac{\|\mathcal{R}(k+1)\|^2}{\|\mathcal{R}(k)\|^2}.$$

Step 5 : If $\mathcal{R}(k) = 0$ or $\mathcal{R}(k) \neq 0$ but $Q_l(k) = 0$, stop. Otherwise $k := k + 1$, go to Step 3.

From Algorithm 2.1, we know that

$$(X_1(k), X_2(k), \cdots, X_5(k)),\ (Q_1(k), Q_2(k), \cdots, Q_5(k)) \in \Phi.$$

Moreover, if $\mathcal{R}(k) = 0$, then $X_l(k)$ is a solution pair of matrix equation (2). However, the residual $\mathcal{R}(k)$ may unequal to zero exactly because of the influences of the roundoff errors. In practical, we regard $\mathcal{R}(k)$ as zero matrix if $\|\mathcal{R}(k)\| < \varepsilon$, in which ε is a small positive number, in this case, the iteration will be stopped.

Now we analysis the properties of Algorithm 2.1.

Lemma 2.2. Suppose that $\{\mathcal{R}(k)\}$, $\{Q_l(k)\}$ $(l = 1, 2, \cdots, 5,\ k = 1, 2, \cdots)$ are the sequences generated by Algorithm 2.1, then

$$tr\left(\mathcal{R}(i)^T\mathcal{R}(j)\right) = 0,\ \sum_{l=1}^{5} tr\left(Q_l(i)^T Q_l(j)\right) = 0,\ i, j = 1, 2, \cdots,\ i \neq j. \tag{4}$$

Proof. Similar to the proof of Lemma 6 in [30].

Lemma 2.2 implies that $\{\mathcal{R}(k)\}$ $(k = 1, 2, \cdots)$ is an orthogonal sequence in matrix space $R^{m\times n}$.

Lemma 2.3. Suppose that the matrix equation (2) is consistent, and $(X_1^*, X_2^*, \cdots, X_5^*)$, is any solution group of which, then for any initial iterative matrix group $(X_1(0), X_2(0), \cdots, X_5(0)) \in \Phi$, the iteration sequences $\{X_l(k)\}$, $\{Q_l(k)\}$, $\{\mathcal{R}(k)\}$ $(l = 1, 2, \cdots, 5)$ generated by Algorithm 2.1 satisfy

$$\sum_{l=1}^{5} tr\left((X_l(i)^* - X_l)^T Q_l(i)\right) = \|\mathcal{R}(i)\|^2, \tag{5}$$

for $i = 0, 1, 2, \cdots$.

Proof. Similar to the proof of Lemma 6 in [30].

From Lemma 2.3, if $Q_l(s) = 0$ $(l = 1, 2, \cdots, 5)$ for some s but $\mathcal{R}(s) \neq 0$, which follows from (5) that matrix equation (2) is not consistent. That is to say, the solvability of Problem 1.1 can be determined automatically by Algorithm 2.1. Based on Lemma 2.2 and 2.3, we obtain the main result of this paper.

Theorem 2.1. If Problem 1.1 is consistent, then for any initial iteration matrix group $(X_1(0), X_2(0), X_3(0), X_4(0), X_5(0)) \in \Phi$, a solution to Problem 1.1 can be obtained by Algorithm 2.1 within finite iteration steps in the absence of roundoff errors.

Proof. If $\mathcal{R}(i) \neq 0$ for $i = 1, 2, \cdots, mn$, it follows from Lemma 2.3 that $Q_l(i) \neq 0$ $(l = 1, 2, \cdots, 5)$, then, by Algorithm 2.1, we get $\mathcal{R}(mn+1)$ and $tr\left(\mathcal{R}(mn+1)^T\mathcal{R}(i)\right) = 0$. Hence $\{\mathcal{R}(i),\ i = 1, 2, \cdots, mn\}$ is an orthogonal basis of matrix space $R^{m\times n}$, which indicates that $\mathcal{R}(mn+1) = 0$ and $(X_1(mn+1), X_2(mn+1), \cdots, X_5(mn+1))$ is a solution of Problem 1.1. This completes the proof.

The following Lemma is restated in [14].

Lemma 2.4. Assume that linear system $Ax = b$ is consistent and $y \in \mathbf{R}(A)$ is a solution of which, then y is its least-norm solution.

Theorem 2.2. Suppose that Problem 1.1 is consistent. For any $H \in R^{m\times n}$, choose the initial iteration matrices

$$X_1(0) = 4A_1^T H B_1^T,\ X_2(0) = 2[A_2^T H B_2^T + B_2 H^T A_2],$$

$$X_3(0) = 2[A_3^T H B_3^T + R_3 A_3^T H B_3^T R_3],$$

$$X_4(0) = A_4^T H B_4^T + B_4 H^T A_4 + R_4[A_4^T H B_4^T + B_4 H^T A_4] R_4,$$

$$X_5(0) = 2[A_5^T H B_5^T + R_5 A_5^T H B_5^T S_5],$$

then the solution group $(\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_5)$ generated by Algorithm 2.1 is the least-norm solution of matrix equation (3).

Proof. From Algorithm 2.1, if we choose the initial iteration matrices $X_l(0)$ as above forms, then the solution group $(\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_5)$ of Problem 1.1 has the form of

$$\widetilde{X}_1(0) = 4A_1^T F B_1^T,\ \widetilde{X}_2(0) = 2[A_2^T F B_2^T + B_2 F^T A_2],$$

$$\widetilde{X}_3(0) = 2[A_3^T F B_3^T + R_3 A_3^T F B_3^T R_3],$$

$$\widetilde{X}_4(0) = A_4^T F B_4^T + B_4 F^T A_4 + R_4[A_4^T F B_4^T + B_4 F^T A_4]R_4,$$

$$\widetilde{X}_5(0) = 2[A_5^T F B_5^T + R_5 A_5^T F B_5^T S_5].$$

for some $F \in R^{m\times n}$. Now it is enough to prove that $(\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_5)$ is the least-norm solution.

Let $\Pi \in R^{mn\times mn}$ be the permutation matrix such that $vec(F^T) = \Pi vec(F)$. Then according to the definition of stretching operator, we have

$$\begin{pmatrix} vec(\widetilde{X}_1) \\ vec(\widetilde{X}_2) \\ vec(\widetilde{X}_3) \\ vec(\widetilde{X}_4) \\ vec(\widetilde{X}_5) \end{pmatrix} = \begin{pmatrix} B_1\otimes A_1^T & B_1\otimes A_1^T & B_1\otimes A_1^T & B_1\otimes A_1^T \\ B_2\otimes A_2^T & B_2\otimes A_2^T & (A_2^T\otimes B_2)\Pi & (A_2^T\otimes B_2)\Pi \\ B_3\otimes A_3^T & B_3\otimes A_3^T & R_3B_3\otimes R_3A_3^T & R_3B_3\otimes R_3A_3^T \\ B_4\otimes A_4^T & R_4B_4\otimes R_4A_4^T & (A_4^T\otimes B_4)\Pi & (R_4A_4^T\otimes R_4B_4)\Pi \\ B_5\otimes A_5^T & B_5\otimes A_5^T & S_5B_5\otimes R_5A_5^T & S_5B_5\otimes R_5A_5^T \end{pmatrix} \begin{pmatrix} f \\ f \\ f \\ f \\ f \end{pmatrix}$$
$$\in \mathbf{R}\left(\begin{pmatrix} B_1^T\otimes A_1 & B_2^T\otimes A_2 & B_3^T\otimes A_3 & B_4\otimes A_4^T & B_5^T\otimes A_5 \\ B_1^T\otimes A_1 & B_2^T\otimes A_2 & B_3^T\otimes A_3 & B_4^TR_4\otimes A_4R_4 & B_5^T\otimes A_5 \\ B_1^T\otimes A_1 & \Pi^T(A_2\otimes B_2^T) & B_3^TR_3\otimes A_3R_3 & \Pi^T(A_4\otimes B_4^T) & B_5^TS_5\otimes A_5R_5 \\ B_1^T\otimes A_1 & \Pi^T(A_2\otimes B_2^T) & B_3^TR_3\otimes A_3R_3 & \Pi^T(A_4R_4\otimes B_4^TR_4) & B_5^TS_5\otimes A_5R_5 \end{pmatrix}^T\right) \tag{6}$$

On the other hand, by Lemma 2.1, matrix equation (2) is equivalent to matrix equation (3). The solvability of matrix equation (3) is equivalent to that of the following linear systems

$$\begin{pmatrix} B_1^T\otimes A_1 & B_2^T\otimes A_2 & B_3^T\otimes A_3 & B_4\otimes A_4^T & B_5^T\otimes A_5 \\ B_1^T\otimes A_1 & B_2^T\otimes A_2 & B_3^T\otimes A_3 & B_4^TR_4\otimes A_4R_4 & B_5^T\otimes A_5 \\ B_1^T\otimes A_1 & \Pi^T(A_2\otimes B_2^T) & B_3^TR_3\otimes A_3R_3 & \Pi^T(A_4\otimes B_4^T) & B_5^TS_5\otimes A_5R_5 \\ B_1^T\otimes A_1 & \Pi^T(A_2\otimes B_2^T) & B_3^TR_3\otimes A_3R_3 & \Pi^T(A_4R_4\otimes B_4^TR_4) & B_5^TS_5\otimes A_5R_5 \end{pmatrix}$$
$$\times \begin{pmatrix} vec(X_1) \\ vec(X_2) \\ vec(X_3) \\ vec(X_4) \\ vec(X_5) \end{pmatrix} = \begin{pmatrix} vec(C) \\ vec(C) \\ vec(C) \\ vec(C) \\ vec(C) \end{pmatrix}. \tag{7}$$

(6) and (7), it follows from Lemma 2.4 that $(\widetilde{X}_1, \widetilde{X}_2, \cdots, \widetilde{X}_5)$ is the least-norm solution group of Problem 1.1.

§3. The solution of Problem 1.2

Without loss of generality, let $(\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_5) \in \Phi$. In fact, for any $X_1 \in R^{p_1 \times q_1}$, $X_2 \in R^{p_2 \times p_2}$, $X_3 \in R^{p_3 \times p_3}$, $X_4 \in R^{p_4 \times p_4}$, $X_5 \in R^{p_5 \times q_5}$, we have

$$\|X_2\|^2 = \|\frac{X_2 + X_2^T}{2} + \frac{X_2 - X_2^T}{2}\|^2 = \|\frac{X_2 + X_2^T}{2}\|^2 + \|\frac{X_2 - X_2^T}{2}\|^2,$$

$$\|X_3\|^2 = \|\frac{X_3 + R_3X_3R_3}{2} + \frac{X_3 - R_3X_3R_3}{2}\|^2 = \|\frac{X_3 + R_3X_3R_3}{2}\|^2 + \|\frac{X_3 - R_3X_3R_3}{2}\|^2,$$

$$\begin{aligned}
&\|X_4\|^2 \\
= &\|\frac{X_4 + X_4^T}{2} + \frac{X_4 - X_4^T}{2}\|^2 = \|\frac{X_4 + X_4^T}{2}\|^2 + \|\frac{X_4 - X_4^T}{2}\|^2 \\
= &\|\frac{X_4 + X_4^T + R_4(X_4 + X_4^T)R_4}{4} + \frac{X_4 + X_4^T - R_4(X_4 + X_4^T)R_4}{4}\|^2 + \|\frac{X_4 - R_4X_4R_4}{2}\|^2 \\
= &\|\frac{X_4 + X_4^T + R_4(X_4 + X_4^T)R_4}{4}\|^2 + \|\frac{X_4 + X_4^T - R_4(X_4 + X_4^T)R_4}{4}\|^2 + \|\frac{X_4 - R_4X_4R_4}{2}\|^2,
\end{aligned}$$

$$\|X_5\|^2 = \|\frac{X_5 + R_5X_5S_5}{2} + \frac{X_5 - R_5X_5S_5}{2}\|^2 = \|\frac{X_5 + R_5X_5S_5}{2}\|^2 + \|\frac{X_5 - R_5X_5S_5}{2}\|^2.$$

Moreover, it is easy to verify that the solution set S_Φ is a closed and convex set in Φ, hence the solution to Problem 1.2 is unique.

Denote $\overline{C} = \sum_{l=1}^{5} A_l\overline{X}_lB_l$, then matrix equation (2) is equivalent to

$$\sum_{l=1}^{5} A_l(X_l - \overline{X}_l)B_l = C - \overline{C}, \tag{8}$$

which indicates that the solution to Problem 1.2 can be obtained by the least-norm solution of matrix equation (8). The least-norm solution can also be derived by Algorithm 2.1. Assume that the least-norm solution group of matrix equation (8), generated by Algorithm 2.1, is $(\overline{\overline{X}}_1, \overline{\overline{X}}_2, \cdots, \overline{\overline{X}}_5)$, then the unique optimal approximation solution group can be represented by $(\widehat{X}_1, \widehat{X}_2, \cdots, \widehat{X}_5) = (\overline{\overline{X}}_1 + \overline{X}_1, \overline{\overline{X}}_2 + \overline{X}_2, \cdots, \overline{\overline{X}}_5 + \overline{X}_5)$.

§4. Numerical experiment

In this section, we offer some numerical experiments to illustrate the efficiency of Algorithm 2.1. All the tests are performed by using MATLAB software in real field. In the experiments, we always choose the initial iterative matrices as zero matrices for convenience, which means the solutions obtained by Algorithm 2.1 is the least-norm solution.

Example 4.1. Given matrices A_l, B_l, $l = 1, 2, \cdots, 5$,

$$A_1 = \begin{bmatrix} -15 & 80 & 0 & 46 & -59 & 93 \\ 45 & -33 & 43 & -63 & 22 & -35 \\ -38 & 21 & -74 & -4 & 83 & -16 \\ -95 & 88 & 82 & -22 & 33 & -13 \\ 82 & 95 & 14 & -33 & 74 & -9 \\ 90 & -96 & 11 & 40 & -67 & -35 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 8 & 7 & -6 & -7 & -2 \\ -3 & 0 & 0 & 3 & -8 & 9 \\ 6 & 1 & -3 & 3 & 7 & 2 \\ -4 & 7 & 8 & -1 & 8 & 1 \\ 2 & 1 & 2 & 8 & 5 & 5 \\ -1 & -4 & 7 & 1 & -9 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -84 & -41 & 72 & -84 & -30 & 72 \\ 58 & -99 & -10 & -4 & 36 & 27 \\ -39 & -53 & 47 & 70 & -20 & 85 \\ -60 & 94 & 12 & -51 & 38 & 10 \\ 77 & 34 & 80 & -42 & 91 & 27 \\ -39 & 82 & -50 & 36 & -4 & 59 \end{bmatrix}, \ B_2 = \begin{bmatrix} -6 & 5 & 4 & -4 & 1 & 7 \\ 1 & 3 & 5 & 7 & 4 & 6 \\ 7 & 0 & 1 & 6 & -8 & 2 \\ 6 & -3 & 2 & 3 & 2 & 9 \\ 7 & 3 & 4 & 2 & -3 & 7 \\ 6 & 7 & -8 & 0 & -9 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 80 & 35 & 44 & 59 & 75 & -89 \\ -79 & -20 & -84 & 52 & -95 & -16 \\ 4 & 56 & 65 & 39 & -96 & -60 \\ -12 & -40 & 61 & -90 & -22 & 36 \\ -91 & -22 & 44 & 25 & -67 & 41 \\ -81 & -11 & 15 & -83 & 19 & -37 \end{bmatrix}, \ B_3 = \begin{bmatrix} 0 & -7 & 2 & 7 & -9 & -9 \\ -9 & 8 & -4 & 1 & 9 & 4 \\ -5 & 3 & -6 & 1 & 2 & 0 \\ 8 & 3 & 2 & -5 & 3 & 6 \\ -5 & 7 & 9 & 4 & 1 & 7 \\ 1 & -5 & 3 & -4 & -8 & -9 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -44 & 71 & 71 & -24 & -72 & -18 \\ 37 & 43 & 1 & 47 & 30 & -32 \\ -92 & 62 & -16 & 14 & 21 & 31 \\ -9 & 12 & 35 & -48 & -96 & -67 \\ 61 & 40 & -64 & -20 & 29 & 81 \\ 86 & 77 & 29 & 65 & -54 & -11 \end{bmatrix}, \ B_4 = \begin{bmatrix} -5 & 5 & 5 & 9 & -5 & -7 \\ 7 & -7 & 7 & 2 & 0 & -9 \\ 4 & -1 & 5 & -3 & 9 & -6 \\ 2 & 6 & -2 & 2 & 4 & -7 \\ 3 & 5 & 9 & 0 & 0 & 4 \\ -7 & -9 & -4 & -6 & -5 & -3 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 25 & -83 & -82 & -16 & -45 & 39 \\ 64 & 2 & -41 & -85 & 19 & 51 \\ -78 & 48 & 17 & -95 & -66 & -28 \\ 69 & -68 & -97 & -91 & -64 & -13 \\ -34 & 2 & -24 & 2 & -85 & -58 \\ 23 & -58 & 96 & 13 & 56 & 97 \end{bmatrix}, \ B_5 = \begin{bmatrix} -8 & 6 & 9 & -6 & -7 & -2 \\ 0 & -4 & 0 & -7 & 0 & 2 \\ -8 & -7 & -8 & 1 & 2 & -9 \\ -9 & 0 & 4 & -5 & -7 & 0 \\ 4 & 0 & -1 & 5 & 0 & 5 \\ -6 & 1 & -8 & -7 & -8 & -8 \end{bmatrix}.$$

For the convenience of making the experiments, let $R_i = diag(-1,-1,-1,-1,1,-1)$, $S_5 = diag(-1,1,1,-1,-1,1)$, $i = 3,4,5$.

Furthermore, because of the influence of roundoff errors, the $\mathcal{R}(k)$ generated by Algorithm 2.1 always unequal to zeros in the iteration processing, we regard $\mathcal{R}(k)$ as zero matrix if $\|\mathcal{R}(k)\| < \epsilon$, where $\epsilon = 1.0e-010$.

By Theorem 2.2, for initial matrices $X_l(0) = 0$ $(l = 1,2,\cdots,5)$, after iterating 45 times, we get the least-norm solution group to Problem 1.1, and the residual $\|\mathcal{R}(45)\| = 3.7092e-011$, and the convergent behavior of the $\mathcal{R}(k)$ in the iterating process is described in Figure 1.

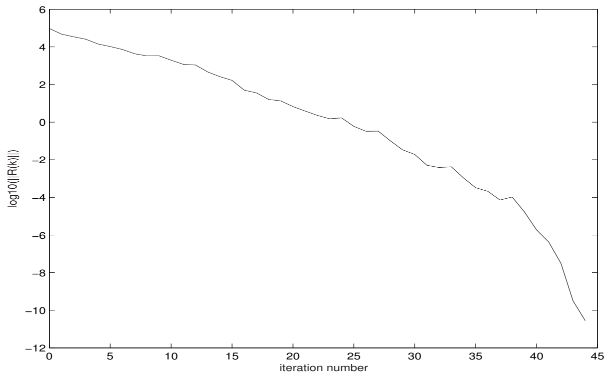

Figure 1. The curve for the Frobenius norm of $\|\mathcal{R}(k)\|$

Moreover, for the above given matrices, suppose that the given matrices $\overline{X}_l$, $l = 1, 2, \cdots, 5$ as follows:

$$\overline{X}_1 = \begin{bmatrix} -15 & 30 & 0 & 46 & -19 & 13 \\ 45 & -33 & 43 & -63 & 22 & -35 \\ -38 & 21 & -34 & -4 & 83 & -16 \\ -95 & -8 & 22 & -27 & 33 & -13 \\ -2 & 55 & 17 & -33 & -4 & -9 \\ -10 & -11 & 11 & 40 & -17 & -35 \end{bmatrix}, \ \overline{X}_2 = \begin{bmatrix} 36 & 57 & -71 & -45 & 23 & 21 \\ 57 & -94 & 15 & -67 & -85 & -34 \\ -71 & 15 & -6 & -24 & -13 & -47 \\ -45 & -67 & -24 & 70 & 33 & -32 \\ 23 & -85 & -13 & 33 & 72 & 93 \\ 21 & -34 & -47 & -32 & 93 & -28 \end{bmatrix},$$

$$\overline{X}_3 = \begin{bmatrix} -8 & 18 & -10 & 14 & 0 & -6 \\ 16 & 8 & 12 & -12 & 0 & -12 \\ -6 & 14 & -14 & -4 & 0 & 18 \\ 2 & 2 & -14 & -4 & 0 & -18 \\ 0 & 0 & 0 & 0 & -6 & 0 \\ -16 & 8 & 2 & -2 & 0 & -10 \end{bmatrix}, \ \overline{X}_4 = \begin{bmatrix} -8 & 6 & 10 & -20 & 0 & -8 \\ 6 & 16 & -2 & 10 & 0 & 22 \\ 10 & -2 & -16 & -14 & 0 & -12 \\ -20 & 10 & -14 & 8 & 0 & -8 \\ 0 & 0 & 0 & 0 & 16 & 0 \\ -8 & 22 & -12 & -8 & 0 & -36 \end{bmatrix},$$

$$\overline{X}_5 = \begin{bmatrix} 14 & 0 & 0 & 12 & -4 & 0 \\ -4 & 0 & 0 & -2 & 6 & 0 \\ 4 & 0 & 0 & 0 & 18 & 0 \\ -16 & 0 & 0 & -2 & -8 & 0 \\ 0 & -14 & -4 & 0 & 0 & -12 \\ -4 & 0 & 0 & -8 & -4 & 0 \end{bmatrix}.$$

Then, by Algorithm 2.1 and after iterating 45 times, we obtain the least-norm solution of matrix equation $\sum_{l=1}^{5} A_l(X_l - \overline{X}_l)B_l = C - \overline{C}$ with $\overline{C} = \sum_{l=1}^{5} A_l\overline{X}_lB_l$, that is

$$X_1^*(45) = \begin{bmatrix} 7.4187 & 2.3654 & 7.6994 & -26.4537 & -0.8903 & 9.4889 \\ -1.1403 & 20.4710 & -8.6811 & 8.2127 & 6.7695 & 30.2093 \\ -22.2824 & -29.0397 & 13.4388 & 13.3700 & 3.4800 & -2.5437 \\ 45.7184 & 19.0782 & -2.6727 & -1.5547 & 0.7860 & 10.3316 \\ -28.9806 & -9.4720 & 4.3301 & 25.8083 & 22.9224 & -8.9168 \\ 14.0796 & 32.0582 & -19.6352 & -30.4605 & -25.1264 & 24.0960 \end{bmatrix},$$

$$X_2^*(45) = \begin{bmatrix} -17.3093 & -9.2962 & 2.6959 & 0.9381 & -13.2859 & -17.2062 \\ -9.2962 & 87.4075 & 35.9680 & 18.7095 & 48.6826 & 15.0530 \\ 2.6959 & 35.9680 & 4.0843 & 12.0422 & 4.4637 & -7.0350 \\ 0.9381 & 18.7095 & 12.0422 & -43.4020 & 3.1580 & 13.6863 \\ -13.2859 & 48.6826 & 4.4637 & 3.1580 & 5.2112 & -20.8418 \\ -17.2062 & 15.0530 & -7.0350 & 13.6863 & -20.8418 & 25.4151 \end{bmatrix},$$

$$X_3^*(45) = \begin{bmatrix} -7.0614 & -6.4590 & 5.5163 & -1.3336 & 0 & 11.5782 \\ -1.1300 & 13.9465 & 16.7789 & -11.0282 & 0 & -1.1532 \\ 1.7971 & 10.1066 & -8.4409 & 16.0665 & 0 & -20.2646 \\ -6.6603 & -14.3232 & 11.7888 & -16.0541 & 0 & 23.4535 \\ 0 & 0 & 0 & 0 & 5.8469 & 0 \\ 23.9338 & -14.0819 & -18.5068 & -4.6412 & 0 & -2.0883 \end{bmatrix},$$

$$X_4^*(45) = \begin{bmatrix} -2.4318 & 20.5025 & -1.2077 & 7.9609 & 0 & 3.7445 \\ 20.5025 & -7.2216 & 1.0598 & 5.2355 & 0 & -9.5683 \\ -1.2077 & 1.0598 & 6.9559 & -2.2644 & 0 & -8.3176 \\ 7.9609 & 5.2355 & -2.2644 & 24.1933 & 0 & 0.9316 \\ 0 & 0 & 0 & 0 & -39.9411 & 0 \\ 3.7445 & -9.5683 & -8.3176 & 0.9316 & 0 & -24.4600 \end{bmatrix},$$

$$X_5^*(45) = \begin{bmatrix} -24.6346 & 0 & 0 & -4.7582 & -15.2988 & 0 \\ -2.7290 & 0 & 0 & 10.9070 & -2.3425 & 0 \\ 14.4704 & 0 & 0 & -27.2356 & 18.8535 & 0 \\ 15.8764 & 0 & 0 & 3.2471 & 24.5090 & 0 \\ 0 & -0.3068 & 20.8887 & 0 & 0 & 31.0194 \\ -9.9946 & 0 & 0 & -9.7552 & -10.0312 & 0 \end{bmatrix}.$$

Hence, the unique optimal approximation solution group of Problem 1.2 is

$$\widehat{X}_1 = \begin{bmatrix} -7.5813 & 32.3654 & 7.6994 & 19.5463 & -19.8903 & 22.4889 \\ 43.8597 & -12.5290 & 34.3189 & -54.7873 & 28.7695 & -4.7907 \\ -60.2824 & -8.0397 & -20.5612 & 9.3700 & 86.4800 & -18.5437 \\ -49.2816 & 11.0782 & 19.3273 & -28.5547 & 33.7860 & -2.6684 \\ -30.9806 & 45.5280 & 21.3301 & -7.1917 & 18.9224 & -17.9168 \\ 4.0796 & 21.0582 & -8.6352 & 9.5395 & -42.1264 & -10.9040 \end{bmatrix},$$

$$\widehat{X}_2 = \begin{bmatrix} 18.6907 & 47.7038 & -68.3041 & -44.0619 & 9.7141 & 3.7938 \\ 47.7038 & -6.5925 & 50.9680 & -48.2905 & -36.3174 & -18.9470 \\ -68.3041 & 50.9680 & -1.9157 & -11.9578 & -8.5363 & -54.0350 \\ -44.0619 & -48.2905 & -11.9578 & 26.5980 & 36.1580 & -18.3137 \\ 9.7141 & -36.3174 & -8.5363 & 36.1580 & 77.2112 & 72.1582 \\ 3.7938 & -18.9470 & -54.0350 & -18.3137 & 72.1582 & -2.5849 \end{bmatrix},$$

$$\widehat{X}_3 = \begin{bmatrix} -15.0614 & 11.5410 & -4.4837 & 12.6664 & 0 & 5.5782 \\ 14.8700 & 21.9465 & 28.7789 & -23.0282 & 0 & -13.1532 \\ -4.2029 & 24.1066 & -22.4409 & 12.0665 & 0 & -2.2646 \\ -4.6603 & -12.3232 & -2.1112 & -20.0541 & 0 & 5.4535 \\ 0 & 0 & 0 & 0 & -0.1531 & 0 \\ 7.9338 & -6.0819 & -16.5068 & -6.6412 & 0 & -12.0883 \end{bmatrix},$$

$$\widehat{X}_4 = \begin{bmatrix} -10.4318 & 26.5025 & 8.7923 & -12.0391 & 0 & -4.2555 \\ 26.5025 & 8.7784 & -0.9402 & 15.2355 & 0 & 12.4317 \\ 8.7923 & -0.9402 & -9.0441 & -16.2644 & 0 & -20.3176 \\ -12.0391 & 15.2355 & -16.2644 & 32.1933 & 0 & -7.0684 \\ 0 & 0 & 0 & 0 & -23.9411 & 0 \\ -4.2555 & 12.4317 & -20.3176 & -7.0684 & 0 & -60.4600 \end{bmatrix},$$

$$\widehat{X}_5 = \begin{bmatrix} -10.6346 & 0 & 0 & 7.2418 & -19.2988 & 0 \\ -6.7290 & 0 & 0 & 8.9070 & 3.6575 & 0 \\ 18.4704 & 0 & 0 & -27.2356 & 36.8535 & 0 \\ -0.1236 & 0 & 0 & 1.2471 & 16.5090 & 0 \\ 0 & -14.3068 & 16.8887 & 0 & 0 & 19.0194 \\ -13.9946 & 0 & 0 & -17.7552 & -14.0312 & 0 \end{bmatrix}.$$

In this case, $\|\mathcal{R}(45)\| = 1.4129e{-}010$,and the norm-cove of the residual $\overline{\mathcal{R}}(k) = C - \overline{C} - \sum_{l=1}^{5} A_l(X_l(k) - \overline{X}_l)B_l$ is described in Figure 2.

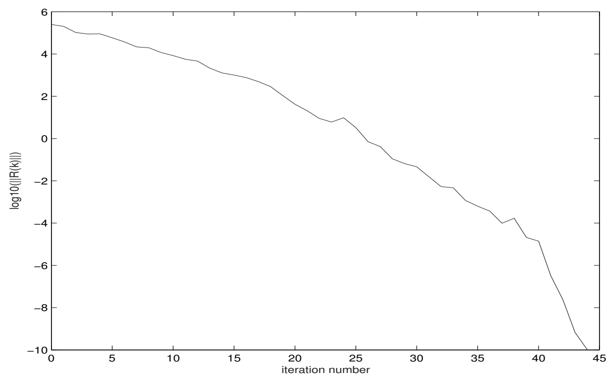

Figure 2. The curve for the Frobenius norm of $\|\overline{\mathcal{R}}(k)\|$.

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A simple proof of the Sophie Germain primes problem along with the Mersenne primes problem and their connection to the Fermat's last conjecture

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Abstract and definitions and statement of results In this paper, we give the proof of a simple Theorem which simultaneously implies the Sophie Germain primes conjecture and the Mersenne primes conjecture, by using only elementary combinatoric, elementary arithmetic congruences, elementary logic, induction and reasoning by reduction to absurd. In addition we also show that the previous two conjectures that we solved elementary were an immediate consequence of the Fermat's last conjecture. Moreover, our paper clearly shows that the Mersenne primes conjecture and the Sophie Germain primes conjecture were only simple arithmetic conjectures, so that strong investigations used in the past to try to solve the previous two conjectures were clearly not welcome. we recall that a Mersenne prime (see [1],[4],[6],[9],[10]) is a prime of the form $M_m = 2^m - 1$, where m is prime, for example M_{13} and M_{19} are Mersenne prime. Mersenne primes are known for some integers $> M_{19}$ and it is conjectured that there are infinitely many Mersenne primes. We recall, (see [2]), that a prime h is called a Sophie Germain prime, if both h and $2h+1$ are prime; the first few Sophie Germain primes are $2, 3, 5, 11, 23, 29, 41, \cdots$ and it is easy to check that 233 is a Sophie Germain prime. Sophie Germain primes are known for some integers > 233 and it is conjectured that there are infinitely many couples of the form $(h, 2h+1)$, where h and $2h+1$ are prime, the Sophie Germain primes conjecture. Finally, we recall ([3],[4],[5],[7],[8],[11]) that the Fermat's last conjecture solved by A. Wiles in a paper of at least 105 pages long (see [11]), and resolved by Ikorong Nemron in a detailled paper of only 19 pages long (see [8]) states that when n is an integer ≥ 3, the equation $x^n + y^n = z^n$ has no non-zero integer solutions for x, y and z, in other words, no three integers of the form $x \geq 1$, $y \geq 1$ and $z \geq 1$ can satisfy the equation $x^n + y^n = z^n$.

Keywords Sophie Germain primes, Mersenne primes.
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§1. Denotations and simple properties

For every integer $n \geq 2$, we define $\mathcal{M}(n)$, m_n and $m_{n,1}$; $\mathcal{H}(n)$, h_n and $h_{n,1}$ as follow:

$$\mathcal{M}(n) = \{x; 1 < x < 2n \text{ and } x \text{ is a Mersenne prime}\},$$

observing by using Abstract and definitions that M_{13} is a Mersenne prime, then it becomes immediate to deduce that for every integer $n \geq M_{13}$, $M_{13} \in \mathcal{M}(n)$, $m_n = \max_{m \in \mathcal{M}(n)} m$, and $m_{n,1} = m_n^{m_n}$; $\mathcal{H}(n) = \{x; 1 < x < 2n$ and x is a Sophie Germain prime$\}$, observing that 233 is a Sophie Germain prime (see Abstract and definitions), then it becomes immediate to deduce that for every integer $n \geq 233$, $233 \in \mathcal{H}(n)$, $h_n = \max_{h \in \mathcal{H}(n)} h$, and $h_{n,1} = h_n^{h_n}$. Using the previous definitions and denotations, let us remark.

Remark 1.1. Let n be an integer $\geq M_{19}$, look at $\mathcal{M}(n)$, m_n and $m_{n,1}$. Then we have the following two simple properties:

(i) $-1 + M_{19} < m_n < m_{n,1}$, $m_{n,1} = m_n^{m_n}$, and $m_{n,1} \geq M_{19}^{M_{19}}$.

(ii) If $m_n < n - 200$, then $n > M_{19}$ and $m_n = m_{n-1}$ and $m_{n,1} = m_{n-1,1}$.

Proof. Property (i) is trivial, indeed, it suffices to use the definition of m_n and $m_{n,1}$, and the fact that $M_{19} \in \mathcal{M}(n)$, note that M_{19} is a Mersenne prime (see Abstract and definitions), since n is an integer $\geq M_{19}$. Property (ii) is immediate, indeed, if $m_n < n - 200$, clearly $n > M_{19}$, use the definition of m_n and observe that $M_{19} \in \mathcal{M}(n)$, since n is an integer $\geq M_{19}$, and so $m_n < n - 200 < 2n - 2$, since $n > M_{19}$ by the previous and $m_n < n - 200$ by the hypothesis, consequently

$$m_n < 2n - 2. \tag{1}$$

Inequality (1) immediately implies that $\mathcal{M}(n) = \mathcal{M}(n-1)$ and therefore

$$m_n = m_{n-1}. \tag{2}$$

Equality (2) immediately implies that $m_{n,1} = m_{n-1,1}$. Property (ii) follows and Remark 1.1 immediately follows.

Remark 1.2. Let n be an integer $\geq M_{19}$, look at $\mathcal{H}(n)$, h_n and $h_{n,1}$. Then we have the following two simple properties:

(i) $232 < h_n < h_{n,1}$, $h_{n,1} = h_n^{h_n}$, and $h_{n,1} \geq 233^{233}$.

(ii) If $h_n < n - 200$, then $n > 233$ and $h_n = h_{n-1}$ and $h_{n,1} = h_{n-1,1}$.

Proof. Property (i) is trivial, it suffices to use the definition of h_n and $h_{n,1}$, and the fact that $233 \in \mathcal{H}(n)$, note that 233 is a Sophie Germain prime (see Abstract and definitions), since n is an integer $\geq M_{19}$ with $M_{19} > 233$. Property (ii) is immediate and is analogous to property (ii) of Remark 1.1, where we replace m_n by h_n, M_{19} by 233, $\mathcal{M}(n)$ by $\mathcal{H}(n)$, $\mathcal{M}(n-1)$ by $\mathcal{H}(n-1)$, m_{n-1} by h_{n-1}, $m_{n,1}$ by $h_{n,1}$ and $m_{n-1,1}$ by $h_{n-1,1}$. Remark 1.2 follows.

From the previous, let us define.

Definition 1.1. For every integer $n \geq 2$, we put

$$\mathcal{MH}(n,1) = \{m_{n,1}\} \bigcup \{h_{n,1}\}.$$

Using Definition 1.1, let us Remark.

Remark 1.3. Let n be an integer $\geq M_{19}$ and consider $\mathcal{MH}(n,1)$. Now let $x_{n,1} \in \mathcal{MH}(n,1)$ and via $x_{n,1}$, look at x_n. Then we have the following:

(i) If $x_{n,1} = m_{n,1}$, then $x_n = m_n$ and we are playing with the Mersenne primes.

(ii) If $x_{n,1} = h_{n,1}$, then $x_n = h_n$ and we are playing with the Sophie Germain primes.

Proof. Immediate, indeed, it suffices to use the definition of x_n and $x_{n,1}$, where $x_{n,1} \in \mathcal{MH}(n,1)$.

Now using Remarks 1.1, 1.2, and 1.3, then the following proposition becomes immediate.

Proposition 1.1. Let n be an integer $\geq M_{19}$ and consider $\mathcal{MH}(n,1)$. Now let $x_{n,1} \in \mathcal{MH}(n,1)$, and via $x_{n,1}$, look at x_n. We have the following two simple properties:

(i) $232 < x_n < x_{n,1}$, $x_{n,1} = x_n^{x_n}$, and $x_{n,1} \geq 233^{233}$.

(ii) If $x_n < n - 200$, then $n > 233$ and $x_n = x_{n-1}$ and $x_{n,1} = x_{n-1,1}$.

Proof. (i) Indeed, let $x_{n,1} \in \mathcal{MH}(n,1)$, if $x_{n,1} = m_{n,1}$ and therefore $x_n = m_n$, use property (i) of Remark 1.1 and apply property (i) of Remark 1.3, if $x_{n,1} = h_{n,1}$ and therefore $x_n = h_n$, use property (i) of Remark 1.2 and apply property (ii) of Remark 1.3. Property (i) follows.

(ii) Indeed, let $x_{n,1} \in \mathcal{MH}(n,1)$ such that $x_n < n - 200$, if $x_{n,1} = m_{n,1}$ and therefore $x_n = m_n$, use property (ii) of Remark 1.1 and apply property (i) of Remark 1.3, if $x_{n,1} = h_{n,1}$ and therefore $x_n = h_n$, use property (ii) of Remark 1.2 and apply property (ii) of Remark 1.3. Property (ii) follows and Proposition 1.1 immediately follows.

Using the definition of $m_{n,1}$ and $h_{n,1}$, then the following remark and corollary become immediate.

Remark 1.4. We have the following three simple properties:

(i) If $\lim_{n\to+\infty} 10m_{n,1} = +\infty$, then there are infinitely many Mersenne primes.

(ii) If $\lim_{n\to+\infty} 10h_{n,1} = +\infty$, then there are infinitely many Sophie Germain primes.

(iii) If $\lim_{n\to+\infty} 10m_{n,1} = +\infty$ and $\lim_{n\to+\infty} 10h_{n,1} = +\infty$, then the Mersenne primes and the Sophie Germain primes are all infinite.

Proof. Properties (i) and (ii) are immediate. Indeed, it suffices to use definitions of $m_{n,1}$ and $h_{n,1}$, and property (iii) follows by using properties (i) and (ii).

Corollary 1.1. If for every integer $n \geq M_{19}$, we have $10m_{n,1} > n - 200$ and $10h_{n,1} > n - 200$, then the Mersenne primes and the Sophie Germain primes are all infinite.

Proof. Clearly, $\lim_{n\to+\infty} 10m_{n,1} = +\infty$ and $\lim_{n\to+\infty} 10h_{n,1} = +\infty$, therefore the Mersenne primes and the Sophie Germain primes are all infinite, by using the previous two equalities and by applying property (iii) of Remark 1.4.

Proposition 1.2. If for every integer $n \geq M_{19}$, and for every $x_{n,1} \in \mathcal{MH}(n,1)$, we have $10x_{n,1} > n - 200$, then the Mersenne primes and the Sophie Germain primes are all infinite.

Proof. Indeed, using the definition of $\mathcal{MH}(n,1)$, we immediately deduce that for every integer $n \geq M_{19}$, $10m_{n,1} > n - 200$ and $10h_{n,1} > n - 200$, therefore the Mersenne primes and the Sophie Germain primes are all infinite, by using the previous two inequalities and by applying Corollary 1.1.

Proposition 1.2 clearly says that: if for every integer $n \geq M_{19}$, and for every $x_{n,1} \in \mathcal{MH}(n,1)$, we have $10x_{n,1} > n - 200$, then the Mersenne primes and the Sophie Germain primes are all infinite, this is what we will do in Section.2, by using only Proposition 1.2, elementary combinatoric, elementary arithmetic congruences, elementary logic, induction and reasoning by reduction to absurd. Proposition 1.2 is stronger than all the investigations which have done on the Mersenne primes and the Sophie Germain primes in the past. Morerover, the reader can easily see that Proposition 1.2 is completely different from all the investigations that have been done on the Mersenne primes and the Sophie Germain primes in the past. So, in Section.2, when we will prove the Mersenne primes conjecture and the Sophie Germain primes conjecture, and the fact that the previous two conjectures are connected to the Fermat's last conjecture, we will not need strong investigations that have been done on the previous two conjectures in the past, and this will not be surprising, since or topic is original, via our original simple remarks and proposition stated above.

§2. Proofs of stated results

In this section, the definition of $\mathcal{MH}(n,1)$ (see Definition 1.1) is fundamental and crucial. Now let us recall.

Recalls and Denotations:

(i) We recall that a statement $S(n)$ is an assertion which can be true or which can be false. In this paper, if $S(n)$ is a statement $k(n)$, we will simply write $S(n) =: k(n)$. So $S(n) =: k(n)$ means $S(n)$ is statement $k(n)$. For example, let $x_{n,1} \in \mathcal{MH}(n,1)$, then $S(n) =: 10x_{n,1} > n+71$ means $S(n)$ is statement $10x_{n,1} > n+71$.

Example 2.1. Let n be an integer $\geq M_{19}$ and consider $x_{n,1} \in \mathcal{MH}(n,1)$ (see Definition 1.1), now let $S(n)$ be the following statement. $S(n) =: 10x_{n,1} > n+71$. Then, $S(n)$ is false, if and only if $10x_{n,1} \leq n+71$.

(ii) We also recall that assertion E and assertion E' are equivalent and we denote by $E \Leftrightarrow E'$, if E and E' are simultaneously true or if E and E' are simultaneously false.

Example 2.2. Let n be an integer $\geq M_{19}$, consider $x_{n,1} \in \mathcal{MH}(n,1)$ (see Definition 1.1), and let $S(n)$ and $S'(n)$ be the following two statements. $S(n) =: 10x_{n,1} > n+71$ and $S'(n) =: 10x_{n,1} > n+70$. If $10x_{n,1} \leq n+70$, then $S(n)$ and $S'(n)$ are equivalent, to see that, it suffices to observe that $10x_{n,1} \leq n+71$, since $10x_{n,1} \leq n+70$, consequently, $S(n)$ and $S'(n)$ are simultaneously false, and therefore $S(n)$ and $S'(n)$ are equivalent.

Example 2.3. Let n be an integer $\geq M_{19}$, consider $x_{n,1} \in \mathcal{MH}(n,1)$, and let statements $S(n)$ and $S'(n)$ defined above. If $10x_{n,1} > n+71$, then $S(n)$ and $S'(n)$ are equivalent, to see that, it suffices to observe that $10x_{n,1} > n+70$, since $10x_{n,1} > n+71$, consequently, $S(n)$ and $S'(n)$ are simultaneously true, and therefore $S(n)$ and $S'(n)$ are equivalent.

(iii) Finally, we recall that assertion F and assertion F' are not equivalent and we denote by $F \not\Leftrightarrow F'$ if F is true and F' is false or if F is false and F' true.

Example 2.4. Let n be an integer $\geq M_{19}$ and consider $x_{n,1} \in \mathcal{MH}(n,1)$ (see Definition 1.1). Now look at statements $S(n)$ and $S'(n)$ defined in Example 2.2 of Recall (i). If $10x_{n,1} \leq n+71$ and if $10x_{n,1} > n+70$, then $S(n)$ and $S'(n)$ are not equivalent, to see that, it suffices to observe that $S(n)$ is false, since $10x_{n,1} \leq n+71$ and $S'(n)$ is true, since $10x_{n,1} > n+70$, consequently $S(n)$ and $S'(n)$ are not equivalent, since $S(n)$ is false and $S'(n)$ is true.

Example 2.5. Let n be an integer $\geq M_{19}$ and consider $x_{n,1} \in \mathcal{MH}(n,1)$ (see Definition 1.1). Now look at statements $S(n)$ and $S'(n)$ defined in Example 2.2 of Recall (i). If $10x_{n,1} = n+71$, then $S(n)$ and $S'(n)$ are not equivalent, to see that, it suffices to observe that $S'(n)$ is true, since $10x_{n,1} = n+71$ and so $10x_{n,1} > n+70$ and $S(n)$ is false, since $10x_{n,1} = n+71$ and so $10x_{n,1} \leq n+71$, consequently $S(n)$ and $S'(n)$ are not equivalent, since $S(n)$ is false and $S'(n)$ is true.

Having made the previous elementary recalls, let us define.

Definitions 2.1. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n+71$ and $1 \leq \lambda'(n) \leq n+71$. Now consider $x_{n,1} \in \mathcal{MH}(n,1)$. Then $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ are statements defined as follow. $Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$, if and only if $10x_{n,1} > n+71$, it is immediate that the previous clearly says that $Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n)$, if and only if $10x_{n,1} \leq n+71$, $Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$, if and only if statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true, it is trivial that the previous clearly says that $Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n)$, if and only if statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is false.

For every integer $n \geq M_{19}$, for every $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n+71$ and $1 \leq \lambda'(n) \leq n+71$, and for every $x_{n,1} \in \mathcal{MH}(n,1)$, it is easy to see that statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ are well defined, it is immediate to see that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ can be true or can be false, and it is also immediate to see that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ can be true or can be false. Now using Definitions 2.1, then the following remark becomes immediate.

Remark 2.1. Let n be an integer $\geq M_{19}$ and look at $x_{n,1} \in \mathcal{MH}(n,1)$ (see Definition 1.1), suppose that $10x_{n,1} = n+71$. Now let $\gamma(n)$ and $\gamma'(n)$, where $\gamma(n)$ and $\gamma'(n)$ are integers such that $1 \leq \gamma(n) \leq n+71$ and $1 \leq \gamma'(n) \leq n+71$, and look at statements $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and $Y(x_{n,1}, \gamma(n), \gamma'(n))$ introduced in Definitions 2.1. We have the following three elementary properties:

(i) If $Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n)$, then $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false.

(ii) If $Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n)$, then $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true.

(iii) $Z(x_{n,1}, \gamma(n), \gamma'(n)) \not\Leftrightarrow Y(x_{n,1}, \gamma(n), \gamma'(n))$.

Proof. Property (i) is immediate, indeed, observe by the definition of statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ that

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n), \text{ if and only if } 10x_{n,1} > n+71. \quad (3)$$

That being so, if statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n),$$

remarking via the hypotheses that $10x_{n,1} = n+71$, clearly $10x_{n,1} \leq n+71$, now using the previous inequality and (3), then it becomes trivial to deduce that statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false, otherwise, $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true, and using (3), then we clearly deduce that $10x_{n,1} > n+71$. A contradiction, since $10x_{n,1} = n+71$, by the hypotheses.

Property (ii) is trivial, indeed, it is trivial by using the definition of statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ that

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n), \text{ if and only if } 10x_{n,1} \leq n+71. \quad (4)$$

That being so, if statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n),$$

remarking via the hypotheses that $10x_{n,1} = n + 71$, clearly $10x_{n,1} \leq n + 71$, now using the previous inequality and (4), then it becomes trivial to deduce that statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true, otherwise, $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false, and using (4), then we clearly deduce that it is false that $10x_{n,1} \leq n + 71$, therefore, $10x_{n,1} > n + 71$. A contradiction, since $10x_{n,1} = n + 71$, by the hypotheses.

(iii) $Z(x_{n,1}, \gamma(n), \gamma'(n)) \not\Leftrightarrow Y(x_{n,1}, \gamma(n), \gamma'(n))$. Otherwise, we reason by reduction to absurd

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) \Leftrightarrow Y(x_{n,1}, \gamma(n), \gamma'(n)), \tag{5}$$

and we are going to distinguish two cases, namely case where $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true and case where $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false.

Case 2.1. Statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true, application of property (i). In this case, using the definition of $Y(x_{n,1}, \gamma(n), \gamma'(n))$ (see Definitions 2.1), then we immediately deduce that statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Y(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n). \tag{6}$$

That being so, we observe the following:

Observation (i). Statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ are simultaneously true. Indeed, remarking by (5) that $Z(x_{n,1}, \gamma(n), \gamma'(n)) \Leftrightarrow Y(x_{n,1}, \gamma(n), \gamma'(n))$, and recalling that we are in the case where $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true, then, using the previous, it becomes trivial to deduce that $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and $Y(x_{n,1}, \gamma(n), \gamma'(n))$, are simultaneously true. Observation (i) follows.

Observation (ii). Look at statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$, then $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form $Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n)$. Otherwise, using the definition of statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ (see Definitions 2.1), then we immediately deduce that statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n). \tag{7}$$

Now look at statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ and remark by (6) that $Y(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Y(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n). \tag{8}$$

That being so, using (7) and (8), then it becomes trivial to see that statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ are not simultaneously true, and this contradicts Observation (i). Observation (ii) follows.

Having made the previous two elementary observations, look at $Z(x_{n,1}, \gamma(n), \gamma'(n))$, observing by using Observation (i), that $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true and remarking that $Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n)$ (use Observation (ii)), then using the previous, it becomes trivial to deduce that

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) \text{ is true and } Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \Leftrightarrow 10x_{n,1} > \gamma'(n), \tag{9}$$

(9) clearly contradicts property (i) of Remark 2.1. Case 2.1 follows.

Case 2.2. Statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false, application of property (ii) of Remark 2.1. In this case, using the definition of $Y(x_{n,1}, \gamma(n), \gamma'(n))$ (see Definitions 2.1), then we immediately deduce that statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Y(x_{n,1}, \gamma(n), \gamma'(n)) =: \ 10x_{n,1} > \gamma(n) \ \not\Leftrightarrow \ 10x_{n,1} > \gamma'(n). \tag{10}$$

That being so, we observe the following:

Observation (iii). Statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ are simultaneously false. Indeed, remarking by (5) that $Z(x_{n,1}, \gamma(n), \gamma'(n)) \Leftrightarrow Y(x_{n,1}, \gamma(n), \gamma'(n))$, and recalling that we are in the case where $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false, then, using the previous, it becomes trivial to deduce that $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and $Y(x_{n,1}, \gamma(n), \gamma'(n))$, are simultaneously false. Observation (iii) follows.

Observation (iv). Look at statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$, then $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form $Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n)$. Otherwise, using the definition of statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ (see Definitions 2.1), then we immediately deduce that statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \ \Leftrightarrow \ 10x_{n,1} > \gamma'(n). \tag{11}$$

Now look at statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ and remark by (10) that $Y(x_{n,1}, \gamma(n), \gamma'(n))$ is of the form

$$Y(x_{n,1}, \gamma(n), \gamma'(n)) =: \ 10x_{n,1} > \gamma(n) \ \not\Leftrightarrow \ 10x_{n,1} > \gamma'(n). \tag{12}$$

That being so, using (11) and (12), then it becomes trivial to see that statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ and statement $Y(x_{n,1}, \gamma(n), \gamma'(n))$ are not simultaneously false, and this contradicts Observation (iii). Observation (iv) follows.

Having made the previous two elementary observations, look at statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$, observing by Observation (iii) that $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false and noticing that $Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n)$ (use Observation (iv)), then using the previous, it becomes trivial to deduce that

$$Z(x_{n,1}, \gamma(n), \gamma'(n)) \text{ is false and } Z(x_{n,1}, \gamma(n), \gamma'(n)) =: 10x_{n,1} > \gamma(n) \not\Leftrightarrow 10x_{n,1} > \gamma'(n), \tag{13}$$

(13) clearly contradicts property (ii) of Remark 2.1. Case 2.2 follows.

That being so, using Case 2.1 (case where statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is true), or using Case 2.2 (case where statement $Z(x_{n,1}, \gamma(n), \gamma'(n))$ is false), we have a contradiction, so in all the cases, we have a contradiction. Property (iii) follows and Remark 2.1 immediately follows.

Now we are quasily ready to state the elementary Theorem which implies stated results. Before, let us introduce the following last definitions:

Definitions 2.2. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), look at statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1, and define $\epsilon(\lambda(n), \lambda'(n))$ as follow: if $Z(x_{n,1}, \lambda(n), \lambda'(n)) \Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n))$, then $\epsilon(\lambda(n), \lambda'(n)) = 0$, and if $Z(x_{n,1}, \lambda(n), \lambda'(n)) \not\Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n))$, then $\epsilon(\lambda(n), \lambda'(n)) = 1$. It is trivial to see that for every integer $n \geq M_{19}$, $\epsilon(\lambda(n), \lambda'(n))$ is well defined. That being so, let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1) and look at $\epsilon(\lambda(n), \lambda'(n))$ defined just above. Then statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is defined as follow:

$$S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) =: Y(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is false},$$

implies that

$$\lambda(n) + \lambda'(n) \equiv \epsilon(\lambda(n), \lambda'(n)) \bmod [2].$$

It is clear that the previous says that $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is statement, $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is false, implies that $\lambda(n) + \lambda'(n) \equiv \epsilon(\lambda(n), \lambda'(n)) \bmod [2]$.

For every integer $n \geq M_{19}$, for every $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$, for every $x_{n,1} \in \mathcal{MH}(n, 1)$ and for every $\epsilon(\lambda(n), \lambda'(n))$, it is immediate to see that statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is well defined (recall that $\epsilon(\lambda(n), \lambda'(n))$ is introduced in Definitions 2.2). Now using Definitions 2.2, then the following two remarks become immediate to prove:

Remark 2.2. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), look at statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1. Now consider $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2 and look at statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ introduced in Definitions 2.2. Now suppose that $10x_{n,1} = n + 71$. We have the following four elementary properties:

(i) $\epsilon(\lambda(n), \lambda'(n)) = 1$.

(ii) If statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is false, then statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n)$.

(iii) If statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true, then statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form $Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n)$.

(iv) If statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is false and if statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is true, then $\lambda(n) + \lambda'(n) \equiv 1 \bmod [2]$.

Proof. Property (i) is immediate, indeed, let statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$, observing via the hypotheses that $10x_{n,1} = n + 71$, then using property (iii) of Remark 2.1, where we replace $\gamma(n)$ by $\lambda(n)$ and $\gamma'(n)$ by $\lambda'(n)$), it becomes trivial to deduce that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \not\Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n)). \tag{14}$$

Now using (14) and the definition of $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2, then it becomes trivial to deduce that $\epsilon(\lambda(n), \lambda'(n)) = 1$. Property (i) follows.

Property (ii) is also immediate, indeed, if statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is false, then using Definitions 2.1, it becomes trivial to deduce that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form

$$Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n). \tag{15}$$

Now let statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$, observing via the hypotheses that $10x_{n,1} = n + 71$, then using property (iii) of Remark 2.1, it becomes trivial to deduce that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \not\Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n)). \tag{16}$$

Recalling that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is false and using (16), then it becomes trivial to deduce that

$$\text{Statement } Y(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is true,} \tag{17}$$

since $Z(x_{n,1}, \lambda(n), \lambda'(n)) \nLeftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n))$ by (16) and since statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is supposed to be false. Now using (17) and (15), then it becomes trivial to deduce that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form

$$Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \nLeftrightarrow 10x_{n,1} > \lambda'(n).$$

Property (ii) follows.

Property (iii) is trivial, indeed, if statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true, then $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \nLeftrightarrow 10x_{n,1} > \lambda'(n). \tag{18}$$

Otherwise, we reason by reduction to absurd, using (18), then it becomes trivial to deduce that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n). \tag{19}$$

Recalling that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true and using (19), then it becomes immediate to deduce that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is true and } Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n). \tag{20}$$

Now observe by the definition of statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1 that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n), \text{ if and only if } 10x_{n,1} > n+71. \tag{21}$$

That being so, using (20) and (21), then it becomes trivial to deduce that $10x_{n,1} > n + 71$, we have a contradiction, since $10x_{n,1} = n + 71$, by the hypotheses. Property (iii) follows.

Property (iv) is also trivial, indeed, if statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is false and if statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is true, then using the previous and the definition of statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, introduced in Definitions 2.2, it becomes trivial to deduce that

$$\lambda(n) + \lambda'(n) \equiv \epsilon(\lambda(n), \lambda'(n)) \bmod [2]. \tag{22}$$

Now look at $\epsilon(\lambda(n), \lambda'(n))$, remarking by property (i) that $\epsilon(\lambda(n), \lambda'(n)) = 1$ and using (22), then it becomes trivial to deduce that $\lambda(n) + \lambda'(n) \equiv 1 \bmod [2]$. Property (iv) follows and Remark 2.2 immediatelly follows.

Remark 2.3. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), look at statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1. Now consider $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2 and look at statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ introduced in Definitions 2.2. Now suppose that $10x_{n,1} > n + 71$. We have the following five elementary properties:

(i) Statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$.

(ii) Statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$.

(iii) Statements $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ and $Z(x_{n,1}, \lambda(n), \lambda'(n))$ are simultaneously true.

(iv) $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n)) \Leftrightarrow Z(x_{n,1}, \lambda(n), \lambda'(n))$.

(v) $\epsilon(\lambda(n), \lambda'(n)) = 0$.

Proof. Property (i) is immediate, indeed, let $\lambda(n)$ and $\lambda'(n)$, note that $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$, noticing via the hypotheses that $10x_{n,1} > n + 71$, and using the previous inequality, then it becomes trivial to deduce that

$$10x_{n,1} > \lambda(n) \text{ and } 10x_{n,1} > \lambda'(n). \tag{23}$$

That being so, we observe the following:

Observation (v). $10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$. Indeed, noticing by (23) that $10x_{n,1} > \lambda(n)$ and $10x_{n,1} > \lambda'(n)$, then it becomes trivial to deduce that $10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$. Observation (v) follows.

Observation (vi). Statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n).$$

Otherwise, we reason by reduction to absurd, it becomes trivial to deduce that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n). \tag{24}$$

By using the definition of statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1, it is immediate to see that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n), \text{ if and only if } 10x_{n,1} \leq n + 71. \tag{25}$$

Now, using (24) and (25), then we immediately deduce that $10x_{n,1} \leq n + 71$, a contradiction, since $10x_{n,1} > n + 71$, by the hypotheses. Observation (vi) follows.

Observation (vii). Statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true. Otherwise, we reason by reduction to absurd $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is false, and using Observation (vi), then we immediately deduce that $10x_{n,1} > \lambda(n) \not\Leftrightarrow 10x_{n,1} > \lambda'(n)$ and this contradicts Observation (v). Observation (vii) follows.

The previous trivial observations made, using Observation (vii) and Observation(vi), then it becomes immediate to deduce that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n).$$

Property (i) follows.

Property (ii) is also immediate, indeed, look at statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$, remarking by using property (i) that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true, then it becomes trivial to deduce that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is of the form

$$Y(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n). \tag{26}$$

Now noticing by using again property (i) that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is true and } Z(x_{n,1}, \lambda(n), \lambda'(n)) =: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n), \quad (27)$$

then, using (27) and (26), then it becomes trivial to deduce that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Y(x_{n,1}, \lambda(n), \lambda'(n)))$ $=: 10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$. Property (ii) follows.

Property (iii) is trivial, indeed, look at statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ (see Definitions 2.2), noticing by using property (ii) that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true, then using the definition of statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, it becomes trivial to deduce that

$$\text{Statement } S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \text{ is true.} \quad (28)$$

Now observe by using properties (ii) and (i) that

$$\text{Statement } Y(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is true and statement } Z(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is true.} \quad (29)$$

That being so, using (28) and (29), then it becomes trivial to deduce that statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ and statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ and statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ are simultaneously true. Property (iii) follows.

Property (iv) follows immediately by using property (iii).

Property (v) is trivial, indeed, observing that $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n)) \Leftrightarrow Z(x_{n,1}, \lambda(n), \lambda'(n))$ (use property (iv)), then in particular, we have clearly

$$Y(x_{n,1}, \lambda(n), \lambda'(n)) \Leftrightarrow Z(x_{n,1}, \lambda(n), \lambda'(n)). \quad (30)$$

Now using (30) and the definition of $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2, then it becomes trivial to deduce that $\epsilon(\lambda(n), \lambda'(n)) = 0$. Property (v) follows and Remark 2.3 immediately follows.

The previous simple definitions and remarks made, now the following Theorem immediately implies stated results.

Theorem 2.1. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n+71$ and $1 \leq \lambda'(n) \leq n+71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), look at statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1. Now consider $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2 and look at statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ introduced in Definitions 2.2. Then the following two properties (i) and (ii) are simultaneously satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$:

(i) $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n)) \Leftrightarrow Z(x_{n,1}, \lambda(n), \lambda'(n))$.

(ii) $10x_{n,1} > n + 71$.

We are going to prove simply Theorem 2.1. But before doing so, let us propose the following four simple propositions.

Proposition 2.1. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), look at statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $Y(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in definitions 2.1. Now consider $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2 and look at statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ introduced in Definitions 2.2. We have the following two trivial properties:

(i) If $10x_{n,1} > n + 71$, then Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$.

(ii) If $n \leq 233^{233}$, then Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$.

Proof. Property (i) is immediate, indeed, let n be an integer $\geq M_{19}$ and let $x_{n,1} \in \mathcal{MH}(n, 1)$, if $10x_{n,1} > n + 71$, then, using property (iv) of Remark 2.3, we immediately deduce that

$$S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \Leftrightarrow Y(x_{n,1}, \lambda(n), \lambda'(n)) \Leftrightarrow Z(x_{n,1}, \lambda(n), \lambda'(n)). \tag{31}$$

Now using (31) and the fact that $10x_{n,1} > n+71$, then it becomes trivial to see that properties (i) and (ii) of Theorem 2.1 are simultaneously satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, therefore Theorem 2.1 is clearly satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$.

Property (ii) is also immediate, indeed let n be an integer $\geq M_{19}$ and let $x_{n,1} \in \mathcal{MH}(n, 1)$, observing by using property (i) of Proposition 1.1 that

$$x_{n,1} \geq 233^{233}, \tag{32}$$

if $n \leq 233^{233}$, then using (32), we immediately deduce that

$$x_{n,1} \geq 233^{233} \geq n, \tag{33}$$

(33) immediately implies that $x_{n,1} \geq n$ and consequently

$$10x_{n,1} \geq 10n. \tag{34}$$

Now remarking via the hypotheses that $n \geq M_{19}$ and using inequality (34), it becomes immediate to deduce that $10x_{n,1} \geq 10n > n + 71$, so

$$10x_{n,1} > n + 71. \tag{35}$$

Consequently, Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, by using inequality (35) and property (i). Property (ii) follows, and Proposition 2.1 follows.

Proposition 2.2. Let n be an integer $\geq M_{19}$, and let $\lambda(n)$ and $\lambda'(n)$, where $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$. Consider $x_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), and via $x_{n,1}$, look at x_n (see Definition 1.1 and Remark 1.3 of Section.1 for the definition of x_n). If $x_n \geq n - 200$, then Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$.

Proof. Indeed, let n be an integer $\geq M_{19}$ and let $x_{n,1} \in \mathcal{MH}(n, 1)$, now, via $x_{n,1}$, look at x_n. Observe by using property (i) of Proposition 1.1 that

$$x_{n,1} = x_n^{x_n} \text{ and } x_{n,1} \geq 233^{233}. \tag{36}$$

That being so, if $x_n \geq n - 200$, then using (36) and the fact that $n \geq M_{19}$, it becomes trivial to deduce that $x_{n,1} > -1 + x_n^{x_n} > -2 + (n - 200)^{n-200} > n + 71$, consequently

$$x_{n,1} > n + 71. \tag{37}$$

Therefore, Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, by using inequality (37) and property (i) of Proposition 2.1. Proposition 2.2 follows.

From Propositions 2.1 and 2.2, it comes:

Proposition 2.3. Suppose that Theorem 2.1 is false. Then there exists $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ such that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum.

Proof. Immediate, by observing that $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n + 71$ and $1 \leq \lambda'(n) \leq n + 71$.

Proposition 2.4.(Application of Proposition 2.3) Suppose that Theorem 2.1 is false, and let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ be a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum, such a $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ exists, by using Proposition 2.3. Now, via $x_{n,1}$, look at x_n (see Definition 1.1 and Remark 1.3 of Section.1 for the definition of x_n). Then we have the following three properties:

(i) $10x_{n,1} \leq n + 71$ and $n > 233^{233}$.

(ii) $x_n < n - 200$.

(iii) $x_{n,1} = x_{n-1,1}$.

Proof. (i) $10x_{n,1} \leq n + 71$. Otherwise, we reason by reduction to absurd $10x_{n,1} > n + 71$, now using the previous inequality and property (i) of Proposition 2.1, then it becomes immediate to deduce that Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, and we have a contradiction, since in particular $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1. Having proved this fact, we have $n > 233^{233}$. Otherwise, we reason by reduction to absurd, clearly $n \leq 233^{233}$, now using the previous inequality and property (ii) of Proposition 2.1, then it becomes immediate to deduce that Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, and we have a contradiction, since in particular $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1. Property (i) follows.

(ii) We have $x_n < n - 200$. Otherwise, we reason by reduction to absurd, clearly $x_n \geq n-200$, now using the previous inequality and Proposition 2.2, then it becomes trivial to deduce that Theorem 2.1 is satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, and we have a contradiction, since in particular $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1. Property (ii) follows.

(iii) Indeed, observing by using property (ii) that $x_n < n - 200$, then using the previous inequality and property (i) of Proposition 1.1, it becomes trivial to deduce that $x_{n,1} = x_{n-1,1}$. Property (iii) follows, and Proposition 2.4 immediately follows.

Now, we are ready to give an elementary proof stated results, but before, let us propose the following three elementary propositions:

Proposition 2.5. The elementary using of the minimality of n. Suppose that Theorem 2.1 is false, and let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ be a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum, such a $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ exists, by using Proposition 2.3. Then $10x_{n,1} = n + 71$.

Proof. Indeed, via $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, look at $(n-1, x_{n-1,1}, \epsilon(\lambda(n-1), \lambda'(n-1)))$, observing by using property (i) of Proposition 2.4 that $n > 233^{233}$, clearly $n - 1 > -1 + 233^{233} > M_{19}$ and $n - 1 < n$, then, by the minimality of n, $(n-1, x_{n-1,1}, \epsilon(\lambda(n-1), \lambda'(n-1)))$ satisfies Theorem 2.1, therefore, properties (i) and (ii) are simultaneously satisfied by $(n-1, x_{n-1,1}, \epsilon(\lambda(n-1), \lambda'(n-1)))$, in particular property (ii) of Theorem 2.1 is satisfied by $(n-1, x_{n-1,1}, \epsilon(\lambda(n-1), \lambda'(n-1)))$, and consequently $10x_{n-1,1} > (n-1) + 71$. The previous inequality clearly says that

$$10x_{n-1,1} > n + 70. \tag{38}$$

Now, observing by property (iii) of Proposition 2.4 that $x_{n,1} = x_{n-1,1}$ and using the previous equality, then it becomes trivial to deduce that inequality (38) clearly says that

$$10x_{n,1} > n + 70. \tag{39}$$

Noticing that $10x_{n,1}$ and $n + 70$ are integers, then it becomes trivial to deduce that inequality (39) clearly says that

$$10x_{n,1} \geq n + 71. \tag{40}$$

That being so, observe by using property (i) of Proposition 2.4 that

$$10x_{n,1} \leq n + 71. \tag{41}$$

Now using (40) and (41), then it becomes trivial to deduce that $10x_{n,1} = n + 71$. Proposition 2.5 follows.

Proposition 2.6. Let n be an integer $\geq M_{19}$, and let $\gamma(n)$ and $\gamma'(n)$, where $\gamma(n)$ and $\gamma'(n)$ are integers such that $1 \leq \gamma(n) \leq n + 71$ and $1 \leq \gamma'(n) \leq n + 71$. Consider $u_{n,1} \in \mathcal{MH}(n, 1)$ (see Definition 1.1), look at statements $Z(u_{n,1}, \gamma(n), \gamma'(n))$ and $Y(u_{n,1}, \gamma(n), \gamma'(n))$ introduced in Definitions 2.1. Consider $\epsilon(\gamma(n), \gamma'(n))$ introduced in Definitions 2.2 and let statement $S(u_{n,1}, \epsilon(\gamma(n), \gamma'(n)))$ introduced in Definitions 2.2. Now look at $(n, u_{n,1}, \epsilon(\gamma(n), \gamma'(n)))$ and suppose that $10u_{n,1} = n + 71$. Then we have the following two trivial properties:

(i) $Y(u_{n,1}, \gamma(n), \gamma'(n)) \not\Leftrightarrow Z(u_{n,1}, \gamma(n), \gamma'(n))$.

(ii) Property (i) of Theorem 2.1 is not satisfied by $(n, u_{n,1}, \epsilon(\gamma(n), \gamma'(n)))$.

Proof. (i) Indeed, noticing via the hypotheses that $10u_{n,1} = n + 71$ and using property (iii) of Remark 2.1, where we replace $x_{n,1}$ by $u_{n,1}$, then it becomes trivial to deduce that

$$Y(u_{n,1}, \gamma(n), \gamma'(n)) \not\Leftrightarrow Z(u_{n,1}, \gamma(n), \gamma'(n)).$$

Property (i) follows.

(ii) Property (i) of Theorem 2.1 is not satisfied by $(n, u_{n,1}, \epsilon(\gamma(n), \gamma'(n)))$, otherwise, we reason by reduction absurd, clearly

$$S(u_{n,1}, \epsilon(\gamma(n), \gamma'(n))) \Leftrightarrow Y(u_{n,1}, \gamma(n), \gamma'(n)) \Leftrightarrow Z(u_{n,1}, \gamma(n), \gamma'(n)), \tag{42}$$

since property (i) of Theorem 2.1 is satisfied by $(n, u_{n,1}, \epsilon(\gamma(n), \gamma'(n)))$. Using (42), then it trivially follows that $Y(u_{n,1}, \gamma(n), \gamma'(n)) \Leftrightarrow Z(u_{n,1}, \gamma(n), \gamma'(n))$ and this contradicts property (i). Property (ii) follows and Proposition 2.6 immediately follows.

From Proposition 2.5 and Proposition 2.6, it comes:

Proposition 2.7. The elementary using of the maximality of $\lambda(n) + \lambda'(n)$. An obvious application of Proposition 2.5 and Proposition 2.6. Suppose that Theorem 2.1 is false, and let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ be a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum, such a $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ exists, by using Proposition 2.3. Then $\lambda(n) = \lambda'(n) = n + 71$.

Proof. Otherwise, we reason by reduction to absurd, clearly

$$\lambda(n) \neq n+71 \text{ or } \lambda'(n) \neq n+71, \tag{43}$$

and we observe the following:

Observation (viii). Look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$. Then $\lambda(n)+\lambda'(n) \leq 2n+141$. Indeed, recalling that $\lambda(n)$ and $\lambda'(n)$ are integers such that $1 \leq \lambda(n) \leq n+71$ and $1 \leq \lambda'(n) \leq n+71$, then, using the previous and using (43), it becomes trivial to deduce that $\lambda(n)+\lambda'(n) \leq 2n+141$. Observation (viii) follows.

Observation (ix). Look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, recall that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1 with n minimum and $\lambda(n)+\lambda'(n)$ maximum. Now let $\alpha(n)$ and $\alpha'(n)$ be integers such that $\alpha(n) = \alpha'(n) = n+71$, and look at $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$. Then

$$\alpha(n)+\alpha'(n) > \lambda(n)+\lambda'(n) \text{ and } (n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n))) \text{ satisfies Theorem 2.1.}$$

Indeed, it is immediate that

$$\alpha(n)+\alpha'(n) = 2n+142, \tag{44}$$

since $\alpha(n) = \alpha'(n) = n+71$. Now remarking by using Observation (viii) that $\lambda(n)+\lambda'(n) \leq 2n+141$ and using equality (44), then it becomes trivial to deduce that

$$\alpha(n)+\alpha'(n) > \lambda(n)+\lambda'(n). \tag{45}$$

That being so, look at $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$, noticing by (45) that $\alpha(n)+\alpha'(n) > \lambda(n)+\lambda'(n)$, then, by the maximality of $\lambda(n)+\lambda'(n)$, clearly $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$ is not a counter-example to Theorem 2.1, since $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1 with n minimum and $\lambda(n)+\lambda'(n)$ maximum, and $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$ is such that $\alpha(n)+\alpha'(n) > \lambda(n)+\lambda'(n)$, so, by the maximality of $\lambda(n)+\lambda'(n)$, $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$ is not a counter-example to Theorem 2.1. Consequently, $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$ satisfies Theorem 2.1. Observation (ix) follows.

Having made the previous two simple obervations, look at $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$, observing by Observation (ix) that $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$ satisfies Theorem 2.1, then property (i) and property (ii) of Theorem 2.1 are simultaneously satisfied by $(n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n)))$, in particular

$$\text{property (i) of Theorem 2.1 is satisfied by } (n, x_{n,1}, \epsilon(\alpha(n), \alpha'(n))), \tag{46}$$

(46) clearly contradicts property (ii) of Proposition 2.6, by remarking that $10x_{n,1} = n+71$ (use Proposition 2.5), and by replacing in Proposition 2.6, $u_{n,1}$ by $x_{n,1}$, $\gamma(n)$ by $\alpha(n)$ and $\gamma'(n)$ by $\alpha'(n)$. Proposition 2.7 follows.

The previous simple propositions made, we now prove simply Theorem 2.1.

Proof of Theorem 2.1. Otherwise, we reason by reduction to absurd, let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ be a counter-example to Theorem 2.1 with n minimum and $\lambda(n)+\lambda'(n)$ maximum, such a $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ exists, by using Proposition 2.3. We observe the following:

Observation (x). Look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$. Then Property (i) of Theorem 2.1 is not satisfied by $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$.

Indeed, look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, observing by Proposition 2.5, that $10x_{n,1} = n+71$, where $n \geq M_{19}$ and $x_{n,1} \in \mathcal{MH}(n, 1)$, then using the previous, it becomes immediate to deduce that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ satisfies all the hypotheses of Proposition 2.6, therefore, $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ satisfies the conclusion of Proposition 2.6, in particular, $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ satisfies property (ii) of Proposition 2.6, and consequently

$$\text{property (i) of Theorem 2.1 is not satisfied by } (n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n))),$$

by replacing in Proposition 2.6, $u_{n,1}$ by $x_{n,1}$, $\gamma(n)$ by $\lambda(n)$ and $\gamma'(n)$ by $\lambda'(n)$. Observation (x) follows.

Observation (xi). Look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$. Then $10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$.

Indeed look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, recall that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum, noticing by Proposition 2.7 that

$$\lambda(n) = \lambda'(n) = n + 71, \tag{47}$$

and remarking by Proposition 2.5 that

$$10x_{n,1} = n + 71, \tag{48}$$

then, using (47) and (48), then it becomes trivial to deduce that

$$10x_{n,1} = \lambda(n) \text{ and } 10x_{n,1} = \lambda'(n). \tag{49}$$

Using (49), then it becomes trivial to deduce that

$$10x_{n,1} \leq \lambda(n) \textit{ and } 10x_{n,1} \leq \lambda'(n). \tag{50}$$

It is trivial to see that (50) immediately implies that

$$10x_{n,1} \leq \lambda(n) \Leftrightarrow 10x_{n,1} \leq \lambda'(n). \tag{51}$$

Now look at (51), then using the definition of the relation "$\Leftrightarrow$" (see property (ii) of Recalls and Denotations for the definition of $E \Leftrightarrow E'$), it becomes trivial to deduce that (51) clearly implies that $10x_{n,1} > \lambda(n) \Leftrightarrow 10x_{n,1} > \lambda'(n)$. Observation (xi) follows.

Observation (xii) Look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$. Then $\lambda(n) + \lambda'(n) \equiv 0 \bmod [2]$.

Indeed look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ recall that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum, noticing by Proposition 2.7 that $\lambda(n) = \lambda'(n) = n + 71$, then it becomes trivial to deduce that

$$\lambda(n) + \lambda'(n) = 2n + 142. \tag{52}$$

Since it is immediate that $2n + 142$ is even, then, using equality (52) and the previous, it becomes trivial to deduce that $\lambda(n) + \lambda'(n) \equiv 0 \bmod [2]$. Observation (xii) follows.

Observation (xiii). Let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ and consider $\epsilon(\lambda(n), \lambda'(n))$. Then $\epsilon(\lambda(n), \lambda'(n)) = 1$.

Indeed, look at $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, observing by Proposition 2.5 that $10x_{n,1} = n + 71$, where $n \geq M_{19}$ and $x_{n,1} \in \mathcal{MH}(n, 1)$, then, using the previous, it becomes immediate to deduce that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ satisfies all the hypotheses of Proposition 2.6, therefore, $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ satisfies the conclusion of Proposition 2.6, in particular, $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ satisfies property (i) of Proposition 2.6, and consequently

$$Y(x_{n,1}, \lambda(n), \lambda'(n))) \not\Leftarrow Z(x_{n,1}, \lambda(n), \lambda'(n)), \tag{53}$$

by replacing in Proposition 2.6, $u_{n,1}$ by $x_{n,1}$, $\gamma(n)$ by $\lambda(n)$ and $\gamma'(n)$ by $\lambda'(n)$. Now using (53) and the definition of $\epsilon(\lambda(n), \lambda'(n))$ introduced in Definitions 2.2, then it becomes trivial to deduce that $\epsilon(\lambda(n), \lambda'(n)) = 1$. Observation (xiii) follows.

Observation (xiv). Let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ recall that $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is a counter-example to Theorem 2.1 with n minimum and $\lambda(n) + \lambda'(n)$ maximum. Now look at statements $Y(x_{n,1}, \lambda(n), \lambda'(n))$ and $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is introduced in Definition 2.1 and $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is introduced in Definitions 2.2. Then, $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \Rightarrow Y(x_{n,1}, \lambda(n), \lambda'(n))$.

Otherwise, we reason by reduction to absurd, clearly

$$Y(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is false.} \tag{54}$$

Now noticing (under the hypothese) that $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is true, and using (54) via the definition of $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, then it becomes trivial to deduce that

$$\lambda(n) + \lambda'(n) \equiv \epsilon(\lambda(n), \lambda'(n)) \bmod [2], \tag{55}$$

since statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is false by (54) and since statement $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is supposed to be true. That being so, look at $\epsilon(\lambda(n), \lambda'(n))$, observing by Observation (xiii) that $\epsilon(\lambda(n), \lambda'(n)) = 1$ and using the previous equality, then it becomes trivial to deduce that congruence (55) clearly says that $\lambda(n) + \lambda'(n) \equiv 1 \bmod [2]$, and this contradicts Observation (xii). So $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \Rightarrow Y(x_{n,1}, \lambda(n), \lambda'(n))$. Observation (xiv) follows.

Observation (xv). Let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$. Now look at statements $Y(x_{n,1}, \lambda(n), \lambda'(n))$ and $Z(x_{n,1}, \lambda(n), \lambda'(n))$ introduced in Definitions 2.1. Then $Y(x_{n,1}, \lambda(n), \lambda'(n)) \Rightarrow Z(x_{n,1}, \lambda(n), \lambda'(n))$.

Otherwise, we reason by reduction to absurd, clearly

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is false.} \tag{56}$$

That being so, look at statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$, then we have the following elementary two facts:

Fact 2.1. Statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Y(x_{n,1}, \lambda(n), \lambda'(n))) =: 10x_{n,1} > \lambda(n) \not\Leftarrow 10x_{n,1} > \lambda'(n)$. Indeed observing by Proposition 2.5 that $10x_{n,1} = n + 71$, where $n \geq M_{19}$ and $x_{n,1} \in \mathcal{MH}(n, 1)$, and noticing by (56) that $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is false, then, using the previous and property (ii) of Remark 2.2, it becomes trivial to deduce that statement $Y(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form

$$Y(x_{n,1}, \lambda(n), \lambda'(n))) =: 10x_{n,1} > \lambda(n) \not\Leftarrow 10x_{n,1} > \lambda'(n).$$

Fact 2.1 follows.

Fact 2.2. $10x_{n,1} > \lambda(n) \;\; \not\Leftrightarrow \;\; 10x_{n,1} > \lambda'(n)$. Indeed, this fact is an immediate consequence of Fact 2.1. Fact 2.2 follows.

Having made the previous elementary two Facts, then it becomes trivial to see that Fact 2.2 clearly contradicts Observation (xi). So $Y(x_{n,1}, \lambda(n), \lambda'(n)) \;\Rightarrow\; Z(x_{n,1}, \lambda(n), \lambda'(n))$. Observation (xv) follows.

Observation (xvi). Let $(n, x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$. Now look at statements $Z(x_{n,1}, \lambda(n), \lambda'(n))$ and $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is introduced in Definitions 2.1 and $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ is introduced in Definitions 2.2. Then $Z(x_{n,1}, \lambda(n), \lambda'(n)) \Rightarrow S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$.

Otherwise, we reason by reduction to absurd, clearly

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \text{ is true and } S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \text{ is false.} \tag{57}$$

That being so, look at statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$, then we have the following elementary two facts:

Fact 2.3. Statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Z(x_{n,1}, \lambda(n), \lambda'(n)) \;=:\; 10x_{n,1} > \lambda(n) \;\; \not\Leftrightarrow \;\; 10x_{n,1} > \lambda'(n)$. Indeed observing by using (57) that $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true and noticing by Proposition 2.5 that $10x_{n,1} = n + 71$, where $n \geq M_{19}$ and $x_{n,1} \in \mathcal{MH}(n, 1)$, then, using the previous and property (iii) of Remark 2.2, it becomes trivial to deduce that statement $Z(x_{n,1}, \lambda(n), \lambda'(n))$ is true and is of the form $Z(x_{n,1}, \lambda(n), \lambda'(n)) \;=:\; 10x_{n,1} > \lambda(n) \;\; \not\Leftrightarrow \;\; 10x_{n,1} > \lambda'(n)$. Fact 2.3 follows.

Fact 2.4. $10x_{n,1} > \lambda(n) \;\; \not\Leftrightarrow \;\; 10x_{n,1} > \lambda'(n)$. Indeed, this fact is an immediate consequence of Fact 2.3. Fact 2.4 follows.

Having made the previous elementary two Facts, then it becomes trivial to see that Fact 2.4 clearly contradicts Observation (xi). So $Z(x_{n,1}, \lambda(n), \lambda'(n)) \;\Rightarrow\; S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$ and Observation (xvi) follows.

These seven simple observations made, look at statements $S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, $Y(x_{n,1}, \lambda(n), \lambda'(n))$ and $Z(x_{n,1}, \lambda(n), \lambda'(n))$, and observe by Observation (xiv) that

$$S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \;\Rightarrow\; Y(x_{n,1}, \lambda(n), \lambda'(n)). \tag{58}$$

Now noticing by Observation (xv) that

$$Y(x_{n,1}, \lambda(n), \lambda'(n)) \;\Rightarrow\; Z(x_{n,1}, \lambda(n), \lambda'(n)), \tag{59}$$

and remarking by Observation (xvi) that

$$Z(x_{n,1}, \lambda(n), \lambda'(n)) \;\Rightarrow\; S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))), \tag{60}$$

then, using (58), (59) and (60), then it becomes trivial to deduce that

$$S(x_{n,1}, \epsilon(\lambda(n), \lambda'(n))) \;\Leftrightarrow\; Y(x_{n,1}, \lambda(n), \lambda'(n)) \;\Leftrightarrow\; Z(x_{n,1}, \lambda(n), \lambda'(n)), \tag{61}$$

(61) clearly says that property (i) of Theorem 2.1 is satisfied by $(x_{n,1}, \epsilon(\lambda(n), \lambda'(n)))$, and this contradicts Observation (x). Theorem 2.1 follows.

Theorem 2.1 immediately implies the Mersenne primes problem and the Sophie Germain primes problem, and their connection to the Fermat's last conjecture.

Theorem 2.2. The Mersenne primes and the Sophie Germain primes, are all infinite.

Proof. Observe by using property (ii) of Theorem 2.1 that for every integer $n \geq M_{19}$ and for every $x_{n,1} \in \mathcal{MH}(n,1)$, we have $10x_{n,1} > n+71$, now using the previous inequality, then it becomes trivial to deduce that for every integer $n \geq M_{19}$ and for every $x_{n,1} \in \mathcal{MH}(n,1)$, we have $10x_{n,1} > n+71 > n-200$, and therefore,

$$\text{for every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 10x_{n,1} > n-200. \tag{62}$$

Consequently, the Mersenne primes and the Sophie Germain primes are all infinite, by using (62) and Proposition 1.2.

Definition 2.3. We say that e is wiles, if e is an integer ≥ 3 and if the equation $x^e + y^e = z^e$ has no solution in integers ≥ 1, for example, it is known many years ago, and it is not very difficult to prove that if $e = 3$, then e is wiles. The Fermat's last conjecture solved by Wiles in a paper of at least 105 pages long (see [11]), and resolved by Ikorong Nemron in a detailled paper of only 19 pages long (see [8]) asserts that every integer $e \geq 3$ is wiles. Now, for every integer $n \geq 3$, we define $\mathcal{W}(n)$ and w_n as follow: $\mathcal{W}(n) = \{x, 2 < x \leq n \text{ and } x \text{ is wiles}\}$, and $w_n = 2 \max_{w \in \mathcal{W}(n)} w$ clearly $3 \in \mathcal{W}(n)$ and consequently $w_n \geq 6$.

Using Definition 2.3, then it comes.

Remark 2.4. The following are equivalent:

(i) The Fermat's last conjecture is true.

(ii) For every integer $n \geq 3$, we have $w_n = 2n$.

Proof. Immediate, by using the definition of w_n and the definition of the Fermat's last conjecture.

Theorem 2.3. The Mersenne primes problem and the Sophie Germain primes problem were only immediate consequence of the Fermat's last conjecture.

Proof. Observe by using property (ii) of Theorem 2.1 that

$$\text{For every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 10x_{n,1} > n+71, \tag{63}$$

(63) clearly says that

$$\text{For every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 20x_{n,1} > 2n+142. \tag{64}$$

Now, since it is immediate to see that $w_n \leq 2n$, then, using (64) and the fact that $w_n \leq 2n$, it becomes trivial to deduce that

$$\text{For every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 20x_{n,1} > w_n. \tag{65}$$

Now look at (65) and suppose that the Fermat's last conjecture is true, then, using Remark 2.4 and (65), then it becomes trivial to deduce that

$$\text{For every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 20x_{n,1} > 2n, \tag{66}$$

(66) clearly says that

$$\text{For every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 10x_{n,1} > n > n-200,$$

and therefore,

$$\text{For every integer } n \geq M_{19} \text{ and for every } x_{n,1} \in \mathcal{MH}(n,1), \text{ we have } 10x_{n,1} > n - 200. \tag{67}$$

Consequently, the Mersenne primes and the Sophie Germain primes are all infinite, by using (67) and Proposition 1.2. So, the Mersenne primes problem and the Sophie Germain primes problem that we have solved elementary, via Theorem 2.1 and Theorem 2.2, were only immediate consequence of the Fermat's last conjecture.

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Uniqueness of solutions of linear integral equations of the first kind with two variables

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Abstract This paper is dedicated to the research of the problem of uniqueness of solutions for linear integral equations of the first kind with two variables. Here the operator generated by the kernels is not the compact operator.

Keywords Linear, inteqral equations, first kind,with two variables, solution and uniqueness.

§1. Introduction

We consider the integral equation

$$\begin{aligned} &\int_a^b K(t,x,y)u(t,y)dy + \int_{t_0}^t H(t,x,s)u(s,x)ds + \int_{t_0}^t \int_a^x C(t,x,s,y)u(s,y)dyds \\ = \ & f(t,x), \quad (t,x) \in G = \left\{(t,x) \in R^2 : t_0 \le t \le T, a \le x \le b\right\}, \end{aligned} \tag{1}$$

where

$$K(t,x,y) = \begin{cases} A(t,x,y), & t_0 \le t \le T, \quad a \le y \le x \le b, \\ B(t,x,y), & t_0 \le t \le T, \quad a \le x \le y \le b. \end{cases} \tag{2}$$

$A(t,x,y)$, $B(t,x,y)$, $H(t,x,s)$, $C(t,x,s,y)$ are given continuous functions, respectively, on the domains

$$\begin{aligned} G_1 &= \{(t,x,y) : t_0 \le t \le T, a \le y \le x \le b\}; \\ G_2 &= \{(t,x,y) : t_0 \le t \le T, a \le x \le y \le b\}; \\ G_3 &= \{(t,x,s) : t_0 \le s \le t \le T, a \le x \le b\}; \\ G_4 &= \{(t,x,s,y) : t_0 \le s \le t \le T, a \le y \le x \le b\}. \end{aligned}$$

$u(t,x)$ and $f(t,x)$ are the desired and given functions respectively, $(t,x) \in G$.

Various issues concerning of integral equations of the first kind were studied in [1-9]. More specifically, fundamental results for Fredholm integral equations of the first kind were obtained in [7], where regularizing operators in the sense of M. M. Lavrent'ev vere constructed for solutions of linear Fredholm integral equations of the first kind. For linear Volterra integral equations of the first and third kinds with smooth kernels, the existence of a multiparameter family of solutions was proven in [8]. The regularization and uniqueness of solutions to systems of nonlinear Volterra integral equations of the first kind were investigated in [5,6]. In this paper, while analyzing the uniqueness of solutions to the equation (1).

§2. Integral equations

Using $A(t,x,y)$ and $B(t,x,y)$ we define the following function:

$$P(t,x,y) = A(t,x,y) + B(t,y,x),\ (t,x,y) \in G_1. \tag{3}$$

Assume that the following conditions are satisfied:

(i) $P(t,b,a) \in C[t_0,T],\ P(t,b,a) \geq 0$ for all $t \in [t_0,T],\ P_y^{'}(s,y,a) \in C(G),\ P_y^{'}(s,y,a) \leq 0$ for all $(s,y) \in G,\ P_z^{'}(s,b,z) \in C(G),\ P_z^{'}(s,b,z) \geq 0$ for all $(s,z) \in G,\ P_{zy}^{''}(s,y,z) \in C(G_1),\ P_{zy}^{''}(s,y,z) \leq 0$ for all $(s,y,z) \in G_1,\ H(T,y,t_0) \in C[a,b],\ H(T,y,t_0) \geq 0$ for all $y \in [a,b]$, $H_s^{'}(s,y,t_0) \in C(G),\ H_s^{'}(s,y,t_0) \leq 0$ for all $(s,y) \in G$, $H_\tau^{'}(T,y,\tau) \in C(G),\ H_\tau^{'}(T,y,\tau) \geq 0$ for all $(y,\tau) \in G$, $H_{\tau s}^{''}(s,y,\tau) \in C(G_3),\ H_{\tau s}^{''}(s,y,\tau) \leq 0$ for all $(s,y,\tau) \in G_3$;

(ii) $C(T,b,t_0,a) \geq 0,\ C_s^{'}(s,b,t_0,a) \in C[t_0,T],\ C_s^{'}(s,b,t_0,a) \leq 0$ for all $s \in [t_0,T]$, $C_\tau^{'}(T,b,\tau,a) \in C[t_0,T],\ C_\tau^{'}(T,b,\tau,a) \geq 0$ for all $\tau \in [t_0,T]$, $C_y^{'}(T,y,t_0,a) \in C[a,b],\ C_y^{'}(T,y,t_0,a) \leq 0$ for all $y \in [a,b]$, $C_z^{'}(T,b,t_0,z) \in C[a,b],\ C_z^{'}(T,b,t_0,z) \geq 0$ for all $z \in [a,b]$, $C_{sy}^{''}(s,y,t_0,a) \in C(G),\ C_{sy}^{''}(s,y,t_0,a) \geq 0$ for all $(s,y) \in G,\ C_{\tau y}^{''}(T,y,\tau,t_0) \in C(G),\ C_{\tau y}^{''}(T,y,\tau,t_0) \leq 0$ for all $(y,\tau) \in G,\ C_{zs}^{''}(s,b,t_0,z) \in C(G),\ C_{zs}^{''}(s,b,t_0,z) \leq 0$ for all $(s,z) \in G,\ C_{\tau z}^{''}(T,b,\tau,z) \in C(G),\ C_{\tau z}^{''}(T,b,\tau,z) \geq 0$ for all $(\tau,z) \in G$, $C_{\tau sy}^{'''}(s,y,\tau,a) \in C(G_3),\ C_{\tau sy}^{'''}(s,y,\tau,a) \geq 0$ for all $(s,y,\tau) \in G_3$, $C_{\tau zs}^{'''}(s,b,\tau,z) \in C(G_3),\ C_{\tau zs}^{'''}(s,b,\tau,z) \leq 0$ for all $(s,z,\tau) \in G_3$, $C_{zsy}^{'''}(s,y,t_0,z) \in C(G_1),\ C_{zsy}^{'''}(s,y,t_0,z) \geq 0$ for all $(s,y,z) \in G_1,\ C_{\tau zy}^{'''}(T,y,\tau,z) \in C(G_1),\ C_{\tau zy}^{'''}(T,y,\tau,z) \leq 0$ for all $(\tau,y,z) \in G_1,\ C_{\tau zsy}^{IV}(s,y,\tau,z) \in C(G_4),\ C_{\tau zsy}^{IV}(s,y,\tau,z) \geq 0$ for all $(s,y,\tau,z) \in G_4$, $C_{\tau s}^{''}(s,b,\tau,a) \in C(G_5),\ C_{\tau s}^{''}(s,b,\tau,a) \leq 0$ for all $(s,\tau) \in G_5 = \{(s,\tau) : t_0 \leq \tau \leq s \leq T\}$, $C_{zy}^{''}(T,y,t_0,z) \in C(G_6),\ C_{zy}^{''}(T,y,t_0,z) \leq 0$ for all $(y,z) \in G_6 = \{(y,z) : a \leq z \leq y \leq b\}$;

(iii) For almost all $(s,y,\tau,z) \in G_4$ the quadratic form

$$\begin{aligned}
& L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\
= \ & \frac{1}{(s-t_0)(y-a)}\Big\{-P_y^{'}(s,y,a)\alpha_1^2 - H_s^{'}(s,y,t_0)\alpha_2^2 - 2C(s,y,t_0,a)\alpha_1\alpha_2 \\
& -(s-t_0)\Big[H_{\tau s}^{''}(s,y,\tau)\alpha_3^2 + 2C_\tau^{'}(s,y,\tau,a)\alpha_3\alpha_1\Big] \\
& -(y-a)\Big[2C_z^{'}(s,y,t_0,z)\alpha_2\alpha_4 + P_{zy}^{''}(s,y,z)\alpha_4^2\Big] \\
& -2(s-t_0)(y-a)C_{\tau z}^{''}(s,y,\tau,z)\alpha_3\alpha_4\Big\},
\end{aligned}$$

is non-negative, i.e., $L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \geq 0$ for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in R$;

(iv) If for almost all $(s,y,\tau,z) \in G_4,\ L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = 0$, then it follow that $\alpha_1 = 0$, or $\alpha_2 = 0$, or $\alpha_3 = 0$ or $\alpha_4 = 0$.

Theorem 2.1. Let conditions (i)-(iv) be satisfied. Then the solution of the equation (1) is unique in $L_2(G)$.

Proof. Taking into account (2) from (1), we obtain

$$\begin{aligned} &\int_a^x A(t,x,y)u(t,y)dy+\int_x^b B(t,x,y)u(t,y)dy+\int_{t_0}^t H(t,x,s)u(s,x)dx \\ &+\int_{t_0}^t\int_a^x C(t,x,s,y)u(s,y)dyds \\ = \;& f(t,x), \quad (t,x)\in G. \end{aligned} \tag{4}$$

Taking the multiplication of both sides of the equation (4) with $u(t,s)$, integrating the results on G, we obtain

$$\begin{aligned} &\int_a^b\int_{t_0}^T\int_a^y A(s,y,z)u(s,z)u(s,y)dzdsdy+\int_a^b\int_{t_0}^T\int_y^b B(s,y,z)u(s,z)u(s,y)dzdsdy \\ &+\int_a^b\int_{t_0}^T\int_{t_0}^s H(s,y,\tau)u(\tau,y)u(s,y)d\tau dsdy \\ &+\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)dzd\tau dsdy \\ = \;& \int_a^b\int_{t_0}^T f(s,y)u(s,y)dsdy. \end{aligned} \tag{5}$$

Using the Dirichlet formula and taking into account (3) from (5), we have

$$\begin{aligned} &\int_{t_0}^T\int_a^b\int_a^y P(s,y,z)u(s,z)u(s,y)dzdyds+\int_a^b\int_{t_0}^T\int_{t_0}^s H(s,y,\tau)u(\tau,y)u(s,y)d\tau dsdy \\ &+\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)dzd\tau dsdy \\ = \;& \int_a^b\int_{t_0}^T f(s,y)u(s,y)dsdy. \end{aligned} \tag{6}$$

Integrating by parts and using the Dirichlet formula we obtain

$$\begin{aligned} &\int_{t_0}^T\int_a^b\int_a^y P(s,y,z)u(s,z)u(s,y)dzdyds \\ = \;& -\int_{t_0}^T\int_a^b\int_a^y P(s,y,z)\frac{\partial}{\partial z}\left(\int_z^y u(s,\nu)d\nu\right)dzu(s,y)dyds \\ = \;& \frac{1}{2}\int_{t_0}^T\int_a^b P(s,y,a)\left[\frac{\partial}{\partial y}\left(\int_a^y u(s,\nu)d\nu\right)^2\right]dyds \\ &+\frac{1}{2}\int_{t_0}^t\int_a^b\int_z^b P_z^{'}(s,y,z)\left[\frac{\partial}{\partial y}\left(\int_z^y u(s,\nu)d\nu\right)^2\right]dydzds \\ = \;& \frac{1}{2}\int_{t_0}^T P(s,b,a)\left(\int_a^b u(s,\nu)d\nu\right)^2 ds-\frac{1}{2}\int_{t_0}^T\int_a^b P_y^{'}(s,y,a)\left(\int_a^y u(s,\nu)d\nu\right)^2 dyds \\ &+\frac{1}{2}\int_{t_0}^T\int_a^b P_y^{'}(s,b,y)\left(\int_y^b u(s,\nu)d\nu\right)^2 dyds \\ &-\frac{1}{2}\int_{t_0}^T\int_a^b\int_a^y P_{zy}^{''}(s,y,z)\left(\int_z^y u(s,\nu)d\nu\right)^2 dzdyds, \end{aligned} \tag{7}$$

where $P_y^{'}(s,b,y) = [P_z^{'}(s,b,z)]|_{z=y}$. Similarly integrating by parts and using the Dirichlet formula we have

$$\begin{aligned}
&\int_a^b\int_{t_0}^T\int_{t_0}^s H(s,y,\tau)u(\tau,y)u(s,y)d\tau dsdy\\
= & -\int_a^b\int_{t_0}^T\int_{t_0}^s H(s,y,\tau)\frac{\partial}{\partial\tau}\left(\int_\tau^s u(\xi,y)d\xi\right)\times d\tau u(s,y)dsdy\\
= & \frac{1}{2}\int_a^b\int_{t_0}^T H(s,y,t_0)\left[\frac{\partial}{\partial s}\left(\int_{t_0}^s u(\xi,y)d\xi\right)^2\right]dsdy\\
& +\frac{1}{2}\int_a^b\int_{t_0}^T\int_\tau^T H_\tau^{'}(s,y,\tau)\left[\frac{\partial}{\partial s}\left(\int_\tau^s u(\xi,y)d\xi\right)^2\right]dsd\tau dy\\
= & \frac{1}{2}\int_a^b H(T,y,t_0)\left(\int_{t_0}^T u(\xi,y)d\xi\right)^2 dy-\frac{1}{2}\int_a^b\int_{t_0}^T H_s^{'}(s,y,t_0)\left(\int_{t_0}^s u(\xi,y)d\xi\right)^2 dsdy\\
& +\frac{1}{2}\int_a^b\int_{t_0}^T H_s^{'}(T,y,s)\left(\int_s^T u(\xi,y)d\xi\right)^2 dsdy\\
& -\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s H_{\tau s}^{''}(s,y,\tau)\left(\int_\tau^s u(\xi,y)d\xi\right)^2 d\tau dsdy,
\end{aligned} \tag{8}$$

where $H_s^{'}(T,y,s) = (H_\tau^{'}(T,y,\tau))|_{\tau=s}$.

Further we use the following formula

$$C\nu_{\tau z}^{''} = (C\nu)_{\tau z}^{''} - (C_\tau^{'}\nu)_z^{'} - (C_z^{'}\nu)_\tau^{'} + C_{\tau z}^{''}\nu.$$

Then integrating by parts and using the Dirichlet formula, we obtain

$$\begin{aligned}
&\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)dzd\tau dsdy\\
= & \int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)\frac{\partial^2}{\partial\tau\partial z}\left(\int_\tau^s\int_z^y u(\xi,\nu)d\nu d\xi\right)dzd\tau u(s,y)dsdy\\
= & \int_a^b\int_{t_0}^T C(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)d\nu d\xi\right)u(s,y)dsdy\\
& +\int_a^b\int_{t_0}^T\int_\tau^T C_\tau^{'}(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)d\nu d\xi\right)u(s,y)dsd\tau dy\\
& +\int_a^b\int_{t_0}^T\int_z^b C_z^{'}(s,y,t_0,z)\left(\int_{t_0}^s\int_z^y u(\xi,\nu)d\nu d\xi\right)u(s,y)dydsdz\\
& +\int_a^b\int_{t_0}^T\int_\tau^T\int_z^b C_{\tau z}^{''}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,\nu)d\nu d\xi\right)u(s,y)dydsd\tau dz.
\end{aligned} \tag{9}$$

Further we apply the following formula:

$$C\nu\nu_{sy}^{''} = \frac{1}{2}(C\nu^2)_{sy}^{''} - \frac{1}{2}(C_s^{'}\nu^2)_y^{'} - \frac{1}{2}(C_y^{'}\nu^2)_s^{'} + \frac{1}{2}C_{sy}^{''}\nu^2 - C\nu_y^{'}\nu_s^{'}$$

and using the Dirichlet formula we obtain

$$\int_a^b \int_{t_0}^T \int_{t_0}^s \int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)dzd\tau dsdy$$

$$= \frac{1}{2}C(T,b,t_0,a)\left(\int_a^b \int_{t_0}^T u(\xi,\nu)d\xi d\nu\right)^2 - \frac{1}{2}\int_{t_0}^T C_s^{'}(s,b,t_0,a)\left(\int_a^b \int_{t_0}^s u(\nu,\xi)d\nu d\xi\right)^2 ds$$

$$-\frac{1}{2}\int_a^b C_y^{'}(T,y,t_0,a)\left(\int_a^y \int_{t_0}^T u(\nu,\xi)d\nu d\xi\right)^2 dy$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T C_{sy}^{''}(s,y,t_0,a)\left(\int_{t_0}^s \int_a^y u(\xi,\nu)d\nu d\xi\right)^2 dsdy$$

$$-\int_a^b \int_{t_0}^T C(s,y,t_0,a)\left(\int_{t_0}^s u(\xi,y)d\xi\right)\left(\int_a^y u(s,\nu)d\nu\right)dsdy$$

$$+\frac{1}{2}\int_{t_0}^T C_s^{'}(T,b,s,a)\left(\int_s^T \int_a^b u(\xi,\nu)d\nu d\xi\right)^2 ds$$

$$-\frac{1}{2}\int_{t_0}^T \int_{t_0}^s C_{\tau s}^{''}(s,b,\tau,a)\left(\int_\tau^s \int_a^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau ds$$

$$-\frac{1}{2}\int_a^b \int_{t_0}^T C_{sy}^{''}(T,y,s,t_0)\left(\int_s^T \int_a^y u(\xi,\nu)d\nu d\xi\right)^2 dsdy$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T \int_{t_0}^s C_{\tau sy}^{'''}(s,y,\tau,a)\left(\int_\tau^s \int_a^y u(\xi,\nu)d\nu d\xi\right)^2 d\tau dsdy$$

$$-\int_a^b \int_{t_0}^T \int_{t_0}^s C_\tau^{'}(s,y,\tau,a)\left(\int_\tau^s u(\xi,y)d\xi\right)\left(\int_a^y u(s,\nu)d\nu\right)d\tau dsdy$$

$$+\frac{1}{2}\int_a^b C_y^{'}(T,b,t_0,y)\left(\int_{t_0}^T \int_y^b u(\xi,\nu)d\nu d\xi\right)^2 dy$$

$$-\frac{1}{2}\int_a^b \int_{t_0}^T C_{ys}^{''}(s,b,t_0,y)\left(\int_{t_0}^s \int_y^b u(\xi,\nu)d\nu d\xi\right)^2 dsdy$$

$$-\frac{1}{2}\int_a^b \int_a^y C_{zy}^{''}(T,y,t_0,z)\left(\int_{t_0}^T \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzdy$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T \int_a^y C_{zsy}^{'''}(s,y,t_0,z)\left(\int_{t_0}^s \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzdsdy$$

$$-\int_a^b \int_{t_0}^T \int_a^y C_z^{'}(s,y,t_0,z)\left(\int_{t_0}^s u(\xi,y)d\xi\right)\left(\int_z^y u(s,\nu)d\nu\right)dzdsdy$$

$$+\frac{1}{2}\int_a^b \int_{t_0}^T C_{\tau z}^{''}(T,b,\tau,z)\left(\int_\tau^T \int_z^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau dz$$

$$-\frac{1}{2}\int_a^b \int_{t_0}^T \int_a^y C_{szy}^{'''}(T,y,s,z)\left(\int_s^T \int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzdsdy$$

$$-\frac{1}{2}\int_a^b \int_{t_0}^T \int_{t_0}^s C_{\tau ys}^{'''}(s,b,\tau,y)\left(\int_\tau^s \int_y^b u(\xi,\nu)d\nu d\xi\right)^2 d\tau dsdy$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C^{(IV)}_{\tau zsy}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,\nu)d\nu d\xi\right)^2 dzd\tau dsdy$$
$$-\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C^{''}_{\tau z}(s,y,\tau,z)\left(\int_\tau^s u(\xi,y)d\xi\right)\left(\int_z^y u(s,\nu)d\nu\right)dzd\tau dsdy, \qquad (10)$$

where $C^{'}_s(T,b,s,a)=(C^{'}_\tau(T,b,\tau,a))|_{\tau=s}, C^{'}_y(T,b,t_0,y)=(C^{'}_z(T,b,t_0,z))|_{z=y}$.

Taking into account (7), (8) and (9), from (6) we obtain

$$\frac{1}{2}C(T,b,t_0,a)\left(\int_a^b\int_{t_0}^T u(\xi,\nu)d\xi d\nu\right)^2$$
$$+\frac{1}{2}\int_{t_0}^T\Bigg\{P(s,b,a)\left(\int_a^b u(s,\nu)d\nu\right)^2-C^{'}_s(s,b,t_0,a)\left(\int_a^b\int_{t_0}^s u(\nu,\xi)d\nu d\xi\right)^2$$
$$+C^{'}_s(T,b,s,a)\left(\int_s^T\int_a^b u(\xi,\nu)d\xi d\nu\right)^2\Bigg\}ds$$
$$+\frac{1}{2}\int_a^b\Bigg\{H(T,y,t_0)\left(\int_{t_0}^T u(\xi,y)d\xi\right)^2-C^{'}_y(T,y,t_0,a)\left(\int_a^y\int_{t_0}^T u(\nu,\xi)d\nu d\xi\right)^2$$
$$+C^{'}_y(T,b,t_0,y)\left(\int_{t_0}^T\int_y^b u(\xi,\nu)d\nu d\xi\right)^2\Bigg\}dy$$
$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y\Bigg\{L(s,y,\tau,z,\int_a^y u(s,\nu)d\nu,\int_{t_0}^s u(\xi,y)d\xi,\int_\tau^s u(\xi,y)d\xi,\int_z^y u(s,\nu)d\nu)$$
$$+\frac{1}{(s-t_0)(y-a)}\left[P^{'}_y(s,b,y)\left(\int_y^b u(s,\nu)d\nu\right)^2+H^{'}_s(T,y,s)\left(\int_s^T u(\xi,y)d\xi\right)^2\right]$$
$$+\frac{1}{(s-t_0)(y-a)}\Bigg[C^{''}_{sy}(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2$$
$$-C^{''}_{sy}(T,y,s,a)\left(\int_s^T\int_a^y u(\xi,\nu)d\nu d\xi\right)^2-C^{''}_{ys}(s,b,t_0,y)\left(\int_{t_0}^s\int_y^b u(\xi,\nu)d\nu d\xi\right)^2$$
$$+C^{''}_{sy}(T,b,s,y)\left(\int_s^T\int_y^b u(\xi,\nu)d\nu d\xi\right)^2\Bigg]$$
$$+\frac{1}{y-a}\left[C^{'''}_{\tau sy}(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)d\nu d\xi\right)^2-C^{'''}_{\tau ys}(s,b,\tau,y)\left(\int_\tau^s\int_y^b u(\xi,\nu)d\nu d\xi\right)^2\right]$$
$$+\frac{1}{s-t_0}\Bigg[C^{'''}_{zsy}(s,y,t_0,z)\left(\int_{t_0}^s\int_z^y u(\xi,\nu)d\nu d\xi\right)^2$$
$$-C^{'''}_{szy}(T,y,s,z)\left(\int_s^T\int_z^y u(\xi,\nu)d\nu d\xi\right)^2\Bigg]$$
$$+C^{(IV)}_{\tau syz}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,\nu)d\nu d\xi\right)^2\Bigg\}dzd\tau dsdy$$

$$
\begin{aligned}
& -\frac{1}{2}\int_{t_0}^{T}\int_{t_0}^{s} C''_{\tau s}(s,b,\tau,a)\left(\int_{\tau}^{s}\int_{a}^{b} u(\xi,\nu)d\nu d\xi\right)^2 d\tau ds-\frac{1}{2}\int_{a}^{b}\int_{a}^{y} C''_{zy}(T,y,t_0,z) \\
& \times\left(\int_{t_0}^{T}\int_{z}^{y} u(\xi,\nu)d\nu d\xi\right)^2 dzdy \\
= & \int_{a}^{b}\int_{t_0}^{T} f(s,y)u(s,y)dsdy. \qquad (11)
\end{aligned}
$$

Let $f(t,x)=0$ for all $(t,x)\in G$. Then by virtue of conditions (i)-(iv), from (10) we obtain $\int_a^y u(s,\nu)d\nu=0$ for all $(s,y)\in G$, or $\int_{t_0}^s u(\xi,y)d\xi=0$ for all $(s,y)\in G$. Here $u(t,x)=0$ for all $(t,x)\in G$. The Theorem 2.2 is proven.

Example 2.1. We consider the equation (1) for

$$A(t,x,y)=m_0(t)\beta_0(x)\gamma_0(y), B(t,x,y)=m_0(t)m_1(x)\beta_0(y),$$

$$H(t,x,s)=m_2(t)\beta_1(x)\beta_2(s), C(t,x,s,y)=\gamma_1(t)\gamma_2(x)\gamma_3(s)\gamma_4(y),$$

where $m_0(t),\ m_2(t),\ m_2'(t),\ \beta_2(t),\ \beta_2'(t),\ \gamma_1(t),\ \gamma_1'(t),\ \gamma_3(t),\ \gamma_3'(t)\in C[t_0,T]$, $\beta_0(x),\ \beta_0'(x),\ \gamma_0(x),\ \gamma_0'(x),\ m_1(x),\ m_1'(x),\ \beta_1(x),\ \gamma_2(x),\ \gamma_2'(x),\ \gamma_4(x),\ \gamma_4'(x)\in C[a,b]$, $m_0(t)\geq 0,\ m_2(t)\geq 0,\ m_2'(t)\leq 0,\ \beta_2(t)\geq 0,\ \beta_2'(t)\geq 0,\ \gamma_1(t)\geq 0,\ \gamma_1'(t)\leq 0,\ \gamma_3(t)\geq 0$ and $\gamma_3'(t)\geq 0$ for all $t\in[t_0,T],\ \beta_0(x)\geq 0,\ \beta_0'(x)\leq 0,\ \gamma_0(x)+m_1(x)\geq 0,\ \gamma_0'(x)+m_1'(x)\geq 0,\ \beta_1(x)\geq 0,\ \gamma_2(x)\geq 0,\ \gamma_2'(x)\leq 0,\ \gamma_4(x)\geq 0,\ \gamma_4'(x)\geq 0$ for all $x\in[a,b],\ \gamma_3(t_0)=0,\ \gamma_4(a)=0,\ \gamma_0(a)+m_1(a)\neq 0,\ m_0(t)\beta_0'(x)<0$ for almost $(t,x)\in G,\ m_0(s)\beta_0'(y)[\gamma_0(z)+m_1(z)]m_2'(s)\beta_1(y)\beta_2'(\tau)-(s-t_0)\times(y-a)\left[\gamma_1(s)\gamma_2(y)\gamma_3'(\tau)\gamma_4'(z)\right]^2\geq 0$ for all $(s,y,\tau,z)\in G_4$.

In this case

$$P(t,x,y)=m_0(t)\beta_0(x)[\gamma_0(y)+m_1(y)], (t,x,y)\in G_1,$$

$$
\begin{aligned}
& L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \\
= & \frac{1}{(s-t_0)(y-a)}\Big\{-m_0(s)\beta_0'(y)[\gamma_0(a)+m_1(a)]\alpha_1^2 \\
& -m_2'(s)\beta_1(y)\beta_2(t_0)\alpha_2^2-[(s-t_0)m_2'(s)\beta_1(y)\beta_2'(\tau)\alpha_3^2+(y-a)m_0(s)\beta_0'(y) \\
& \times[\gamma_0(z)+m_1(z)]\alpha_4^2+2(s-t_0)(y-a)\gamma_1(s)\gamma_2(y)\gamma_3'(\tau)\gamma_4'(z)\alpha_3\alpha_4]\Big\}, \quad (s,y,\tau,z)\in G_4.
\end{aligned}
$$

Then the conditions (i)-(iv) are satisfied.

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On two inequalities for the composition of arithmetic functions

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Abstract Let f, g be arithmetic functions satisfying certain conditions. We prove the inequalities $f(g(n)) \leq 2n - \omega(n) \leq 2n - 1$ and $f(g(n)) \leq n + \omega(n) \leq 2n - 1$ for any $n \geq 1$, where $\omega(n)$ is the number of distinct prime factors of n. Particular cases include $f(n) =$ Smarandache function, $g(n) = \sigma(n)$ or $g(n) = \sigma^*(n)$.

Keywords Arithmetic functions, inequalities.
AMS Subject Classification: 11A25.

§1. Introduction

Let $S(n)$ be the Smarandache (or Kempner-Smarandache) function, i.e., the function that associates to each positive integer n the smallest positive integer k such that $n|k!$. Let $\sigma(n)$ denote the sum of distinct positive divisors of n, while $\sigma^*(n)$ the sum of distinct unitary divisors of n (introduced for the first time by E. Cohen, see e.g. [7] for references and many informations on this and related functions). Put $\omega(n) =$ number of distinct prime divisors of n, where $n > 1$. In paper [4] we have proved the inequality

$$S(\sigma(n)) \leq 2n - \omega(n), \tag{1}$$

for any $n > 1$, with equality if and only if $\omega(n) = 1$ and $2n - 1$ is a Mersenne prime.

In what follows we shall prove the similar inequality

$$S(\sigma^*(n)) \leq n + \omega(n), \tag{2}$$

for $n > 1$. Remark that $n + \omega(n) \leq 2n - \omega(n)$, as $2\omega(n) \leq n$ follows easily for any $n > 1$. On the other hand $2n - \omega(n) \leq 2n - 1$, so both inequalities (1) and (2) are improvements of

$$S(g(n)) \leq 2n - 1, \tag{3}$$

where $g(n) = \sigma(n)$ or $g(n) = \sigma^*(n)$.

We will consider more general inequalities, for the composite functions $f(g(n))$, where f, g are arithmetical functions satisfying certain conditions.

§2. Main results

Lemma 2.1. For any real numbers $a \geq 0$ and $p \geq 2$ one has the inequality

$$\frac{p^{a+1}-1}{p-1} \leq 2p^a - 1, \tag{4}$$

with equality only for $a = 0$ or $p = 2$.

Proof. It is easy to see that (4) is equivalent to

$$(p^a - 1)(p-2) \geq 0,$$

which is true by $p \geq 2$ and $a \geq 0$, as $p^a \geq 2^a \geq 1$ and $p - 2 \geq 0$.

Lemma 2.2. For any real numbers $y_i \geq 2$ $(1 \leq i \leq r)$ one has

$$y_1 + \ldots + y_r \leq y_1 \ldots y_r \tag{5}$$

with equality only for $r = 1$.

Proof. For $r = 2$ the inequality follows by $(y_1 - 1)(y_2 - 1) \geq 1$, which is true, as $y_1 - 1 \geq 1$, $y_2 - 1 \geq 1$. Now, relation (5) follows by mathematical induction, the induction step $y_1 \ldots y_r + y_{r+1} \leq (y_1 \ldots y_r)y_{r+1}$ being an application of the above proved inequality for the numbers $y_1' = y_1 \ldots y_r$, $y_2' = y_{r+1}$.

Now we can state the main results of this paper.

Theorem 2.1. Let $f,\ g : \mathbb{N} \to \mathbb{R}$ be two arithmetic functions satisfying the following conditions:

(i) $f(xy) \leq f(x) + f(y)$ for any $x, y \in \mathbb{N}$.

(ii) $f(x) \leq x$ for any $x \in \mathbb{N}$.

(iii) $g(p^\alpha) \leq 2p^\alpha - 1$, for any prime powers p^α (p prime, $\alpha \geq 1$).

(iv) g is multiplicative function.

Then one has the inequality

$$f(g(n)) \leq 2n - \omega(n),\ n > 1. \tag{6}$$

Theorem 2.2. Assume that the arithmetical functions f and g of Theorem 2.1 satisfy conditions (i), (ii), (iv) and

(iii)' $g(p^\alpha) \leq p^\alpha + 1$ for any prime powers p^α.

Then one has the inequality

$$f(g(n)) \leq n + \omega(n),\ n > 1. \tag{7}$$

Proof of Theorem 2.1. As $f(x_1) \leq f(x_1)$ and

$$f(x_1 x_2) \leq f(x_1) + f(x_2),$$

it follows by mathematical induction, that for any integers $r \geq 1$ and $x_1, \ldots, x_r \geq 1$ one has

$$f(x_1 \ldots x_r) \leq f(x_1) + \ldots + f(x_r). \tag{8}$$

Let now $n = p_1^{\alpha_1} \dots p_r^{\alpha_r} > 1$ be the prime factorization of n, where p_i are distinct primes and $\alpha_i \geq 1$ $(i = 1, \dots, r)$. Since g is multiplicative, by inequality (8) one has

$$f(g(n)) = f(g(p_1^{\alpha_1}) \dots g(p_r^{\alpha_r})) \leq f(g(p_1^{\alpha_1})) + \dots + f(g(p_r^{\alpha_r})).$$

By using conditions (ii) and (iii), we get

$$f(g(n)) \leq g(p_1^{\alpha_1}) + \dots + g(p_r^{\alpha_r}) \leq 2(p_1^{\alpha_1} + \dots + p_r^{\alpha_r}) - r.$$

As $p_i^{\alpha_i} \geq 2$, by Lemma 2.2 we get inequality (6), as $r = \omega(n)$.

Proof of Theorem 2.2. Use the same argument as in the proof of Theorem 2.1, by remarking that by (iii)'

$$f(g(n)) \leq (p_1^{\alpha+1} + \dots + p_r^{\alpha_r}) + r \leq p_1^{\alpha_1} \dots p_r^{\alpha_r} + r = n + \omega(n).$$

Remark 2.1. By introducing the arithmetical function $B^1(n)$ (see [7], Ch.IV.28)

$$B^1(n) = \sum_{p^\alpha \| n} p^\alpha = p_1^{\alpha_1} + \dots + p_r^{\alpha_r}.$$

(i.e., the sum of greatest prime power divisors of n), the following stronger inequalities can be stated:

$$f(g(n)) \leq 2B^1(n) - \omega(n), \tag{6'}$$

(in place of (6)); as well as:

$$f(g(n)) \leq B^1(n) + \omega(n), \tag{7'}$$

(in place of (7)).

For the average order of $B^1(n)$, as well as connected functions, see e.g. [2], [3], [8], [7].

§3. Applications

1. First we prove inequality (1).

Let $f(n) = S(n)$. Then inequalities (i), (ii) are well-known (see e.g. [1], [6], [4]). Put $g(n) = \sigma(n)$. As $\sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1}$, inequality (iii) follows by Lemma 2.1. Theorem 2.1 may be applied.

2. Inequality (2) holds true.

Let $f(n) = S(n)$, $g(n) = \sigma^*(n)$. As $\sigma^*(n)$ is a multiplicative function, with $\sigma^*(p^\alpha) = p^\alpha + 1$, inequality (iii)' holds true. Thus (2) follows by Theorem 2.2.

3. Let $g(n) = \psi(n)$ be the Dedekind arithmetical function, i.e., the multiplicative function whose value of the prime power p^α is

$$\psi(p^\alpha) = p^{\alpha-1}(p+1).$$

Then $\psi(p^\alpha) \leq 2p^\alpha - 1$ since

$$p^\alpha + p^{\alpha-1} \leq 2p^\alpha - 1;\ p^{\alpha-1} + 1 \leq p^\alpha;\ p^{\alpha-1}(p-1) \geq 0,$$

which is true, with strict inequality.

Thus Theorem 2.1 may be applied for any function f satisfying (i) and (ii).

4. There are many functions satisfying inequalities (i) and (ii) of Theorems 2.1 and 2.2.

Let $f(n) = \log \sigma(n)$.

As $\sigma(mn) \leq \sigma(m)\sigma(n)$ for any $m,\ n \geq 1$, relation (i) follows. The inequality $f(n) \leq n$ follows by $\sigma(n) \leq e^n$, which is a consequence of e.g. $\sigma(n) \leq n^2 < e^n$ (the last inequality may be proved e.g. by induction).

Remark 3.1. More generally, assume that $F(n)$ is a submultiplicative function, i.e., satisfying

$$F(mn) \leq F(m)F(n) \text{ for } m, n \geq 1. \tag{i'}$$

Assume also that

$$F(n) \leq e^n. \tag{ii'}$$

Then $f(n) = \log F(n)$ satisfies relations (i) and (ii).

5. Another nontrivial function, which satisfies conditions (i) and (ii) is the following

$$f(n) = \begin{cases} p, & \text{if } n = p \text{ (prime)}, \\ 1, & \text{if } n = \text{composite or } n = 1. \end{cases} \tag{9}$$

Clearly, $f(n) \leq n$, with equality only if $n = 1$ or $n =$ prime. For $y = 1$ we get $f(x) \leq f(x) + 1 = f(x) + f(1)$, when $x,\ y \geq 2$ one has

$$f(xy) = 1 \leq f(x) + f(y).$$

6. Another example is

$$f(n) = \Omega(n) = \alpha_1 + \ldots + \alpha_r, \tag{10}$$

for $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, i.e., the total number of prime factors of n. Then $f(mn) = f(m) + f(n)$, as $\Omega(mn) = \Omega(m) + \Omega(n)$ for all $m,\ n \geq 1$. The inequality $\Omega(n) < n$ follows by $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r} \geq 2^{\alpha_1 + \ldots + \alpha_r} > \alpha_1 + \ldots + \alpha_r$.

7. Define the additive analogue of the sum of divisors function σ, as follows: If $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ is the prime factorization of n, put

$$\Sigma(n) = \Sigma \left(\frac{p^{\alpha+1} - 1}{p - 1} \right) = \sum_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}. \tag{11}$$

As $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1}-1}{p-1}$, and $\frac{p^{\alpha+1}-1}{p-1} > 2$, clearly by Lemma 2.2 one has

$$\Sigma(n) \leq \sigma(n). \tag{12}$$

Let $f(n)$ be any arithmetic function satisfying condition (ii), i.e., $f(n) \leq n$ for any $n \geq 1$. Then one has the inequality:

$$f(\Sigma(n)) \leq 2B^1(n) - \omega(n) \leq 2n - \omega(n) \leq 2n - 1 \tag{13}$$

for any $n > 1$.

Indeed, by Lemma 2.1, and Remark 2.1, the first inequality of (13) follows. Since $B^1(n) \leq n$ (by Lemma 2.2), the other inequalities of (13) will follow. An example:

$$S(\Sigma(n)) \leq 2n - 1, \tag{14}$$

which is the first and last term inequality in (13).

It is interesting to study the cases of equality in (14). As $S(m) = m$ if and only if $m = 1$, 4 or p (prime) (see e.g. [1], [6], [4]) and in Lemma 2.2 there is equality if $\omega(n) = 1$, while in Lemma 2.1, as $p = 2$, we get that n must have the form $n = 2^\alpha$. Then $\Sigma(n) = 2^{\alpha+1} - 1$ and $2^{\alpha+1} - 1 \neq 1$, $2^{\alpha+1} - 1 \neq 4$, $2^{\alpha+1} - 1 =$ prime, we get the following theorem:

There is equality in (14) iff $n = 2^\alpha$, where $2^{\alpha+1} - 1$ is a prime.

In paper [5] we called a number n almost f-perfect, if $f(n) = 2n - 1$ holds true. Thus, we have proved that n is almost $S \circ \Sigma$-perfect number, iff $n = 2^\alpha$, with $2^{\alpha+1} - 1$ a prime (where "$\circ$" denotes composition of functions).

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On the equation $x^8 + y^3 = z^{2k}$

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Abstract In this article, by utilizing Pell equations and elliptic curves, we prove that when $k = 2, 4$ the Diophantine equation $x^8 + y^3 = z^{2k}$ has no solution for relatively prime positive integers $x,\ y,\ z$.

Keywords Diophantine equations, elliptic curves.
2000 Mathematics Subject Classification: 11D99, 11H52.

§1. Introduction

Fermat's Last Theorem states that if $n > 2$, then the equation

$$a^n + b^n = c^n$$

has no solution in nonzero integers a, b, c. This theorem had been studied for hundred years and A. Wiles completed the proof of Fermat's Last Theorem with the aid of elliptic curves. The first instances of elliptic curves occur in the works of Diophantus and Fermat.

An elliptic curve is a curve given by an equation

$$E : y^2 = f(x)$$

for a cubic or quartic polynomial of x.

Elliptic curves are plane curves that are the locus of points satisfying a cubic equation in two variables. If the elliptic curve is defined on Euclidean plane, the points which are related to this elliptic curve will be affine rational points corresponding to the solutions of the equation [8].

In this article, we consider the general Diophantine equation $x^p + y^q = z^r$. This Diophantine equation in integers $p > 1,\ q > 1,\ r > 1$ and $x,\ y,\ z$ is a generalization of the well-known Fermat equation $x^n + y^n = z^n$.

There are a vast amount of results related to the equation $Ax^p + By^q = Cz^r$. A special case of interest is when $A = B = C = 1$. In many such cases the solution set has been found. Below it can be listed the exponent triples (p, q, r) of solved equations together with the non-trivial solutions $(xyz \neq 0)$.

Bruin [2] determined all the solutions of the equation $x^p + y^q = z^r$ where $(p, q, r) \in \{(2, 4, 6), (2, 6, 4), (4, 6, 2), (2, 8, 3)\}$. Bruin applied elliptic Chabauty method and other methods to prove that the only solutions of $x^8 + y^3 = z^2$ are non-zero relatively prime integers $(\pm 1, 2, \pm 3)$ and $(\pm 43, 96222, \pm 30042907)$. Bruin [3] also found that the integer solutions $x,\ y,\ z$ such that $(p, q, r) = (3, 9, 2)$ and $\gcd(x, y, z) = 1$.

Beukers [1] gave a partial solution of $x^2 + y^8 = z^3$ by using stepwise descent methods.

Poonen [5] researched the solutions of the equation for $(p, q, r) \in \{(5, 5, 2), (9, 9, 2), (5, 5, 3)\}$.

In the light of these studies we consider the equation $x^8 + y^3 = z^{2k}$. In order to calculate some values related to these equations, we use GP/Pari [9] is widely used computer algebra system designed for fast computations in number theory. This programme is originally (1985-1996) developed at Universite Bordeaux I by a team led by Henri Cohen.

§2. The equation $x^8 + y^3 = z^4$

In our main general equation $x^8 + y^3 = z^{2k}$, firstly we consider the case $k = 2$ and we give our main theorem.

Theorem 2.1. The Diophantine equation $x^8 + y^3 = z^4$ has no relatively prime positive integer solutions.

Proof. Suppose that there are infinitely many primitive solutions to the Diophantine equation

$$x^8 + y^3 = z^4. \tag{1}$$

Now we rewrite equation (1) as

$$y^3 = (z^2 - x^4)(z^2 + x^4).$$

If x or z is even, then the other one must be odd. Therefore the two factors on the right hand side are coprime, so we have integers $m,\ n$ such that

$$m^3 = z^2 + x^4, \qquad n^3 = z^2 - x^4.$$

From these we get integers x and z of the form

$$z^2 = \frac{m^3 + n^3}{2}, \tag{2}$$

$$x^4 = \frac{m^3 - n^3}{2}. \tag{3}$$

We have two different cases for the above equations. The first one is that m and n are both odd. The second one is that m and n are both even.

The case: m and n are both odd.

If m and n are both odd, then x^4 and z^2 are clearly integers. There are several ways of constructing an infinitely family with this property.

(i) If $m = 2k + 1$ and $n = 2k - 1$, then the equation (2) becomes

$$z^2 = 8k^3 + 6k.$$

If we write $k = \dfrac{K}{2}$, we get

$$z^2 = K^3 + 3K.$$

This implies an elliptic curve in reduced Weierstrass form.

By using GP/Pari, it is obvious that the curve is minimal and the rational points of this curve are $(K, z) \in \{(0, 0), (1, 2), (3, 6), (12, 42)\}$.

Now we try to find possible situations. For $(K, z) = (0, 0)$, we have $k = 0$ and then $z = 0$. Since $m = 2k+1$, $n = 2k-1$ and $k = 0$, we get $m = 1$ and $n = -1$. From equation (3), we find that $x = \pm 1$. Therefore we find $(x, y, z) \in \{(1, -1, 0), (-1, -1, 0)\}$. This gives a contradiction. By checking the other (K, z) values, one can see that there is no solution for the equation (1) with $xyz \neq 0$.

(ii) If $m = 2k + 1$ and $n = 2k - 1$, then the equation (3) becomes

$$x^4 - 12k^2 = 1.$$

Setting $x^2 = a$, the equation becomes a Pell equation such that

$$a^2 - 12k^2 = 1.$$

By the usual methods we have the solutions with the form

$$a_n + \sqrt{12}k_n = (7 + 2\sqrt{12})^n, \quad n = 0, 1, 2, 3, \cdots$$

from this $k_n = 0, 2, 28, 390, \cdots$ and in general by the recurrence relation

$$k_{n+2} = 14k_{n+1} - k_n, \qquad \text{for } n = 0, 1, 2, 3, \cdots .$$

Obviously the other variable of the Pell equation has the same recurrence relation. By checking the values of general recurrence for the equation $x^2 = a$, we can not obtain any value of a which implies a square of an integer. So we can not find an integer solution for x.

The case: m and n are both even.

If m and n are both even, then x^4 and z^2 are clearly integer.

(i) If $m = 2k + 2$ and $n = 2k - 2$, then the equation (2) becomes

$$z^2 = 8k^3 + 24k.$$

If we write $k = \dfrac{K}{2}$, then we have

$$z^2 = K^3 + 12K.$$

This implies an elliptic curve in reduced Weierstrass form. From GP/Pari, this curve has only one affine rational point, namely $(K, z) = (0, 0)$. Hence we find the solution as $z = 0$. This is not a solution for equation (1).

(ii) If $m = 2k + 2$ and $n = 2k - 2$, then the equation (3) becomes

$$x^4 = 24k^2 + 8.$$

Setting $x^2 = a$, the equation becomes a Pell equation such that

$$a^2 - 24k^2 = 8.$$

For this equation it is easy to see that a is even. So let $a = 2A$. Then we obtain equation below

$$A^2 - 6k^2 = 2.$$

We see that A must be even, too. So let $A = 2B$. From this equality, we obtain

$$2B^2 = 3k^2 + 1.$$

Now we consider both sides of the equation above for $\bmod 8$. As squares are congruent to 0, 4 or 1 $(\bmod 8)$, the left hand side is congruent to 2 or 0 $(\bmod 8)$, whereas the right hand side is congruent to 1, 4 or 5 $(\bmod 8)$ and this gives a contradiction. This means that our main equation $x^4 = 24k^2 + 8$ has no solution in integers.

This completes the proof.

§3. The equation $x^8 + y^3 = z^8$

Now we consider our main equation for $k = 4$.

Theorem 3.1. The Diophantine equation $x^8 + y^3 = z^8$ has no relatively prime positive integer solutions.

Proof. Suppose that there are infinitely many primitive solutions to the Diophantine equation

$$x^8 + y^3 = z^8. \tag{4}$$

Now we rewrite equation (4) as

$$y^3 = z^8 - x^8,$$

$$y^3 = (z^4 - x^4)(z^4 + x^4).$$

If x or z is even, then the other one must be odd. Therefore the two factors on the right hand side are coprime, so we have integers $m,\ n$ such that

$$m^3 = z^4 + x^4, \qquad n^3 = z^4 - x^4.$$

From these we get integers $x,\ z$ of the form

$$z^4 = \frac{m^3 + n^3}{2}, \tag{5}$$

$$x^4 = \frac{m^3 - n^3}{2}. \tag{6}$$

From equation (5) we get $2z^4 = m^3 + n^3$. This equation is solved by S. Quinning and W. Yunkui. They gave only trivial solution with $(m, n) = 1$ and $z > 0$ for the equation $2z^4 = m^3 + n^3$.

If we consider the trivial solution, we find that $x = 0$ from the equation (6). Hence our main equation has no solution with $xyz \neq 0$.

This completes the proof.

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Arithmetic-geometric alternate sequence

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Abstract This paper will present a special sequence of numbers related to arithmetic and geometric sequence of numbers. The formulas for the general term a_n and the sum of the first n terms, denoted by S_n, are given respectively.

Keywords Sequence of numbers with alternate common difference and ratio, general term a_n, the sum of the first n terms denoted by S_n.

§1. Introduction

Definition 1.1. A sequence of numbers $\{a_n\}$ is called an arithmetic-geometric alternate sequence of numbers if the following conditions are satisfied:

(i) for any $k \in \mathbb{N}$, $\dfrac{a_{2k}}{a_{2k-1}} = r$,

(ii) for any $k \in \mathbb{N}$, $a_{2k+1} - a_{2k} = d$,

where r and d are called the common ratio and common difference of the sequence $\{a_n\}$ respectively.

Example 1.1. The number sequence 1, 1/2, 3/2, 3/4, 7/4, 7/8, 15/8, 15/16, 31/16, 31/32, $\cdots$ is an arithmetic-geometric sequence of numbers with alternate common ratio and difference, where $r = 1/2$ and $d = 1$.

Obviously, the number sequence $\{a_n\}$ has the following form:

$$a_1,\ a_1r,\ a_1r + d,\ (a_1r + d)r,\ (a_1r + d)r + d,\ ((a_1r + d)r + d)r,\ \cdots$$

§2. Main results

Theorem 2.1. The formula of the general term of the sequence $\{a_n\}$ is

$$a_n = a_1 r^{\lfloor \frac{n}{2} \rfloor} + \left(\frac{1 - r^{\lfloor \frac{n}{2} \rfloor}}{1 - r}\right) d + \left(\left\lfloor \frac{n-1}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor\right) d. \tag{1}$$

Proof. We prove this theorem using induction on n.

Obviously, (1) holds for $n = 1$, 2, and 3. Suppose (1) holds when $n = k$, then

$$a_k = a_1 r^{\lfloor \frac{k}{2} \rfloor} + \left(\frac{1 - r^{\lfloor \frac{k}{2} \rfloor}}{1 - r}\right) d + \left(\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) d.$$

We need to show that P$(k+1)$ also holds for any $k \in \mathbb{N}$.

(i) If $k = 2m-1$, where $m \in \mathbb{N}$, then $a_{k+1} = a_k \cdot r$,

$$\begin{aligned}
a_{k+1} &= \left(a_1 r^{\left\lfloor \frac{k}{2} \right\rfloor} + \left(\frac{1 - r^{\left\lfloor \frac{k}{2} \right\rfloor}}{1-r}\right) d + \left(\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) d\right) \cdot r \\
&= a_1 r^{\left\lfloor \frac{2m-1}{2} \right\rfloor + 1} + \left(\frac{1 - r^{\left\lfloor \frac{2m-1}{2} \right\rfloor}}{1-r}\right) dr + \left(\left\lfloor \frac{2m-1-1}{2} \right\rfloor - \left\lfloor \frac{2m-1}{2} \right\rfloor\right) dr \\
&= a_1 r^{(m-1)+1} + \left(\frac{1-r^{m-1}}{1-r}\right) dr \\
&= a_1 r^m + \left(\frac{1-r^{m-1}}{1-r}\right) dr \\
&= a_1 r^m + (1 + r + r^2 + \ldots + r^{m-3} + r^{m-2}) dr \\
&= a_1 r^m + (1 + r + r^2 + r^3 + \ldots + r^{m-2} + r^{m-1}) d - d \\
&= a_1 r^{\left\lfloor \frac{k+1}{2} \right\rfloor} + \left(\frac{1 - r^{\left\lfloor \frac{k+1}{2} \right\rfloor}}{1-r}\right) d + \left(\left\lfloor \frac{(k+1)-1}{2} \right\rfloor - \left\lfloor \frac{k+1}{2} \right\rfloor\right) d.
\end{aligned}$$

So, P$(k+1)$ holds for $k = 2m-1$.

(ii) If $k = 2m$, where $m \in \mathbb{N}$, then $a_{k+1} = a_k + d$,

$$\begin{aligned}
a_{k+1} &= a_1 r^{\left\lfloor \frac{k}{2} \right\rfloor} + \left(\frac{1 - r^{\left\lfloor \frac{k}{2} \right\rfloor}}{1-r}\right) d + \left(\left\lfloor \frac{k-1}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor\right) d + d \\
&= a_1 r^m + \left(\frac{1-r^m}{1-r}\right) d + ((m-1) - m) d + d \\
&= a_1 r^{\left\lfloor \frac{k+1}{2} \right\rfloor} + \left(\frac{1 - r^{\left\lfloor \frac{k+1}{2} \right\rfloor}}{1-r}\right) d + \left(\left\lfloor \frac{(k+1)-1}{2} \right\rfloor - \left\lfloor \frac{k+1}{2} \right\rfloor\right) d.
\end{aligned}$$

So, P$(k+1)$ holds for $k = 2m$.

Therefore, (1) holds when $n = k+1$. This proves the theorem.

Lemma 2.1. For any integer $m > 0$,

$$\sum_{i=1}^{n} r^{e_i} = m - 1 + rm\left(\frac{1 - r^{e_n - 1}}{1-r}\right) + (n + 1 - me_n)\, r^{e_n}, \text{ where } e_i = \left\lfloor \frac{i}{m} \right\rfloor.$$

Proof. Let $e_i = \left\lfloor \frac{i}{m} \right\rfloor$ and $q = e_n = \left\lfloor \frac{n}{m} \right\rfloor$,

$$\begin{aligned}
\sum_{i=1}^{n} r^{e_i} &= r^{e_1} + r^{e_2} + \ldots + r^{e_{m-1}} + r^{e_m} + r^{e_{m+1}} + r^{e_{m+2}} + \ldots + r^{e_{2m-1}} + \\
&\quad r^{e_{2m}} + r^{e_{2m+1}} + \ldots + r^{e_{qm-1}} + r^{e_{qm}} + r^{e_{qm+1}} + \ldots + r^{e_{n-1}} + r^{e_n} \\
&= \sum_{i=1}^{m-1} r^{e_i} + \sum_{i=m}^{2m-1} r^{e_i} + \ldots + \sum_{i=m(q-1)}^{mq-1} r^{e_i} + \sum_{i=mq}^{n} r^{e_i}
\end{aligned}$$

$$
\begin{aligned}
&= m-1+\sum_{i=m}^{2m-1} r^{e_i}+\ldots+\sum_{i=m(q-1)}^{mq-1} r^{e_i}+\sum_{i=mq}^{n} r^{e_i} \\
&= m-1+\sum_{j=1}^{q-1}\left(\sum_{i=jm}^{(j+1)m-1} r^{e_i}\right)+\sum_{i=mq}^{n} r^{e_i} \\
&= m-1+rm\sum_{j=1}^{q-1} r^{j-1}+\sum_{i=mq}^{n} r^{e_i} \\
&= m-1+rm\left(\frac{1-r^{e_n-1}}{1-r}\right)+(n+1-me_n)\, r^{e_n}.
\end{aligned}
$$

Theorem 2.2. The formula for the sum of the first n terms of the sequence is given by

$$
S_n=\frac{nd}{1-r}-de_n+\left(a-\frac{d}{1-r}\right)\left(2r\left(\frac{1-r^{e_n-1}}{1-r}\right)+(n-2e_n+1)\, r^{e_n}+1\right),
$$

where $e_n=\left\lfloor\frac{n}{2}\right\rfloor$.

Proof. Let $e_i=\left\lfloor\frac{i}{2}\right\rfloor$ and $q=e_n=\left\lfloor\frac{n}{2}\right\rfloor$,

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n}\left(a_1 r^{e_i}+\left(\frac{1-r^{e_i}}{1-r}\right)d+\left(\left\lfloor\frac{i-1}{2}\right\rfloor-\left\lfloor\frac{i}{2}\right\rfloor\right)d\right) \\
&= \sum_{i=1}^{n}\left(\frac{d}{1-r}+\left(a-\frac{d}{1-r}\right)r^{e_i}-\frac{1}{2}\left(1+(-1)^n\right)d\right) \\
&= \frac{d}{1-r}\sum_{i=1}^{n}1+\left(a-\frac{d}{1-r}\right)\sum_{i=1}^{n} r^{e_i}-\frac{1}{2}d\sum_{i=1}^{n}\left(1+(-1)^n\right) \\
&= \frac{nd}{1-r}+\left(a-\frac{d}{1-r}\right)\left(1+2r\left(\frac{1-r^{e_n-1}}{1-r}\right)+(n+1-2e_n)\, r^{e_n}\right)-de_n.
\end{aligned}
$$

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Evaluation of certain indefinite integrals

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Abstract In this paper we have evaluate certain indefinite integrals involving Hypergeometric function. The results represent here are assume to be new.

Keywords Pochhammer symbol, gaussian hypergeometric function.
2000 Mathematics Subject Classification: 33C05, 33C45, 33C15, 33D50, 33D60.

§1. Introduction and preliminaries

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,k)=(b)_k=\frac{\Gamma(b+k)}{\Gamma(b)}=\begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k=1,2,3,\cdots \\ 1 \qquad ; & \text{if } k=0 \\ k! \qquad ; & \text{if } b=1,\ k=1,2,3,\cdots \end{cases} \tag{1}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

Generalized Gaussian Hypergeometric Function: Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{array}{lr} a_1,a_2,\cdots,a_A; & \\ & z \\ b_1,b_2,\cdots,b_B; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_kz^k}{(b_1)_k(b_2)_k\cdots(b_B)_kk!}$$

or

$${}_AF_B\left[\begin{array}{lr} (a_A); & \\ & z \\ (b_B); & \end{array}\right]\equiv{}_AF_B\left[\begin{array}{lr} (a_j)_{j=1}^A; & \\ & z \\ (b_j)_{j=1}^B; & \end{array}\right]=\sum_{k=0}^{\infty}\frac{((a_A))_kz^k}{((b_B))_kk!}, \tag{2}$$

where denominator parameters $b_1,b_2,\cdots,b_B$ are neither zero nor negative integers and A, B are non-negative integers.

§2. Main indefinite integrals

$$\int \frac{\cos x \cos(\frac{\pi x}{2a})}{\sqrt{(1-\cos x)}}\, dx$$
$$= \frac{1}{\sqrt{1-\cos x}}\; a \sin\frac{x}{2}\Bigg[\frac{1}{(a+\pi)(3a+\pi)}\; e^{\frac{-\iota(a+\pi)x}{2a}}$$
$$\times\Bigg\{(3a+\pi)\,{}_2F_1\left(\begin{matrix}1, -\frac{a+\pi}{2a}; & \\ & e^{\iota x}\\ \frac{a-\pi}{2a}; & \end{matrix}\right) - (a+\pi)e^{\frac{\iota(2a+\pi)x}{a}}\,{}_2F_1\left(\begin{matrix}1, \frac{3a+\pi}{2a}; & \\ & e^{\iota x}\\ \frac{5a+\pi}{2a}; & \end{matrix}\right)\Bigg\}$$
$$+\frac{1}{(\pi-3a)(\pi-a)}\; e^{-\frac{\iota(\pi-3a)x}{2a}}\;\Bigg\{(\pi-a)\,{}_2F_1\left(\begin{matrix}1, \frac{3a-\pi}{2a}; & \\ & e^{\iota x}\\ \frac{5a-\pi}{2a}; & \end{matrix}\right)$$
$$-(\pi-3a)\; e^{\frac{\iota(\pi-2a)x}{a}}\,{}_2F_1\left(\begin{matrix}1, \frac{\pi-a}{2a}; & \\ & e^{\iota x}\\ \frac{a+\pi}{2a}; & \end{matrix}\right)\Bigg\}\Bigg] + \text{Constant}. \tag{3}$$

$$\int \frac{\sinh x \cos(\frac{\pi x}{2a})}{\sqrt{(1-\sin x)}}\, dx$$
$$= \frac{1}{2\sqrt{1-\sin x}}\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\times\Bigg[\frac{1}{5a^2-2\pi a+\pi^2}\Bigg\{(1-\iota)a\Bigg(\cosh\Big(\frac{2a+(\pi-a)\iota}{2a}\Big)x$$
$$-\sinh\Big(\frac{2a+(\pi-a)\iota}{2a}\Big)x\Bigg)(\pi-a+2a\iota)\,{}_2F_1\left(\begin{matrix}1, \frac{a-\pi+2a\iota}{2a}; & \\ & \sin x - \iota\cos x\\ \frac{3a-\pi+2a\iota}{2a}; & \end{matrix}\right)$$
$$-(\pi-a-2a\iota)\Big(\sinh(2x)+\cosh(2x)\Big)\,{}_2F_1\left(\begin{matrix}1, \frac{a-\pi-2a\iota}{2a}; & \\ & \sin x - \iota\cos x\\ \frac{3a-\pi-2a\iota}{2a}; & \end{matrix}\right)\Bigg\}\Bigg]$$
$$+\Bigg[\frac{1}{((2+\iota)a+\iota\pi)(\pi+(1+2\iota)a)}(1+\iota)a\; e^{\frac{\iota(\pi+(1+2\iota)a)x}{2a}}$$
$$\times\Bigg\{(\pi+(1+2\iota)a)e^{2x}\,{}_2F_1\left(\begin{matrix}1, \frac{(1-2\iota)a+\pi}{2a}; & \\ & -\iota e^{\iota x}\\ \frac{(3-2\iota)a+\pi}{2a}; & \end{matrix}\right)$$
$$-(\pi+(1-2\iota)a)\,{}_2F_1\left(\begin{matrix}1, \frac{(1+2\iota)a+\pi}{2a}; & \\ & -\iota e^{\iota x}\\ \frac{(3+2\iota)a+\pi}{2a}; & \end{matrix}\right)\Bigg\}\Bigg] + \text{Constant}. \tag{4}$$

$$\int \frac{\cosh x \cos(\frac{\pi x}{2a})}{\sqrt{(1-\sin x)}}\; dx$$
$$= \frac{1}{\sqrt{1-\sin x}}\Big(\frac{1-\iota}{2}\Big)a\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)$$
$$\Bigg[\frac{1}{5a^2+2\pi a+\pi^2}\Bigg\{e^{\frac{\iota(\pi+(1+2\iota)a)x}{2a}}\Bigg((\pi+a+2a\iota)e^{2x}{}_2F_1\left(\begin{array}{ll} 1, \frac{a+\pi-2a\iota}{2a}; & \\ & -\iota e^{\iota x} \\ \frac{3a+\pi-2a\iota}{2a}; & \end{array}\right)$$
$$+(\pi+a-2a\iota){}_2F_1\left(\begin{array}{ll} 1, \frac{a+\pi-2a\iota}{2a}; & \\ & -\iota e^{\iota x} \\ \frac{3a+\pi-2a\iota}{2a}; & \end{array}\right)\Bigg)\Bigg\}\Bigg]$$
$$+\Bigg[\frac{1}{((2+\iota)a+\iota\pi)(\pi+(1+2\iota)a)}(1+\iota)a\; e^{\frac{\iota(\pi+(1+2\iota)a)x}{2a}}$$
$$\times\Bigg\{(\pi+(1+2\iota)a)e^{2x}{}_2F_1\left(\begin{array}{ll} 1, \frac{(1-2\iota)a+\pi}{2a}; & \\ & -\iota e^{\iota x} \\ \frac{(3-2\iota)a+\pi}{2a}; & \end{array}\right)$$
$$-(\pi+(1-2\iota)a){}_2F_1\left(\begin{array}{ll} 1, \frac{(1+2\iota)a+\pi}{2a}; & \\ & -\iota e^{\iota x} \\ \frac{(3+2\iota)a+\pi}{2a}; & \end{array}\right)\Bigg\}\Bigg]$$
$$-\Bigg[\Bigg\{\iota\sin\Bigg(\frac{(-\pi+(1+2\iota)a)x}{2a}\Bigg)+\cos\Bigg(\frac{(-\pi+(1+2\iota)a)x}{2a}\Bigg)\Bigg\}$$
$$\times\Bigg\{(\pi-(1-2\iota)a){}_2F_1\left(\begin{array}{ll} 1, \frac{a+2a\iota-\pi}{2a}; & \\ & \sin x-\iota\cos x \\ \frac{3a+2a\iota-\pi}{2a}; & \end{array}\right)$$
$$+(\pi-(1+2\iota)a)\Big(\sinh(2x)+\cosh(2x)\Big){}_2F_1\left(\begin{array}{ll} 1, \frac{a-2a\iota-\pi}{2a}; & \\ & \sin x-\iota\cos x \\ \frac{3a-2a\iota-\pi}{2a}; & \end{array}\right)\Bigg\}\Bigg]$$
$$+\text{Constant.} \tag{5}$$

$$\int \frac{\cos x \cosh(\frac{\pi x}{2a})}{\sqrt{(1-\sin x)}}\; dx = \frac{2a\{\pi\cos x\sinh(\frac{\pi x}{2a})+a(\sin x-1)\cosh(\frac{\pi x}{2a})\}}{(a^2+\pi^2)\sqrt{1-sinx}}+\text{Constant.} \tag{6}$$

$$\int \frac{\cos x \cos(\frac{\pi x}{2a})}{\sqrt{(1-\sin x)}}\; dx$$
$$= \frac{1}{\sqrt{1-\sin x}}\Big(\frac{1}{2}+\frac{\iota}{2}\Big)a\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)$$

$$\times\Bigg[-\iota\Bigg\{\frac{e^{\frac{\iota(\pi-a)x}{2a}}\,{}_2F_1\Big(1,\frac{1}{2}(\frac{\pi}{a}-1);\frac{a+\pi}{2a};-\iota e^{\iota x}\Big)}{\pi-a}+\frac{e^{\frac{\iota(\pi+3a)x}{2a}}\,{}_2F_1\Big(1,\frac{1}{2}(\frac{\pi}{a}+3);\frac{5a+\pi}{2a};-\iota e^{\iota x}\Big)}{\pi+3a}\Bigg\}$$
$$+\frac{1}{(\pi-3a)}\Big\{\sin\Big(\frac{(\pi-3a)x}{2a}\Big)+\iota\cos\Big(\frac{(\pi-3a)x}{2a}\Big)\Big\}$$
$$\times{}_2F_1\Big(1,\frac{3}{2}-\frac{\pi}{2a};\frac{5a-\pi}{2a};\sin x-\iota\cos x\Big)$$
$$+\frac{\Big(\sin\Big(\frac{(a+\pi)x}{2a}\Big)+\iota\cos\Big(\frac{(a+\pi)x}{2a}\Big)\Big){}_2F_1\Big(1,-\frac{a+\pi}{2a};\frac{a-\pi}{2a};\sin x-\iota\cos x\Big)}{a+\pi}\Bigg]+\text{Constant}. \tag{7}$$

$$\int\frac{\sin x\cos(\frac{\pi x}{2a})}{\sqrt{(1-\cos x)}}\,dx$$
$$=\frac{1}{\sqrt{1-\sin x}}\,\iota a\sin\frac{x}{2}\Bigg[\frac{1}{(a+\pi)(3a+\pi)}\,e^{\frac{\iota(a+\pi)x}{2a}}$$
$$\times\Bigg\{(3a+\pi){}_2F_1\Big(1,-\frac{a+\pi}{2a};\frac{a-\pi}{2a};e^{\iota x}\Big)+(a+\pi)\,e^{\frac{\iota(2a+\pi)x}{a}}\,{}_2F_1\Big(1,\frac{3a+\pi}{2a};\frac{5a+\pi}{2a};e^{\iota x}\Big)\Bigg\}$$
$$+\frac{1}{(\pi-3a)(\pi-a)}\,e^{-\frac{\iota(\pi-3a)x}{2a}}\Bigg\{(a-\pi){}_2F_1\Big(1,\frac{3a-\pi}{2a};\frac{5a-\pi}{2a};e^{\iota x}\Big)$$
$$-(\pi-3a)\,e^{\frac{\iota(\pi-2a)x}{a}}\,{}_2F_1\Big(1,\frac{\pi-a}{2a};\frac{a+\pi}{2a};e^{\iota x}\Big)\Bigg\}\Bigg]+\text{Constant}. \tag{8}$$

$$\int\frac{\sin x\cos(\frac{\pi x}{2a})}{1-\cos x}\,dx$$
$$=\frac{1}{\pi^2-4\pi a^2}\,a\,e^{-\frac{\iota\pi x}{2a}}\Bigg[(\pi^2-4a^2)\,{}_2F_1\Big(1,-\frac{\pi}{2a};\frac{2a-\pi}{2a};e^{\iota x}\Big)$$
$$-(\pi^2-4a^2)\,e^{\frac{\iota\pi x}{a}}\,{}_2F_1\Big(1,\frac{\pi}{2a};\frac{2a+\pi}{2a};e^{\iota x}\Big)+\pi\Bigg\{(2a+\pi)\,e^{\iota x}{}_2F_1\Big(1,-\frac{2a-\pi}{2a};\frac{4a-\pi}{2a};e^{\iota x}\Big)$$
$$-(\pi-2a)\,e^{\frac{\iota(a+\pi)x}{a}}\,{}_2F_1\Big(1,-\frac{2a+\pi}{2a};\frac{4a+\pi}{2a};e^{\iota x}\Big)\Bigg\}\Bigg]+\text{Constant}. \tag{9}$$

$$\int\frac{\sin x\cos(\frac{\pi x}{2a})}{\sqrt{(1-\sin x)}}\,dx$$
$$=\frac{1}{\sqrt{1-\sin x}}\Big(\frac{1}{2}+\frac{\iota}{2}\Big)a\Big(\cos\frac{x}{2}-\sin\frac{x}{2}\Big)$$
$$\times\Bigg[-\iota\Bigg\{\frac{e^{\frac{\iota(\pi-a)x}{2a}}\,{}_2F_1\Big(1,\frac{1}{2}(\frac{\pi}{a}-1);\frac{a+\pi}{2a};-\iota e^{\iota x}\Big)}{\pi-a}-\frac{e^{\frac{\iota(\pi+3a)x}{2a}}\,{}_2F_1\Big(1,\frac{1}{2}(\frac{\pi}{a}+3);\frac{5a+\pi}{2a};-\iota e^{\iota x}\Big)}{\pi+3a}\Bigg\}$$
$$+\frac{1}{(\pi-3a)}\Big\{\cos\Big(\frac{(\pi-3a)x}{2a}\Big)-\iota\sin\Big(\frac{(\pi-3a)x}{2a}\Big)\Big\}$$

$$\times {}_2F_1\left(1, \frac{3}{2} - \frac{\pi}{2a}; \frac{5a-\pi}{2a}; \sin x - \iota\cos x\right)$$
$$\left. - \frac{\left(\cos\left(\frac{(a+\pi)x}{2a}\right) - \iota\sin\left(\frac{(a+\pi)x}{2a}\right)\right) {}_2F_1\left(1, -\frac{a+\pi}{2a}; \frac{a-\pi}{2a}; \sin x - \iota\cos x\right)}{a+\pi}\right]$$
$$+\text{Constant}. \tag{10}$$

§3. Derivation of the integrals

Applying the same method which is used in [4], integrals will be established.

§4. Applications

The integrals which are presented here are very special integrals. These are applied in the field of engineering and other allied sciences.

§5. Conclusion

In our work we have established certain indefinite integrals involving Hypergeometric function. However, one can establish such type of integrals which are very useful for different field of engineering and sciences by involving these integrals. Thus we can only hope that the development presented in this work will stimulate further interest and research in this important area of classical special functions.

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On Theta paracompactness by Grills

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Abstract The aim of this paper is to highlight the concept of paracompactness in the light of θ-open sets via Grills. Some of their properties and charecterizations are investigated.

Keywords Grill, grill Topology, $\mathcal{G}$ θ-paracompactness, $\mathcal{G}$ θ-regularity.
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§1. Introduction

Choquet [3] introduced the concept of grills in 1947. The idea of grills was found to be very useful device like nets and filters. Also for the investigations of many topological notions like compactifications, proximity spaces, theory of grill topology was used.

The notion of paracompactness in ideals was initiated by Hamlett et al [6] in the year 1997. B. Roy and M. N. Mukherjee [10] extended the concept of para compactness in terms of grills. Following their work we formulate the new definition of θ-paracompactness via grills. Also we attempted to achieve a general form of the well known Michaels theorem on regular paracompact spaces perticularly for θ-open sets.

§2. Preliminaries

Definition 2.1.[3] A colletion $\mathcal{G}$ of nonempty subsets of a set X is called a grill if
(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ implies that $B \subseteq \mathcal{G}$, and
(ii) $A \cup B \in \mathcal{G}$ $(A, B \subseteq X)$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.2.[8] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. We define a mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$, denoted by $\Phi_{\mathcal{G}}(A, \tau)$ or simply $\Phi(A)$, is called the operator associated with the grill $\mathcal{G}$ and the topology τ, and is defined by $\Phi(A)= \{x \in X\colon\ A \cap U \in \mathcal{G},\ \forall\ U \in \tau(x)\}$.

Definition 2.3.[8] The topology τ of a topological space (X, τ) is said to be suitable for a grill $\mathcal{G}$ on X if for any $A \subseteq X$, $A \backslash \Phi(A) \notin \mathcal{G}$.

Definition 2.4.[8] A grill $\mathcal{G}$ is called a μ grill if any arbitrary family $\{A_\alpha\colon\ \alpha \in \Lambda\}$ of subsets of X, $\cup_\alpha A_\alpha \in \mathcal{G}$ then $A_\alpha \in \mathcal{G}$ for atleast one $\alpha \in \Lambda$.

Definition 2.5.[7] A topological space (X, τ) is said to be $\mathcal{G}$ θ-regular if for any θ-closed set F in X with $x \notin F$, there exist disjoint θ-open sets U and V such that $x \in U$, and $F \backslash V \notin \mathcal{G}$.

Definition 2.6.[15] A paracompact space (X, τ) is a Hausdorff space with the property that every open cover of X has an open locally finite refiniment.

Definition 2.7.[15] In a Lindelof space (X, τ), for every open cover there exists a subcover of X, which is having countable collection of open sets.

§3. $\mathcal{G}$-paracompactness through θ-open sets

Definition 3.1. Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then the space X is said to be θ-paracompact with respect to the grill or simply $\mathcal{G}$-θ-paracompact if every θ-open cover $\mathcal{U}$=$\{U_\alpha : \alpha \in \lambda\}$ of X has a precise locally finite θ-open refiniment $\mathcal{U}^*$ such that $X \backslash \cup\, \mathcal{U}^* \notin \mathcal{G}$. Also a cover has a precise refinement means, there exists a collection $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of subsets of X such that $V_\alpha \subseteq U_\alpha$, for all $\alpha \in \Lambda$.

Remark 3.1. (i) Every θ-paracompact space X is $\mathcal{G}$-θ-paracompact, for every grill $\mathcal{G}$ on X.

(ii) For the grill $\mathcal{G} = \mathcal{P}(X) \backslash \emptyset$, the concepts of θ-paracompactness and $\mathcal{G}$-θ-paracompactness coincide for any space X, where $\mathcal{P}(X)$ denotes the power set of X.

(iii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two grills on a space with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\mathcal{G}_2$-θ-paracompactness of $X \Rightarrow$ $\mathcal{G}_1$-θ-paracompactness of X. Moreover, it may so happen that a space X is $\mathcal{G}_1$-θ-paracompact as well as $\mathcal{G}_2$-θ-paracompact while the grills $\mathcal{G}_1$ and $\mathcal{G}_2$ are non-comparable.

(iv) Considering $\mathcal{G}$ θ paracompactness, refinement need not be a cover.

Theorem 3.1. Let $\mathcal{G}$ be a μ grill on a topological space (X, τ). Then $(X, \tau_{\mathcal{G}})$ is $\mathcal{G}$-θ-paracompact if (X, τ) is so.

Proof. Let us consider a cover $\mathcal{W}$ of X by basic θ-open sets of $(X, \tau_{\mathcal{G}})$, given by $\mathcal{W}$=$\{W_\alpha : \alpha \in \Lambda\}$, where for each $\alpha \in \Lambda$, $\mathcal{W}_\alpha$=$U_\alpha \backslash\, A_\alpha$ with $U_\alpha \in \tau$ also θ-open set and $A_\alpha \notin \mathcal{G}$. Then $\mathcal{U}$ = $\{U_\alpha : \alpha \in \Lambda\}$ is a τ θ-open cover of X. By $\mathcal{G}$-θ-paracompactness of (X, τ), $\mathcal{U}$ has a τ locally finite τ θ-open precise refinement $\mathcal{V}$=$\{V_\alpha : \alpha \in \Lambda\}$ such that $X \backslash (\cup_{\alpha \in \Lambda} V_\alpha) \notin \mathcal{G}$. It suffices to show that $\mathcal{W}^* = \{\, V_\alpha \backslash A_\alpha\colon \alpha \in \Lambda\}$ is a precise $\tau_{\mathcal{G}}$ -locally finite $\tau_{\mathcal{G}}$ θ open refinement of $\mathcal{W}$.

$\mathcal{W}^*$ is a $\tau_{\mathcal{G}}$ θ-open precise refinement of $\mathcal{W}$. Also $\mathcal{W}$ is τ-locally finite and $\tau \subseteq \tau_{\mathcal{G}}$, $\mathcal{V}$ is $\tau_{\mathcal{G}}$-locally finite, and hence $\mathcal{W}^*$ is $\tau_{\mathcal{G}}$-locally finite. It thus remains to show that $X \backslash \cup_{\alpha \in \Lambda} (V_\alpha \backslash A_\alpha) \notin \mathcal{G}$. Then, $X \backslash \cup_{\alpha \in \Lambda} (V_\alpha \backslash A_\alpha) = X \backslash [\cup_{\alpha \in \Lambda} (V_\alpha \cap A_\alpha^C)] = \cap_{\alpha \in \Lambda} [X \backslash (V_\alpha \cap (X \backslash A_\alpha))] = \cap_{\alpha \in \Lambda} [(X \backslash V_\alpha) \cup A_\alpha]$ $= [\cap_{\alpha \in \Lambda} (X \backslash V_\alpha)] \cup [\cap_{\alpha \in \Lambda} A_\alpha] = [\cup_\Lambda ((\cap_{\alpha \in \Lambda_1} (X \backslash V_\alpha)) \cap (\cap_{\beta \in \Lambda_2} A_\alpha))]$, where

$$\Lambda_1 \cup \Lambda_2 = \Lambda \text{ and } \Lambda_1 \cap \Lambda_2 = \emptyset. \tag{1}$$

The union over Λ stands for all possible partition (1) holds. Now $\cap_{\alpha \in \Lambda} (X \backslash V_\alpha) = X \backslash (\cup \mathcal{V}) \notin \mathcal{G}$ and $\cap_{\alpha \in \Lambda} A_\alpha \notin \mathcal{G}$ (since $A_\alpha \notin \mathcal{G}$ for each α). Furthermore, for any partition $\{\Lambda_1, \Lambda_2\}$ of Λ with the property (1), $\cap_{\alpha \in \Lambda_1} (X \backslash V_\alpha)] \cap [\cap_{\beta \in \Lambda_2} A_\beta] \subseteq [\cap_{\beta \in \Lambda_2} A_\beta] \notin \mathcal{G}$. Thus $\cup_\Lambda [(\cap_{\alpha \in \Lambda_1} (X \backslash V_\alpha)) \cap (\cap_{\beta \in \Lambda_2} A_\beta)] \notin \mathcal{G}$. Hence the result.

Theorem 3.2. Let $\mathcal{G}$ be a grill on a space (X, τ) such that $\tau \backslash \{\emptyset\} \subseteq \mathcal{G}$. If τ is suitable for $\mathcal{G}$ and $(X, \tau_{\mathcal{G}})$ is $\mathcal{G}$-θ-paracompact, then (X, τ) is $\mathcal{G}$-θ-paracompact.

Proof. Let $\mathcal{U}$=$\{U_\alpha : \alpha \in \Lambda\}$ be a τ θ-open cover of X. Then $\mathcal{U}$ is a $\tau_{\mathcal{G}}$ θ-open cover of X. Hence $\mathcal{U}$ is a $\tau_{\mathcal{G}}$-locally finite precise refinement $\{V_\alpha \backslash A_\alpha\colon \alpha \in \Lambda,\ V_\alpha \in \tau \text{ and } A_\alpha \notin \mathcal{G}\}$ such that

$$X \backslash \cup_{\alpha \in \Lambda} (V_\alpha \backslash A_\alpha) \notin \mathcal{G}. \tag{2}$$

We now show that $\mathcal{V}=\{V_\alpha : \alpha \in \Lambda\}$ is τ locally finite. In fact, for each $x \in X$ there exists some $U \in \tau_{\mathcal{G}}$ such that $U \cap (V_\alpha \backslash A_\alpha)=\emptyset$, for all $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_n$(assumption). But $U = V\backslash A$, where $V \in \tau$ and $A \notin \mathcal{G}$. Thus for any $\alpha \neq \alpha_1, \alpha_2, \cdots, \alpha_n$, $(V\backslash A) \cap (V_\alpha \backslash A_\alpha) = \emptyset$, That is $(V \cap V_\alpha)\backslash (A \cap A_\alpha) = \emptyset$. Then either $V \cap V_\alpha = \emptyset$ or $(V \cap V_\alpha)(\neq \emptyset) \subseteq A \cup A_\alpha$. We claim that $V \cap V_\alpha = \emptyset$. For otherwise, $V \cap V_\alpha$ is nonempty $\tau\theta$-open set $\Rightarrow V \cap V_\alpha \in \mathcal{G} \Rightarrow A \cup A_\alpha \in \mathcal{G}$ a contradiction. Thus $\mathcal{V}$ is τ locally finite.

Again, $V_\alpha \backslash A_\alpha \subseteq U_\alpha$ and $V_\alpha \backslash A_\alpha \subseteq V_\alpha \Rightarrow V_\alpha \backslash A_\alpha \subseteq U_\alpha \cap V_\alpha \Rightarrow \cup_{\alpha\in\Lambda}(V_\alpha \backslash A_\alpha) \subseteq \cup_{\alpha\in\Lambda}(U_\alpha \cap V_\alpha) \Rightarrow X\backslash \cup_{\alpha\in\Lambda}(V_\alpha \backslash A_\alpha \supseteq X \backslash \cup_{\alpha\in\Lambda}(U_\alpha \cap V_\alpha)$ and hence by (2), $X\backslash\cup_{\alpha\in\Lambda}(U_\alpha \cap V_\alpha) \notin \mathcal{G}$. Now $\mathcal{W} = \{U_\alpha \cap V_\alpha : \alpha \in \Lambda\}$ is τ locally finite $\tau\theta$-open precise refiniment of $\mathcal{U}$ such that $X\backslash(\cup\mathcal{W}) \notin \mathcal{G}$. Thus (X, τ) is $\mathcal{G}$-θ-paracompact.

Corollary 3.1. Let (X, τ) be a topological space and $\mathcal{G}$ a μ-grill on X such that $\tau\backslash\{\emptyset\} \subseteq \mathcal{G}$ and τ is suitable for $\mathcal{G}$. Then (X, τ) is $\mathcal{G}$ θ-compact iff $(X, \tau_{\mathcal{G}})$ is $\mathcal{G}$-θ-paracompact.

Corollary 3.2.[11] For any topological space X, $\mathcal{G}_\delta = \{A \subseteq X : intclA \neq \emptyset\}$ is a grill on X.

A weaker form of paracompactness is almost paracompactness and the definition for almost θ-paracompactness is,

Definition 3.2. A topological space (X, τ) is said to be almost θ-paracompact if every θ-open cover $\mathcal{U}$ of X has a locally finite θ-open refinement $\mathcal{U}^*$ such that $X\backslash cl(\cup\, \mathcal{U}^*) = \emptyset$.

Theorem 3.3. A topological space (X, τ) is almost θ-para compact iff X is $\mathcal{G}_\delta$ θ-paracompact.

Proof. Let $\mathcal{U}$ be an θ-open cover of an almost θ-paracompact space (X, τ). Then there exists a precise locally finite θ-open refinement $\mathcal{U}^*$ of $\mathcal{U}$ such that $X\backslash cl(\mathcal{U}^*) = \emptyset$. We claim that $X\backslash(\cup\, \mathcal{U}^*) \notin \mathcal{G}$. For otherwise, $X\backslash(\cup\, \mathcal{U}^*) \in \mathcal{G} \Rightarrow intcl(X\backslash(\cup\, \mathcal{U}^*)) \neq \emptyset \Rightarrow X\backslash clint(\cup\, \mathcal{U}^*) \neq \emptyset \Rightarrow X\backslash cl(\cup\, \mathcal{U}^*) \neq \emptyset$, a contradiction. Thus (X, τ) is a $\mathcal{G}_\delta$ θ-paracompact.

We now prove a stronger converse that whenever $\mathcal{G}$ is any grill on X with $\tau\backslash\{\emptyset\} \subseteq \mathcal{G}$, then the almost θ-paracompactness of (X, τ) is implied by the $\mathcal{G}\theta$-paracompactness of X. We first observe that for such a grill $\mathcal{G}$, we have $intA = \emptyset$ whenever $A(\subseteq X) \notin \mathcal{G}$. Now let $\mathcal{U}$ be an θ-open cover of X. Then by the definition of $\mathcal{G}\theta$-paracompactness there exists a precise locally finite θ-open refinement $\mathcal{U}^*$ of $\mathcal{U}$ such that $X\backslash(\cup\, \mathcal{U}^*) \notin \mathcal{G}$. Thus int $(X\backslash(\cup\, \mathcal{U}^*)) = \emptyset$, That is $X = cl(\cup\mathcal{U}^*)$, proving (X, τ) to be almost θ-paracompact.

§4. Principal grill $[A]$, its regularity and θ-paracompactness

Definition 4.1.[11] Let X be a nonempty set and $(\emptyset \neq)A \subseteq X$. Then the principal $[A]$ is defined as $[A] = \{B \subseteq X : A \cap B \neq \emptyset\}$.

Remark 4.1. In the grill topological space X, if $\mathcal{G}=[X]$, then $[X]$-θ-paracompactness reduces simply to θ-paracompactness.

Definition 4.2. $\mathcal{G}$ is a grill on a topological space (X, τ), the space X is said to be $\mathcal{G}$-θ regular if for each θ-closed subset F of X and each $x \in X\backslash F$, there exist disjoint θ-open sets U and V such that $x \in U$ and $F\backslash V \notin \mathcal{G}$.

Remark 4.2. From the above two definitions the principal grill $[X]$ generated by X is, in fact, $\mathcal{P}(X)\backslash\{\emptyset\}$ and hence a space (X, τ) is $[X]$-θ-regular iff (X, τ) is θ-regular.

Remark 4.3. Every regular space is $\mathcal{G}$-θ-reguler for any grill on X.

Theorem 4.1. Let X be any nonempty subset of a space (X,τ). Then (X,τ) is $[A]$-θ-regular iff for each θ-closed subset F of X and each $x\notin F$, there exist disjoint θ-open sets U and V such that $x\in U$ and $F\cap A\subseteq V$.

Proof. Let (X,τ) be a $[A]$-θ-regular and F a θ-closed subset of X and $x\in X\backslash F$. Then there exist disjoint θ-open sets U and V such $x\in U$ and $F\backslash V\notin[A]$. Now, $F\backslash V\notin[A]$ $\Rightarrow(F\backslash V)\cap A=\emptyset\Rightarrow F\cap A\cap(X\backslash V)=\emptyset\Rightarrow F\cap A\subseteq V$.

Conversely, let the given condition hold and let F be a θ-closed subset of X with $x\in X\backslash F$. Then there exist disjoint θ-open sets U and V such that $x\in U$ and $F\cap A\subseteq V$. Now, $F\cap A\subseteq V$ $\Rightarrow F\cap A\cap(X\backslash V)=\emptyset\Rightarrow F\cap(X\backslash V)\notin[A]\Rightarrow(F\backslash V)\notin[A]$.

We modify the E. Micheal's theorem for θ-open sets.

Theorem 4.2. Let $\mathcal{G}$ be a grill on a space (X,τ). If X is $\mathcal{G}$-θ-paracompact and ${}_{\theta}T_2$ space, then X is $\mathcal{G}$-θ-regular.

Proof. Let F be a θ-closed subset of X and $y\in X\backslash F$. Then the Hausdorffnes of X implies that for each $x\in F$, there exist disjoint θ-open sets G_x and H_x such that $y\in G_x$ and $x\in H_x$. Clearly $y\notin clH_x$. Then $\mathcal{U}=\{H_x:x\in F\}\cup\{X\backslash F\}$ is an θ-open cover of X. Thus there exists a precise locally finite θ-open refinement $\mathcal{U}^*=\{H_x^{'}:x\in F\}\cup\{W\}$ such that $H_x^{'}\subseteq H_x$ for each $x\in F$, $W\subseteq X\backslash F$ and $X\backslash(\cup\,\mathcal{U}^*)\notin\mathcal{G}$. Let $G=X\backslash\cup\{clH_x^{'}:x\in F\}$. Then G and H are two nonempty disjoint θ-open sets, such that $y\in G$, $F\backslash H\notin\mathcal{G}$. Hence (X,τ) is $\mathcal{G}$-θ-regular.

Corollary 4.1. Let A be a nonempty subset of a space (X,τ). If X is an $[A]$-θ-paracompact Hausdorff space, then it is $[A]$-θ-regular.

Corollary 4.2. A θ-paracompact space is θ-regular. The proof is immediate.

Lemma 4.1. For a nonempty subset A of a Hausdorff space (X,τ), let X be $[A]$-θ-paracompact. Then for each $x\in X$ and each θ-open set U containing x, there exists a θ-open neighbourhood V of x such that $clV\backslash U\subseteq X\backslash A$. That is $(clV\backslash U)\cap A=\emptyset$, and hence $clV\cap A\subseteq U$.

Proof. Let $x\in X$ and U be an θ-open neighbourhood of x. Then $X\backslash U$ is a θ-closed subset of X, not containing x. As (X,τ) is $[A]$-θ-regular, by theorem 4.3, there exists two disjoint θ-open sets G and V such that $x\in V$ and $(X\backslash U)\cap A\subseteq G$. Now, $G\cap clV=\emptyset\Rightarrow[(X\backslash U)\cap A]\cap clV=\emptyset\Rightarrow(X\backslash U)\cap A\cap clV=\emptyset\Rightarrow clV\cap(X\backslash U)\subseteq X\backslash A$. That is $clV\backslash U\subseteq X\backslash A$ and hence the proof.

Theorem 4.3. Let (X,τ) be an $[A]$-θ-paracompact, Hausdorff space for some nonempty subset A of X and $\mathcal{U}=\{U_\alpha:\alpha\in\Lambda\}$ be a θ-open cover of X. Then there exists a precise locally finite θ-open refinement $\{G_\alpha:\alpha\in\Lambda\}$ of $\mathcal{U}$ such that $A\subseteq\cup\{G_\alpha:\alpha\in\Lambda\}$ and $clG_\alpha\cap A\subseteq U_\alpha\cap A$.

Proof. Let $\mathcal{U}=\{U_\alpha:\alpha\in\Lambda\}$ be a θ-open cover of X. Then by the Lemma 4.1, for each $\alpha\in\Lambda$ and each $x\in U_\alpha$, there exists $V_{\alpha,x}\in\tau$ with $x\in V_{\alpha,x}$ such that $clV_{\alpha,x}\cap A\subseteq U_\alpha$. Now $\mathcal{V}=\{V_{\alpha,x}:\alpha\in\Lambda\}$ is a θ-open cover of X.

Hence by $[A]$-θ-paracompactness of X, there exists a precise locally finite θ-open refinement $\mathcal{W}=\{W_{\alpha,x}:x\in U_\alpha,\ \alpha\in\Lambda\}$ of $\mathcal{V}$ such that $X\backslash(\cup\,\{W_{\alpha,x}:x\in U_\alpha,\ \alpha\in\Lambda\})\notin$[A]. That is $A\subseteq\cup\{W_{\alpha,x}:x\in\Lambda\}$. Now, for any $x\in U_\alpha$ and $\alpha\in\Lambda$, $W_{\alpha,x}\subseteq V_{\alpha,x}\Rightarrow clW_{\alpha,x}\cap A\subseteq clV_{\alpha,x}\cap A\subseteq U_\alpha\cap A$. Let $G_\alpha=\cup_{x\in U_\alpha}W_{\alpha,x}$ for each $\alpha\in\Lambda$, $W_\alpha\subseteq V_{\alpha,x}\Rightarrow clW_{\alpha,x}\cap A\subseteq clV_{\alpha,x}\cap A\subseteq U_\alpha\cap A$. Let $G_\alpha=\cup_{x\in\Lambda}W_{\alpha,x}$ for each $\alpha\in\Lambda$. Then clearly $\{G_\alpha:\alpha\in\Lambda\}$ is a precise locally finite θ-open refinement of $\mathcal{U}$, and $clG_\alpha=cl(\cup_{x\in\Lambda}W_{\alpha,x})=\cup_{x\in\Lambda}clW_{\alpha,x}$. So, $(clG_\alpha)\cap A=\cup_{x\in\Lambda}(clW_{\alpha,x}\cap A)\subseteq U_\alpha\cup A$.

Theorem 4.4. Let (X, τ) be a Hausdorff space and A a dense subset of X. Then the following statements are equivalent:

(i) (X, τ) is $[A]$-θ-paracompact.

(ii) Each θ-open cover of X has a precise locally finite refinement that covers A and consists of sets which are not necessarily closed or open.

(iii) For each θ-open cover $\mathcal{U}$=$\{U_\alpha : \alpha \in \Lambda\}$ of X, there exists a locally finite θ-closed cover $\{F_\alpha : \alpha \in \Lambda\}$ of X such that $F_\alpha \cap A \subseteq U_\alpha$ for each $\alpha \in \Lambda$.

Proof. (i)$\Rightarrow$(ii) It is trivial.

(ii)$\Rightarrow$(iii) Let $\{U_\alpha : \alpha \in \Lambda\}$ be a θ-open cover of X. Then for any $x \in X$, there exists some $U_{\alpha(x)} \in \mathcal{U}$ such that $x \in U_{\alpha(x)}$. Then by Lemma 4.1, there exists some $H_x \in \tau$ with $x \in H_x$ such that $clH_x \cap A \subseteq U_{\alpha(x)}$. Thus $\mathcal{H}$= $\{H_x : x \in X\}$ is a θ-open cover of X, and hence there is a precise locally finite refinement $\{A_x : x \in X\}$ of $\mathcal{H}$ such that $A \subseteq \{A_x$: $x \in X\}$. Since $\{A_x : x \in X\}$ is locally finite, so is $\{clA_x : x \in X\}$. Thus $\cup\{clA_x : x \in X\}$= $cl[\cup\{A_x : x \in X\}] \supseteq clA = X \Rightarrow X = \cup\{clA_x : x \in X\}$. As $A_x \subseteq H_x \Rightarrow clA_x \subseteq clH_x$ $\Rightarrow clA_x \cap A \subseteq clH_x \cap A \subseteq U_{\alpha(x)}$.

For each $\alpha \in \Lambda$, set F_α= $\cup\{clA_x : \alpha$=$\alpha(x)\}$. Then F_α is θ-closed for each $\alpha \in \Lambda$, as it is a union of locally finite θ-closed sets. Thus $\{F_\alpha : \alpha \in \Lambda\}$ is locally finite and a cover of X. Finally $F_\alpha \cap A$= $\cup\{clA_x : \alpha$=$\alpha(x)\}\cap$A= $\cup\{clA_x \cap A : \alpha$=$\alpha(x)\} \subseteq U_\alpha \cap A$, for each $\alpha \in \Lambda$.

(iii)$\Rightarrow$(i) Let $\mathcal{U}$= $\{U_\alpha : \alpha \in \Lambda\}$ be a θ-open cover of X. Let $\{F_\alpha : \alpha \in \Lambda\}$ be a locally finite θ-closed cover of X such that $F_\alpha \cap A \subseteq U_\alpha$ for each $\alpha \in \Lambda$. For any $x \in X$, there exists $V_x \in \tau$ with $x \in V_x$ such that $V_x \cap F_\alpha \neq \emptyset$ for atmost finitely many $\alpha \in \Lambda$. Now, $\mathcal{V}$=$\{V_x : x \in X\}$ is a cover of X. So there exists a locally finite θ-closed cover $\{B_x : x \in X\}$ such that $B_x \cap A \subseteq V_x$, for all $x \in X$. Thus $\{B_x \cap A : x \in A\}$ is a cover of A.

Let us now consider $U(F_\alpha)$= $X \backslash \cup\{B_x : B_x \cap F_\alpha \cap A$=$\emptyset\}$. We first note that $U(F_\alpha)$ is θ-open for each $\alpha \in \Lambda$. Now, $F_\alpha \cap A \subseteq U(F_\alpha)$. In fact, $y \in F_\alpha \cap A$ and $y \notin U(F_\alpha) \Rightarrow y \in F_\alpha \cap A$ and $y \in B_{y'}$ for some $y' \in X \Rightarrow B_{y'} \cap F_\alpha \cap A$=$\emptyset$. But $y \in F_\alpha \cap A$ and $y \in B_{y'} \Rightarrow y \in B_{y'} \cap F_\alpha \cap A$, a contradiction.

We show that $\{U(F_\alpha : \alpha \in \Lambda)\}$ is locally finite. Each $x \in X$ has some θ-open neighbourhood W intersecting finitely many B_x's, say $B_{x_1}, B_{x_2}, \cdots, B_{x_n}$. Then W is contained in $\cup_{i=1}^{n} B_{x_i}$ (since $\{B_x : \alpha \in \Lambda\}$ is a θ-cover of X). Now $B_x \cap U(F_\alpha) \neq \emptyset \Rightarrow B_x \cap F_\alpha \cap A \neq \emptyset$. Each $B_x \cap A$ is contained in V_x, where V_x intersects atmost finitely many $F_\alpha \Rightarrow B_x \cap A$ intersects atmost finitely many $F_\alpha \Rightarrow$ each set B_x intersects atmost finitely many $U(F_\alpha) \Rightarrow W$ intersects atmost finitely many $U(F_\alpha)$. Thus $\{U(F_\alpha) : \alpha \in \Lambda\}$ is locally finite. Also $\{U(F_\alpha) : \alpha \in \Lambda\}$ covers A, because $F_\alpha \cap A \subseteq U_\alpha$ and $\{F_\alpha \cap U_\alpha : \alpha \in \Lambda\}$ is a θ-cover of A.

Let $\mathcal{U}^*$= $\{U(F_\alpha) \cap U_\alpha : \alpha \in \Lambda\}$. Then $\mathcal{U}^*$ is a precise locally finite θ-open refinement of $\mathcal{U}$. Thus $F_\alpha \cap A \subseteq U_\alpha \cap U(F_\alpha)$, for all $\alpha \in \Lambda \Rightarrow A \subseteq \cup_{\alpha\in\Lambda}(F_\alpha \cap A) \subseteq \cup_{\alpha\in\Lambda}(U_\alpha \cap U(F_\alpha)) \Rightarrow$ $A \subseteq \cup\, \mathcal{U}^* \Rightarrow A \cap (X \backslash (\cup\, \mathcal{U}^*))$=$\emptyset \Rightarrow$ X$\backslash(\cup\, \mathcal{U}^*) \notin [A]$, Thus (X, τ) to be $[A]$-θ-paracompact.

Corollary 4.3. In a regular space X, the following are equivalent:

(i) X is θ-paracompact.

(ii) Every θ-open cover of X has a locally finite refinement consisting of sets not necessarily θ-open or θ-closed.

(iii) Each θ-open cover of X has a closed locally finite refinement.

Theorem 4.5. Let $\mathcal{G}$ and $\mathcal{G}^{'}$ be two grills respectively on two topological spaces (X, τ) and $(Y, \tau^{'})$. Let $f : (X, \tau) \rightarrow (Y, \tau^{'})$ be a θC homeomorphism and $f(\mathcal{G}) \supseteq \mathcal{G}^{'}$. If (X, τ) is $\mathcal{G}\theta$-paracompact then $(Y, \tau^{'})$ is $\mathcal{G}^{'}\theta$-paracompact. Here $f(\mathcal{G})$ stands for $\{f(G) : G \in \mathcal{G}\}$ which is clearly a grill in Y.

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a θ-open cover of Y. Then by continuity and surjectiveness of f, $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is a θ-open cover of X. Hence by $\mathcal{G}\theta$-compactness of (X, τ), there exists a locally finite precise θ-open refinement $\{W_\alpha : \alpha \in \Lambda\}$ of $\{f^{-1}(V_\alpha)\}$ such that $X \backslash \cup_{\alpha \in \Lambda} W_\alpha \notin \mathcal{G}$. Since f is an θC homeomorphism, f is bijective, f and f^{-1} both are θ-irresolute maps. We have $\{f(W_\alpha) : \alpha \in \Lambda\}$ is an θ-open precise refinement of $\{V_\alpha : \alpha \in \Lambda\}$ in $(Y, \tau^{'})$. We note $\{W_\alpha : \alpha \in \Lambda\}$ is locally finite as f is a homeomorphism. Now, as $X \backslash \cup_{\alpha \in \Lambda} W_\alpha \notin \mathcal{G}$, $Y \backslash \cup_{\alpha \in \Lambda} f(W_\alpha) \notin f(\mathcal{G})$ and hence $Y \backslash \cup_{\alpha \in \Lambda} f(W_\alpha) \notin \mathcal{G}^{'}$. Thus $(Y, \tau^{'})$ is $\mathcal{G}^{'}\theta$-paracompact.

Corollary 4.4. Let (X, τ) and $(Y, \tau^{'})$ be two topological spaces, $A (\neq \emptyset) \subseteq X$, and $f : (X, \tau) \rightarrow (Y, \tau^{'})$ a homeomorphism. If (X, τ) is $[A]$-θ-paracompact then $(Y, \tau^{'})$ is $[f(A)]\theta$-paracompact.

Proof. If we put $[f(A)] = f([A])$ then we can get the result using the previous theorem.

Let $A = X$ in the previous theorem we get the next Corollary.

Corollary 4.5. Let (X, τ) and $(Y, \tau^{'})$ be two topological spaces. $f : (X, \tau) \rightarrow (Y, \tau^{'})$ a homeomorphism. If (X, τ) is θ-paracompact then $(Y, \tau^{'})$ is θ-paracompact.

§5. Relations among other compactness with $\mathcal{G}$-paracompact ness through θ-open sets

Definition 5.1. A space X is T-Lindelof if for every θ-open cover there exists a θ-open subcover for X, which is having countable collection of θ-open sets.

Theorem 5.1. Every $\mathcal{G}$ θ-regular, T-Lindelof space is $\mathcal{G}$ θ-paracompact.

Proof. Let (X, τ) be a θ-regular, T-Lindelof space. Let $\mathcal{U}$ be a θ-open cover of X. Since (X, τ) is a T-Lindelof, there exists a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ that covers X. Then $\mathcal{V}$ is a θ-open refiniment of $\mathcal{U}$. Since (X, τ) is θ-regular the space (X, τ) is $\mathcal{G}$ θ-paracompact.

Remark 5.1. (i) Every $\mathcal{G}$ θ-compact space is $\mathcal{G}$ θ-paracompact.

(ii) Every ${}_\theta T_2$, $\mathcal{G}$ θ-paracompact space is θ-normal.

Remark 5.2. From the above results we have the following implications:

$$\begin{array}{ccc}
\boxed{\text{Compact}} & \Longrightarrow & \boxed{\mathcal{G}\text{-Compact}} \\
\Downarrow +T_2 & & \Downarrow \\
\boxed{\text{Paracompact}} & \Longrightarrow & \boxed{\mathcal{G}\text{-paracompact}} \\
\Downarrow & & \Downarrow \\
\boxed{\theta\text{-paraCompact}} & \Longrightarrow & \boxed{\mathcal{G}\text{-}\theta\text{-paraCompact}} \\
 & & +_\theta T_2
\end{array}$$

$\swarrow$ $\searrow$ $\swarrow$ $\searrow$

$\boxed{\theta\text{-regular}}$ $\boxed{\theta\text{-normal}}$ $\boxed{\mathcal{G}\ \theta\text{-regular}}$ $\boxed{\mathcal{G}\theta\text{-normal}}$

Note 5.1. From [5], [14], we have $\tau_\delta \subseteq \tau_\theta \subseteq \tau$ and that $\tau_\theta = \tau$ if and only if (X, τ) is regular. So, if the space is regular the concept of θ-paracompactness coincide with paracompactness.

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On the closed form of $Z(p.2^k),\ p = 2^q - 1$

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Abstract This paper gives the closed form expression of $Z(p.2^k)$ for all positive integral values of k, in the particular case when p is a prime of the form $p = 2^q - 1$, where $Z(.)$ is the pseudo Smarandache function.

Keywords Pseudo Smarandache function, closed form.

§1. Introduction

The pseudo Smarandache function, $Z(n)$, introduced by Kashihara [1], is as follows:

Definition 1.1. For any integer $n \geq 1$, the pseudo Smarandache function $Z(n)$ is the smallest positive integer m such that $1 + 2 + \cdots + m \equiv \frac{m(m+1)}{2}$ is divisible by n. Thus,

$$Z(n) = \min\{m : m \in \mathbb{Z}^+,\ n | \frac{m(m+1)}{2}\},\ n \geq 1,$$

where $\mathbb{Z}^+$ is the set of all positive integers.

Some of the properties satisfied by $Z(n)$ are given in Majumdar [2], which also gives the explicit forms of $Z(n)$ in some particular cases. It seems that there is no single closed form expression of $Z(n)$.

Of particular interest is the values of $Z(p.2^k)$, where p is a prime and $k \in \mathbb{Z}^+$. Majumdar [2] gives the explicit forms of $Z(p.2^k)$ for $p = 3, 5, 7, 11, 13, 17, 19, 31$. In this paper, we derive the explicit form of $Z(p.2^k)$ when p is a prime of the form $p = 2^q - 1$. This is given in the next section.

§2. Closed form expression of $Z(p.2^k),\ p = 2^q - 1$

First note that, for any integer $a \geq 1$,

$$2^{q(a+1)} - 1 = 2^q(2^{qa} - 1) + 2^q - 1.$$

Therefore, it follows by induction on a that p divides $2^{qa} - 1$ for any integer $a \geq 1$.

The closed form expression of $Z(p.2^k)$, when $p = 2^q - 1$, is given in the theorem below.

Theorem 2.1. Let p be a prime of the form $p = 2^q - 1, q \geq 1$. Then

$$Z(p.2^k) = \begin{cases} (p-1)2^k, & \text{if } q \text{ divides } k, \\ 2^{k+q-i}, & \text{if } q \text{ divides } k - i,\ 1 \leq i \leq q{-}1. \end{cases}$$

Proof. First note that, if $p = 2^q - 1$ is prime, then by the Cataldi-Fermat Theorem, q must be a prime (see, for example, Theorem 4 in Daniel Shanks [3]).
Now, by definition,

$$Z(p.2^k) = \min\{m : p.2^k | \frac{m(m+1)}{2}\} = \min\{m : p.2^{k+1} | m(m+1)\}. \tag{1}$$

Here, 2^{k+1} must divide one of m and $m+1$, and p must divide the other. We now consider all the possible cases below:

Case (1): When k is of the form $k = qa$ for some integer $a \geq 1$. Let $p = 2P+1$. Now, since

$$P.2^{k+1} + 1 = 2P(2^k - 1) + (2P + 1),$$

it follows that p divides $P.2^{k+1} + 1$, so that $p.2^{k+1}$ divides $P.2^{k+1}(P.2^{k+1} + 1)$. Therefore, the minimum m in (1) can be taken as $P.2^{k+1}$, and hence,

$$Z(p.2^k) = P.2^{k+1} = (p-1)2^k.$$

Case (2): When k is of the form $k = qa + 1$ for some integer $a \geq 0$. Here,

$$2^{q-2}.2^{k+1} - 1 = 2^q(2^{qa} - 1) + 2^q - 1,$$

so that, p divides $2^{k+q-1} - 1$ and hence, $p.2^{k+1}$ divides $2^{k+q-1}(2^{k+q-1} - 1)$. Thus, in this case, the minimum m in (1) may be taken as $2^{k+q-1} - 1$, so that $Z(p.2^k) = 2^{k+q-1} - 1$.

Case (3): When k is of the form $k = qa + 2$ for some integer $a \geq 0$. In this case, since

$$2^{q-3}.2^{k+1} - 1 = 2^q(2^{qa} - 1) + 2^q - 1,$$

it follows that, $p.2^{k+1}$ divides $2^{k+q-2}(2^{k+q-2} - 1)$, and hence, $Z(p.2^k) = 2^{k+q-2} - 1$.

$\vdots$

Case (q) : When k is of the form $k = qa + q - 1$ for some integer $a \geq 0$. Here,

$$2^{k+1} - 1 = 2^{q(a+1)} - 1,$$

so that $p.2^{k+1}$ divides $2^{k+1}(2^{k+1} - 1)$, and consequently, $Z(p.2^k) = 2^{k+1} - 1$.

All these complete the proof of the theorem.

§3. Some special cases

Some special cases of Theorem 2.1 are $Z(3.2^k)$ (corresponding to $q = 2$), $Z(7.2^k)$ (corresponding to $q = 3$), and $Z(31.2^k)$ (corresponding to $q = 4$). The explicit forms of $Z(3.2^k)$, $Z(7.2^k)$ and $Z(31.2^k)$ are given below.

Corollary 3.1. For any integer $k \geq 1$,

$$Z(3.2^k) = \begin{cases} 2^{k+1} - 1, & \text{if k is odd,} \\ 2^{k+1}, & \text{if k is even.} \end{cases}$$

Proof. Since this case corresponds to $q = 2$, q divides k if and only if k is even. The result then follows from Theorem 2.1 immediately.

Corollary 3.2. For any integer $k \geq 1$,

$$Z(7.2^k) = \begin{cases} 3.2^{k+1}, & \text{if } 3|k, \\ 2^{k+2} - 1, & \text{if } 3|(k-1), \\ 2^{k+1} - 1, & \text{if } 3|(k-2). \end{cases}$$

Proof. This case corresponds to $q = 3$, and so, there are three possibilities, namely, k is one of the three forms $k = 3a$, $3a + 1$, $3a + 2$. Then, appealing to Theorem 2.1, we get the desired expression for $Z(7.2^k)$.

Corollary 3.3. For any integer $k \geq 1$,

$$Z(31.2^k) = \begin{cases} 15.2^{k+1}, & \text{if } 5|k, \\ 2^{k+4} - 1, & \text{if } 5|(\mathrm{k}-1), \\ 2^{k+3} - 1, & \text{if } 5|(\mathrm{k}-2), \\ 2^{k+2} - 1, & \text{if } 5|(\mathrm{k}-3), \\ 2^{k+1} - 1, & \text{if } 5|(\mathrm{k}-4). \end{cases}$$

Proof. Here, k can be one of the five forms $k = 5a$, $5a + 1$, $5a + 2$, $5a + 3$, $5a + 4$. When $k = 5a$, by Theorem 2.1, $Z(31.2^k) = 30.2^k = 15.2^{k+1}$. Similarly, the other four cases follow from Theorem 2.1.

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A note on BF-algebras with intuitionistic L-fuzzy p-ideals

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Abstract The notion of Fuzzy BF-subalgebras was developed by A. Borumand Saeid and M. A. Rezvani in 2009. Recently, in 2010, we introduced the notion of Intuitionistic L-fuzzy ideals and in 2011 the notion of Intuitionistic L-fuzzy H-ideal of BF-algebras. This paper deals with the idea of Intuitionistic L-fuzzy p-ideal of a BF-algebra and some results on it.

Keywords BF-algebra, ideal, p-ideal, intuitionistic L-fuzzy subset, intuitionistic L-fuzzy B F-ideal, intuitionistic L-fuzzy p-ideal of a BF-algebra.

§1. Introduction

The study of fuzzy subsets and their application to mathematical concepts has reached to what is commonly called as fuzzy mathematics. Fuzzy algebra is an important branch in it.

In 1965, Lofti A. Zadeh [14] introduced the notion of a fuzzy subset and Goguen [4] generalized the notion of fuzzy subset of X to that of an L-fuzzy subset. Then Intuitionistic Fuzzy Subset was defined by K. T. Atanassov [2] in 1986.

Y. Imai and K. Iseki [5] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Then Neggers and H. S. Kim [10] introduced B-algebras. Recently, in 2007, Andrzej Walendziak [1] defined a BF-algebra.

Since then many research work have been introduced using fuzzy subsets and Intuitionistic fuzzy sets in the various classes of abstract algebraic structures like $BCI/BCK/BCC$ algebras. Recently fuzzy BF-subalgebras of were developed by A. Borumand Saeid and M. A. Rezvani [3] in 2009.

Motivated by these, we have introduced Intuitionistic L-fuzzy ideals of BF-algebras [9] and Intuitionistic L-fuzzy H-ideal of BF-algebras [6]. In this paper, we investigate Intuitionistic L-fuzzy p-ideal of a BF-algebra and establish some of their basic properties.

§2. Preliminaries

In this section the basic definitions of a BF-Algebra and Intuitionistic L-Fuzzy subset are recalled. We start with,

Definition 2.1.[1] A BF-algebra is a non-empty set X with a constant 0 and a single binary operation $*$ satisfying the following axioms:

(i) $x * x = 0$.

(ii) $x * 0 = x$.

(iii) $0 * (x * y) = y * x$, for all $x, y \in X$.

Example 2.1. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following table :

$*$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

Then $(X, *, 0)$ is a BF-algebra.

Definition 2.2. A binary relation "$\leq$" on X can be defined as $x \leq y$ if and only if $x * y = 0$.

Definition 2.3.[3] A non-empty subset S of a BF-algebra X is said to be a subalgebra if $x * y \in S$, $\forall x, y \in S$.

Definition 2.4.[3] A non-empty subset I of a BF-algebra X is said to be an ideal of X if

(i) $0 \in I$.

(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$, $\forall\ x, y \in X$.

Definition 2.5.[1] An ideal I of X is called closed if $0 * x \in I$, $\forall\ x \in I$.

Definition 2.6.[6] A non-empty subset I of a BF-algebra X is said to be a H-ideal of X if

(i) $0 \in I$.

(ii) $x * (y * z) \in I$ and $y \in I \Rightarrow x \in I$, $\forall\ x, y, z \in X$.

Definition 2.7.[6] A H-ideal I of X is called closed if $0 * x \in I$, $\forall\ x \in X$.

Definition 2.8.[11] A non-empty subset A of a BF-algebra X is called a p-ideal of X, if

(i) $0 \in A$.

(ii) $(x * z) * (y * z) \in A$ and $y \in A \ \Rightarrow\ x \in A$, $\forall\ x, y, z \in X$.

Definition 2.9. A p-ideal I of X is called closed if $0 * x \in I$, $\forall\ x \in X$.

Definition 2.10.[2] Let $(L, \leq)$ be a complete lattice with least element 0 and greatest element 1 and an involutive order reversing operation $N : L \to L$. Then an Intuitionistic L-fuzzy subset ($ILFS$) A in a non-empty set X is defined as an object of the form

$$A = \{< x, \mu_A(x), \nu_A(x) > / x \in X\},$$

where $\mu_A : X \to L$ is the degree membership and $\nu_A : X \to L$ is the degree of nonmembership of the element $x \in X$ satisfying $\mu_A(x) \leq N(\nu_A(x))$.

Definition 2.11.[8] An $ILFS$ A in a BF-algebra X with the degree membership $\mu_A : X \to L$ and degree of nonmembership $\nu_A : X \to L$ is said to have Sup-Inf property if for any subset T of X there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t) \text{ and } \nu_A(x_0) = \inf_{t \in T} \nu_A(t).$$

Definition 2.12.[8] Let $f : X \to Y$ be a function and A and B be the $ILFS$ of X and Y respectively where $A = \{< x, \mu_A(x), \nu_A(x) > |x \in X\}$ and $B = \{< x, \mu_B(x), \nu_B(x) > |x \in Y\}$. Then the image of A under f is defined as $f(A) = \left\{< y, \mu_{f(A)}(y), \nu_{f(A)} > |y \in Y\right\}$ such that

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z), & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\nu_{f(A)}(y) = \begin{cases} \inf\limits_{z \in f^{-1}(y)} \nu_A(z), & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.13.[8] Let $f : X \to Y$ be a function and A and B be the intuitionistic L-fuzzy subsets of X and Y respectively such that $A = \{< x, \mu_A(x), \nu_A(x) > |x \in X\}$ and $B = \{< x, \mu_B(x), \nu_B(x) > |x \in Y\}$. Then the inverse image of B under f is defined as $f^{-1}(B) = \left\{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) > |x \in X\right\}$ such that $\mu_{f^{-1}(B)}(x) = \mu_{(B)}(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_{(B)}(f(x)),\ \forall x \in X$.

Definition 2.14.[7] Let $(X, *_X, 0_X), (Y, *_Y, 0_Y)$ be two BF-algebras. The cartesian product of X and Y is defined to be the set

$$X \times Y = \{(x, y)/x \in X,\ y \in Y\}.$$

In $X \times Y$ we define the product $*_{X \times Y}$ as follows:

$$(x_1, y_1) *_{X \times Y} (x_2, y_2) \ = \ (x_1 *_X x_2, y_1 *_Y y_2).$$

One can easily verify that the cartesian product of two BF-algebras is again a BF-algebra.

§3. Intuitionistic L-fuzzy p-ideal

This section introduces the notion of Intuitionistic L-fuzzy p-ideal of a BF-algebra X. Here after X represents a BF-algebra, unless otherwise soecified. We start with,

Definition 3.1. An $ILFS$ A in a BF-algebra X is said to be an Intuitionistic L-fuzzy p-ideal (ILF-p-ideal) of X if

(i) $\mu_A(0) \geq \mu_A(x)$.

(ii) $\nu_A(0) \leq \nu_A(x)$.

(iii) $\mu_A(x) \geq \mu_A((x * z) * (y * z)) \wedge \mu_A(y)$.

(iv) $\nu_A(x) \leq \nu_A((x * z) * (y * z)) \vee \nu_A(y),\ \forall\ x, y, z \in X$.

Example 3.1. Consider the BF-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table given below.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$A = \{< x, \mu_A(x), \nu_A(x) >| \, x \in X\}$ is the $ILFS$ of X defined as

$$\mu_A(x) = \begin{cases} 1 \ , \ x = 0, 1, \\ 0.5 \ , \ x = 2, 3. \end{cases} \text{ and } \nu_A(x) = \begin{cases} 0 \ , \ x = 0, 1, \\ 0.5 \ , \ x = 2, 3. \end{cases}$$

is an ILF-p-ideal of X.

Definition 3.2. An $ILFS$ A in a BF-algebra X is said to be an Intuitionistic L-fuzzy closed p-ideal ($ILFC$-p-ideal) of X if

(i) $\mu_A(x) \geq \mu_A((x * z) * (y * z)) \wedge \mu_A(y)$.

(ii) $\nu_A(x) \leq \nu_A((x * z) * (y * z)) \vee \nu_A(y)$.

(iii) $\mu_A(0 * x) \geq \mu_A(x)$.

(iv) $\nu_A(0 * x) \leq \nu_A(x), \ \forall \ x, y, z \in X$.

Example 3.2. Consider the BF-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table given below.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$A = \{< x, \mu_A(x), \nu_A(x) >| \, x \in X\}$ is the $ILFS$ of X defined as

$$\mu_A(x) = \begin{cases} 0.8 \ , \ x = 0, 1, \\ 0.1 \ , \ x = 2, 3. \end{cases} \text{ and } \nu_A(x) = \begin{cases} 0.1 \ , \ x = 0, 1, \\ 0.5 \ , \ x = 2, 3. \end{cases}$$

is an $ILFC$-p-ideal of X.

Proposition 3.1. Every $ILFC$-p-ideal is an ILF-p-ideal.

Proof. It is clear.

The converse of the above proposition is not true, in general, as seen from the following.

Example 3.3. Consider the BF-algebra $X = \{0, 1, 2, 3\}$ with the Cayley table given below.

$*$	0	1	2	3
0	0	3	0	1
1	1	0	1	3
2	2	3	0	1
3	3	1	3	0

$A = \{< x, \mu_A(x), \nu_A(x) >| \; x \in X\}$ is the $ILFS$ of X defined as

$$\mu_A(x) = \begin{cases} 0.6 \; , \; x = 0, 1, \\ 0.2 \; , \; x = 2, 3. \end{cases} \text{ and } \; \nu_A(x) = \begin{cases} 0.2 \; , \; x = 0, 1, \\ 0.5 \; , \; x = 2, 3. \end{cases}$$

is an ILF-p-ideal of X but not $ILFC$-p-ideal, since $\mu_A(0 * 1) < \mu_A(1)$ and $\nu_A(0 * 1) > \nu_A(1)$.

Proposition 3.2. If A is Intuitionistic L-fuzzy p-ideal of X with $x \leq y$ for any $x, \; y \in X$ then $\mu_A(x) \geq \mu_A(y)$ and $\nu_A(x) \leq \nu_A(y)$.

That is μ_A is order-reversing and ν_A is order-preserving.

Proof. Let $x, \; y, \; z \in X$ such that $x \leq y \leq z$. Then by the partial ordering $\leq$ defined in X, we have $x * y = 0$ and $y * z = 0$.

Thus

$$\begin{aligned} \mu_A(x) &\geq \mu_A((x * y) * (y * z)) \wedge \mu_A(y) \\ &\geq \mu_A((0 * 0) \wedge \mu_A(y)) \\ &= \mu_A(0) \wedge \mu_A(y) \\ &= \mu_A(y). \end{aligned}$$

And

$$\begin{aligned} \nu_A(x) &\leq \nu_A((x * y) * (y * z)) \vee \nu_A(y) \\ &\leq \nu_A((0 * 0) \vee \nu_A(y)) \\ &= \nu_A(0) \vee \nu_A(y) \\ &= \nu_A(y). \end{aligned}$$

This completes the proof.

Proposition 3.3. A is an $ILFS$ of X. A is in ILF-ideal of X is ILF-p-ideal of X if and only if $\mu_A(x) \geq \mu_A((0) * (0 * x))$ and $\nu_A(x) \leq \nu_A((0) * (0 * x))$.

Theorem 3.1. If A is $ILFC$-p-ideal of X, then the sets $J = \{x \in X \; ; \; \mu_A(x) = \mu_A(0)\}$ and $K = \{x \in X \; ; \; \nu_A(x) = \nu_A(0)\}$ are p-ideals of X.

Proof. Clearly $0 \in J$ and $0 \in K$. Hence $J \neq \phi$ and $K \neq \phi$.

Let $(x * y) * (y * z) \in J$ and $y \in J$.

$\Rightarrow \; \mu_A((x * y) * (y * z)) = \mu_A(y) = 0$.

$\Rightarrow \; \mu_A(x) \geq \mu_A((x * y) * (y * z)) \wedge \mu_A(y) = 0 \wedge 0 = 0$.

But $\mu_A(0) \geq \mu_A(x)$.

$\Rightarrow \; \mu_A(x) = 0$.

$\Rightarrow\ x \in J$.

Hence J is p-ideal of X.

Similarly K is p-ideal of X.

Theorem 3.2. Intersection of any two Intuitionistic L-fuzzy p-ideals of X is also an Intuitionistic L-fuzzy p-idea of X.

Proof. Let A and B be any two Intuitionistic L-fuzzy p-ideals of X. Let $A = \{< x, \mu_A(x), \nu_A(x) >| \ x \in X\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >| \ x \in X\}$. Take $C = A \cap B = \{< x, \mu_C(x), \nu_C(x) >| \ x \in X\}$, where $\mu_C(x) = \mu_A(x) \wedge \mu_B(x)$ and $\nu_C(x) = \nu_A(x) \vee \nu_B(x)$. Let $x,\ y \in X$. Now $\mu_C(0) = \mu_A(0) \wedge \mu_B(0) \geq \mu_A(x) \wedge \mu_B(x) = \mu_C(x)$ and $\nu_C(0) = \nu_A(0) \vee \nu_B(0) \leq \nu_A(x) \vee \nu_B(x) = \nu_C(x)$.

$$\begin{aligned}
\mu_C(x) &= \mu_A(x) \wedge \mu_B(x) \\
&\geq (\mu_A((x*z)*(y*z)) \wedge \mu_A(y)) \wedge (\mu_B((x*z)*(y*z)) \wedge \mu_B(y)) \\
&= (\mu_A((x*z)*(y*z)) \wedge \mu_B((x*z)*(y*z))) \wedge (\mu_A(y) \wedge \mu_B(y)) \\
&= \mu_C((x*z)*(y*z)) \wedge \mu_C(y).
\end{aligned}$$

Similarly $\nu_C(x) \leq \nu_C((x*z)*(y*z)) \vee \nu_C(y)$.

This completes the proof.

The above theorem can be generalized as follows.

Theorem 3.3. The intersection of a family of Intuitionistic L-fuzzy p-ideals of X is an Intuitionistic L-fuzzy p-ideal of X.

Analogously we prove the following.

Theorem 3.4. Intersection of any two Intuitionistic L-fuzzy closed p-ideal of X is also an Intuitionistic L-fuzzy closed p-ideal of X and hence the intersection of a family of Intuitionistic L-fuzzy closed p-ideal of X is also an Intuitionistic L-fuzzy closed p-ideal of X.

Theorem 3.5. An $ILFS$ $A = \{< x, \mu_A(x), \nu_A(x) >| \ x \in X\}$ is an ILF-p-ideal of X if and only if the L-fuzzy subsets μ_A and $\bar{\nu_A}$ are L-fuzzy p-ideals of X.

Proof. Let $A = \{< x, \mu_A(x), \nu_A(x) >| \ x \in X\}$ be an ILF-p-ideal of X. Then clearly μ_A is a L-fuzzy p-ideal of X. Now $\bar{\nu_A}(0) = 1 - \nu_A(0) \geq 1 - \nu_A(x) = \bar{\nu_A}(x)$.

And for all $x,\ y,\ z \in X,\ \nu_A(x) \leq \nu_A((x*z)*(y*z)) \vee \nu_A(y)$.

$\Rightarrow\ 1 - \bar{\nu_A}(x) \leq [1 - \bar{\nu_A}((x*z)*(y*z))] \vee [1 - \bar{\nu_A}(y)]$.

$\Rightarrow\ \bar{\nu_A}(x) \geq 1 - \{[1 - \bar{\nu_A}((x*z)*(y*z))] \vee [1 - \bar{\nu_A}(y)]\}$.

$\Rightarrow\ \bar{\nu_A}(x) \geq (1 - \nu_A((x*z)*(y*z))) \wedge (1 - \nu_A(y))$.

$\Rightarrow\ \bar{\nu_A}(x) \geq \bar{\nu_A}((x*z)*(y*z)) \wedge \bar{\nu_A}(y)$.

$\therefore\ \bar{\nu_A}$ is a L-fuzzy p-ideal of X.

Conversely, assume μ_A and $\bar{\nu_A}$ are L-fuzzy p-ideals of X.

Hence to prove $A = \{< x, \mu_A(x), \nu_A(x) >| \ x \in X\}$ is an ILF-p-ideal of X it is enough to prove $\nu_A(0) \leq \nu_A(x)$ and $\nu_A(x) \leq \nu_A((x*z)*(y*z)) \vee \nu_A(y),\ \forall\ x, y, z \in X$.

For, $1 - \nu_A(0) = \bar{\nu_A}(0) \geq \bar{\nu_A}(x) = 1 - \nu_A(x)\ \Rightarrow\ \nu_A(0) \leq \nu_A(x)$.

Also

$$\begin{aligned} 1-\nu_A(x)=\bar{\nu_A}(x) &\geq \bar{\nu_A}((x*z)*(y*z))\wedge\bar{\nu_A}(y) \\ &= (1-\nu_A((x*z)*(y*z)))\wedge(1-\nu_A(y)) \\ &= 1-[\nu_A((x*y)*(y*z))\vee\nu_A(y)]. \end{aligned}$$

$\Rightarrow\ \nu_A(x)\leq\nu_A((x*z)*(y*z))\vee\nu_A(y)),\ \forall\ x,y,z\in X.$

This completes the proof.

Using this theorem we have the following.

Theorem 3.6. An $ILFS$ $A=\{<x,\mu_A(x),\nu_A(x)>|\ x\in X\}$ is an ILF-p-ideal ($ILFC$-p-ideal) of X if and only if

(i) $\Box A=\{<x,\mu_A(x),\bar{\mu_A}(x)>\ \forall\ x\in X\}$ and

(ii) $\Diamond A=\{<x,\bar{\nu_A}(x),\nu_A(x)>\ \forall\ x\in X\}$ are also ILF-p-ideals ($ILFC$-p-ideals) of X.

§4. Homomorphism on intuitionistic L-fuzzy p-ideal

In this section the homomorphic properties of an image and pre-image of an Intuitionistic L-fuzzy p-Ideal of a BF-algebra has been verified.

Definition 4.1.[8] A function $f:X\to Y$ of BF-algebras is said to be homomorphism on X if $f(x*y)=f(x)*f(y),\ \forall\ x,y\in X.$

Remark 4.1. If $f:X\to Y$ is a homomorphism on BF-algebras then $f(0_X)=0_Y$.

Definition 4.2.[8] A function $f:X\to Y$ of BF-algebras is said to be anti-homomorphism on X if $f(x*y)=f(y)*f(x),\ \forall\ x,y\in X.$

Theorem 4.1. Let f be a homomorphism on BF-algebras X onto Y and A be an ILF-p-ideal of X with Sup-Inf property. Then the image of A, $f(A)=\{<y,\mu_{f(A)}(y),\nu_{f(A)}(y)>|\ y\in Y\}$ is an ILF-p-ideal of Y.

Proof. Let $a,\ b,\ c\in Y$ with $x_0\in f^{-1}(a),\ y_0\in f^{-1}(b)$ and $z_0\in f^{-1}(c)$ such that

$$\mu_A(x_0)=\sup_{t\in f^{-1}(a)}\mu_A(t)\ ;\ \mu_A(y_0)=\sup_{t\in f^{-1}(b)}\mu_A(t)\ ;\ \mu_A(z_0)=\sup_{t\in f^{-1}(c)}\mu_A(t)$$

and

$$\nu_A(x_0)=\inf_{t\in f^{-1}(a)}\nu_A(t)\ ;\ \nu_A(y_0)=\inf_{t\in f^{-1}(b)}\nu_A(t)\ ;\ \nu_A(z_0)=\inf_{t\in f^{-1}(c)}\nu_A(t).$$

By Definitions 2.11 and 2.12 we have the following:

$$\mu_{f(A)}(0)=\sup_{t\in f^{-1}(0)}\mu_A(t)\geq\mu_A(0)\geq\mu_A(x_0)=\sup_{t\in f^{-1}(a)}\mu_A(t)=\mu_{f(A)}(a)$$

and

$$\nu_{f(A)}(0)=\inf_{t\in f^{-1}(a)}\nu_A(t)\leq\nu_A(0)\leq\nu_A(x_0)=\inf_{t\in f^{-1}(a)}\nu_A(t)=\nu_{f(A)}(a).$$

Now

$$\begin{aligned}\mu_{f(A)}((a*c)*(b*c)) \wedge \mu_{f(A)}(b) &= \sup_{t\in f^{-1}((a*c)*(b*c))} \mu_A(t) \wedge \sup_{t\in f^{-1}(b)} \mu_A(t) \\ &\leq \mu_A((x_0 * z_0) * (y_0 * z_0)) \wedge \mu_A(y_0) \\ &\leq \mu_A(x_0) \\ &= \sup_{t\in f^{-1}(a)} \mu_A(t) \\ &= \mu_{f(A)}(a).\end{aligned}$$

$$\begin{aligned}\nu_{f(A)}((a*c)*(b*c)) \vee \nu_{f(A)}(b) &= \inf_{t\in f^{-1}((a*c)*(b*c))} \nu_A(t) \vee \inf_{t\in f^{-1}(b)} \nu_A(t) \\ &\geq \nu_A((x_0 * z_0) * (y_0 * z_0)) \vee \nu_A(y_0) \\ &\geq \nu_A(x_0) \\ &= \inf_{t\in f^{-1}(a)} \nu_A(t) \\ &= \nu_{f(A)}(a).\end{aligned}$$

Hence the image $f(A) = \big\{< y, \mu_{f(A)}(y), \nu_{f(A)}(y) >| \ y \in Y\big\}$ is an ILF-p-ideal of Y.

Theorem 4.2. Let f be a homomorphism from BF-algebras X onto Y and A be an $ILFC$-p-ideal of X with Sup-Inf property. Then the image of A, $f(A) = \{< y, \mu_{f(A)}(y), \nu_{f(A)}(y) >| \ y \in Y\}$ is an $ILFC$-p-ideal of Y.

Proof. Let $x \in Y$ with $x_0 \in f^{-1}(x)$ such that

$$\mu_A(x_0) = \sup_{t\in f^{-1}(x)} \mu_A(t) \ ; \ \nu_A(x_0) = \inf_{t\in f^{-1}(x)} \nu_A(t).$$

Then

$$\begin{aligned}\mu_{f(A)}(x) &= \sup_{t\in f^{-1}(x)} \mu_A(t) \leq \mu_A(x_0) \leq \mu_A(0 * x_0) \\ &= \sup_{t\in f^{-1}(0*x)} \mu_A(t) \\ &= \mu_{f(A)}(0 * x)\end{aligned}$$

and

$$\begin{aligned}\nu_{f(A)}(x) &= \inf_{t\in f^{-1}(x)} \nu_A(t) \geq \nu_A(x_0) \geq \nu_A(0 * x_0) \\ &= \inf_{t\in f^{-1}(0*x)} \nu_A(t) \\ &= \nu_{f(A)}(0 * x).\end{aligned}$$

Hence by the above theorem the image $f(A) = \{< y, \mu_{f(A)}(y), \nu_{f(A)}(y) >| \ y \in Y\}$ is an ILF-p-ideal of Y.

Theorem 4.3. Let f be a homomorphism from BF-algebras X onto Y and B be an ILF-p-ideal of Y. Then the inverse image of B, $f^{-1}(B) = \big\{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) >| \ x \in X\big\}$ is an ILF-p-ideal of X.

Proof. Let $x,\ y \in X$. Now it is clear that

$$\mu_{f^{-1}(B)}(0) = \mu_B(f(0)) \geq \mu_B(f(x)) = \mu_{f^{-1}(B)}(x)$$

and

$$\nu_{f^{-1}(B)}(0) = \nu_B(f(0)) \leq \nu_B(f(x)) = \nu_{f^{-1}(B)}(x).$$

Then

$$\begin{aligned} \mu_{f^{-1}(B)}(x) &= \mu_B(f(x)) \\ &\geq \mu_B((f(x) * f(z)) * (f(y) * f(z))) \wedge \mu_B(f(y)) \\ &= \mu_{f^{1}(B)}((x * z) * (y * z)) \wedge \mu_{f^{-1}(B)}(y). \end{aligned}$$

Also

$$\begin{aligned} \nu_{f^{-1}(B)}(x) &= \nu_B(f(x)) \\ &\leq \nu_B((f(x) * f(z)) * (f(y) * f(z))) \vee \nu_B(f(y)) \\ &= \nu_{f^{-1}(B)}((x * z) * (y * z)) \vee \nu_{f^{-1}(B)}(y). \end{aligned}$$

Then the inverse image of B, $f^{-1}(B) = \{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) >|\ x \in X\}$ is an ILF-p-ideal of X.

Theorem 4.4. Let f be a homomorphism from BF-algebras X onto Y and B be an $ILFC$-p-ideal of Y. Then the inverse image of B, $f^{-1}(B) = \{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) >|\ x \in X\}$ is an $ILFC$-p-ideal of X.

Proof. Let $x \in X$. Then

$$\begin{aligned} \mu_{f^{-1}(B)}(0 * x) &= \mu_B(f(0 * x)) \\ &= \mu_B(f(0) * f(x)) \\ &\geq \mu_B(f(x)) \\ &= \mu_{f^{-1}(B)}(x). \end{aligned}$$

Also

$$\begin{aligned} \nu_{f^{-1}(B)}(0 * x) &= \nu_B(f(0 * x)) \\ &= \nu_B(f(0) * f(x)) \\ &\leq \nu_B(f(x)) \\ &= \nu_{f^{-1}(B)}(x). \end{aligned}$$

Hence by the above theorem the inverse image $f^{-1}(B) = \{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) >|\ x \in X\}$ is an $ILFC$-p-ideal of X.

In the similar way we can prove the following:

Theorem 4.5. Let f be an anti-homomorphism from X onto Y and A be an ILF-p-ideal of X with Sup-Inf property. Then the image of A, $f(A) = \{< y, \mu_{f(A)}(y), \nu_{f(A)}(y) >|\ y \in Y\}$ is an ILF-p-ideal of Y.

Theorem 4.6. Let f be an anti-homorphism from X onto Y and B be an ILF-p-ideal of Y. Then the inverse image of B, $f^{-1}(B) = \{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) >| \ x \in X\}$ is an ILF-p-ideal of X.

Theorem 4.7. Let f be an anti-homorphism from X onto Y and A be an $ILFC$-p-ideal of X with Sup-Inf property. Then the image of A, $f(A) = \{< y, \mu_{f(A)}(y), \nu_{f(A)}(y) >| \ y \in Y\}$ is an $ILFC$-p-ideal of Y.

Theorem 4.8. Let f be an anti-homomorphism from X onto Y and B be an $ILFC$-p-ideal of Y. Then the inverse image of B, $f^{-1}(B) = \{< x, \mu_{f^{-1}(B)}(x), \nu_{f^{-1}(B)}(x) >| \ x \in X\}$ is an $ILFC$-p-ideal of X.

§5. Product on intuitionistic L-fuzzy p-ideal

In this section the Cartisean product of two Intuitionistic L-fuzzy p-Ideal has been defined and some results are also proved using the product.

Definition 5.1. Let A and B be any two $ILFS$ of X. The Cartesian product of A and B is defined as $A \times B = (X \times X, \mu_A \times \mu_B, \nu_A \times \nu_B)$ with $(\mu_A \times \mu_B)(x, y) = \mu_A(x) \wedge \mu_B(y)$ and $(\nu_A \times \nu_B)(x, y) = \nu_A(x) \vee \nu_B(y)$ where $\mu_A \times \mu_B : X \times X \to L$ and $\nu_A \times \nu_B : X \times X \to L$, $\forall \ x, y \in X$.

Definition 5.2. Let A and B be any two $ILFS$ of X and Y respectively. The Cartesian product of A and B is defined as $A \times B = (X \times Y, \mu_A \times \mu_B, \nu_A \times \nu_B)$ with $(\mu_A \times \mu_B)(x, y) = \mu_A(x) \wedge \mu_B(y)$ and $(\nu_A \times \nu_B)(x, y) = \nu_A(x) \vee \nu_B(y)$ where $\mu_A \times \mu_B : X \times Y \to L$ and $\nu_A \times \nu_B : X \times Y \to L$, $\forall \ x \in X \ ; \ y \in Y$.

Theorem 5.1. Let A and B be any two Intuitionistic L-fuzzy p-ideals of X and Y respectively. Then $A \times B$ is an Intuitionistic L-fuzzy p-ideal of $X \times Y$.

Proof. Take $(x, y) \in X \times Y$. Then

$$\begin{aligned} (\mu_A \times \mu_B)(0, 0) &= \mu_A(0) \wedge \mu_B(0) \\ &\geq \mu_A(x) \wedge \mu_B(y) \quad \forall \ x \in X \ ; \ y \in Y \\ &= (\mu_A \times \mu_B)(x, y). \end{aligned}$$

And

$$\begin{aligned} (\nu_A \times \nu_B)(0, 0) &= \nu_A(0) \vee \nu_B(0) \\ &\leq \nu_A(x) \vee \nu_B(x) \quad \forall \ x \in X \ ; \ y \in Y \\ &= (\nu_A \times \nu_B)(x, y) \end{aligned}$$

Take (x_1, y_1), (x_2, y_2) and $(x_3, y_3) \in X \times Y$, $\forall \ x_i \in X : y_i \in Y \ ; \ i = 1, 2, 3$. Then

$$\begin{aligned} & (\mu_A \times \mu_B)(x_1, y_1) \\ = \ & \mu_A(x_1) \wedge \mu_B(y_1) \\ \geq \ & (\mu_A((x_1 * x_3) * (x_2 * x_3)) \wedge \mu_A(x_2)) \wedge (\mu_B((y_1 * y_3) * (y_2 * y_3)) \wedge \mu_B(y_2)) \\ \geq \ & (\mu_A((x_1 * x_3) * (x_2 * x_3)) \wedge \mu_B((y_1 * y_3) * (y_2 * y_3))) \wedge (\mu_A(x_2) \wedge \mu_B(y_2)) \\ = \ & (\mu_A \times \mu_B)[((x_1 * x_3) * (x_2 * x_3)), ((y_1 * y_3) * (y_2 * y_3))] \wedge (\mu_A \times \mu_B)(x_2, y_2) \\ = \ & (\mu_A \times \mu_B)[((x_1, y_1) * (x_3, y_3)) * ((x_2, y_2) * (x_3, y_3))] \wedge (\mu_A \times \mu_B)(x_2, y_2). \end{aligned}$$

And

$$\begin{aligned}
& (\nu_A \times \nu_B)(x_1, y_1) \\
= \;& \nu_A(x_1) \vee \nu_B(y_1) \\
\leq \;& (\nu_A((x_1 * x_3) * (x_2 * x_3)) \vee \nu_A(x_2)) \vee (\nu_B((y_1 * y_3) * (y_2 * y_3)) \vee \nu_B(y_2)) \\
\leq \;& (\nu_A((x_1 * x_3) * (x_2 * x_3)) \vee \nu_B((y_1 * y_3) * (y_2 * y_3))) \vee (\nu_A(x_2) \vee \nu_B(y_2)) \\
= \;& (\nu_A \times \nu_B)[((x_1 * x_3) * (x_2 * x_3)), ((y_1 * y_3) * (y_2 * y_3))] \vee (\nu_A \times \nu_B)(x_2, y_2) \\
= \;& (\nu_A \times \nu_B)[((x_1, y_1) * (x_3, y_3)) * ((x_2, y_2) * (x_3, y_3))] \vee (\nu_A \times \nu_B)(x_2, y_2).
\end{aligned}$$

This completes the proof.

Theorem 5.2. Let A and B be any two Intuitionistic L-fuzzy closed p-ideals of X and Y respectively. Then $A \times B$ is an Intuitionistic L-fuzzy closed p-ideal of $X \times Y$.

Proof. For any $(x, y) \in X \times Y$ we have

$$\begin{aligned}
(\mu_A \times \mu_B)((0,0) * (x, y)) &= (\mu_A \times \mu_B)(0 * x, 0 * y) \\
&= \mu_A(0 * x) \wedge \mu_B(0 * y) \\
&\geq \mu_A(x) \wedge \mu_B(y) \\
&= (\mu_A \times \mu_B)(x, y)
\end{aligned}$$

and

$$\begin{aligned}
(\nu_A \times \nu_B)((0,0) * (x, y)) &= (\nu_A \times \nu_B)(0 * x, 0 * y) \\
&= \nu_A(0 * x) \vee \nu_B(0 * y) \\
&\leq \nu_A(x) \vee \nu_B(y) \\
&= (\nu_A \times \nu_B)(x, y).
\end{aligned}$$

Thus $A \times B$ is an Intuitionistic L-fuzzy closed p-ideal of $X \times Y$.

Theorem 5.3. Let A and B be any two Intuitionistic L-fuzzy p-ideals of X and Y. In the Intuitionistic L-fuzzy p-ideal $A \times B$ of $X \times Y$, we have

(i) $\mu_A(0) \geq \mu_B(y)$ and $\mu_B(0) \geq \mu_A(x)$.

(ii) $\nu_A(0) \leq \nu_B(y)$ and $\nu_B(0) \leq \nu_A(x)$, $\forall\ x \in X\ ;\ y \in Y$.

Proof. Assume $\mu_B(y) > \mu_A(0)$ and $\mu_A(x) > \mu_B(0)$ for some $x \in X\ ;\ y \in Y$. Then

$$(\mu_A \times \mu_B)(x, y) = \mu_A(x) \wedge \mu_B(y) \geq \mu_B(0) \wedge \mu_A(0) = (\mu_A \times \mu_B)(0, 0).$$

Which is a $\Rightarrow\Leftarrow$.

Similarly,assume $\nu_A(y) < \nu_B(0)$ and $\nu_B(x) < \nu_A(0)$ for some $x \in X\ ;\ y \in Y$. Then

$$(\nu_A \times \nu_B)(x, y) = \nu_A(x) \vee \nu_B(y) \leq \nu_B(0) \vee \nu_A(0) = (\nu_A \times \nu_B)(0, 0).$$

Which is a $\Rightarrow\Leftarrow$.

Hence proved.

Theorem 5.4. If $A \times B$ is an Intuitionistic L-fuzzy p-ideal of $X \times Y$, then either A is Intuitionistic L-fuzzy p-ideal of X or B is Intuitionmistic L-fuzzy p-ideal of Y.

Proof. Now by above theorem if we take $\mu_A(0) \geq \mu_B(y)$ and $\nu_A(0) \leq \nu_B(y)$ then

$$(\mu_A \times \mu_B)(0, y) = \mu_A(0) \wedge \mu_B(y) = \mu_B(y)$$

and

$$(\nu_A \times \nu_B)(0, y) = \nu_A(0) \vee \nu_B(y) = \nu_B(y). \tag{1}$$

Since $A \times B$ is an Intuitionistic L-fuzzy p-ideal of $X \times Y$,

$$\begin{aligned} & (\mu_A \times \mu_B)(x_1, y_1) \\ \geq \; & (\mu_A \times \mu_B)[((x_1, y_1) * (x_3, y_3)) * ((x_2, y_2) * (x_3, y_3))] \wedge (\mu_A \times \mu_B)(x_2, y_2). \end{aligned} \tag{2}$$

Putting $x_1 = x_2 = x_3 = 0$ in (2) we get,

$$\begin{aligned} & (\mu_A \times \mu_B)(0, y_1) \\ \geq \; & (\mu_A \times \mu_B)[((0, y_1) * (0, y_3)) * ((0, y_2) * (0, y_3))] \wedge (\mu_A \times \mu_B)(0, y_2). \\ & (\mu_A \times \mu_B)(0, y_1) \\ \geq \; & (\mu_A \times \mu_B)[0, ((y_1 * y_3) * (y_2 * y_3))] \wedge (\mu_A \times \mu_B)(0, y_2). \end{aligned} \tag{3}$$

Using equation (1) in (3), we have,

$$\mu_B(y_1) \geq \mu_B((y_1 * y_3) * (y_2 * y_3)) \wedge \mu_B(y_2).$$

In the similar way we can prove

$$\nu_B(y_1) \leq \nu_B((y_1 * y_3) * (y_2 * y_3)) \vee \nu_B(y_2).$$

This proves B is Intuitionistic L-fuzzy p-ideal of Y.

This completes the proof.

Theorem 5.5. For any Intuitionistic L-fuzzy p-ideals A and B of X and Y respectively. $A \times B$ is an Intuitionistic L-fuzzy p-ideal of $X \times Y$ if and only if $(\mu_A \times \mu_B)(x, y)$ and $(\nu_A \bar{\times} \nu_B)(x, y)$ are L-fuzzy p-ideals of $X \times Y$.

Proof. Let $A \times B$ is an Intuitionistic L-fuzzy p-ideal of $X \times Y$.

Clearly $(\mu_A \times \mu_B)(x, y) = \mu_A(x) \wedge \mu_B(y)$ is L-fuzzy p-ideal of $X \times Y$.

We have $(\nu_A \times \nu_B)(x, y) = \nu_A(x) \vee \nu_B(y)$.

$\Rightarrow$ $1 - (\bar{\nu_A} \times \bar{\nu_B})(x, y) = (1 - \bar{\nu_A}(x)) \vee (1 - \bar{\nu_B}(y))$.

$\Rightarrow$ $1 - \{(1 - \bar{\nu_A}(x)) \vee (1 - \bar{\nu_B}(y))\} = (\bar{\nu_A} \times \bar{\nu_B})(x, y)$.

$\Rightarrow$ $(\bar{\nu_A} \times \bar{\nu_B})(x, y) = \bar{\nu_A}(x) \wedge \bar{\nu_B}(y)$ is L-fuzzy p-ideal of $X \times Y$.

Conversely, assume $(\mu_A \times \mu_B)(x, y)$ and $(\nu_A \bar{\times} \nu_B)(x, y)$ are L-fuzzy p-ideals of $X \times Y$.

Now $A \times B = (X \times Y, \mu_A \times \mu_B, \nu_A \times \nu_B)$.

Since $(\bar{\nu_A} \times \bar{\nu_B})(x, y) = \bar{\nu_A}(x) \wedge \bar{\nu_B}(y) \;\Rightarrow\; (\nu_A \times \nu_B)(x, y) = \nu_A(x) \vee \nu_B(y)$.

We can easily observe that $A \times B$ is an Intuitionistic L-p-ideal of $X \times Y$.

Theorem 5.6. For any $ILFS$ A and B, A and B are Intuitionistic L-fuzzy p-ideals of X and Y respectively, if and only if

(i) $\Box(A \times B) = (X \times X, \mu_A \times \mu_B, \bar{\mu_A} \times \bar{\mu_B})$ and

(ii) $\Diamond(A \times B) = (X \times X, \bar{\nu_A} \times \bar{\nu_B}, \nu_A \times \nu_B)$ are Intuitionistic L-fuzzy p-ideals of $X \times Y$.

Proof. Since $(\mu_A \times \mu_B)(x, y) = \mu_A(x) \wedge \mu_B(y) \;\Rightarrow\; (\bar{\mu_A} \times \bar{\mu_B})(x, y) = \bar{\mu_A}(x) \vee \bar{\mu_B}(y)$ and $(\nu_A \times \nu_B)(x, y) = \nu_A(x) \vee \nu_B(y) \Rightarrow\; (\bar{\nu_A} \times \bar{\nu_B})(x, y) = \bar{\nu_A}(x) \wedge \bar{\nu_B}(y)$, the proof is clear.

§6. Conclusion

In this article, we have introduced the notion of Intuitionistic L-fuzzy p-ideal of BF-Algebras. In [1], it is proved that if $(A, *, 0)$ is a BF-algebra, than A is a BG-algebra. Hence it is clear that all the results proved in this paper for a BF-algebra are valid for a BG-algebra.

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Some results on Smarandache groupoids

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Abstract In this paper we prove some results towards classifying Smarandache groupoids which are in $Z^*(n)$ and not in $Z(n)$ when n is even and n is odd.

Keywords Groupoids, Smarandache groupoids.

§1. Introduction and preliminaries

In [3] and [4], W. B. Kandasamy defined new classes of Smarandache groupoids using Z_n. In this paper we prove some theorems for construction of Smarandache groupoids according as n is even or odd.

Definition 1.1. A non-empty set of elements G is said to form a groupoid if in G is defined a binary operation called the product denoted by $*$ such that $a * b \in G$, $\forall\ a,\ b \in G$.

Definition 1.2. Let S be a non-empty set. S is said to be a semigroup if on S is defined a binary operation $*$ such that

(i) for all $a,\ b \in S$ we have $a * b \in S$ (closure).

(ii) for all $a,\ b,\ c \in S$ we have $a * (b * c) = (a * b) * c$ (associative law).

$(S, *)$ is a semi-group.

Definition 1.3. A Smarandache groupoid G is a groupoid which has a proper subset $S \subset G$ which is a semi-group under the operation of G.

Example 1.1. Let $(G, *)$ be a groupoid on the set of integer modulo 6, given by the following table.

*	0	1	2	3	4	5
0	0	5	0	5	0	5
1	1	3	1	3	1	3
2	2	4	2	4	2	4
3	3	1	3	1	3	1
4	4	2	4	2	4	2
5	5	0	5	0	5	0

Here, $\{0,5\},\{1,3\},\{2,4\}$ are proper subsets of G which are semigroups under $*$.

Definition 1.4. Let $Z_n = \{0,1,2,\cdots,n-1\}$, $n \geq 3$. For $a,b \in Z_n\backslash\{0\}$ define a binary operation $*$ on Z_n as: $a*b = ta+ub \pmod n$ where t,u are 2 distinct elements in $Z_n\backslash\{0\}$ and $(t,u)=1$. Here "$+$" is the usual addition of two integers and "ta" mean the product of the two integers t and a.

Elements of Z_n form a groupiod with respect to the binary operation. We denote these groupoids by $\{Z_n(t,u),*\}$ or $Z_n(t,u)$ for fixed integer n and varying $t,u \in Z_n\backslash\{0\}$ such that $(t,u)=1$. Thus we define a collection of groupoids $Z(n)$ as follows

$$Z(n) = \{Z_n(t,u),*| \text{ for integers } t,u \in Z_n\backslash\{0\} \text{ such that } (t,u)=1\}.$$

Definition 1.5. Let $Z_n = \{0,1,2,\cdots,n-1\}$, $n \geq 3$. For a, $b \in Z_n\backslash\{0\}$, define a binary operation $*$ on Z_n as: $a*b = ta+ub \pmod n$ where t, u are two distinct elements in $Z_n\backslash\{0\}$ and t and u need not always be relatively prime but $t \neq u$. Here "$+$" is usual addition of two integers and "ta" means the product of two integers t and a.

For fixed integer n and varying $t,u \in Z_n\backslash\{0\}$ s.t $t \neq u$ we get a collection of groupoids $Z^*(n)$ as: $Z^*(n) = \{Z_n(t,u),*| \text{ for integers } t,u \in Z_n\backslash\{0\} \text{ such that } t \neq u\}$.

Remarks 1.1. (i) Clearly, $Z(n) \subset Z^*(n)$.

(ii) $Z^*(n)\backslash Z(n) = \Phi$ for $n = p+1$ for prime $p = 2,\ 3$.

(iii) $Z^*(n)\backslash Z(n) \neq \Phi$ for $n \neq p+1$ for prime p.

We are interested in Smarandache Groupoids which are in $Z^*(n)$ and not in $Z(n)$ i.e., $Z^*(n)\backslash Z(n)$.

§2. Smarandache groupoids when n is even

Theorem 2.1. Let $Z_n(t,lt) \in Z^*(n)\backslash Z(n)$. If n is even, $n > 4$ and for each $t = 2,3,\cdots,\frac{n}{2}-1$ and $l = 2,3,4,\cdots$ such that $lt < n$, then $Z_n(t,lt)$ is Smarandache groupoid.

Proof. Let $x = \frac{n}{2}$.

Case 1. t is even.

$x*x = xt + ltx = (l+1)tx \equiv 0 \bmod n$.

$x*0 = xt \equiv 0 \bmod n$.

$0*x = lxt \equiv 0 \bmod n$.

$0*0 = 0 \bmod n$.

$\therefore$ $\{0,x\}$ is semigroup in $Z_n(t,lt)$.

$\therefore$ $Z_n(t,lt)$ is Smarandache groupoid when t is even.

Case 2. t is odd.

(a) If l is even.

$x*x = xt + ltx = (l+1)tx \equiv x \bmod n$.

$\{x\}$ is semigroup in $Z_n(t,lt)$.

$\therefore$ $Z_n(t,lt)$ is Smarandache groupoid when t is odd and l is even.

(b) If l is odd then $(l+1)$ is even.
$x * x = xt + ltx = (l+1)tx \equiv 0 \bmod n$.
$x * 0 = xt \equiv x \bmod n$.
$0 * x = ltx \equiv x \bmod n$.
$0 * 0 \equiv 0 \bmod n$.
$\Rightarrow \{0, x\}$ is semigroup in $Z_n(t, lt)$.
$\therefore Z_n(t, lt)$ is Smarandache groupoid when t is odd and l is odd.

Theorem 2.2. Let $Z_n(t, u) \in Z^*(n) \backslash Z(n)$, n is even $n > 4$ where $(t, u) = r$ and $r \neq t, u$ then $Z_n(t, u)$ is Smarandache groupoid.

Proof. Let $x = \frac{n}{2}$.

Case 1. Let r be even i.e t and u are even.
$x * x = tx + ux = (t+u)x \equiv 0 \bmod n$.
$0 * x = ux \equiv 0 \bmod n$.
$x * 0 = tx \equiv 0 \bmod n$.
$0 * 0 = 0 \bmod n$.
$\{0, x\}$ is semigroup in $Z_n(t, lt)$.
$\therefore Z_n(t, lt)$ is Smarandache groupoid when t is even and u is even.

Case 2. Let r be odd.

(a) when t is odd and u is odd,
$\Rightarrow t + u$ is even.
$x * x = tx + ux = (t+u)x \equiv 0 \bmod n$.
$x * 0 = tx \equiv x \bmod n$.
$0 * x \equiv ux \equiv x \bmod n$.
$0 * 0 \equiv 0 \bmod n$.
$\{0, x\}$ is a semigroup in $Z_n(t, u)$.
$\therefore Z_n(t, u)$ is Smarandache groupoid when t is odd and u is odd.

(b) when t is odd and u is even,
$\Rightarrow t + u$ is odd.
$x * x = tx + ux = (t+u)x \equiv x \bmod n$.
$\{x\}$ is a semigroup in $Z_n(t, u)$.
$\therefore Z_n(t, u)$ is Smarandache groupoid when t is odd and u is even.

(c) when t is even and u is odd,
$\Rightarrow t + u$ is odd.
$x * x = tx + ux = (t+u)x \equiv x \bmod n$.
$\{x\}$ is a semigroup in $Z_n(t, u)$.
$\therefore Z_n(t, u)$ is Smarandache groupoid when t is even and u is odd.

By the above two theorems we can determine Smarandache groupoids in $Z^*(n) \backslash Z(n)$ when n is even and $n > 4$.

We find Smarandache groupoids in $Z^*(n) \backslash Z(n)$ for $n = 22$ by Theorem 2.1.

t	$\boldsymbol{l}$	$\boldsymbol{lt < 22}$	$\boldsymbol{Z_n(t, lt)}$	**Proper subset which is semigroup**	**Smarandache groupoid in** $\boldsymbol{Z^*(n)\backslash Z(n)}$
	2	4	$Z_{22}(2,4)$	$\{0,11\}$	$Z_{22}(2,4)$
	3	6	$Z_{22}(2,6)$	$\{0,11\}$	$Z_{22}(2,6)$
	4	8	$Z_{22}(2,8)$	$\{0,11\}$	$Z_{22}(2,8)$
2	5	10	$Z_{22}(2,10)$	$\{0,11\}$	$Z_{22}(2,10)$
	6	12	$Z_{22}(2,12)$	$\{0,11\}$	$Z_{22}(2,12)$
	7	14	$Z_{22}(2,14)$	$\{0,11\}$	$Z_{22}(2,14)$
	8	16	$Z_{22}(2,16)$	$\{0,11\}$	$Z_{22}(2,16)$
	9	18	$Z_{22}(2,18)$	$\{0,11\}$	$Z_{22}(2,18)$
	10	20	$Z_{22}(2,20)$	$\{0,11\}$	$Z_{22}(2,20)$
	2	6	$Z_{22}(3,6)$	$\{11\}$	$Z_{22}(3,6)$
	3	9	$Z_{22}(3,9)$	$\{0,11\}$	$Z_{22}(3,9)$
3	4	12	$Z_{22}(3,12)$	$\{11\}$	$Z_{22}(3,12)$
	5	15	$Z_{22}(3,15)$	$\{0,11\}$	$Z_{22}(3,15)$
	6	18	$Z_{22}(3,18)$	$\{11\}$	$Z_{22}(3,18)$
	7	21	$Z_{22}(3,21)$	$\{0,11\}$	$Z_{22}(3,21)$
	2	8	$Z_{22}(4,8)$	$\{0,11\}$	$Z_{22}(4,8)$
4	3	12	$Z_{22}(4,12)$	$\{0,11\}$	$Z_{22}(4,12)$
	4	16	$Z_{22}(4,16)$	$\{0,11\}$	$Z_{22}(4,16)$
	5	20	$Z_{22}(4,20)$	$\{0,11\}$	$Z_{22}(4,20)$
	2	10	$Z_{22}(5,10)$	$\{11\}$	$Z_{22}(5,10)$
5	3	15	$Z_{22}(5,15)$	$\{0,11\}$	$Z_{22}(5,15)$
	4	20	$Z_{22}(5,20)$	$\{11\}$	$Z_{22}(5,20)$
	2	12	$Z_{22}(6,12)$	$\{0,11\}$	$Z_{22}(6,12)$
6	3	18	$Z_{22}(6,18)$	$\{0,11\}$	$Z_{22}(6,18)$
	2	14	$Z_{22}(7,14)$	$\{11\}$	$Z_{22}(7,14)$
7	3	21	$Z_{22}(7,21)$	$\{0,11\}$	$Z_{22}(7,21)$
8	2	16	$Z_{22}(8,16)$	$\{0,11\}$	$Z_{22}(8,16)$
9	2	18	$Z_{22}(9,18)$	$\{11\}$	$Z_{22}(9,18)$
10	2	20	$Z_{22}(10,20)$	$\{0,11\}$	$Z_{22}(10,20)$

Next, we find Smarandache groupoids in $Z^*(n)\backslash Z(n)$ for $n = 22$ by Theorem 2.2.

t	u	$(t,u)=r$ $r\neq t,u$	$Z_n(t,u)$	**Proper subset which is semigroup**	**Smarandache groupoid in**$Z^*(n)\backslash Z(n)$
	6	(4,6)=2	$Z_{22}(4,6)$	$\{0,11\}$	$Z_{22}(4,6)$
4	10	(4,10=2	$Z_{22}(4,10)$	$\{0,11\}$	$Z_{22}(4,10)$
	14	(4,14)=2	$Z_{22}(4,14)$	$\{0,11\}$	$Z_{22}(4,14)$
	18	(4,18)=2	$Z_{22}(4,18)$	$\{0,11\}$	$Z_{22}(4,18)$
	8	(6,8)=2	$Z_{22}(6,8)$	$\{0,11\}$	$Z_{22}(6,8)$
	9	(6,9)=3	$Z_{22}(6,9)$	$\{11\}$	$Z_{22}(6,9)$
	10	(6,10)=2	$Z_{22}(6,10)$	$\{0,11\}$	$Z_{22}(6,10)$
6	14	(6,14)=2	$Z_{22}(6,14)$	$\{0,11\}$	$Z_{22}(6,14)$
	16	(6,16)=2	$Z_{22}(6,16)$	$\{0,11\}$	$Z_{22}(6,16)$
	20	(6,20)=2	$Z_{22}(6,20)$	$\{0,11\}$	$Z_{22}(6,20)$
	21	(6,21)=3	$Z_{22}(6,21)$	$\{11\}$	$Z_{22}(6,21)$
	10	(8,10)=2	$Z_{22}(8,10)$	$\{0,11\}$	$Z_{22}(8,10)$
	12	(8,12)=4	$Z_{22}(8,12)$	$\{0,11\}$	$Z_{22}(8,12)$
8	14	(8,14)=2	$Z_{22}(8,14)$	$\{0,11\}$	$Z_{22}(8,14)$
	18	(8,18)=2	$Z_{22}(8,18)$	$\{0,11\}$	$Z_{22}(8,18)$
	20	(8,20)=4	$Z_{22}(8,20)$	$\{0,11\}$	$Z_{22}(8,20)$
9	21	(9,21)=3	$Z_{22}(9,21)$	$\{0,11\}$	$Z_{22}(9,21)$
	12	(10,12)=2	$Z_{22}(10,12)$	$\{0,11\}$	$Z_{22}(10,12)$
	14	(10,14)=2	$Z_{22}(10,14)$	$\{0,11\}$	$Z_{22}(10,14)$
10	16	(10,16)=2	$Z_{22}(10,16)$	$\{0,11\}$	$Z_{22}(10,16)$
	18	(10,18)=2	$Z_{22}(10,18)$	$\{0,11\}$	$Z_{22}(10,19)$
	14	(12,14)=2	$Z_{22}(12,14)$	$\{0,11\}$	$Z_{22}(12,14)$
	15	(12,15)=3	$Z_{22}(12,15)$	$\{11\}$	$Z_{22}(12,15)$
12	16	(12,16)=4	$Z_{22}(12,16)$	$\{0,11\}$	$Z_{22}(12,16)$
	18	(12,18)=6	$Z_{22}(12,18)$	$\{0,11\}$	$Z_{22}(12,18)$
	20	(12,20)=4	$Z_{22}(12,20)$	$\{0,11\}$	$Z_{22}(12,20)$
	21	(12,21)=3	$Z_{22}(12,21)$	$\{11\}$	$Z_{22}(12,21)$
	16	(14,16)=2	$Z_{22}(14,16)$	$\{0,11\}$	$Z_{22}(14,16)$
	18	(14,18)=2	$Z_{22}(14,18)$	$\{0,11\}$	$Z_{22}(14,18)$
14	20	(14,20)=2	$Z_{22}(14,20)$	$\{0,11\}$	$Z_{22}(14,20)$
	21	(14,21)=7	$Z_{22}(14,21)$	$\{11\}$	$Z_{22}(14,21)$
15	20	(15,20)=5	$Z_{22}(15,20)$	$\{11\}$	$Z_{22}(15,20)$
	18	(16,18)=2	$Z_{22}(16,18)$	$\{0,11\}$	$Z_{22}(16,18)$
16	20	(16,20)=4	$Z_{22}(16,20)$	$\{0,11\}$	$Z_{22}(16,20)$
18	20	(18,20)=2	$Z_{22}(18,20)$	$\{0,11\}$	$Z_{22}(18,20)$

§3. Smarandache groupoids when n is odd

Theorem 3.1. Let $Z_n(t,u) \in Z^*(n)\backslash Z(n)$. If n is odd, $n > 4$ and for each $t = 2, \cdots, \frac{n-1}{2}$, and $u = n - (t-1)$ such that $(t,u) = r$ then $Z_n(t,u)$ is Smarandache groupoid.

Proof. Let $x \in \{0, \cdots, n-1\}$.

$$x * x = xt + xu = (n+1)x \equiv x \bmod n.$$

$\therefore$ $\{x\}$ is semigroup in Z_n.
$\therefore$ $Z_n(t,u)$ is Smarandanche groupoid.

By the above theorem we can determine the Smarandache groupoids in $Z^*(n)\backslash Z(n)$ when n is odd and $n > 4$.

Also we note that all $\{x\}$ where $x \in \{0, \cdots, n-1\}$ are proper subsets which are semigroups in $Z_n(t,u)$.

Let us consider the examples when n is odd. We will find the Smarandache groupoids in $Z^*(n)\backslash Z(n)$ by Theorem 3.1.

n	**t**	$u = n - (t-1)$	$(t,u) = r$	$Z_n(t,u)$ **Smarandache groupoid (S.G.) in** $Z^*(n)\backslash Z(n)$
5	2	4	$(2,4) = 2$	$Z_5(2,4)$ is S.G. in $Z^*(5)\backslash Z(5)$
7	2	6	$(2,6) = 3$	$Z_7(2,6)$ is S.G. in $Z^*(7)\backslash Z(7)$
9	2	8	$(2,8) = 2$	$Z_9(2,8)$ is S.G. in $Z^*(9)\backslash Z(9)$
	4	6	$(4,6) = 2$	$Z_9(4,6)$ is S.G. in $Z^*(9)\backslash Z(9)$
	2	10	$(2,10) = 2$	$Z_{11}(2,10)$ is S.G. in $Z^*(11)\backslash Z(11)$
11	3	9	$(3,9) = 3$	$Z_{11}(3,9)$ is S.G. in $Z^*(11)\backslash Z(11)$
	4	8	$(4,8) = 4$	$Z_{11}(4,8)$ is S.G. in $Z^*(11)\backslash Z(11)$
	2	12	$(2,12) = 2$	$Z_{13}(2,12)$ is S.G. in $Z^*(13)\backslash Z(13)$
13	4	10	$(4,10) = 2$	$Z_{13}(4,10)$ is S.G. in $Z^*(13)\backslash Z(13)$
	6	8	$(6,8) = 2$	$Z_{13}(6,8)$ is S.G. in $Z^*(13)\backslash Z(13)$
	2	14	$(2,14) = 2$	$Z_{15}(2,14)$ is S.G. in $Z^*(15)\backslash Z(15)$
15	4	12	$(4,12) = 4$	$Z_{15}(4,12)$ is S.G. in $Z^*(15)\backslash Z(15)$
	6	10	$(6,10) = 2$	$Z_{15}(6,10)$ is S.G. in $Z^*(15)\backslash Z(15)$
	2	16	$(2,16) = 2$	$Z_{17}(2,16)$ is S.G. in $Z^*(17)\backslash Z(17)$
	3	15	$(3,15) = 3$	$Z_{17}(3,15)$ is S.G. in $Z^*(17)\backslash Z(17)$
17	4	14	$(4,14) = 2$	$Z_{17}(4,14)$ is S.G. in $Z^*(17)\backslash Z(17)$
	6	12	$(6,12) = 6$	$Z_{17}(6,12)$ is S.G. in $Z^*(17)\backslash Z(17)$
	8	10	$(8,10) = 2$	$Z_{17}(8,10)$ is S.G. in $Z^*(17)\backslash Z(17)$

n	t	$u = n - (t - 1)$	$(t, u) = r$	$Z_n(t, u)$ Smarandache groupoid (S.G.) in $Z^*(n)\backslash Z(n)$
19	2	18	$(2, 18) = 2$	$Z_{19}(2, 18)$ is S.G. in $Z^*(19)\backslash Z(19)$
	4	16	$(4, 16) = 4$	$Z_{19}(4, 16)$ is S.G. in $Z^*(19)\backslash Z(19)$
	5	15	$(5, 15) = 5$	$Z_{19}(5, 15)$ is S.G. in $Z^*(19)\backslash Z(19)$
	6	14	$(6, 14) = 2$	$Z_{19}(6, 14)$ is S.G. in $Z^*(19)\backslash Z(19)$
	8	12	$(8, 12) = 4$	$Z_{19}(8, 12)$ is S.G. in $Z^*(19)\backslash Z(19)$
21	2	20	$(2, 20) = 2$	$Z_{21}(2, 20)$ is S.G. in $Z^*(21)\backslash Z(21)$
	4	18	$(4, 18) = 2$	$Z_{21}(4, 18)$ is S.G. in $Z^*(21)\backslash Z(21)$
	6	16	$(6, 16) = 2$	$Z_{21}(6, 16)$ is S.G. in $Z^*(21)\backslash Z(21)$
	8	14	$(8, 14) = 2$	$Z_{21}(8, 14)$ is S.G. in $Z^*(21)\backslash Z(21)$
	10	12	$(10, 12) = 2$	$Z_{21}(10, 12)$ is S.G. in $Z^*(21)\backslash Z(21)$

Open Problems:

1. Let n be a composite number. Are all groupoids in $Z^*(n)\backslash Z(n)$ Smarandache groupoids?
2. Which class will have more number of Smarandache groupoids in $Z^*(n)\backslash Z(n)$?
   (a) When $n + 1$ is prime.
   (b) When n is prime.

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On weakly S-quasinormal subgroups of finite groups[1]

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Abstract In this paper, we investigate the influence of E-supplemented and SS-quasinormal subgroups on the structure of finite groups. Some recent results are generalized and unified.

Keywords E-supplemented, SS-quasinormal, p-nilpotent, supersoluble.

§1. Introduction

All groups considered in this article are finite. A subgroup H of a group G is said to be S-quasinormal in G if H permutes with every Sylow subgroups of G. This concept was introduced by Kegel in [5]. There has been many generalizations of S-quasinormal subgroups in the literature.

Definition 1.1.[8] A subgroup H of a group G is said to be an SS-quasinormal subgroup of G if there is a subgroup B such that $G = HB$ and H permutes with every Sylow subgroup of B.

Definition 1.2.[1] A subgroup H of G is said to be S-quasinormally embedded in G if for each prime p dividing $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G.

In 2012, C. Li proposed the following concept which covers properly both S-quasinormally embedding property and Skiba's weakly S-supplementation (see [15]).

Definition 1.3.[6] A subgroup H is said to be E-supplemented in G if there is a subgroup K of G such that $G = HK$ and $H \cap K \leq H_{eG}$, where H_{eG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormally embedded in G.

There are examples to show that E-supplemented subgroups are not SS-quasinormal subgroups and in general the converse is also false. The aim of this article is to prove Theorem 3.2. As its applications, some known results are generalized and unified.

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§2. Preliminaries

Lemma 2.1.[6] Let H be a E-supplemented subgroup of a group G.

(i) If $H \leq L \leq G$, then H is E-supplemented in L.

(ii) If $N \lhd G$ and $N \leq H \leq G$, then H/N is E-supplemented in G/N.

(iii) If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is E-supplemented in G/N.

Lemma 2.2.[8] Let H be an ss-quasinormal subgroup of a group G.

(i) If $H \leq L \leq G$, then H is ss-quasinormal in L.

(ii) If $N \trianglelefteq G$, then HN/N is ss-quasinormal in G/N.

Lemma 2.3.[3] If H is a subgroup of G with $|G : H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.

Lemma 2.4.[3] Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent for some prime p. Then

(i) G has a normal Sylow p-subgroup P and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(iii) The exponent of P is p or 4.

Lemma 2.5.[13] Suppose that P is a p-subgroup of G contained in $O_p(G)$. If P is s-quasinormally embedded in G, then P is s-quasinormal in G.

Lemma 2.6.[4] Let G be a group and $N \leq G$.

(i) If $N \trianglelefteq G$, then $F^*(N) \leq F^*(G)$.

(ii) If $G \neq 1$, then $F^*(G) \neq 1$. In fact, $F^*(G)/F(G) = \mathrm{Soc}\ (F(G)C_G(F(G))/F(G))$.

(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is soluble, then $F^*(G) = F(G)$.

§3. Main results

Theorem 3.1. Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is either E-supplemented or SS-quasinormal in G, then G is p-nilpotent.

Proof. Let H be a maximal subgroup of P. We will prove H is E-supplemented in G.

If H is SS-quasinormal in G, then there is a subgroup $B \leq G$ such that $G = HB$ and $HX = XH$ for all $X \in \mathrm{Syl}(B)$. From $G = HB$, we obtain $|B : H \cap B|_p = |G : H|_p = p$, and hence $H \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $H \cap B$. Thus $S \nsubseteq H$ for all $S \in \mathrm{Syl}_p(B)$ and $HS = SH$ is a Sylow p-subgroup of G. In view of $|P : H| = p$ and by comparison of orders, $S \cap H = B \cap H$, for all $S \in \mathrm{Syl}_p(B)$. Therefore

$$B \cap H = \bigcap_{b \in B} (S^b \cap H) = \leq \bigcap_{b \in B} S^b = O_p(B).$$

We claim that B has a Hall p'-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. Since p is the smallest prime dividing $|G|$, we have $B/O_p(B)$ is p-nilpotent, and hence B is p-soluble. So B has a Hall p'-subgroup. Thus the claim holds.

Now, let K be a p'-subgroup of B, $\pi(K) = \{p_2, \cdots, p_s\}$ and $P_i \in \mathrm{Syl}_{p_i}(K)$. By the condition, H permutes with every P_i and so H permutes with the subgroup $< P_2, \cdots, P_s >= K$. Thus $HK \leq G$. Obviously, K is a Hall p'-subgroup of G and HK is a subgroup of index p in G. Let $M = HK$ and so $M \trianglelefteq G$ by Lemma 2.3. It follows that H is normally embedded in G, and so E-supplemented in G.

Since every maximal subgroup of P is E-supplemented in G, we have G is p-nilpotent by [6, Theorem 3.2].

Corollary 3.1. Let p be the smallest prime dividing the order of a group G and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either E-supplemented or SS-quasinormal in G, then G is p-nilpotent.

Proof. By Lemmas 2.1 and 2.2, every maximal subgroup of P is either E-supplemented or SS-quasinormal in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \lhd G$. If $H_{p'} \neq 1$, then we consider $G/H_{p'}$. It is easy to see that $G/H_{p'}$ satisfies all the hypotheses of our corollary for the normal subgroup $H/H_{p'}$ of $G/H_{p'}$ by Lemmas 2.1 and 2.2. Now by induction, we see that $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, we may let K/H be the normal p-complement of G/H. By Schur–Zassenhaus Theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. By Theorem 3.1, K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G, i.e., G is p-nilpotent.

Corollary 3.2. Suppose that every maximal subgroup of any Sylow subgroup of a group G is either E-supplemented or SS-quasinormal in G. Then G is a Sylow tower group of supersoluble type.

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. By Theorem 3.1, G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 2.1 and 2.2, U satisfies the hypothesis of the corollary. It follows by induction that U, and hence G is a Sylow tower group of supersoluble type.

Theorem 3.2. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either E-supplemented or SS-quasinormald in G.

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

By Lemmas 2.1 and 2.2, every maximal subgroup of any Sylow subgroup of H is either E-supplemented or SS-quasinormald in H. By Corollary 3.2, H is a Sylow tower group of supersoluble type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. We consider G/P. From Lemmas 2.1 and 2.2, it is easy to see that $(G/P, H/P)$ satisfies the hypothesis of the Theorem. By the minimality of G, we have $G/P \in \mathcal{F}$. If the maximal P_1 of P is SS-quasinormal in G, then P_1 is S-quasinormal in G by Lemma 2.5. Thus every maximal subgroup of P is E-supplemented in G. Applying Theorem [6, Theorem 4.1], $G \in \mathcal{F}$, a contradiction.

Corollary 3.3. If every maximal subgroup of any Sylow subgroup of a group G is either E-supplemented or SS-quasinormal in G, then G is supersoluble.

Theorem 3.3. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. If there is a normal subgroup H of a group G such that $G/H \in \mathcal{F}$ and every cyclic subgroup of H with prime order or order 4 is either E-supplemented or SS-quasinormald in G, then $G \in \mathcal{F}$.

Proof. Suppose that the assertion is false and let (G, H) be a counterexample for which $|G||H|$ is minimal. Let K be any proper subgroup of H. By Lemmas 2.1 and 2.2, the hypothesis of the theorem still holds for (K, K). By the choice of G, K is supersoluble. By [2, Theorem 3.11.9], H is soluble. Since $G/H \in \mathcal{F}$, $G^{\mathcal{F}} \leq H$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nsubseteq M$ (that is, M is an $\mathcal{F}$-abnormal maximal subgroup of G). Then $G = MH$. We claim that the hypothesis holds for $(M, M \cap H)$. In fact, $M/M \cap H \cong MH/H = G/H \in \mathcal{F}$. Let $< x >$ be any cyclic subgroup of $M \cap H$ with prime order or order 4. It is clear that $< x >$ is also a cyclic subgroup of H with prime order or order 4. By Lemmas 2.1 and 2.2, $< x >$ is either E-supplemented or SS-quasinormald in M. Therefore the hypothesis holds for $(M, M \cap H)$. By the choice of G, $M \in \mathcal{F}$. Then, by [2, Theorem 3.4.2], $G^{\mathcal{F}}$ is a p-group, where $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G. In view of Lemma 2.5, every cyclic subgroup of $G^{\mathcal{F}}$ with prime order and order 4 of is E-supplemented in G. Applying [6, Theorem 4.3], $G \in \mathcal{F}$, a contradiction.

Corollary 3.4. If every cyclic subgroup of a group G with prime order or order 4 is either E-supplemented or SS-quasinormald in G, then G is supersoluble.

Theorem 3.4. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if one of the following conditions holds:

(i) every maximal subgroup of any Sylow subgroup of $F^*(H)$ is either E-supplemented or SS-quasinormald in G.

(ii) every cyclic subgroup of any Sylow subgroup of $F^*(H)$ with prime order or order 4 is either E-supplemented or SS-quasinormald in G.

Proof. We only need to prove the "if" part. If the condition (1) holds, then every maximal subgroup of any Sylow subgroup of $F^*(H)$ is either E-supplemented or SS-quasinormald in $F^*(H)$ by Lemmas 2.1 and 2.2. From Corollary 3.3, we have that $F^*(H)$ is supersoluble. In particular, $F^*(H)$ is soluble. By Lemma 2.6, $F^*(H) = F(H)$. Since S-quasinormal subgroup is E-supplemented subgroup, it follows that every maximal subgroup of any Sylow subgroup of $F^*(H)$ is E-supplemented in G by Lemma 2.5. Applying Theorem [6, Theorem 4.5], $G \in \mathcal{F}$, If the condition (2) holds, then we have also $G \in \mathcal{F}$ using similar arguments as above.

Corollary 3.5. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if one of the following conditions holds:

(i) every maximal subgroup of any Sylow subgroup of $F(H)$ is either E-supplemented or SS-quasinormald in G.

(ii) every cyclic subgroup of any Sylow subgroup of $F(H)$ with prime order or order 4 is either E-supplemented or SS-quasinormald in G.

§4. Some applications

Corollary 4.1.[14] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are S-quasinormal in G, then $G \in \mathcal{F}$.

Corollary 4.2.[16] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c-normal in G, then $G \in \mathcal{F}$.

Corollary 4.3.[17] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.

Corollary 4.4.[9] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are weakly S-supplemented in G, then $G \in \mathcal{F}$.

Corollary 4.5.[7] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is SS-quasinormal in G.

Corollary 4.6.[12] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. If there is a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are π-quasinormally embedded in G, then $G \in \mathcal{F}$.

Corollary 4.7.[11] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is S-quasinormal in G, then $G \in \mathcal{F}$.

Corollary 4.8.[16] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is c-normal in G, then $G \in \mathcal{F}$.

Corollary 4.9.[17] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is c-supplemented in G, then $G \in \mathcal{F}$.

Corollary 4.10.[12] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. If there is a normal subgroup H such that $G/H \in \mathcal{F}$ and the subgroups of prime order or order 4 of $F^*(H)$ are π-quasinormally embedded in G, then $G \in \mathcal{F}$.

Corollary 4.11.[10] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 weakly S-supplemented in G.

Corollary 4.12.[12] Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is SS-quasinormal in G.

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Connected dominating sets in unit disk graphs[1]

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Abstract A subset of vertices in a graph is called a dominating set if every vertex is either in the subset or adjacent to a vertex in the subset. A dominating set is connected if it induces a connected subgraph. A subset of vertices in a graph is independent if no two vertices are connected by an edge. Many constructions for approximating the minimum connected dominating set are based on the construction of a maximal independent set. Let $|mis(G)|$ and $|mcds(G)|$ be the size of a maximum independent set and the size of a minimum connected dominating set in the same graph G respectively. In [Theoretical Computer Science 352 (2006) 1-7] Wu et al by showing that $|mis(G)| \leq 3.8|mcds(G)| + 1.2$, they have really shown the relation between $|mis(G)|$ and $|mcds(G)|$ plays an important role in establishing the performance ratio of those approximation algorithms. They have also conjectured "the neighbor area of a 4-star subgraph in a unit disk graph contains at most 20 independent vertices". In this paper we show that $|mis(G)| \leq 3.5|mcds(G)|$ for all unit disk graphs and improve the conjecture.

Keywords Connected dominating set, independent set, unit disk graph.
2000 Mathematical Subject Classification: 05C69.

§1. Introduction

Let $G = (V, E)$ be a graph. A subset of vertices in a graph G is called a dominating set if every vertex is either in the subset or adjacent to a vertex in the subset. A dominating set is connected if it induces a connected subgraph. A subset $I \subseteq V$ is called independent if all vertices are not connected. It is also called maximal independent set if it cannot be extended by the addition of any other vertex from the graph without violating the independence property. An independent set is a maximal if and only if it is a dominating set. A maximum independent set (mis) is a maximum cardinality subset of V such that there is no edge between any two vertices of it. Given $G = (V, E)$ and a vertex v, we use $N(v)$ to denote the set of vertices adjacent to v; as the neighborhood of v, see [4]. Let $|mis(G)|$ and $|mcds(G)|$ be the size of a maximum independent set and the size of a minimum connected dominating set in the same graph G respectively. A graph is a unit disk graph (UDG) if its vertices can be drown as circular disks of equal radius in the plane in such a way that there is an edge between two vertices if and only if the two disks have non-empty intersection. (It is assumed that the tangent circles

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intersect.) Without lose of generality, the radius of each circle (disk) is assumed to be 1. Unit disk graphs ($UDGs$) are probably the most prominent class of graphs used to model the wireless networks and have been used to model broadcast networks [2,3] and optimal facility location [5]. A connected dominating set can be used as a virtual backbone in wireless sensor networks to improve communication and storage performance [1]. In this note we improve the Theorem 1 and Conjecture of [6].

§2. Preliminaries

The followings are useful.

Lemma 2.1. The neighborhood of a vertex in unit disk graph, contains at most five independent vertices.

Proof. Assume v is a vertex of unit disk graph G which has six independent vertices. Let $v_1, \cdots, v_6$ be independent vertices adjacent to v, and let $v_1, \cdots, v_6$ lie counter-clockwise around v. There are two vertices v_i and v_j such that $\angle v_i v v_j \leq 60°$. It means that the distance between v_i and v_j is at most 1, which implies that v_i and v_j are adjacent and contradicting our assumption.

Lemma 2.2.[6] The neighbor area of two adjacent vertices contains at most eight independent vertices.

Theorem 2.1.[6] For any unit disk graph G, $|mis(G)| \leq 3.8|mcds(G)| + 1.2$.

The following conjecture has been proposed in [6].

Conjecture 2.1.[6] The neighbor area of a 4-star subgraph in a unit disk graph contains at most 20 independent vertices.

§3. Main results

Here we improve the Theorem 2.1 and Conjecture 2.1.

Lemma 3.1. The neighborhood of two adjacent vertices in unit disk graph G contains at most seven independent vertices.

Proof. Let u and v be two adjacent vertices in unit disk graph G. We consider two steps to prove this lemma.

State 1. We first assume that u has five independent vertices, $u_1, \cdots, u_5$ in its neighborhood. We show that $N(v)$ cannot contains more than two independent vertices, v_1, v_2 from themselves and from $u_1, \cdots, u_5$. Let $u_1, \cdots, u_5$ lie counter-clockwise in $N(u)$. By Lemma 2.1, we know that for independence u_i, u_j should have $\angle u_i u u_j > 60°$. Without lose of generality assume that every angles $\angle u_1 u u_2$, $\angle u_2 u u_3$, $\angle u_3 u u_4$ and $\angle u_4 u u_5$ are $(60 + \varepsilon_1)°$. Then we have $\angle u_1 u u_5 = (120 - 4\varepsilon_1)°$. Let v_1, v_2 be two independent vertices that lie counter-clockwise in $N(v)$. It is easy to see that for independence v_1 and v_2 from $u_1, \cdots, u_5$, we should have $\angle v_1 v v_2 < (120 - 4\varepsilon_1)°$. (Note that in case that the quadrilateral $u_1 v_1 u v$ is a parallelogram, the disks corresponding to u_1 and v_1 are tangent and therefore are adjacent, and while the quadrilateral $u_2 v_5 u v$ is a parallelogram, disks corresponding to u_5 and v_2 are tangent.) Since

$\angle v_1vv_2 < (120 - 4\varepsilon_1)^\circ$, it follows that no independent vertex from v_1, v_2 and $u_1, \cdots, u_5$ can adjacent to v.

State 2. Now assume that there are four independent vertices $u_1, \cdots, u_4$ which lie counter-clockwise in $N(u)$, we show that $N(v)$ contains at most three independent vertices from themselves and from $u_1, \cdots, u_4$. Without lose of generality assume that the angles $\angle u_1uu_2$, $\angle u_2uu_3$ and $\angle u_3uu_4$ are $(60+\varepsilon_2)^\circ$. Therefore $\angle u_1uu_4 = (180-3\varepsilon_2)^\circ$. Let v_1 and v_2 be two independent vertices, lie counter-clockwise in $N(v)$, similar to State 1, we see that for independence $v_1,\ v_2$ from $u_1, \cdots, u_4$, we should have $\angle v_1vv_2 < (180 - 3\varepsilon_2)^\circ$, thus we follow $N(v)$ cannot contains more than one independent vertex from $v_1,\ v_2$ and $u_1, \cdots, u_4$ as desired.

Theorem 3.1. For any unit disk graph G, $|mis(G)| \leq 3.5|mcds(G)|$.

Proof. According to the Lemma 3.1, every neighborhood of two adjacent vertices in unit disk graph G contains at most seven independent vertices. That is, any of these two adjacent vertices dominates some of these independent vertices, and this set is the minimum set that dominates these seven independent vertices. Thus both of them have to be member of a minimum dominating set. In particular, for any two members of a minimum connected dominating set ($mcds$), there are at most seven independent vertices in maximum independent set, that is $|mis| \leq 3.5|mcds|$.

As an immediate result of Theorem 3.1 we have.

Corollary 3.1. For any unit disk graph G, the cardinality of a maximal independent set is at most $3.5|mcds(G)|$.

Proof. Note that the cardinality of any maximal independent set is smaller than or equal to the maximum independent set, so the result holds.

Remark 3.1. Unit disk graph has been presented in [4] and elsewhere. A unit disk graph is a disk with radius one. A unit disk graph is associated with a set of unit disks in the Euclidean plane. Each vertex is the center of a unit disk. An edge exists between two vertices u and v if and only if $\mid uv \mid \leq 1$ where $\mid uv \mid$ is the Euclidean distance between u and v. This means that two vertices are connected by an edge if and only if u's disk covers v and v's disk cover u. We call a unit disk (including its boundary) at center v, the neighbor area of v, denoted by $N(v)$. Therefore, two vertices u and v are said to be adjacent if $\mid uv \mid \leq 1$ and independent if $\mid uv \mid > 1$.

Using Remark 3.1 and the proof of the Lemma 3.1, the Conjecture 2.1 is improved as follows.

Conjecture 3.1. The neighbor area of a 4-star subgraph in a unit disk graph contains at most 14 independent vertices.

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