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Connectedness in topology of intuitionistic fuzzy rough sets

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Abstract In this paper Type-1 connectedness and Type-2 connectedness in topology of intuitionistic fuzzy rough sets are defined and properties of two types of connectedness are studied.

Keywords Fuzzy subsets, rough sets, instuitionistic fuzzy sets, intuitionistic fuzzy rough s-ets, fuzzy topology, Type-1 connectedness, Type-2 connectedness.
2000 Mathematics Subject Classification: 54A40

§1. Introduction

After introducing the idea of fuzzy subsets by Lotfi Zadeh [22] different variations/generalisations of fuzzy subsets were made by several authors. Pawlak [20] introduced the idea of rough sets. Nanda [19] and Çoker [7] gave the definition of fuzzy rough sets. Atanassov [2] introduced the idea of intuitionistic fuzzy sets. Combining the ideas of fuzzy rough sets and intuitionistic fuzzy sets T. K. Mandal and S. K. Samanta [16] introduced the concept of intuitionistic fuzzy rough sets (briefly call $IFRS$, definition 1.15). On the other hand fuzzy topology (we call it topology of fuzzy subsets) was first introduced by C. L. Chang [4]. Later many authors dealt with the idea of fuzzy topology of different kinds of fuzzy sets. M. K. Chakroborti and T. M. G. Ahsanullah [3] introduced the concept of fuzzy topology on fuzzy sets. T. K. Mondal and S. K. Samanta introduced the topology of interval valued fuzzy sets in [18] and the topology of interval valued intuitionistic fuzzy sets in [17]. In [11], we introduced the concept of topology of intuitionistic fuzzy rough sets and study its various properties. In defining topology on an $IFRS$ from the parent space, we observed that two topologies are induced on the $IFRS$ and accordingly two types of continuity are defined. In [12], we studied the connectedness in topology of intuitionistic fuzzy sets.

In this paper, we have defined Type-1 connectedness and Type-2 connectedness in topology of intuitionistic fuzzy rough sets and studied properties of two types of connectedness.

§2. Preliminaries

Unless otherwise stated, we shall consider $(V, \mathcal{B})$ to be a rough universe where V is a nonempty set and $\mathcal{B}$ is a Boolean subalgebra of the Boolean algebra of all subsets of V. Also consider a rough set $X = (X_L, X_U) \in \mathcal{B}^2$ with $X_L \subset X_U$.

Moreover we assume that $\mathcal{C}_X$ be the collection of all $IFRSs$ in X.

Definition 2.1.[19] A fuzzy rough set (briefly FRS) in X is an object of the form $A = (A_L, A_U)$, where A_L and A_U are characterized by a pair of maps $A_L : X_L \to \mathcal{L}$ and $A_U : X_U \to \mathcal{L}$ with $A_L(x) \le A_U(x)$, $\forall x \in X_L$, where $(\mathcal{L}, \le)$ is a fuzzy lattice (i.e. complete and completely distributive lattice whose least and greatest elements are denoted by 0 and 1 respectively with an involutive order reversing operation $' : \mathcal{L} \to \mathcal{L}$).

Definition 2.2.[19] For any two fuzzy rough sets $A = (A_L,\ A_U)$ and $B = (B_L,\ B_U)$ in X,

(i) $A \subset B$ iff $A_L(x) \le B_L(x),\ \forall x \in X_L$ and $A_U(x) \le B_U(x),\ \forall x \in X_U$;

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

If $\{A_i : i \in J\}$ be any family of fuzzy rough sets in X, where $A_i = (A_{iL},\ A_{iU})$, then

(iii) $E = \bigcup_i A_i$, where $E_L(x) = \vee A_{iL}(x),\ \forall x \in X_L$ and $E_U(x) = \vee A_{iU}(x),\ \forall x \in X_U$;

(iv) $F = \bigcap_i A_i$, where $F_L(x) = \wedge A_{iL}(x),\ \forall x \in X_L$ and $F_U(x) = \wedge A_{iU}(x),\ \forall x \in X_U$.

Definition 2.3.[16] If A and B are fuzzy sets in X_L and X_U respectively where $X_L \subset X_U$. Then the restriction of B on X_L and the extension of A on X_U (denoted by $B_{>L}$ and $A_{<U}$ respectively) are defined by $B_{>L}(x) = B(x)$, $\forall x \in X_L$ and

$$A_{<U}(x) = \begin{cases} A(x), & \forall x \in X_L; \\ \vee_{\xi \in X_L}\{A(\xi)\}, & \forall x \in X_U - X_L. \end{cases}$$

Complement of an FRS $A = (A_L, A_U)$ in X are denoted by $\bar{A} = ((\bar{A})_L, (\bar{A})_U)$ and is defined by

$$(\bar{A})_L(x) = (A_{U>L})'(x),\ \forall x \in X_L \text{ and } (\bar{A})_U(x) = (A_{L<U})'(x),\ \forall x \in X_U.$$

For simplicity we write $(\bar{A}_L, \bar{A}_U)$ instead of $((\bar{A})_L, (\bar{A})_U)$.

Theorem 2.1.[16] If $A,\ B,\ C,\ D$ and $B_i,\ i \in J$ are $FRSs$ in X, then

(i) $A \subset B$ and $C \subset D$ implies $A \cup C \subset B \cup D$ and $A \cap C \subset B \cap D$,

(ii) $A \subset B$ and $B \subset C$ implies $A \subset C$,

(iii) $A \cap B \subset A,\ B \subset A \cup B$,

(iv) $A \cup (\bigcap_i B_i) = \bigcap_i (A \cup B_i)$ and $A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)$,

(v) $A \subset B \Rightarrow \bar{A} \supset \bar{B}$,

(vi) $\overline{\bigcup_i B_i} = \bigcap_i \bar{B}_i$ and $\overline{\bigcap_i B_i} = \bigcup_i \bar{B}_i$.

Theorem 2.2.[16] If A be any FRS in X, $\tilde{0} = (\tilde{0}_L, \tilde{0}_U)$ be the null FRS and $\tilde{1} = (\tilde{1}_L, \tilde{1}_U)$ be the whole FRS in X, then

(i) $\tilde{0} \subset A \subset \tilde{1}$,

(ii) $\bar{\tilde{0}} = \tilde{1},\ \bar{\tilde{1}} = \tilde{0}$.

Notation 2.1. Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : V \to V_1$ be a mapping. If $f(\lambda) \in \mathcal{B}_1,\ \forall \lambda \in \mathcal{B}$, then f maps $(V, \mathcal{B})$ to $(V_1, \mathcal{B}_1)$ and it is denoted by $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$. If $f^{-1}(\mu) \in \mathcal{B}, \forall \mu \in \mathcal{B}_1$, then it is denoted by $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Definition 2.4.[16] Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$.

Let $A = (A_L, A_U)$ be a FRS in X. Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L)$, $Y_U = f(X_U)$. The image of A under f, denoted by $f(A) = (f(A_L), f(A_U))$ and is defined by

$$f(A_L)(y) = \vee\{A_L(x) : x \in X_L \cap f^{-1}(y)\},\ \forall y \in Y_L;$$

and

$$f(A_U)(y) = \vee\{A_U(x) : x \in X_U \cap f^{-1}(y)\},\ \forall y \in Y_U.$$

Next let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Let $B = (B_L, B_U)$ be a FRS in Y, where $Y = (Y_L, Y_U) \in \mathcal{B}_1^2$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}^2$, where $X_L = f^{-1}(Y_L),\ X_U = f^{-1}(Y_U)$. Then the inverse image of B, under f, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ and is defined by

$$f^{-1}(B_L)(x) = B_L(f(x)),\ \forall x \in X_L \text{ and } f^{-1}(B_U)(x) = B_U(f(x)),\ \forall x \in X_U.$$

Theorem 2.3.[16] If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then for all $FRSs$ A, A_1 and A_2 in X, we have

(i) $f(\bar{A}) \supset \overline{f(A)}$,

(ii) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

Theorem 2.4.[16] If $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all $FRSs$ B, $B_i,\ i \in J$ in Y we have

(i) $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$,

(ii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$,

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathcal{B}_2) \to (V_1, \mathcal{B}_1)$, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any FRS C in Z where $Z = (Z_L, Z_U) \in \mathcal{B}_2^2$ is a rough set. $g \circ f$ is the composition of g and f,

(iv) $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$,

(v) $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$.

Theorem 2.5.[16] If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all FRS A in X and B in Y, we have

(i) $B = f(f^{-1}(B))$,

(ii) $A \subset f^{-1}(f(A))$.

Definition 2.5.[16] If A and B are two $FRSs$ in X with $B \subset \bar{A}$ and $A \subset \bar{B}$, then the ordered pair (A, B) is called an intuitionistic fuzzy rough set (briefly $IFRS$) in X.

The condition $A \subset \bar{B}$ and $B \subset \bar{A}$ are called intuitionistic condition (briefly IC).

Definition 2.6.[16] Let $P = (A, B)$ and $Q = (C, D)$ be two $IFRSs$ in X. Then

(i) $P \subset Q$ iff $A \subset C$ and $B \supset D$,

(ii) $P = Q$ iff $P \subset Q$ and $Q \subset P$,

(iii) The complement of P in X, denoted by P', is defined by $P' = (B, A)$,

(iv) For $IFRSs$ $P_i = (A_i, B_i)$ in X, $i \in J$, define $\bigcup_{i\in J} P_i = (\bigcup_{i\in J} A_i,\ \bigcap_{i\in J} B_i)$ and $\bigcap_{i\in J} P_i = (\bigcap_{i\in J} A_i,\ \bigcup_{i\in J} B_i)$.

Theorem 2.6.[16] Let $P = (A, B),\ Q = (C, D),\ R = (E, F)$ and $P_i = (A_i, B_i),\ i \in J$ be $IFRSs$ in X, then

(i) $P \cap P = P = P \cup P$,

(ii) $P \cap Q = Q \cap P$, $P \cup Q = Q \cup P$,

(iii) $(P \cap Q) \cap R = P \cap (Q \cap R)$, $(P \cup Q) \cup R = P \cup (Q \cup R)$,

(iv) $P \cap Q \subset P,\ Q \subset P \cup Q$,

(v) $P \subset Q$ and $Q \subset R \Rightarrow P \subset R$,

(vi) $P_i \subset Q,\ \forall i \in J \Rightarrow \bigcup_{i\in J} P_i \subset Q$,

(vii) $Q \subset P_i,\ \forall i \in J \Rightarrow Q \subset \bigcap_{i\in J} P_i$,

(viii) $Q \cup (\bigcap_{i\in J} P_i) = \bigcap_{i\in J}(Q \cup P_i)$,

(ix) $Q \cap (\bigcup_{i\in J} P_i) = \bigcup_{i\in J}(Q \cap P_i)$,

(x) $(P')' = P$,

(xi) $P \subset Q \Leftrightarrow Q' \subset P'$,

(xii) $(\bigcup_{i\in J} P_i)' = \bigcap_{i\in J} P'$ and $(\bigcap_{i\in J} P_i)' = \bigcup_{i\in J} P'_i$.

Definition 2.7.[16] $0^* = (\tilde{0}, \tilde{1})$ and $1^* = (\tilde{1}, \tilde{0})$ are respectively called null $IFRS$ and whole $IFRS$ in X. Clearly $(0^*)' = 1^*$ and $(1^*)' = 0^*$.

Theorem 2.7.[16] If P be any $IFRS$ in X, then $0^* \subset P \subset 1^*$.

Slightly changing the definition of the image of an $IFRS$ under f given by Samanta and Mondal [16] we have given the following:

Definition 2.8.[11] Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping. Let $P = (A, B)$ be an $IFRS$ in $X(= (X_L, X_U))$ and $Y = f(X) \in \mathcal{B}_1^2$, where $Y_L = f(X_L)$ and $Y_U = f(X_U)$.

Then we define image of P, under f by $f(P) = (\breve{f}(A), \hat{f}(B))$, where $\breve{f}(A) = (f(A_L), f(A_U))$, $A = (A_L, A_U)$ and $\hat{f}(B) = (C_L, C_U)$ (where $B = (B_L, B_U)$) is defined by

$$C_L(y) = \wedge\{B_L(x) : x \in X_L \cap f^{-1}(y)\},\ \forall y \in Y_L,$$

$$C_U(y) = \begin{cases} \wedge\{B_U(x) : x \in X_L \cap f^{-1}(y)\}, & \forall y \in Y_L; \\ \wedge\{B_U(x) : x \in X_U \cap f^{-1}(y)\}, & \forall y \in Y_U - Y_L. \end{cases}$$

Definition 2.9.[16] Let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Let $Q = (C, D)$ be an $IFRS$ in Y, where $Y = (Y_L, Y_U) \in \mathcal{B}_1^2$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}^2$, where $X_L = f^{-1}(Y_L)$ and $X_U = f^{-1}(Y_U)$. Then the inverse image $f^{-1}(Q)$ of Q, under f, is defined by $f^{-1}(Q) = (f^{-1}(C), f^{-1}(D))$, where $f^{-1}(C) = (f^{-1}(C_L), f^{-1}(C_U))$ and $f^{-1}(D) = (f^{-1}(D_L), f^{-1}(D_U))$.

The following three theorems of Samanta and Mondal [16] are also valid for this modified definition of functional image of an $IFRS$.

Theorem 2.8. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping. Then for all $IFRSs$ P and Q, we have

(i) $f(P') \supset (f(P))'$,

(ii) $P \subset Q \Rightarrow f(P) \subset f(Q)$.

Theorem 2.9. Let $f : V \to V_1$ be such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all $IFRSs$ $R,\ S$ and $R_i,\ i \in J$ in Y,

(i) $f^{-1}(R') = (f^{-1}(R))'$,

(ii) $R \subset S \Rightarrow f^{-1}(R) \subset f^{-1}(S)$,

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, \mathcal{B}_2) \to (V_1, \mathcal{B}_1)$, then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ for any $IFRS$ W in Z, where $Z = (Z_L, Z_U) \in \mathcal{B}_2^2$ is a rough set, $g \circ f$ is the composition of g and f,

(iv) $f^{-1}(\bigcup_{i\in J} R_i) = \bigcup_{i\in J} f^{-1}(R_i)$,

(v) $f^{-1}(\bigcap_{i\in J} R_i) = \bigcap_{i\in J} f^{-1}(R_i)$.

Theorem 2.10. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all $IFRS$ P in X and R in Y, we have

(i) $R = f(f^{-1}(R))$,

(ii) $P \subset f^{-1}(f(P))$.

Theorem 2.11.[11] If P and Q be two $IFRSs$ in X and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then $f(P \cup Q) = f(P) \cup f(Q)$.

Theorem 2.12.[11] If P, Q be two $IFRSs$ in X and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping, then $f(P \cap Q) \subset f(P) \cap f(Q)$.

Note 2.1. If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be one-one, then clearly $f(P \cap Q) = f(P) \cap f(Q)$. But in general $f(P \cap Q) \neq f(P) \cap f(Q)$.

Definition 2.10.[11] Let $X = (X_L, X_U)$ be a rough set and τ be a family of $IFRSs$ in X such that

(i) $0^*,\ 1^* \in \tau$,

(ii) $P \cap Q \in \tau,\ \forall P,\ Q \in \tau$,

(iii) $P_i \in \tau,\ i \in \Delta \Rightarrow \bigcup_{i\in\Delta} P_i \in \tau$.

Then τ is called a topology of $IFRSs$ in X and (X, τ) is called a topological space of $IFRSs$ in X.

Every member of τ is called open $IFRS$. An $IFRS$ C is called closed $IFRS$ if $C' \in \tau$. Let $\mathcal{F}$ denote the collection of all closed $IFRSs$ in (X, τ). If $\tau_I = \{0^*, 1^*\}$, then τ_I is a topology of $IFRSs$ in X. This topology is called the indiscrete topology. The discrete topology of $IFRSs$ in X contains all the $IFRSs$ in X.

Theorem 2.13.[11] The collection $\mathcal{F}$ of all closed $IFRSs$ satisfies the following properties:

(i) $0^*,\ 1^* \in \mathcal{F}$,

(ii) $P,\ Q \in \mathcal{F} \Rightarrow P \cup Q \in \mathcal{F}$,

(iii) $P_i \in \mathcal{F},\ i \in \Delta \Rightarrow \bigcap_{i\in\Delta} P_i \in \mathcal{F}$.

Definition 2.11.[11] Let P be an $IFRS$ in X. The closure of P in (X, τ), denoted by $cl_\tau P$, is defined by the intersection of all closed $IFRSs$ in (X, τ) containing P.

Clearly $cl_\tau P$ is the smallest closed $IFRS$ containing P and P is closed iff $P = cl_\tau P$.

Definition 2.12.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then $f : (X, \tau) \to (Y, u)$ is said to be IFR continuous if $f^{-1}(Q) \in \tau,\ \forall Q \in u$.

Unless otherwise stated we consider (X, τ) and (Y, u) be topological spaces of $IFRSs$ and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$.

Theorem 2.14.[11] The following statements are equivalent:

(i) $f : (X, \tau) \to (Y, u)$ is IFR continuous,

(ii) $f^{-1}(Q)$ is closed $IFRS$ in (X, τ), for every closed $IFRS$ Q in (Y, u),

(iii) $f(cl_\tau P) \subset cl_u(f(P))$, for every $IFRS$ P in X.

Definition 2.13.[11] Let $P \in \mathcal{C}_X$. Then a subfamily T of $\mathcal{C}_X$ is said to be a topology on P if

(i) $Q \in T \Rightarrow Q \subset P$,

(ii) $0^*,\ P \in T$,

(iii) $P_1,\ P_2 \in T \Rightarrow P_1 \cap P_2 \in T$,

(iv) $P_i \in T,\ i \in \Delta \Rightarrow \bigcup_{i\in\Delta} P_i \in T$.

Then (P, T) is called a subspace topology of (X, τ).

Theorem 2.15.[11] Let τ be a topology of $IFRSs$ in X and let $P \in \mathcal{C}_X$. Then $\tau_1 = \{P \cap R : R \in \tau\}$ is a topology on P. Every member of τ_1 is called open $IFRS$ in (P, τ_1). If $Q \in \tau_1$, then Q'_P is called a closed $IFRS$ in (P, τ_1), where $Q'_P = P \cap Q'$. We take 0^* as closed $IFRS$ also in (P, τ_1). Let $C_1 = \{Q'_P = P \cap Q' : Q \in \tau_1\} \cup \{0^*\}$.

Theorem 2.16.[11] C_1 is closed under arbitrary intersection and finite union.

Remark 2.1. Clearly the collection $\tau_2 = \{(S'_P)'_P = (P \cap P') \cup S : S \in \tau_1\} \cup \{0^*\}$ forms a topology of $IFRSs$ on P of which C_1 is a family of closed $IFRSs$ [11]. But C_1 is also a family of closed $IFRSs$ in (P, τ_1). Thus $\exists$ two topologies of $IFRSs$ τ_1 and τ_2 on P. τ_1 is called the first subspace topology of (X, τ) on P and τ_2 is called the second subspace topology of (X, τ) on P. We briefly write, τ_1 and τ_2 are first and second topologies respectively on P, where there is no confusion about the topological space (X, τ) of $IFRSs$.

Definition 2.14.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ and $P \in \mathcal{C}_X$. Let τ_1 and u_1 are first topologies on P and $f(P)$ respectively. Then $f : (P, \tau_1) \to (f(P), u_1)$ is said to be IFR_1 continuous if $P \cap f^{-1}(Q) \in \tau_1,\ \forall Q \in u_1$.

Theorem 2.17.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ in X and Y respectively and $P \in \mathcal{C}_X$ and let τ_1 and u_1 be first topologies on P and $f(P)$ respectively. If $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous.

Definition 2.15.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ in X and Y respectively and $P \in \mathcal{C}_X$ and let $\tau_2,\ u_2$ be second topologies on P and $f(P)$ respectively. Then $f : (P, \tau_2) \to (f(P), u_2)$ is said to be IFR_2 continuous if $P \cap (P' \cup f^{-1}(Q)) \in \tau_2,\ \forall Q \in u_2$.

Theorem 2.18.[11] Let (X, τ) and (Y, u) be two topological spaces of $IFRSs$ in X and Y respectively and $P \in \mathcal{C}_X$. Let τ_1 and u_1 be first topologies on P and $f(P)$ respectively and $\tau_2,\ u_2$ be second topologies on P and $f(P)$ respectively. If $f : (P, \tau_1) \to (f(P), u_1)$ is IFR_1 continuous, then $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

Corollary 2.1.[11] If $f : (X, \tau) \to (Y, u)$ is IFR continuous, then $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous, where the symbols have usual meaning.

§3. Connectedness of an $IFRS$

Connectedness in fuzzy topological space has been studied by several authors (for reference see [1], [5], [6], [8], [15], [21]). We studied connectedness in topology of intuitionistic fuzzy sets in [12]. In this section we study two types of connectedness in a topological space of $IFRSs$.

Definition 3.1. Let τ_1 be the first topology on P. An $IFRS$ $Q \subset P$ is said to be nonempty in (P, τ_1) if $Q \neq 0^*$.

Definition 3.2. Let τ_1 be the first topology on P. Two $IFRSs$ Q_1 and Q_2 are said to be disjoint in (P, τ_1) if $Q_1 \cap Q_2 = 0^*$.

Definition 3.3. Let (X, τ) be a topological space of $IFRSs$ in X and let $P \in \mathcal{C}_X$. Then P is said to be Type-1 connected in (X, τ), if P can not be expressed as a union of two nonempty disjoint open sets in (P, τ_1).

Theorem 3.1. Let $f : (P, \tau_1) \rightarrow (f(P), u_1)$ be IFR_1 continuous and P be Type-1 connected in (X, τ), then $f(P)$ is Type-1 connected in (Y, u).

Proof. Let $f : (P, \tau_1) \rightarrow (f(P), u_1)$ be IFR_1 continuous and P be Type-1 connected in (X, τ). We shall show that $f(P)$ is Type-1 connected in (Y, u), where $Y = f(X)$ and u is the topology of $IFRSs$ in Y. If possible, let $f(P)$ be not Type-1 connected in (Y, u).

Therefore $f(P) = Q_1 \cup Q_2$, where $Q_1,\ Q_2 \in u_1$ and $Q_1,\ Q_2$ are nonempty disjoint open sets in (Y, u_1). Therefore

$$\begin{aligned} P &= P \cap f^{-1}(f(P)) \\ &= P \cap f^{-1}(Q_1 \cup Q_2) \\ &= P \cap (f^{-1}(Q_1) \cup f^{-1}(Q_2)) \\ &= (P \cap f^{-1}(Q_1)) \cup (P \cap f^{-1}(Q_2)). \end{aligned} \tag{1}$$

Since $f : (P, \tau_1) \rightarrow (f(P), u_1)$ is IFR_1 continuous and $Q_1, Q_2 \in u_1$, we have $P \cap f^{-1}(Q_1),\ P \cap f^{-1}(Q_2) \in \tau_1$. Now we shall show that $P \cap f^{-1}(Q_1)$ and $P \cap f^{-1}(Q_2)$ are nonempty and disjoint. If possible let $P \cap f^{-1}(Q_1) = 0^*$. Therefore from (1) we have $P = P \cap f^{-1}(Q_2)$ and hence $f^{-1}(Q_2) \supset P$ and consequently $f(f^{-1}(Q_2)) \supset f(P)$, i.e., $Q_2 \supset f(P)$, which contradicts $f(P) = Q_1 \cup Q_2$ and $Q_1,\ Q_2$ are nonempty disjoint.

Thus $P \cap f^{-1}(Q_1) \neq 0^*$. Similarly we can prove that $P \cap f^{-1}(Q_2) \neq 0^*$. Now $(P \cap f^{-1}(Q_1)) \cap (P \cap f^{-1}(Q_2)) = 0^*$. Thus $P \cap f^{-1}(Q_1)$ and $P \cap f^{-1}(Q_2)$ are disjoint. Thus P can be expressed as a union of two nonempty disjoint open sets in (P, τ_1). Thus P can not be Type-1 connected in (X, τ), which is a contradiction. Hence $f(P)$ is Type-1 connected in (Y, u).

Definition 3.4. Let (X, τ) be a topological space of $IFRSs$ in X and let $P \in \mathcal{C}_X$ and let τ_2 be the second topology on P. Then an $IFRS$ $Q \subset P$ is said to be nonempty in (P, τ_2) if $Q \supset P \cap P'$ and $Q \neq P \cap P'$. Two $IFRSs$ Q and R are said to be disjoint in (P, τ_2) if $Q \cap R \subset P \cap P'$.

Definition 3.5. Let (X, τ) be a topological space of $IFRSs$ in X and let $P \in \mathcal{C}_X$. Then P is said to be Type-2 connected in (X, τ) if P can not be expressed as a union of two nonempty disjoint open sets in (P, τ_2).

Note 3.1. If $f : (P, \tau_2) \rightarrow (f(P), u_2)$ is IFR_2 continuous and P is Type-2 connected in (X, τ), then $f(P)$ may not be Type-2 connected in (Y, u). This is shown by the following Example.

Example 3.1. Let $X_L = X_U = V = \{x, y, z, u, v, w\}$, $Y_L = Y_U = V_1 = \{a, b\}$. Define

$f : V \to V_1$ by $f(x) = f(y) = f(z) = a, f(u) = f(v) = f(w) = b$. Let

$$\begin{aligned}
Q &= ((\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.4, w/0.4\}, \{x/0.4, y/0.4, z/0.4, u/0.5, v/0.5, w/0.5\}), \\
&\quad (\{x/0.4, y/0.4, z/0.4, u/0.3, v/0.3, w/0.3\}, \{x/0.5, y/0.5, z/0.5, u/0.4, v/0.4, w/0.4\})), \\
R &= ((\{x/0.4, y/0.4, z/0.4, u/0.3, v/0.3, w/0.3\}, \{x/0.5, y/0.5, z/0.5, u/0.4, v/0.4, w/0.4\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.4, w/0.4\}, \{x/0.4, y/0.4, z/0.4, u/0.5, v/0.5, w/0.5\})), \\
T &= Q \cup R \\
&= ((\{x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\}, \{x/0.5, y/0.5, z/0.5, u/0.5, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.3, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\})), \\
S &= Q \cap R \\
&= ((\{x/0.3, y/0.3, z/0.3, u/0.3, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\}), \\
&\quad (\{x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\}, \{x/0.5, y/0.5, z/0.5, u/0.5, v/0.5, w/0.5\})), \\
M &= ((\{x/0.35, y/0.4, z/0.4, u/0.3, v/0.3, w/0.3\}, \{x/0.45, y/0.5, z/0.5, u/0.4, v/0.4, w/0.4\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.4, w/0.4\}, \{x/0.4, y/0.4, z/0.4, u/0.5, v/0.5, w/0.5\})), \\
N &= ((\{x/0.35, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\}, \{x/0.45, y/0.5, z/0.5, u/0.5, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.3, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\})).
\end{aligned}$$

Clearly $\tau = \{0^*, 1^*, Q, R, T, S, M, N\}$ forms a topology of $IFRSs$ in $X = (X_L, X_U)$. Let

$$\begin{aligned}
P &= ((\{x/0.4, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})).
\end{aligned}$$

Therefore

$$\begin{aligned}
P \cap P' &= ((\{x/0.3, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.4, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})).
\end{aligned}$$

Now

$$\begin{aligned}
\underline{Q} &= P \cap (P' \cup Q) \\
&= ((\{x/0.3, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.4, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) \\
&= P \cap P', \\
\underline{R} &= P \cap (P' \cup R) \\
&= ((\{x/0.4, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) \\
&= P,
\end{aligned}$$

$$\begin{aligned}
\underline{T} &= P \cap (P' \cup T) \\
&= ((\{x/0.4, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) \\
&= P, \\
\underline{S} &= P \cap (P' \cup S) \\
&= ((\{x/0.3, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.4, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) \\
&= P \cap P', \\
\underline{M} &= P \cap (P' \cup M) \\
&= ((\{x/0.35, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.45, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})), \\
\underline{N} &= P \cap (P' \cup N) \\
&= ((\{x/0.35, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.45, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}), \\
&\quad (\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) \\
&= \underline{M},
\end{aligned}$$

$P \cap (P' \cup 0^*) = P \cap P'$,

$P \cap (P' \cup 1^*) = P$.

Thus $\tau_2 = \{0^*, P, P \cap P', \underline{M}\}$ forms a second topology on P. Clearly P is Type-2 connected in (X, τ). Let

$$\begin{aligned}
\tilde{Q} &= ((\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\}), (\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\})), \\
\tilde{R} &= ((\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\}), (\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\})), \\
\tilde{T} &= \tilde{Q} \cup \tilde{R} = ((\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\}), (\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\})), \\
\tilde{S} &= \tilde{Q} \cap \tilde{R} = ((\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\}), (\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\})).
\end{aligned}$$

Clearly $u = \{0^*, 1^*, \tilde{Q}, \tilde{R}, \tilde{T}, \tilde{S}\}$ forms a topology of $IFRSs$ in $Y = (Y_L, Y_U)$. Now

$$f(P) = ((\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\}), (\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\})) = \tilde{T}.$$

Therefore

$$f(P) \cap (f(P))' = ((\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\}), (\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\})) = \tilde{S},$$

$$\begin{aligned}
\underline{\tilde{Q}} &= f(P) \cap ((f(P))' \cup \tilde{Q}) \\
&= ((\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\}), (\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\})), \\
\underline{\tilde{R}} &= f(P) \cap ((f(P))' \cup \tilde{R}) \\
&= ((\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\}), (\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\})), \\
\underline{\tilde{T}} &= f(P) \cap ((f(P))' \cup \tilde{T}) = f(P).
\end{aligned}$$

Since $\tilde{T} = f(P)$, $\underline{\tilde{S}} = f(P) \cap ((f(P))' \cup \tilde{S}) = f(P) \cap (f(P))'$, since $\tilde{S} = f(P) \cap (f(P))'$, $f(P) \cap ((f(P))' \cup 0^*) = f(P) \cap (f(P))'$, $f(P) \cap ((f(P))' \cup 1^*) = f(P)$. Thus $u_2 = \{0^*, f(P) \cap (f(P))', f(P), \underline{\tilde{Q}}, \underline{\tilde{R}}\}$ forms a second topology on $f(P)$. Clearly $f(P) = \underline{\tilde{Q}} \cup \underline{\tilde{R}}$, where $f(P) \cap (f(P))' \subset \underline{\tilde{Q}},\ \underline{\tilde{R}} \subset f(P)$, $\underline{\tilde{Q}},\ \underline{\tilde{R}} \neq f(P)$ and $\underline{\tilde{Q}},\ \underline{\tilde{R}} \neq f(P) \cap (f(P))'$. Also $\underline{\tilde{Q}} \cap \underline{\tilde{R}} = f(P) \cap (f(P))'$. Therefore $f(P)$ can not be Type-2 connected in (Y, u). Since $f^{-1}(0^*) = 0^*,\ f^{-1}(1^*) = 1^*,\ f^{-1}(\tilde{Q}) = Q,\ f^{-1}(\tilde{R}) = R,\ f^{-1}(\tilde{T}) = T,\ f^{-1}(\tilde{S}) = S$, it follows that $f : (X, \tau) \to (Y, u)$ is IFR continuous and hence $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous. Thus IFR_2 continuous image of a Type-2 connected $IFRS$ need not be Type-2 connected $IFRS$.

Definition 3.6. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping and $P \in \mathcal{C}_X$. Then f is said to satisfy disjoint condition with respect to P, if $\forall Q \in \mathcal{C}_X,\ P \cap Q \subset P \cap P' \Rightarrow f(P) \cap f(Q) \subset f(P) \cap (f(P))'$.

Theorem 3.2. If $P \in \mathcal{C}_X$ and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping and is one-one, then $f(P') = (f(P))'$.

The proof is straightforward.

Remark 3.1. The converse of the theorem 3.2 is not true. This can be shown by the following Example.

Example 3.2. Let $V = \{x, y\},\ X_L = X_U = V,\ X = (X_L, X_U);\ V_1 = \{a\},\ Y_L = Y_U = V_1,\ Y = (Y_L, Y_U)$. Let $f : V \to V_1$ be defined by $f(x) = f(y) = a$. Let $P = ((\{x/0.4, y/0.4\},\ \{x/0.4, y/0.4\}),\ (\{x/0.4, y/0.4\},\ \{x/0.4, y/0.4\}))$. Clearly $P \in \mathcal{C}_X$ and $f(P') = (f(P))'$, but f is not one-one.

Theorem 3.3. If $P \in \mathcal{C}_X$ and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping and is one-one, then f satisfies disjoint condition with respect to P.

Proof. Let $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be one-one and $P \in \mathcal{C}_X$. Then $\forall Q \in \mathcal{C}_X,\ P \cap Q \subset P \cap P' \Rightarrow f(P \cap Q) \subset f(P \cap P') \Rightarrow f(P) \cap f(Q) \subset f(P) \cap f(P')$, since f is one-one. $\Rightarrow f(P) \cap f(Q) \subset f(P) \cap (f(P))'$, since f is one-one, $f(P') = (f(P))'$. Thus f satisfies disjoint condition with respect to P.

But the converse of the above theorem is not true, which can be shown by the following Example.

Example 3.3. Let $V = \{x, y\},\ X_L = X_U = V,\ X = (X_L, X_U),\ V_1 = \{a\},\ Y_L = Y_U = V_1,\ Y = (Y_L, Y_U)$. Let $f : V \to V_1$ be defined by $f(x) = f(y) = a$. Let $P = ((\{x/0.4, y/0.4\},\ \{x/0.4, y/0.4\}),\ (\{x/0.4, y/0.4\},\ \{x/0.4, y/0.4\}))$. Then $P \cap P' = P$. $f(P) = ((\{a/0.4\},\ \{a/0.4\}),\ (\{a/0.4\},\ \{a/0.4\}))$. Therefore $f(P) \cap (f(P))' = f(P)$. Clearly $f(P) \cap f(Q) \subset f(P) = f(P) \cap (f(P))',\ \forall Q \in \mathcal{C}_X$. Thus $\forall Q \in \mathcal{C}_X,\ P \cap Q \subset P \cap P' \Rightarrow f(P) \cap f(Q) \subset f(P) \cap (f(P))'$. Thus f satisfies disjoint condition with respect to P. Clearly f is not one-one. Thus f satisfies disjoint condition with respect to P, but f is not one-one.

Theorem 3.4. If P is Type-2 connected in (X, τ) and $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous satisfying disjoint condition with respect to P, then $f(P)$ is Type-2 connected in (Y, u).

Proof. Let P be Type-2 connected in (X, τ) and $f : (P, \tau_2) \to (f(P), u_2)$ be IFR_2 continuous satisfying disjoint condition with respect to P. If possible, let $f(P)$ be not Type-2 connected in (Y, u). Therefore $f(P) = Q_1 \cup Q_2$ for some $Q_1,\ Q_2 \in u_2$, where $Q_1 \cap Q_2 \subset$

$f(P) \cap (f(P))'$ and $Q_1,\ Q_2 \supsetneqq f(P) \cap (f(P))'$.

$$\begin{aligned} P &= P \cap f^{-1}(f(P)) \\ &= P \cap f^{-1}(Q_1 \cup Q_2) \\ &= P \cap (f^{-1}(Q_1) \cup f^{-1}(Q_2)) \\ &= (P \cap f^{-1}(Q_1)) \cup (P \cap f^{-1}(Q_2)) \\ &\subset (P \cap (P' \cup f^{-1}(Q_1))) \cup (P \cap (P' \cup f^{-1}(Q_2))) \\ &\subset P. \end{aligned}$$

Therefore $P = (P \cap (P' \cup f^{-1}(Q_1))) \cup (P \cap (P' \cup f^{-1}(Q_2))) = P_1 \cup P_2$ (say), where $P_1 = P \cap (P' \cup f^{-1}(Q_1)),\ P_2 = P \cap (P' \cup f^{-1}(Q_2)) \in \tau_2$, since $f : (P, \tau_2) \to (f(P), u_2)$ is IFR_2 continuous.

$$\begin{aligned} P_1 \cap P_2 &= P \cap (P' \cup f^{-1}(Q_1)) \cap (P' \cup f^{-1}(Q_2)) \\ &= P \cap (P' \cup (f^{-1}(Q_1) \cap f^{-1}(Q_2))) \\ &= P \cap (P' \cup (f^{-1}(Q_1 \cap Q_2))) \\ &\subset P \cap (P' \cup (f^{-1}(f(P) \cap (f(P))'))) \\ &= P \cap (P' \cup ((f^{-1}(f(P))) \cap (f^{-1}(f(P)))')) \\ &= (P \cap P') \cup (P \cap ((f^{-1}(f(P))) \cap (f^{-1}(f(P)))')) \\ &= (P \cap P') \cup (P \cap (f^{-1}(f(P)))') \\ &= P \cap (P' \cup (f^{-1}(f(P)))') \\ &= P \cap P'. \end{aligned}$$

Therefore $P_1 \cap P_2 \subset P \cap P'$. We claim that $P_1,\ P_2 \supsetneqq P \cap P'$. If possible, let $P_1 = P \cap P'$. Therefore $P \cap (P' \cup f^{-1}(Q_1)) \subset P \cap P'$, i.e., $(P \cap P') \cup (P \cap f^{-1}(Q_1)) \subset P \cap P'$. Therefore $P \cap f^{-1}(Q_1) \subset P \cap P'$. Therefore $f(P) \cap f(f^{-1}(Q_1)) \subset f(P) \cap (f(P))'$, since f satisfies disjoint condition with respect to P. Therefore $f(P) \cap Q_1 \subset f(P) \cap (f(P))'$ and hence $Q_1 \subset f(P) \cap (f(P))'$, since $Q_1 \subset f(P)$. But this contradicts the fact that $Q_1 \supsetneqq f(P) \cap (f(P))'$. Thus $P_1 \supsetneqq P \cap P'$. Similarly we can prove that $P_2 \supsetneqq P \cap P'$. Thus P can be expressed as the union of two nonempty disjoint open sets in (P, τ_2), which contradicts that P is Type-2 connected in (X, τ). Hence $f(P)$ is Type-2 connected in (Y, u).

References

[1] N. Ajmal and J. K. Kohli, Connectedness in fuzzy topological spaces, Fuzzy Sets and Systems, **31**(1989), 369-388.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, **20**(1986), 87-96.

[3] M. K. Chakroborty and T. M. G. Ahsanullah, Fuzzy topology on fuzzy sets and tolerance topology, Fuzzy Sets and Systems, **45**(1992), 103-108.

[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl., **24**(1968), 182-190.

[5] K. C. Chattopadhyay and S. K. Samanta, Fuzzy topology: Fuzzy closure operator, fuzzy compactness and fuzzy connectedness, Fuzzy Sets and Systems, **54**(1993), 207-212.

[6] A. K. Chaudhuri and P. Das, Fuzzy connected sets in fuzzy topological spaces, Fuzzy Sets and Systems, **49**(1992), 223-229.

[7] Doğan Çoker, Fuzzy rough sets and Intuitionistic L-fuzzy sets, Fuzzy Sets and Systems, **96**(1998), 381-383.

[8] D. Çoker and S. ÖzcaǦ, On connectedness in intuitionistic fuzzy topological spaces, to appear in J. Math. and Math. Sci.

[9] D. Dubois and H. Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General systems, **17**(1990), 191-209.

[10] T. E. Gantner, R. C. Steinlage and W. H. Warren, Compactness in fuzzy topological spaces, J. Math. Anal. Appl., **62**(1978), 547-562.

[11] H. Hazra, S. K. Samanta and K. C. Chattopadhyay, Topology of intuitionistic fuzzy rough sets, Scientia Magna, **7**(2011), No. 3, 18-30.

[12] H. Hazra, T. K. Mondal and S. K. Samanta, Connectedness in topology of intuitionistic fuzzy sets, Advanced studies in contemporary Mathematics, **7**(2003), No. 2, 119-134.

[13] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl., **50**(1976), 621-623.

[14] R. Lowen, A comparision of different compactness notions in fuzzy topological spaces, J. Math. Anal. Appl., **64**(1978), 446-454.

[15] P. P. Ming and L. Y. Ming, Fuzzy topology I, Neighbourhood structure of a fuzzy point and Moore-Smith Convergence, J. Math. Anal. Appl., **76**(1980), 571-599.

[16] T. K. Mondal and S. K. Samanta, Intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets, The Journal of fuzzy Mathematics, **9**(2001), No. 3, 561-582.

[17] T. K. Mondal and S. K. Samanta, Topology of interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, **119**(2001), 483-494.

[18] T. K. Mondal and S. K. Samanta, Topology of interval-valued fuzzy sets, Indian J. Pure and Appl. Math., **30**(1999), No. 1, 23-38.

[19] S. Nanda and S. Majumdar, Fuzzy rough sets, Fuzzy Sets and Systems, **45**(1992), 157-160.

[20] Z. Pawlak, Rough sets, Internat. J. Inform. Comput. Sci., **11**(1982), No. 5, 341-356.

[21] S. Saha, Local connectedness in fuzzy setting, Simon Stevin, **61**(1987), 3-13.

[22] L. A. Zadeh, Fuzzy sets, Information and control, **8**(1965), 338-353.

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Some fixed point theorems in asymmetric metric spaces

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Abstract In this work, we define the concept of forward and backward contractions. Then, we prove the *Banach's* contraction principle in asymmetric metric spaces. Also, we prove a fixed point theorem in partially ordered asymmetric metric spaces.

Keywords Asymmetric metric space, fixed point theorems.

§1. Introduction

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$.

In the realms of applied mathematics and materials science we find many recent applications of asymmetric metric spaces, for example, in rate-independent models for plasticity [1], shape-memory alloys [2], and models for material failure [3].

There are other applications of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton-Jacobi equations [4] in mind. The study of asymmetric metrics apparently goes back to Wilson [5]. Following his terminology, asymmetric metrics are often called quasi-metrics.

Author in [6] has completely discussed on asymmetric metric spaces. Also, In [7], Aminpour, Khorshidvandpour and Mousavi have proved some useful results in asymmetric metric spaces. In this work we prove some theorems in asymmetric metric spaces. We start with some elementary definitions from [6].

Definition 1.1. A function $d : X \times X \to \mathbb{R}$ is an asymmetric metric and (X, d) is an asymmetric metric space if

(i) For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$,

(ii) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$. Henceforth, (X, d) shall be an asymmetric metric space.

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Example 1.1. Let $\alpha > 0$. Then $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by

$$d(x,y) = \begin{cases} x - y, & x \geq y; \\ \alpha(y - x), & y > x. \end{cases}$$

is obviously an asymmetric metric.

Definition 1.2. The forward topology τ^+ induced by d is the topology generated by the forward open balls

$$B^+(x,\varepsilon) = \{y \in X : \ d(x,y) < \varepsilon\} \quad \text{for } x \in X \ \ , \ \ \varepsilon > 0.$$

Likewise, the backward topology τ^- induced by d is the topology generated by the backward open balls

$$B^-(x,\varepsilon) = \{y \in X : \ d(y,x) < \varepsilon\} \quad \text{for } x \in X \ \ , \ \ \varepsilon > 0.$$

Definition 1.3. A sequence $\{x_k\}_{k\in\mathbb{N}}$ forward converges to $x_0 \in X$, respectively backward converges to $x_0 \in X$ if and only if

$$\lim_{k\to\infty} d(x_0, x_k) = 0, \text{ respectively, } \lim_{k\to\infty} d(x_k, x_0) = 0.$$

Then we write $x_k \xrightarrow{f} x_0$, $x_k \xrightarrow{b} x_0$ respectively.

Example 1.2. Let $(\mathbb{R}, d)$ be an asymmetric space, where d is as in example 1.1. It is easy to show that the sequence $\left\{x + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ $(x \in X)$ is both forward and backward converges to x.

Definition 1.4. Suppose (X, d_X) and (Y, d_Y) are asymmetric metric spaces. Let $f : X \to Y$ be a function. We say f is forward continuous at $x \in X$, respectively backward continuous, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^+(x, \delta)$ implies $f(y) \in B^+(f(x), \varepsilon)$, respectively $f(y) \in B^-(f(x), \varepsilon)$.

However, note that uniform forward continuity and uniform backward continuity are the same.

Definition 1.5. A set $S \subset X$ is forward compact if every open cover of S in the forward topology has a finite subcover. We say that S is forward relatively compact if $\bar{S}$ is forward compact, where $\bar{S}$ denotes the closure in the forward topology. We say S is forward sequentially compact if every sequence has a forward convergent subsequence with limit in S. Finally, $S \subset X$ is forward complete if every forward Cauchy sequence is forward convergent.

Note that there is a corresponding backward definition in each case, which is obtained by replacing forward with backward in each definition.

Lemma 1.1.[6] Let $d : X \times X \to \mathbb{R}^{\geq 0}$ be an asymmetric metric. If (X, d) is forward sequentially compact and $x_n \xrightarrow{b} x_0$ then $x_k \xrightarrow{f} x_0$.

Lemma 1.2. Let (x_n) be a backward Cauchy sequence in X. If (x_n) has a backward convergent subsequence, then (x_n) converges to the limit of it's subsequence.

As symmetric case.

§2. Banach′s contraction principle

In this section we prove *Banach, s* contraction principle in the sense of asymmetric metric spaces. First of all, we have the following definition. Throughout this section (X, d) denotes an asymmetric metric space, unless the contrary is specified.

Definition 2.1. A mapping $T : X \to X$ is said forward (backward) contraction when there exists $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)(d(Tx, Ty) \leq \alpha d(y, x)),$$

for each $x, y \in X$.

Example 2.1. Let $X = \mathbb{R} \geq 0$ be a nonempty set. Consider the map $d : X \times X \to \mathbb{R} \geq 0$ defined by

$$d(x, y) = \begin{cases} y - x, & y \geq x; \\ \frac{1}{4}(x - y), & x > y. \end{cases}$$

It is easy to show that (X, d) is an asymmetric metric space. Consider $T : X \to X$ by $Tx = \frac{1}{2}x$. Clearly, T is a forward contraction, whereas it is not backward. For this, let α be an arbitrary with $0 < \alpha < 1$. Set $x = 2^{-2\alpha}$ and $y = 2^{-\alpha}$. Then we have

$$d(Tx, Ty) = \frac{1}{2}(2^{-\alpha} - 2^{-2\alpha}), \ d(y, x) = \frac{1}{4}(2^{-\alpha} - 2^{-2\alpha}).$$

Since $0 < \alpha < 1$, $d(Tx, Ty) > \alpha d(y, x)$. Therefore, T is not a backward contraction.

Next, we prove asymmetric version of *Banach′s* contraction principle.

Theorem 2.1. Let (X, d) be a forward complete space. Let $T : X \to X$ be a forward contraction. Also, suppose that forward convergence implies backward convergence in X. Then T has an unique fixed point.

Proof. Choose $x_0 \in X$ and construct the sequence (x_n) as follow:

$$x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_n = T^nx_0, \cdots$$

Since T is a forward contraction we have

$$\begin{aligned} d(x_m, x_{m+1}) = d(Tx_{m-1}, Tx_m) &\leq \alpha d(x_{m-1}, x_m) = \alpha d(Tx_{m-2}, Tx_{m-1}) \\ &\leq \alpha^2 d(x_{m-2}, x_{m-1}) \\ &\leq \cdots \leq \alpha^m d(x_0, x_1). \end{aligned}$$

If $n > m$, then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1})d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \leq \frac{\alpha^m}{1 - \alpha} d(x_0, x_1). \end{aligned}$$

If m is sufficiently large, then the right side of the last inequality approaches to zero, since $0 < \alpha < 1$. Hence (x_n) is a forward Cauchy sequence in X. Since X is forward complete, so $x_n \xrightarrow{f} x \in X$ as $n \to \infty$. By assumption $x_n \xrightarrow{b} x$. Therefore we have

$$d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, x).$$

Now, the right side of last inequality approaches to zero as $n \to \infty$. Hence $d(x, Tx) = 0$ which means $Tx = x$. Now, let $y \in X$ be another fixed point of T, i.e., $Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y).$$

Since $0 < \alpha < 1$, so $d(x, y) = 0$ which implies $x = y$.

The following theorem is an another version of $Banach's$ contraction principle.

Theorem 2.2. Let (X, d) be a forward sequentially compact space and $T : X \to X$ backward contraction. Then T has an unique fixed point.

Proof. Choose $x_0 \in X$ and construct the sequence (x_n) as the proof of theorem 2.1. Then (x_n) is a backward Cauchy sequence in X. Since X is backward sequentially compact, so (x_n) has a backward convergent subsequence, say (x_{n_k}), which $x_{n_k} \xrightarrow{b} x \in X$ as $k \to \infty$. So, $x_n \xrightarrow{b} x$ by lemma 1.2, also by lemma 1.1, $x_n \xrightarrow{f} x$. Now

$$d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, x).$$

The right side of last inequality approaches to zero as $n \to \infty$. Hence $d(x, Tx) = 0$ which implies $Tx = x$. Now, Let $y \in X$ be another fixed point of T, i.e., $Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y).$$

Since $0 < \alpha < 1$, so $d(x, y) = 0$ which implies $x = y$.

Lemma 2.1. Any forward (backward) contraction is forward (backward) continuous. The proof is clear.

Authors in [8] have completely discussed on the fixed point of a nondecreasing continuous mappings on a partially ordered cone metric spaces. We wish to study the fixed points of nondecreasing contraction on a partially ordered asymmetric metric spaces.

Theorem 2.3. Let $(X, \sqsubseteq)$ be a partially ordered set and there exists a metric d on X such that (X, d) to be a backward complete asymmetric metric space. Consider $T : X \to X$ as a forward contraction and nondecreasing w.r.t. $\sqsubseteq$. If there exists $x_0 \in X$ with $x_0 \sqsubseteq Tx_0$ then T has a fixed point.

Proof. If $T(x_0) = x_0$, then there is nothing for proof. Now, let $T(x_0) \neq x_0$. Since $x_0 \sqsubseteq Tx_0$, we can obtain the following sequence by induction

$$x_0 \sqsubseteq T(x_0) \sqsubseteq T^2(x_0) \sqsubseteq T^3(x_0) \sqsubseteq \cdots \sqsubseteq T^n(x_0) \sqsubseteq T^{n+1}(x_0). \tag{1}$$

Also, we have

$$d(T^{n+1}(x_0), T^n(x_0)) \leq \alpha^n d(T(x_0), x_0). \tag{2}$$

For all $n \in \mathbb{N}$, since T is a forward contraction. By (1) and (2) we have

$$\begin{aligned} d(T^{n+2}(x_0), T^{n+1}(x_0)) &= d(T(T^{n+1}(x_0), T(T^n(x_0)) \\ &\leq \alpha d(T^{n+1}(x_0), T^n(x_0)) \\ &\leq \alpha^{n+1} d(f(x_0), x_0). \end{aligned}$$

Now, let $m > n$. Then

$$\begin{aligned} d(T^m(x_0), T^n(x_0)) &\leq d(T^m(x_0), T^{m-1}(x_0)) + \cdots + d(T^{n+1}(x_0), T^n(x_0)) \\ &\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) d(T(x_0), x_0) \\ &= \frac{\alpha^n - \alpha^m}{1-\alpha} d(T(x_0), x_0) \leq \frac{\alpha^n}{1-\alpha} d(T(x_0), x_0). \end{aligned}$$

Now, if n is sufficiently large, then we deduce $\{T^n(x_0)\}$ is a backward cauchy sequence in X. Since X is backward complete, so there exists $y \in X$ such that $T^n(x_0) \xrightarrow{b} y$ as $n \to \infty$. Finally, we prove that y is a fixed point of T. Fixed $\varepsilon > 0$. T is forward continuous by lemma 2.1. Hence there exists $\delta > 0$ so that $d(y, x) < \delta$ implies $d(T(y), T(x)) < \frac{\varepsilon}{2}$. Set $\gamma := \min\left\{\frac{\varepsilon}{2}, \delta\right\}$. Then there exists $N \in \mathbb{N}$ such that

$$d(T^n x_0, y) < \gamma, \text{ for all } n \geq N.$$

Now, we have

$$d(Ty, y) \leq d(Ty, T(T^n(x_0)) + d(T^{n+1}(x_0), y) < \frac{\varepsilon}{2} + \gamma \leq \varepsilon.$$

Since ε was arbitrary, so $d(Ty, y) = 0$ which implies $Ty = y$.

Remark 2.1. Taslim [9], introduced the concept of denseness in asymmetric metric spaces. In particular, she proved that if X is forward and backward compact and $Y \subset X$ both of backward and forward in X, then $\tau^+ = \tau^-$. Therefore, in the case, all of results in the literature go to symmetric case.

References

[1] A. Mainik and A. Mielke, Existence results for energetic models for rate-independent systems, Calc. Var. Partial Differential Equations, **22**(2005), 73-99.

[2] A. Mielke and T. Roubcek, A rate-independent model for inelastic behavior of shape-memory alloys, Multiscale Model. Simul, **1**(2003), 571-597, (electronic).

[3] MO. Rieger and J. Zimmer, Young measure flow as a model for damage, Preprint 11/05, Bath Institute for Complex Systems, Bath, UK, 2005.

[4] A. Mennucci, On asymmetric distances, Technical report, Scuola Normale Superiore, Pisa, 2004.

[5] WA. Wilson, On quasi-metric spaces, Amer. J. Math, **53**(1931), 675-684.

[6] J. Collins and J. Zimmer, An asymmetric Arzela-Ascoli theorem, Topology and it's Applications, **154**(2007), 2312-2322.

[7] AM. Aminpour, S. Khorshidvandpour and M. Mousavi, Some results in asymmetric metric spaces, Mathematica Aeterna, **2**(2012), 533-540.

[8] I. Altun, B. Damjanovic and D. Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, Applied Mathematics Letters, **23**(2010), 310-316.

[9] L. Taslim, On denseness in asymmetric metric spaces, Mathematica Aeterna, **2**(2012), 617-627.

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Schur harmonic convexity of Stolarsky extended mean values

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Abstract In this paper, the Schur harmonic convexity of Stolarsky extended mean values are discussed.

Keywords Stolarsky, harmonic, concavity, convexity.
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§1. Introduction and preliminaries

In literature the importance and applications of means and its inequality to science and technology is explored by eminent researchers see [3]. In [20-28], we studied some results on contra harmonic mean. In [11,41], authors studied the different properties of the so-called Stolarsky (extended) two parameter mean values, is defined as follows:

$$E_{r,s}(a,b)=\begin{cases}\left[\frac{r}{s}\frac{(a^s-b^s)}{(a^r-b^r)}\right]^{\frac{1}{s-r}}, & rs(s-r)\neq 0;\\ exp\left(\frac{-1}{s}+\frac{(a^s lna-b^s lnb)}{(a^s-b^s)}\right), & r=s\neq 0;\\ \left(\frac{(a^s-b^s)}{s(lna-lnb)}\right)^{\frac{1}{s}}, & r=0, s\neq 0;\\ \sqrt{ab}, & r=s=0;\\ a, & a=b>0.\end{cases} \tag{1}$$

Some of the classical two parameter means are special cases.

Here we recall some of the known means which are essential for the paper, the arithmetic mean in weighted form,

$$A_{p,q}(a,b)=pa+qb=A(a,b;p,q);$$

such that $a,b>0$ and $p+q=1$, where p and q are the weights.

The Stolarsky means $S_{p;q}(a,b)$ are C^∞ function on the domain (p,q,a,b), $p,\ q\in R,\ a,\ b>0$. Obviously, Stolarsky means $S_{p;q}(a,b)$ are symmetric with respect to $a,\ b$ and $p,\ q$. Most of the classical two variable means are special cases of $S_{p;q}(a,b)$, Stolarsky mean. For example:

$$E_{1,2}(a,b)=\frac{a+b}{2},\qquad \text{is the Arithmetic mean;}$$

$$E_{0,0}(a,b)=E_{-1,-1}(a,b)=\sqrt{ab},\qquad \text{is the Geometric mean;}$$

$$E_{-1,-2}(a,b)=\frac{2ab}{a+b},\qquad \text{is the Harmonic mean;}$$

$$E_{0,1}(a,b)=\frac{a-b}{lna-lnb},\qquad \text{is the Logarithmic mean;}$$

$$E_{1,1}(a,b)=\frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}},\qquad \text{is the Identric mean;}$$

$$E_{r,2r}(a,b)=\left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}},\qquad \text{is the } r^{th} \text{ Power mean.}$$

The basic properties of Stolarsky means, as well as their comparison theorems, log-convexities, and inequalities are studied in papers [4,9,13,16,17,19,22,23,30-35,44,49,50,51,52,55].

In recent years, the Schur convexity and Schur geometrically convexity of $S_{p;q}(a,b)$ have attracted the attention of a considerable number of mathematicians [5,6,18,36,38,40]. Qi [37] first proved that the Stolarsky means $S_{p;q}(a,b)$ are Schur convex on $(-\infty,0]\times(-\infty,0]$ and Schur concave on $[0,\infty)\times[0,\infty)$ with respect to (p,q) for fixed $a,b>0$ with $a\neq b$. Yang [53] improved Qi's result and proved that Stolarsky means $S_{p;q}(a,b)$ are Schur convex with respect to (p,q) for fixed $a,b>0$ with $a\neq b$ if and only if $p+q<0$ and Schur concave if and only if $p+q>0$.

Qi [36] tried to obtain the Schur convexity of $S_{p;q}(a,b)$ with respect to (a,b) for fixed (p,q) and declared an incorrect conclusion. Shi [40] observed that the above conclusion is wrong and obtained a sufficient condition for the Schur convexity of $S_{p;q}(a,b)$ with respect to (a,b). Chu and Zhang [6] improved Shi's results and gave an necessary and sufficient condition. This perfectly solved the Schur convexity of Stolarsky means with respect to (a,b).

For the Schur geometrically convexity, Chu and Zhang [5] proved that Stolarsky means $S_{p;q}(a,b)$ are Schur geometrically convex with respect to $(a,b)\in(0,\infty)\times(0,\infty)$ if $p+q\geq 0$ and Schur geometrically concave if $p+q\leq 0$. Li [18] also investigated the Schur geometrically convexity of generalized exponent mean $I_p(a,b)$.

The purpose of this paper is to investigate another type of Schur convexity that is the Schur harmonic convexity of Stolarsky means $S_{p;q}(a,b)$, of which the idea is to find the necessary conditions from lemma 2, and then prove these conditions are sufficient.

In [3], the weighted contra harmonic mean is defined on the basis of proportions by,

$$C_{p,q}(a,b)=\frac{pa^2+qb^2}{pa+qb}=C(a,b;p,q),$$

such that $a,\ b>0$ and $p+q=1$, where p and q are the weights.

This work motivates us to introduce a new family of Stolarsky's extended type mean values in weighted forms in two and n variables.

For two variables $a,\ b>0$, $p,\ q\in R$ and $p,\ q$ are the weights, such that $p+q=1$, then consider a new mean in the following form:

$$N_{r,s}(a,b;p,q)=\left[\frac{r^2}{s^2}\frac{C(a^s,b^s;p,q)-A(a^s,b^s;p,q)}{C(a^r,b^r;p,q)-A(a^r,b^r;p,q)}\right]^{\frac{1}{s-r}},$$

which is equivalently,

$$N_{r,s}(a,b;p,q) = \left[\frac{r^2}{s^2}\left(\frac{pa^r+qb^r}{pa^s+qb^s}\right)\left(\frac{pa^2s+qb^2s-(pa^s+qb^s)^2}{pa^2r+qb^2r-(pa^r+qb^r)^2}\right)\right]^{\frac{1}{s-r}},$$

which is equivalently,

$$N_{r,s}(a,b;p,q) = \left[\frac{r^2}{s^2}\left(\frac{pa^r+qb^r}{pa^s+qb^s}\right)\left(\frac{a^s-b^s}{a^r-b^r}\right)^2\right]^{\frac{1}{s-r}}.$$

In [50], the authors introduced the homogeneous function with two parameters r and s by,

$$H_f(a,b;s,r) = \left[\frac{f(a^s,b^s)}{f(a^r,b^r))}\right]^{\frac{1}{s-r}},$$

and studied its monotonicity and deduces some inequalities involving means, where f is a homogeneous function for a and b.

In particular, $f = A$, is the arithmetic mean of a and b.

$$H_A(a,b;s,r) = \begin{cases} \left[\frac{(a^s+b^s)}{(a^r+b^r)}\right]^{\frac{1}{s-r}}, & r \neq s; \\ G_{A,p}(a,b) = a^{\frac{a^s}{a^s+b^s}} b^{\frac{b^s}{a^s+b^s}}, & r = s \neq 0; \\ \sqrt{ab}, & r = s = 0; \\ a, & a = b > 0. \end{cases} \tag{2}$$

Here, $G_{A,s}(a,b) = Z_s(a,b) = Z^{\frac{1}{s}}(a^p,b^p) = Z_s$. $Z(a,b) = a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}$ is named power-exponential mean between two positive numbers a and b.

In weighted form,

$$H_{A(a,b;p,q)}(a,b;s,r) = \begin{cases} \left[\frac{(pa^s+qb^s)}{(pa^r+qb^r)}\right]^{\frac{1}{s-r}}, & r \neq s; \\ G_{A(a,b;p,q),s}(a,b) = a^{\frac{pa^s}{pa^s+qb^s}} b^{\frac{qb^s}{pa^s+qb^s}}, & r = s \neq 0; \\ a^p b^q, & r = s = 0; \\ a, & a = b > 0. \end{cases} \tag{3}$$

In [42], author introduced and studied the various properties and log-convexity results of the class W of weighted two parameter means is given by

$$W_{r,s}(a,b;p,q) = \begin{cases} \left[\frac{r^2}{s^2}\left(\frac{pa^r+qb^r-a^{ps}b^{qs}}{pa^s+qb^s-a^{ps}b^{qs}}\right)\right]^{\frac{1}{s-r}}, & rs(s-r)(a-b) \neq 0; \\ \left[\frac{2}{ln^2(a/b)}\left(\frac{pa^r+qb^r-a^{ps}b^{qs}}{pqs^2}\right)^2\right]^{\frac{1}{s}}, & s(a-b) \neq 0, r = 0; \\ exp\left(\frac{-2}{s} + \frac{pa^s lna + qb^s lnb - (plna + qslnb)a^{ps}b^{qs}}{pa^r+qb^r-a^{ps}b^{qs}}\right), & r = s, s \neq 0; \\ a^{(p+1)/3} b^{(q+1)/3}, & a \neq b, r = s = 0; \\ a, & a = b > 0. \end{cases} \tag{4}$$

The above definitions leads to express the mean values $N_{r,s}(a,b;p,q)$ in the following form:

$$N_{r,s}(a,b;p,q) = \left[\frac{pa^r+qb^r}{pa^s+qb^s}\right]^{\frac{1}{s-r}} \left[\left(\frac{r}{s}\frac{(a^s-b^s)}{(a^r-b^r)}\right)^{\frac{1}{s-r}}\right]^2,$$

which is equivalently

$$N_{r,s}(a,b;p,q) = \left[\frac{f(a^s,b^s;p,q)}{f(a^r,b^r;p,q))}\right]^{\frac{1}{s-r}} E^2_{r,s}(a,b),$$

or

$$N_{r,s}(a,b;p,q) = H_{f=A(a,b;p,q)}(a,b;s,r)E^2_{r,s}(a,b). \tag{5}$$

Here $f = A(a,b;p,q)$ is arithmetic mean in weighted form.

The various properties and identities concerning to $N_{r,s}(a,b;p,q)$ are also listed. The laborious calculations gives the following different cases of the mean value $N_{r,s}(a,b;p,q)$.

$$N_{r,s}(a,b;p,q) = \begin{cases} \left[\frac{r^2}{s^2}\left(\frac{pa^r+qb^r}{pa^s+qb^s}\right)\left(\frac{a^s-b^s}{a^r-b^r}\right)^2\right]^{\frac{1}{s-r}}, & rs(s-r)(a-b) \neq 0; \\ \left[\frac{2}{ln^2(a/b)}\left(\frac{1}{pa^s+qb^s}\right)\left(\frac{a^s-b^s}{s}\right)^2\right]^{\frac{1}{s}}, & s(a-b) \neq 0, r=0; \\ exp\left(\frac{-2}{s} - \frac{pa^s lna+qb^s lnb}{pa^s+qb^s} + 2\frac{a^s lna-b^s lnb}{a^s-b^s}\right), & r=s, s\neq 0; \\ a^{1-p}b^{1-q}, & a\neq b, r=s=0; \\ a, & a=b>0. \end{cases} \tag{6}$$

§2. Definition and properties

Schur convexity was introduced by Schur in 1923 [24], and it has many important applications in analytic inequalities [2,12,54], linear regression [43], graphs and matrices [8], combinatorial optimization [15], information theoretic topics [10], Gamma functions [25], stochastic orderings [39], reliability [14], and other related fields. Recently, Anderson [1] discussed an attractive class of inequalities, which arise from the notation of harmonic convexity. For convenience of readers, we recall some definitions as follows.

We recall the definitions which are essential to develop this paper.

Definition 2.1.[24,46] Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in R^n$,

1. x is majorized by y, (in symbol $x \prec y$). If $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ and $\sum_{i=1}^n x_{[i]} \leq \sum_{i=1}^n y_{[i]}$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in descending order.

2. $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$. Let $\Omega \in R^n (n \geq 2)$. The function $\varphi : \Omega \to R$ is said to be decreasing if and only if $-\varphi$ is increasing.

3. $\Omega \subseteq R^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \cdots, \alpha x_n + \beta y_n)$ for every x and $y \in \Omega$ where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

4. Let $\Omega \subseteq R^n$ the function $\varphi : \Omega \to R$ be said to be a schur convex function on Ω if $x \leq y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a schur concave function on Ω if and only if $-\varphi$ is schur convex.

Definition 2.2.[55] Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in R^n_+$, $\Omega \subseteq R^n$ is called Harmonically convex set if $(x_1^\alpha y_1^\beta, \cdots, x_1^\alpha y_1^\beta) \in \Omega$ for all x and $y \in \Omega$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

Let $\Omega \subseteq R^n_+$, the function $\varphi : \Omega \to R_+$ is said to be schur Harmonically convex function on Ω, if $(lnx_1, \cdots, ln_n) \prec (lny_1, \cdots, lny_n)$ on Ω implies $\varphi(x) \leq \varphi(y)$. Then φ is said to be a schur Harmonically concave function on Ω if and only if $-\varphi$ is schur Harmonically convex.

Definition 2.3.[24,46] Let $\Omega \subseteq R^n$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P, the function $\varphi : \Omega \to R$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Definition 2.4.[24,46] Let $\Omega \subseteq R^n$, $\varphi : \Omega \to R$ is called symmetric and convex function. Then φ is schur convex on Ω.

Lemma 2.1.[55] Let $\Omega \subseteq R^n$ be symmetric with non empty interior Harmonically convex set and let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and

$$(x_1 - x_2)(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}) \geq 0 (\leq 0), \tag{7}$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$, then φ is a schur-Harmonically convex (Schur-Harmonically concave function.

Lemma 2.2.[40] Let $a \leq b$, $u(t) = ta + (1-t)b$, $v(t) = tb + (1-t)a$, $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$, $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$, then

$$\frac{(a+b)}{2} \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)). \tag{8}$$

§3. Main results

In this section, we shall prove some the lemmas required for proving main theorem.

Lemma 3.1. Stolarsky's extended family type means $N_{p,q}(a, b; r, s)$ are schur-Harmonic convex or Schur-Harmonic concave with respect to $(a, b) \in (0, \infty) \times (0, \infty)$ if and only if $g(t) \geq 0$ or $g(t) \leq 0$ for all $t > 0$, where

$g(t) = g_{p,q}(t)$

$$= \begin{cases} \left[\frac{(p-q)\sinh At\{\cosh(p+q)t + 3\cosh(p-q)t\} - (p\sinh Bt + q\sinh Ct)\{3\cosh(p+q)t + \cosh(p-q)t\}}{pq(p-q)}\right], & pq(p-q) \neq 0; \\ \left[\frac{2qt\cosh(2q+1)t - \cosh t(4\sinh 2qt - 6qt)}{q^2}\right], & p = 0, q \neq 0; \\ \left[\frac{2pt\cosh(2p+1)t - \cosh t(4\sinh 2pt - 6pt)}{p^2}\right], & p \neq 0, q = 0; \\ \left[\frac{(3+\cosh 2qt)\sinh(2q+1)t - (2qt\cosh t + \sinh t)(1 + 3\cosh 2qt)}{q^2}\right], & p = q \neq 0; \\ \left[4t^2 \sinh t\right], & p = q \neq 0, \end{cases} \tag{9}$$

and

$$A = p + q + 1, \quad B = p - q + 1, \quad C = p - q - 1, \quad D = p + q, \quad E = p - q. \tag{10}$$

Proof. Let Stolarsky's extended family type mean $N = N_{p,q}(a,b;r,s)$ defined for $pq(p-q) \neq 0$ as

$$N = N_{p,q}(a,b;r,s) = \left[\frac{q^2}{p^2}\left(\frac{ra^p + sb^p}{ra^q + sb^q}\right)\left(\frac{a^p - b^p}{a^q - b^q}\right)^2\right]^{\frac{1}{p-q}}. \quad (11)$$

Let $r = s = \frac{1}{2}$, take log on both sides and differentiate partially with respect to a and multiply by a^2, gives

$$a^2\frac{\partial N}{\partial a} = \frac{N}{p-q}\left[\frac{qa^{q+1}}{a^q + b^q} - \frac{pa^{p+1}}{a^p + b^p} + 2\frac{pa^{p+1}}{a^p - b^p} - 2\frac{qa^{q+1}}{a^q - b^q}\right]. \quad (12)$$

Similarly,

$$b^2\frac{\partial N}{\partial a} = \frac{N}{p-q}\left[\frac{qb^{q+1}}{a^q + b^q} - \frac{pb^{p+1}}{a^p + b^p} + 2\frac{pb^{p+1}}{a^p - b^p} - 2\frac{qb^{q+1}}{a^q - b^q}\right], \quad (13)$$

then,

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) = \frac{(a-b)N}{p-q}\left[\Delta\right], \quad (14)$$

where,

$$\Delta = \left[\frac{q(a^{q+1} - b^{q+1})}{a^q + b^q} - \frac{p(a^{p+1} - b^{p+1})}{a^p + b^p} + 2\frac{p(a^{p+1} + b^{p+1})}{a^p - b^p} - 2\frac{q(a^{q+1} + b^{q+1})}{a^q - b^q}\right].$$

Substituting $\ln\sqrt{a/b} = t$ and using $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$, $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$, we have

$$\Delta = \sqrt{ab}\left[q\frac{\sinh(q+1)t}{\cosh qt} - p\frac{\sinh(p+1)t}{\cosh pt} + 2p\frac{\cosh(p+1)t}{\sinh pt} - 2q\frac{\cosh(q+1)t}{\sinh qt}\right].$$

Using the product into sum formula for hyperbolic functions leads to: For $pq(p-q) \neq 0$

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) = \frac{(pq)N(a-b)\sqrt{ab}}{2\sinh pt\sinh qt\cosh pt\cosh qt}\left[g_{p,q}(t)\right], \quad (15)$$

where,

$$g_{p,q}(t) = \left[\frac{(p-q)\sinh At\{\cosh(p+q)t + 3\cosh(p-q)t\}}{pq(p-q)}\right]$$

$$-\left[\frac{(p\sinh Bt + q\sinh Ct)\{3\cosh(p+q)t + \cosh(p-q)t\}}{pq(p-q)}\right]. \quad (16)$$

In case of $p \neq q = 0$. Since $N_{p,q} \in C^\infty$, we have

$$\frac{\partial N_{p,0}}{\partial a} = \lim_{q\to 0}\frac{\partial N_{p,q}}{\partial a}, \qquad \frac{\partial N_{p,0}}{\partial b} = \lim_{q\to 0}\frac{\partial N_{p,q}}{\partial b},$$

$$\frac{\partial N_{p,p}}{\partial a} = \lim_{q\to p}\frac{\partial N_{p,p}}{\partial a}, \qquad \frac{\partial N_{p,p}}{\partial b} = \lim_{q\to 0}\frac{\partial N_{p,q}}{\partial b},$$

$$\frac{\partial N_{0,0}}{\partial a} = \lim_{p\to 0}\frac{\partial N_{p,q}}{\partial a}, \qquad \frac{\partial N_{0,0}}{\partial b} = \lim_{p\to 0}\frac{\partial N_{p,q}}{\partial b},$$

Thus

$$(a-b)\left(a^2\frac{\partial N_{p,0}}{\partial a}-b^2\frac{\partial N_{p,0}}{\partial b}\right)=\lim_{q\to 0}\left[(a-b)\left(a^2\frac{\partial N_{p,0}}{\partial a}-b^2\frac{\partial N_{p,0}}{\partial b}\right)\right]$$

$$=\lim_{q\to 0}\left(\frac{(pq)N_{p,q}(a-b)\sqrt{ab}}{2\sinh pt\sinh qt\cosh pt\cosh qt}\left[g_{p,q}(t)\right]\right)=\frac{(p)N_{p,0}(a-b)\sqrt{ab}}{2t\sinh pt\cosh pt}\left[g_{p,0}(t)\right].$$

Likewise for $q\neq p=0$,

$$(a-b)\left(a^2\frac{\partial N_{0,q}}{\partial a}-b^2\frac{\partial N_{0,q}}{\partial b}\right)=\lim_{p\to 0}\left[(a-b)\left(a^2\frac{\partial N_{0,q}}{\partial a}-b^2\frac{\partial N_{0,q}}{\partial b}\right)\right]$$

$$=\lim_{p\to 0}\left(\frac{(pq)N_{p,q}(a-b)\sqrt{ab}}{2\sinh pt\sinh qt\cosh pt\cosh qt}\left[g_{p,q}(t)\right]\right)=\frac{(q)N_{0,q}(a-b)\sqrt{ab}}{2t\sinh qt\cosh qt}\left[g_{0,q}(t)\right].$$

For $q=p\neq 0$,

$$(a-b)\left(a^2\frac{\partial N_{p,q}}{\partial a}-b^2\frac{\partial N_{p,q}}{\partial b}\right)=\lim_{q\to p}\left[(a-b)\left(a^2\frac{\partial N_{p,p}}{\partial a}-b^2\frac{\partial N_{p,p}}{\partial b}\right)\right]$$

$$=\lim_{q\to p}\left(\frac{(pq)N_{p,q}(a-b)\sqrt{ab}}{2\sinh^2 pt\cosh^2 pt}\left[g_{p,p}(t)\right]\right)=\frac{(p^2)N_{p,q}(a-b)\sqrt{ab}}{2\sinh pt\cosh pt}\left[g_{p,p}(t)\right].$$

For $q=p=0$,

$$(a-b)\left(a^2\frac{\partial N_{0,0}}{\partial a}-b^2\frac{\partial N_{0,0}}{\partial b}\right)=\lim_{p\to 0}\left[(a-b)\left(a^2\frac{\partial N_{p,p}}{\partial a}-b^2\frac{\partial N_{p,p}}{\partial b}\right)\right]$$

$$=\lim_{p\to 0}\left(\frac{(pq)N_{p,p}(a-b)\sqrt{ab}}{2\sinh^2 pt\cosh^2 pt}\left[g_{p,p}(t)\right]\right)=\frac{N_{0,0}(a-b)\sqrt{ab}}{2t^2}\left[g_{0,0}(t)\right].$$

By summarizing all cases above yield

$$(a-b)\left(a^2\frac{\partial N}{\partial a}-b^2\frac{\partial N}{\partial b}\right)=\begin{cases}\frac{(pq)N(a-b)\sqrt{ab}}{2\sinh pt\sinh qt\cosh pt\cosh qt}\left[g_{p,q}(t)\right], & pq(p-q)\neq 0;\\ \frac{(q)N_{0,q}(a-b)\sqrt{ab}}{2t\sinh qt\cosh qt}\left[g_{0,q}(t)\right], & p=0,q\neq 0;\\ \frac{(p)N_{p,0}(a-b)\sqrt{ab}}{2t\sinh pt\cosh pt}\left[g_{p,0}(t)\right], & p\neq 0,q=0;\\ \frac{(p^2)N_{p,q}(a-b)\sqrt{ab}}{2\sinh pt\cosh pt}\left[g_{p,p}(t)\right], & p=q\neq 0;\\ \frac{N_{0,0}(a-b)\sqrt{ab}}{2t^2}\left[g_{0,0}(t)\right], & p=q\neq 0.\end{cases} \quad (17)$$

Since $(a-b)\left(a^2\frac{\partial N}{\partial a}-b^2\frac{\partial N}{\partial b}\right)$ is symmetric with respect to a and b, without loss of generality we assume $a>b$, then $t=\ln\sqrt{a/b}>0$. It is easy to verify that $\frac{N(a-b)\sqrt{ab}}{2}>0$, $\frac{p}{\sinh pt}$, $\frac{q}{\sinh qt}>0$, if $pq\neq 0$ for $t>0$. Thus by lemma 2 Stolarsky means $S_{p,q}(a,b)$ are Schur harmonic convex (Schur harmonic concave) with respect to $(a,b)\in(0,\infty)\times(0,\infty)$, if and only if $(a-b)\left(a^2\frac{\partial N}{\partial a}-b^2\frac{\partial N}{\partial b}\right)(\geq)(\leq)0$, if and only if $g(t)=g_{p,q}(t)(\geq)(\leq)0$ for all $t>0$. This completes the proof of lemma 3.1.

Lemma 3.2. The function $g(t)=g_{p,q}(t)$ defined by (3.1) and $g'(t)=\frac{\partial g_{p,q}(t)}{\partial t}$ both are symmetric with respect to p and q, and both continuous with respect to p and q on $R\times R$.

Proof. It is easy to check that $g_{p,q}(t)$ and $\frac{\partial g_{p,q}(t)}{\partial t}$ are symmetric with respect to p and q, then $\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{q,p}(t)}{\partial t}$.

By lemma 3.1, we note that $g(t) = g_{p,q}(t)$ is continuous with respect to p and q on $R \times R$. Finally, we prove that $g'(t) = \frac{\partial g_{p,q}(t)}{\partial t}$ is also continuous with respect to p and q on $R \times R$.

A simple calculations yield,

Case i: For $pq(p-q) \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{(p-q)A\cosh At\{\cosh(p+q)t + 3\cosh(p-q)t\}}{pq(p-q)}$$

$$+\frac{(p-q)\sinh At\{(p+q)\sinh(p+q)t + 3(p-q)\sinh(p-q)t\}}{pq(p-q)}$$

$$-\frac{(pB\cosh Bt + qC\cosh Ct)\{3\cosh(p+q)t + \cosh(p-q)t\}}{pq(p-q)}$$

$$\frac{-(p\sinh Bt + q\sinh Ct)\{3(p+q)\sinh(p+q)t + (p-q)\sinh(p-q)t\}}{pq(p-q)}.$$

Case ii: For $q = 0, p \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{2p\cosh(1+2p)t + 2pt(1+2p)\sinh(2p+1)t - 6p\cosh t}{-p^2}$$

$$-\frac{6pt\sinh t - 2(2p+1)\cosh(2p+1)t - (2p-1)\cosh(2p-1)t}{-p^2}.$$

Case iii: For $p = 0, q \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{2q\cosh(1+2q)t + 2qt(1+2q)\sinh(2q+1)t - 6q\cosh t}{-p^2}$$

$$-\frac{6qt\sinh t - 2(2q+1)\cosh(2q+1)t - (2q-1)\cosh(2q-1)t}{-p^2}.$$

Case iv: For $p = q \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{(1+2q)(3+\cosh 2qt)\cosh(2q+1)t - 6q(\sinh 2qt)(2qt\cosh t + \sinh t)}{q^2}$$

$$-\frac{(1+3\cosh 2qt)((1+2q)\cosh t + 2qt\sinh t) + 2q(\sinh 2qt)\sinh(2q+1)t}{q^2}.$$

Case v: For $p = q = 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = 4t^2\cosh t + 8t\sinh t.$$

It is obvious that $\frac{\partial g_{p,q}(t)}{\partial t}$ is continuous with respect to p and q on $R \times R$, again in view of

$$\lim_{q\to 0}\frac{\partial g_{p,q}(t)}{\partial t}=\frac{\partial g_{p,0}(t)}{\partial t}; \qquad \lim_{p\to 0}\frac{\partial g_{p,q}(t)}{\partial t}=\frac{\partial g_{0,q}(t)}{\partial t};$$

$$\lim_{q\to p}\frac{\partial g_{p,q}(t)}{\partial t}=\frac{\partial g_{p,p}(t)}{\partial t}; \qquad \lim_{p\to 0}\frac{\partial g_{p,p}(t)}{\partial t}=\frac{\partial g_{0,0}(t)}{\partial t}.$$

These arguments leads that the above five cases are continuous for all values of p and q. This completes the proof of lemma 3.2.

Lemma 3.3. $\lim_{t\to 0, t>0} t^{-3}g(t)=\frac{-4}{3}(p+q-3)$.

Proof.

It is easy to check that first and second derivatives of $g(t)=0$, at $t=0$. In the case of $pq(p-q)\neq 0$. Applying L-Hospital's rule (three times) yields

$$\lim_{t\to 0,\ t>0}\frac{g_{p,q}(t)}{t^3}=\lim_{t\to 0,\ t>0}\frac{g'_{p,q}(t)}{3t^2}=\cdots=\lim_{t\to 0,\ t>0}\frac{g'''_{p,q}(t)}{6}=\frac{-4}{3}(p+q-3).$$

Similarly, for $p=0,\ q\neq 0$,

$$\lim_{t\to 0,\ t>0}\frac{g_{0,q}(t)}{t^3}=\lim_{t\to 0,\ t>0}\frac{g'_{0,q}(t)}{3t^2}=\cdots=\lim_{t\to 0,\ t>0}\frac{g'''_{0,q}(t)}{6}=\frac{-4}{3}(q-3);$$

for $q=0,\ p\neq 0$,

$$\lim_{t\to 0,\ t>0}\frac{g_{p,0}(t)}{t^3}=\lim_{t\to 0,\ t>0}\frac{g'_{p,0}(t)}{3t^2}=\cdots=\lim_{t\to 0,\ t>0}\frac{g'''_{p,0}(t)}{6}=\frac{-4}{3}(p-3);$$

for $p=q\neq 0$,

$$\lim_{t\to 0, t>0}\frac{g_{p,p}(t)}{t^3}=\lim_{t\to 0,\ t>0}\frac{g'_{p,p}(t)}{3t^2}=\cdots=\lim_{t\to 0,\ t>0}\frac{g'''_{p,p}(t)}{6}=\frac{-4}{3}(2p-3);$$

for $p=q=0$,

$$\lim_{t\to 0,\ t>0}\frac{g_{0,0}(t)}{t^3}=\lim_{t\to 0,\ t>0}\frac{g'_{0,0}(t)}{3t^2}=\cdots=\lim_{t\to 0,\ t>0}\frac{g'''_{0,0}(t)}{6}=4.$$

This completes the proof.

The proof of our main result stated below follows from the above lemmas:

Theorem 3.1. For fixed $(p,q)\in R\times R$,

(1) Stolarsky's extended family type mean means $N_{r,s}(a,b;p,q)$ are Schur harmonic convex with respect to (a,b) if $p+q-3\geq 0$.

(2) Stolarsky's extended family type mean means $N_{r,s}(a,b;p,q)$ are Schur harmonic concave if $p+q-3\leq 0$.

References

[1] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, J. Math. Anal. Appl., **335**(2007), No. 2, 1294-1308.

[2] J. S. Aujla and F. C. Silva, Weak majorization inequalities and convex functions, Linear Algebra Appl., **369**(2003), 217-233.

[3] P. S. Bullen, Handbook of means and their inequalities, Kluwer Acad. Publ., Dordrecht, 2003.

[4] Ch. P. Chen, The monotonicity of the ratio between Stolarsky means, RGMIA Res. Rep. Coll., **11**(2008), No. 4, available online at http://www.staff.vu.edu.au/rgmia/v11n4.asp.

[5] Y. M. Chu, X. M. Zhang and G. D. Wang, The Schur geometrical convexity of the extended mean values, J. Convex. Anal., **15**(2008), No. 4, 707-718.

[6] Y. M. Chu and X. M. Zhang, Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave, J. Math. Kyoto Univ., **48**(2008), No. 1, 229-238.

[7] Y. M. Chu and Y. P. Lv, The Schur harmonic convexity of the Hamy symmetric function and its applications, J. Inequal. Appl., Vol. 2009, Art. ID 838529, 10 pages, available online at http://downloads.hindawi.com/journals/jia/2009/838529.pdf.

[8] G. M. Constantine, Schur convex functions on the spectra of graphs, Disc. Math., **45**(1983), No. 2-3, 181-188.

[9] P. Czinder and Zs. Pes, An extension of the Hermite-Hadmard inequality and an application for Gini and stolarsky means, J. Inequal. Pure Appl. Math., **5**(2004), No. 2, Art. 42, available online at http://jipam.vu.edu.au/images/167-03.pdf.

[10] A. Forcina and A. Giovagnoli, Homogeneity indices and Schur-convex functions, Statistica, **42**(1982), No. 4, 529-542.

[11] G. H. Hardy, J. E. Littlewood and G. Pòlya, Inequalities, 2nd edition, Cambridge University Press, Cambridge, 1959.

[12] G. H. Hardy, J. E. Littlewood and G. Pòlya, Some simple inequalities satisfied by convex functions, Messenger Math., **58**(1929), 145-152.

[13] Peter. A. Hasto, A montonicity property of ratios of symmetric homogeneous means, J. Inequal. Pure Appl. Math., **3**(2002), No. 5, Art. 71, available online at http://jipam-old.vu.edu.au/v3n5/013-02.pdf.

[14] F. K. Hwang, U. G. Rothblum and L. Shepp, Monotone optimal multipartitions using Schur convexity with respect to partial orders, SIAM J. Discrete Math., **6**(1993), No. 4, 533-547.

[15] F. K. Hwang and U. G. Rothblum, Partition optimization with Schur convex sum objective functions, SIAM J. Discrete Math., **18**(2004), No. 3, 512-524.

[16] E. B. Leach and M. C. Sholander, Extended mean values, Amer. Math. Monthly., **85**(1978), 84-90.

[17] E. B. Leach and M. C. Sholander, Extended mean values II, J. Math. Anal. Appl., **92**(1983), 207-223.

[18] D. M. Li and H. N. Shi, Schur convexity and Schur-geometrically concavity of generalized exponent mean, J. Math. Inequal., **3**(2009), No. 2, 217-225.

[19] Zh. Liu, Minkowski's inequality for extended mean values, Proceedings of the Second ISAAC Congress, Int. Soc. Anal. Appl. Comput., Kluwer Acad. Publ., Dordrecht, **1**(1999), 585-592.

[20] V. Lokesha, S. Padmanabhan, K. M. Nagaraja and Yilmaz Simsek, Relation between

Greek means and various other means, General Mathematics, **17**(2009), No. 3, 3-13.

[21] V. Lokesha, K. M. Nagaraja, Naveen Kumar. B and S. Padmanabhan, Oscillatory type mean in Greek means, Int. e-Journal of Engg. Maths Theory and Applications, **9**(2010), No. 3, 18-26.

[22] L. Losonczi and Zs. Pes, Minkowski's inequality for two variable dirence means, Proc. Amer. Math. Soc., **126**(1998), 779-789.

[23] L. Losonczi, Ratio of Stolarsky means: monotonicity and comparison, Publ. Math. Debrecen, **75**(2009), No. 1-2, art. 18.

[24] A. W. Marshall and I. Olkin, Inequalities: theorey of majorization and its applications, New York, Academic Press, 1979.

[25] M. Merkle, Convexity, Schur-convexity and bounds for the gamma function involving the digamma function, Rocky Mountain J. Math., **28**(1998), No. 3, 1053-1066.

[26] K. M. Nagaraja, V. Lokesha and S. Padmanabhan, A simple proof on strengthening and extension of inequalities, Advn. Stud. Contemp. Math., **17**(2008), No. 1, 97-103.

[27] K. M. Nagaraja and P. Siva Kota Reddy, α-Centroidal mean and its dual, Proceedings of the Jangjeon Math. Soc., **15**(2012), No. 2, 163-170.

[28] N. Kumar, K. M. Nagaraja, A. Bayad and M. Saraj, New means and its properties, Proceedings of the Jangjeon Math. Soc., **14**(2010), No. 3, 243-254.

[29] C. P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl., **3**(2000), No. 2, 155-167.

[30] E. Neuman and J. Sdor, Inequalities involving Stolarsky and Gini means, Math. Pannon., **14**(2003), No. 1, 29-44.

[31] E. Neuman and Zs. Pes, On comparison of Stolarsky and Gini means, J. Math. Anal. Appl., **278**(2003), No. 2, 274-284.

[32] Zs. Pes, Inequalities for differences of powers, J. Math. Anal. Appl., **131**(1988), 271-281.

[33] Zs. Pes, Comparison of two variable homogeneous means, General Inequal. 6. Proc. 6-th Internat. Conf. Math. Res. Inst. Oberwolfach, Birkhser Verlag Basel, (1992), 59-69.

[34] F. Qi, Logarithmically convexities of the Extended Mean values, Proc. Amer. Math. Soc., **130**(2002), No. 6, 1787-1796.

[35] F. Qi, The Extended mean values: definition, properties, monotonicities, comparison, con-vexities, generalizations, and applications, RGMIA Res. Rep. Coll., **2**(2002), No. 1, art. 6, available online at http://rgmia.vu.edu.au/v2n5.html.

[36] F. Qi, J. Sdor and S. S. Dragomir, Notes on the Schur-convexity of the extended mean values, Taiwanese J. Math., **9**(2005), No. 3, 411-420.

[37] F. Qi, A note on Schur-convexity of extended mean values, Rocky Mountain J. Math., **35**(2005), No. 5, 1787-1793.

[38] J. Sdor, The Schur-convexity of Stolarsky and Gini means, Banach J. Math. Anal., **1**(2007), No. 2, 212-215.

[39] M. Shaked, J. G. Shanthikumar and Y. L. Tong, Parametric Schur convexity and arrangement monotonicity properties of partial sums, J. Multivariate Anal., **53**(1995), No. 2, 293-310.

[40] H. N. Shi, S. H. Wu and F. Qi, An alternative note on the Schur-convexity of the extended mean values, Math. Inequal. Appl., **9**(2006), No. 2, 219-224.

[41] Slavko Simic, An extension of Stolarsky means, Novi Sad J. Math., **38**(2008), No. 3, 81-89.

[42] Slavko Simic, On weighted Stolarsky means, Sarajevo Journal of Mathematics, **7**(2011), No. 19, 3-9.

[43] C. Stepniak, Stochastic ordering and Schur-convex functions in comparison of linear experiments, Metrika, **36**(1989), No. 5, 291-298.

[44] K. B. Stolarsky, Generalizations of the Logarithmic Mean, Math. Mag., **48**(1975), 87-92.

[45] K. B. Stolarsky, The power and generalized Logarithmic Means, Amer. Math. Monthly., **87**(1980), 545-548.

[46] B. Y. Wang, Foundations of majorization inequalities, Beijing Normal Univ. Press, Beijing, China, 1990. (Chinese)

[47] W. F. Xia, The Schur harmonic convexity of Lehmer means, Int. Math. Forum, **41**(2009), No. 4, 2009-2015.

[48] W. F. Xia and Y. M. Chu, Schur-convexity for a class of symmetric functions and its applications, J. Inequal. Appl., Vol. 2009, Art. ID 493759, 15 pages.

[49] Zh. H. Yang, Simple discriminances of convexity of homogeneous functions and applications, **4**(2004), No. 7, 14-19.

[50] Zh. H. Yang, On the homogeneous functions with two parameters and its monotonicity, J. Inequal. Pure Appl. Math., **6**(2005), No. 4, art. 101, available online at http://jipam.vu.edu.au/images/155-05.pdf.

[51] Zh. H. Yang, On the log convexity of two parameter homogeneous functions, Math. Inequal. Appl., **10**(2007), No. 3, 499-516, available online at http://files.ele-math.com/articles/mia-10-47.pdf.

[52] Zh. H. Yang, On the monotonicity and log-convexity of a four-parameter homogeneous mean, J. Inequal. Appl., Vol. 2008, Art. ID 149286, 2008, available online at http://www.hindawi.com/GetArticle.aspx?doi=10.1155/2008/149286.

[53] Zh. H. Yang, Necessary and sufficient conditions for Schur convexity of Stolarsky and Gini means. Submitted.

[54] X. M. Zhang, Schur-convex functions and isoperimetric inequalities, Proc. Amer. Math. Soc., **126**(1998), No. 2, 461-470.

[55] X. M. Zhang, Geometrically convex functions, Hefei, An hui University Press, 2004.

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Differential sandwich-type results for starlike functions

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Abstract In this paper, we present a generalized criterion for starlike univalent functions. We show that our result unifies some known results of starlike functions. Using the dual concept of differential subordination and superordination, we find some sandwich-type results regarding starlike univalent functions. Mathematica 7.0 is used to plot the images of the unit disk under certain functions.

Keywords differential subordination, differential superordination, starlike function.

§1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (complex plane) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$ is said to be starlike in the open unit disk $\mathbb{E}$ if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. Denote by $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α which is analytically defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{E} \right\},$$

where α $(0 \leq \alpha < 1)$ is a real number. Write $\mathcal{S}^* = \mathcal{S}^*(0)$, the class of univalent starlike w.r.t. the origin.

If f is analytic and g is analytic univalent in the open unit disk $\mathbb{E}$, we say that f is subordinate to g in $\mathbb{E}$ and write as $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$. To derive certain sandwich-type results, we use the dual concept of differential subordination and superordination.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ such that $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z),\ \ \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p(z) \prec q(z)$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, p be analytic univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), \ h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a subordinant of the differential superordination (2), if $q(z) \prec p(z)$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2), is said to be the best subordinant of (2).

The expressions $\dfrac{zf'(z)}{f(z)}$ and $1 + \dfrac{zf''(z)}{f'(z)}$ play an important role in the theory of univalent functions. Several new classes have been introduced and studied by various researchers by combining these expressions in different manners. For example, in 1976, Lewandowski et al. [2] proved the following result.

Theorem 1.1. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \ z \in \mathbb{E},$$

then $f \in \mathcal{S}^*$.

In 2002, Li and Owa [3] generalized and improved the above result by proving the next two results.

Theorem 1.2. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right] > -\frac{\alpha}{2}, \ z \in \mathbb{E},$$

for some $\alpha(\alpha \geq 0)$, then $f \in \mathcal{S}^*$.

Theorem 1.3. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right] > -(1-\alpha)\frac{\alpha^2}{4}, \ z \in \mathbb{E},$$

for some $\alpha(0 < \alpha \leq 2)$, then $f \in \mathcal{S}^*(\alpha/2)$.

Ravichandran et al. [7] improved the above results further and gave the following result.

Theorem 1.4. If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right] > \alpha\beta\left(\beta - \frac{1}{2}\right) + \left(\beta - \frac{\alpha}{2}\right), \ z \in \mathbb{E},$$

for some $\alpha(0 \leq \alpha, \beta \leq 1)$, then $f \in \mathcal{S}^*(\beta)$.

The main objective of this paper is to generalize and improve the results of above nature and obtain certain sandwich-type results for starlike functions. Mathematica 7.0 is used to plot the images of the open unit disk $\mathbb{E}$ under certain functions.

§2. Preliminaries

We shall use the following definition and lemmas to prove our main results.

Definition 2.1.([5], p.21, definition 2.2b) We denote by Q the set of functions p that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where

$$\mathbb{B}(p) = \left\{ \zeta \in \partial\mathbb{E} : \lim_{z \to \zeta} p(z) = \infty \right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 2.1.([5], p.132, theorem 3.4 h) Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that one of the following conditions satisfies:

1. h is convex,
2. Q_1 is starlike.

In addition, assume that $\Re \frac{zh'(z)}{Q_1(z)} > 0,\ z \in \mathbb{E}$. If p is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.2.[1] Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that Q_1 is starlike in $\mathbb{E}$ and $\Re \frac{\theta'(q(z))}{\phi(q(z))} > 0,\ z \in \mathbb{E}$. If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)],$$

then $q(z) \prec p(z)$ and q is the best subordinant.

§3. Main results

In what follows, the value of any complex power taken is the principal one.

Theorem 3.1. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that

1. $\Re\left(1 + \dfrac{zq''(z)}{q'(z)} + (\gamma - 1)\dfrac{zq'(z)}{q(z)}\right) > 0,$
2. $\Re\left(1 + \dfrac{zq''(z)}{q'(z)} + (\gamma - 1)\dfrac{zq'(z)}{q(z)} + (\gamma + 1)q(z) + \dfrac{(1-\alpha)\gamma}{\alpha}\right) > 0$ for all $z \in \mathbb{E}$.

If $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination

$$\left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec \alpha zq'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha)q^{\gamma}(z), \quad (3)$$

where $\alpha,\ \gamma$ are complex numbers with $\alpha \neq 0$, then $\dfrac{zf'(z)}{f(z)} \prec q(z)$ and q is the best dominant.

Proof. On writing $p(z) = \dfrac{zf'(z)}{f(z)}$, the subordination (3) becomes:

$$\alpha z p'(z) p^{\gamma-1}(z) + \alpha p^{\gamma+1}(z) + (1-\alpha) p^{\gamma}(z) \prec \alpha z q'(z) q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha) q^{\gamma}(z). \quad (4)$$

Define the functions θ and ϕ as under:

$$\theta(w) = \alpha w^{\gamma+1} + (1-\alpha) w^{\gamma} \ \text{ and } \ \phi(w) = \alpha w^{\gamma-1}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0, \ w \in \mathbb{D}$. Setting the functions Q_1 and h as follows:

$$Q_1(z) = zq'(z)\phi(q(z)) = \alpha z q'(z) q^{\gamma-1}(z),$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = \alpha z q'(z) q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha) q^{\gamma}(z).$$

A little calculation yields

$$\frac{zQ_1'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma-1)\frac{zq'(z)}{q(z)},$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma-1)\frac{zq'(z)}{q(z)} + (\gamma+1)q(z) + \frac{(1-\alpha)\gamma}{\alpha}.$$

In view of the given conditions 1 and 2, we have that Q_1 is starlike in $\mathbb{E}$ and $\Re \dfrac{zh'(z)}{Q_1(z)} > 0, \ z \in \mathbb{E}$. Thus conditions 2 of lemma 2.1, is satisfied. In view of (4), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof follows from lemma 2.1.

Theorem 3.2. Let $q, \ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that

1. $\Re \left(1 + \dfrac{zq''(z)}{q'(z)} + (\gamma-1)\dfrac{zq'(z)}{q(z)}\right) > 0,$
2. $\Re \left((\gamma+1)q(z) + \dfrac{(1-\alpha)\gamma}{\alpha}\right) > 0$ for all $z \in \mathbb{E}$.

If $f \in \mathcal{A}, \ \dfrac{zf'(z)}{f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ with $\dfrac{zf'(z)}{f(z)} \neq 0, \ z \in \mathbb{E}$, satisfies the differential superordination

$$\alpha z q'(z) q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha) q^{\gamma}(z) \prec \left(\frac{zf'(z)}{f(z)}\right)^{\gamma} \left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) = h(z), \quad (5)$$

where $\alpha, \ \gamma$ are complex numbers with $\alpha \neq 0$ and h is univalent in $\mathbb{E}$, then $q(z) \prec \dfrac{zf'(z)}{f(z)}$ and q is the best subordinant.

Proof. Setting $p(z) = \dfrac{zf'(z)}{f(z)}$, the superordination (5) becomes:

$$\alpha z q'(z) q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha) q^{\gamma}(z) \prec \alpha z p'(z) p^{\gamma-1}(z) + \alpha p^{\gamma+1}(z) + (1-\alpha) p^{\gamma}(z). \quad (6)$$

By defining the functions $\theta,\ \phi$ and Q_1 same as in case of theorem 3.1 and observing that

$$\frac{\theta'(q(z))}{\phi(q(z))} = (\gamma+1)q(z) + \frac{(1-\alpha)\gamma}{\alpha}.$$

The use of lemma 2.2 along with (6) completes the proof on the same lines as in case of theorem 3.1.

On combining theorem 3.1 and theorem 3.2, we obtain the following sandwich-type theorem.

Theorem 3.3. Suppose $\alpha,\ \gamma$ are complex numbers with $\alpha \neq 0$ and suppose that $q_1, q_2\ (q_1(z) \neq 0, q_2(z) \neq 0,\ z \in \mathbb{E})$ are univalent functions in $\mathbb{E}$ such that q_1 satisfies the conditions 1 and 2 of theorem 3.2 and q_2 follows the conditions 1 and 2 of theorem 3.1. If $f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \in \mathcal{H}[q_1(0), 1] \cap Q$ with $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential sandwich-type condition

$$\alpha z q_1'(z) q_1^{\gamma-1}(z) + \alpha q_1^{\gamma+1}(z) + (1-\alpha)q_1^{\gamma}(z) \prec h(z) = \left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)$$

$$\prec \alpha z q_2'(z) q_2^{\gamma-1}(z) + \alpha q_2^{\gamma+1}(z) + (1-\alpha)q_2^{\gamma}(z),$$

where h is univalent in $\mathbb{E}$, then $q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$. Moreover q_1 and q_2 are, respectively, the best subordinant and the best dominant.

§4. Deductions

If we consider the dominant $q(z) = \frac{1+(1-2\lambda)z}{1-z},\ 0 \leq \lambda < 1$, a little calculation yields that this dominant satisfies the conditions of theorem 3.1 in following particular cases. Select $\gamma = 1$ in theorem 3.1, we get the following result.

Corollary 4.1. Suppose that $\alpha\ (0 < \alpha \leq 1)$ is a real number and if $f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec \frac{2\alpha(1-\lambda)z}{(1-z)^2} + \alpha\left(\frac{1+(1-2\lambda)z}{1-z}\right)^2 + (1-\alpha)\frac{1+(1-2\lambda)z}{1-z},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z}, \quad \text{i.e.}\ \ f \in \mathcal{S}^*(\lambda),\ 0 \leq \lambda < 1.$$

Note that the above corollary gives the result of Kwon [6] for $\alpha = 1$.

Example 4.1. We compare the result of above corollary with the result of Ravichandran et al. [7] by considering the following particular cases. We see that the above corollary extends the result of Ravichandran et al. [7], stated in theorem 1.4. Write $\alpha = 1/2,\ \beta = 0$ in theorem 1.4, we obtain:

If $f \in \mathcal{A}$ satisfies

$$\Re\left[\frac{zf'(z)}{f(z)}\left(1 + \frac{1}{2}\frac{zf''(z)}{f'(z)}\right)\right] > -\frac{1}{4},\ z \in \mathbb{E}, \tag{7}$$

then $f \in \mathcal{S}^*$.

For $\alpha = 1/2,\ \lambda = 0$ in above corollary, we get:
If $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)}\left(1+\frac{1}{2}\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+2z}{(1-z)^2} = F(z), \tag{8}$$

then $f \in \mathcal{S}^*$.

The image of the open unit disk $\mathbb{E}$ under F (given by (8)) is the shaded region in Figure 4.1. Therefore, in view of (8), the differential operator $\dfrac{zf'(z)}{f(z)}\left(1+\dfrac{1}{2}\dfrac{zf''(z)}{f'(z)}\right)$ takes the values in entire shaded region in Figure 4.1 to ensure the starlikeness of f whereas from (7), we see that f is starlike in $\mathbb{E}$, when the operator $\dfrac{zf'(z)}{f(z)}\left(1+\dfrac{1}{2}\dfrac{zf''(z)}{f'(z)}\right)$ takes values in the portion right to the vertical dashing line at $\Re(w) = -\dfrac{1}{4}$. Therefore, the result in (8) is an improvement over the result given in (7).

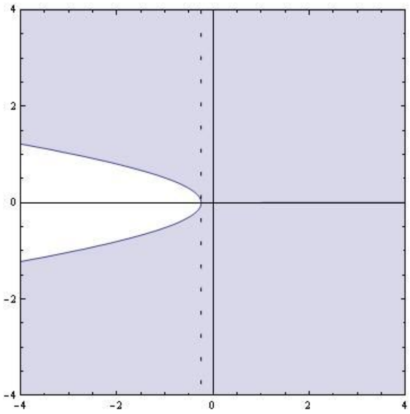

Figure 4.1

On writing $\gamma = -1$ in theorem 3.1, we get:

Corollary 4.2. Let α be a real number such that $\alpha \in (-\infty, 0) \cup [1, \infty)$ and let $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfy

$$\frac{1+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + \frac{(1-\alpha)(1-z)}{1+(1-2\lambda)z} + \frac{2\alpha(1-\lambda)z}{(1+(1-2\lambda)z)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z}, \quad \text{i.e.} \quad f \in \mathcal{S}^*(\lambda),\ 0 \leq \lambda < 1.$$

Taking $\gamma = 0$ in theorem 3.1, we obtain:

Corollary 4.3. If $f \in \mathcal{A}, \dfrac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies

$$1+\alpha\frac{zf''(z)}{f'(z)} \prec 1-\alpha+\frac{\alpha(1+(1-2\lambda)z)}{1-z} + \frac{2\alpha(1-\lambda)z}{(1-z)(1+(1-2\lambda)z)},$$

where α is a non-zero complex number. Then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z},$$

i.e. $f \in \mathcal{S}^*(\lambda),\ 0 \le \lambda < 1$.

For $\alpha = 1,\ \lambda = 1/2$ in above corollary, we find below the result justifying the well-known result Marx [4] and Strohhäcker [9] that $\mathcal{K} \subset \mathcal{S}^*(1/2)$.

Corollary 4.4. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \ne 0$ in $\mathbb{E}$, satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1-z},\ z \in \mathbb{E}.$$

When we consider the dominant $q(z) = \dfrac{1+az}{1-z},\ -1 < a \le 1$, a little calculation yields that it satisfies the conditions of theorem 3.1 in following special cases and consequently we obtain the next results. Setting $\gamma = 1$ in theorem 3.1, we have the following result.

Corollary 4.5. Suppose that α $(0 < \alpha \le 1)$ is a real number and if $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \ne 0$ in $\mathbb{E}$, satisfies

$$\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right) \prec \frac{\alpha(1+a)z}{(1-z)^2} + \alpha\left(\frac{1+az}{1-z}\right)^2 + (1-\alpha)\frac{1+az}{1-z},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z},\ z \in \mathbb{E},\ -1 < a \le 1.$$

Writing $\gamma = -1$ in theorem 3.1, we have the following result.

Corollary 4.6. Let α be a real number such that $\alpha \in (-\infty, 0) \cup [1, \infty)$ and let $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \ne 0$ in $\mathbb{E}$, satisfy

$$\frac{1+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + \frac{(1-\alpha)(1-z)}{1+az} + \frac{\alpha(1+a)z}{(1+az)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z},\ z \in \mathbb{E},\ -1 < a \le 1.$$

Setting $\gamma = 0$ in theorem 3.1, we get:

Corollary 4.7. Suppose that α is a non-zero complex number and if $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \ne 0$ in $\mathbb{E}$, satisfies

$$1 + \alpha\frac{zf''(z)}{f'(z)} \prec 1 - \alpha + \frac{\alpha(1+az)}{1-z} + \frac{\alpha(1+a)z}{(1-z)(1+az)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z},\ z \in \mathbb{E},\ -1 < a \le 1.$$

Note that for $\alpha = 1$, the above three corollaries reduce to the results of Singh and Gupta. [8]

§5. Sandwich-type results

In this section, we apply theorem 3.3 to find certain sandwich-type results which give the best subordinant and the best dominant for $\frac{zf'(z)}{f(z)}$. By selecting the subordinant $q_1(z) = 1+az$ and the dominant $q_2(z) = 1 + bz,\ 0 < a < b$, in theorem 3.3, we deduce, below, some criteria for starlike functions. Keeping $\gamma = 1$ in theorem 3.3, we obtain:

Corollary 5.1. Suppose $\alpha,\ a,\ b$ are real numbers such that $0 < \alpha \leq 1,\ 0 < a < b < 1$. If $f \in \mathcal{A}$ is such that $\frac{zf'(z)}{f(z)} \in \mathcal{H}[1, 1] \cap Q$, with $\frac{zf'(z)}{f(z)} \neq 0$ and satisfies the condition

$$1 + (1 + 2\alpha)az + \alpha a^2 z^2 \prec \frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right) \prec 1 + (1 + 2\alpha)bz + \alpha b^2 z^2,\ z \in \mathbb{E},$$

where $\frac{zf'(z)}{f(z)}\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)$ is univalent in $\mathbb{E}$, then

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz.$$

Example 5.1. For $\alpha = 1, a = 1/10, b = 9/10$ and f same as in above corollary, we obtain:

$$1 + \frac{3}{10}z + \frac{1}{100}z^2 \prec \frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + \frac{27}{10}z + \frac{81}{100}z^2, \tag{9}$$

then

$$1 + \frac{1}{10}z \prec \frac{zf'(z)}{f(z)} \prec 1 + \frac{9}{10}z. \tag{10}$$

We show the above results pictorially. In view of (9) and (10), when the operator $\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right)$ takes values in the light shaded portion of Figure 5.1, then $\frac{zf'(z)}{f(z)}$ takes values in the light shaded portion of Figure 5.2 and hence f is starlike in $\mathbb{E}$.

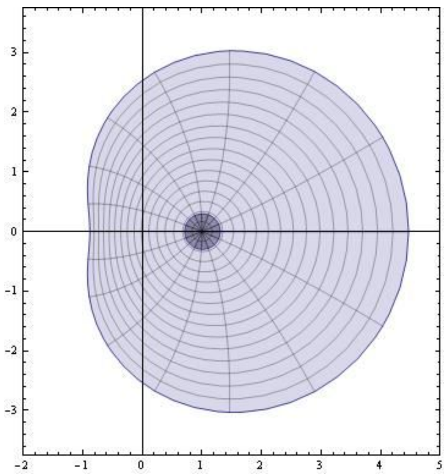

Figure 5.1

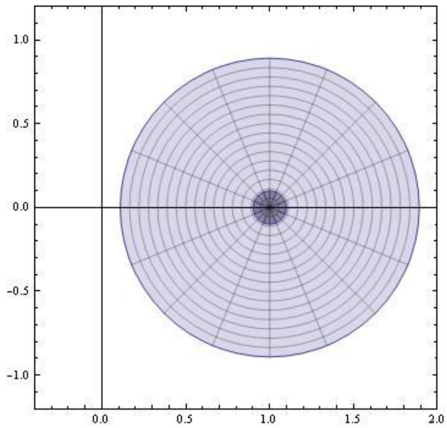

Figure 5.2

Writing $\gamma = -1$ in theorem 3.3, we obtain:

Corollary 5.2. Let $\alpha,\ a,\ b$ be real numbers such that $\alpha \in (-\infty, 0)\cup(1,\infty),\ 0 < a < b < 1$. If $f \in \mathcal{A}$ is such that $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,1] \cap Q$, with $\dfrac{zf'(z)}{f(z)} \neq 0$ and $\dfrac{1+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)}$ is univalent in $\mathbb{E}$. Then

$$\alpha + \frac{1-\alpha}{1+az} + \frac{\alpha a z}{(1+az)^2} \prec \frac{1+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + \frac{1-\alpha}{1+bz} + \frac{\alpha b z}{(1+bz)^2},\ z \in \mathbb{E},$$

implies

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz,\ z \in \mathbb{E}.$$

Writing $\gamma = 0$ in theorem 3.3, we get:

Corollary 5.3. Suppose α is a non-zero complex number and $a,\ b$ are real numbers such that $0 < a < b < 1$. If $f \in \mathcal{A}$ is such that $\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,1] \cap Q$, with $\dfrac{zf'(z)}{f(z)} \neq 0$ and $1+\alpha\dfrac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$. Then

$$1 + \alpha a z + \frac{\alpha a z}{1+az} \prec 1 + \alpha\frac{zf''(z)}{f'(z)} \prec 1 + \alpha b z + \frac{\alpha b z}{1+bz},\ z \in \mathbb{E},$$

implies

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz,\ z \in \mathbb{E}.$$

Example 5.2. For $\alpha = 1, a = 1/4, b = 3/4$ and f same as in above corollary, we obtain:

$$1 + \frac{1}{4}z + \frac{z}{4+z} \prec 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{3}{4}z + \frac{3z}{4+3z}, \tag{11}$$

then

$$1 + \frac{1}{4}z \prec \frac{zf'(z)}{f(z)} \prec 1 + \frac{3}{4}z. \tag{12}$$

In view of (11) and (12), we see that whenever the operator $\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)$ takes values in the light shaded portion of Figure 5.3, then $\frac{zf'(z)}{f(z)}$ takes values in the light shaded portion of Figure 5.4 and hence f is starlike in $\mathbb{E}$.

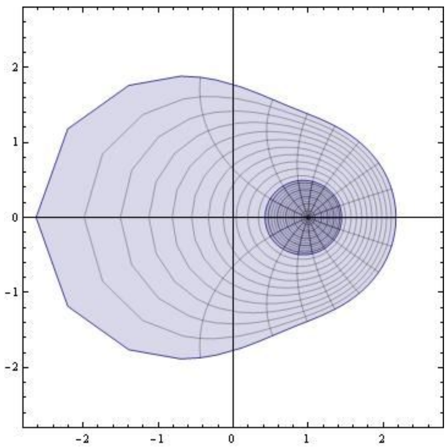

Figure 5.3

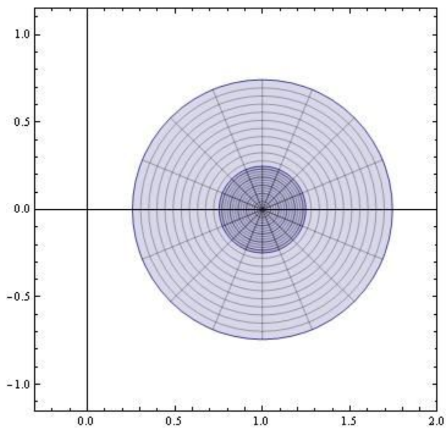

Figure 5.4

References

[1] T. Bulboacă, Classes of first order differential superordinations, Demonstratio Math, **35**(2002), No. 2, 287-292.

[2] Z. Lewandowski, S. Miller and E. Zlotkiewicz, Generating functions for some classes of univalent functions, Pro. Amer. Math. Soc., **56**(1976), 111-117.

[3] J. L. Li and S. Owa, Sufficient conditions for starlikeness, Indian J. Pure and Appl. Math., **33**(2002), 313-318.

[4] A. Marx, Untersuchungen *ü*ber schlichte Abbildungen, Math. Ann., **107**(1932/1933), 40-65.

[5] S. S. Miller and P. T. Mocanu, Differential subordinations: theory and applications, Marcel Dekker, New York and Basel, (2000).

[6] Oh Sang Kwon, Sufficient conditions for Starlikeness and strongly-Starlikeness, Commun. Korean. Math. Soc., **22**(2007), No. 1, 19-26.

[7] V. Ravichandran, C. Selvaraj and R. Rajalaksmi, Sufficient conditions for starlike functions of order α, J. Ineq. Pure and Appl. Math., **3**(2002), No. 5, 1-6.

[8] S. Singh and S. Gupta, Some applications of a first order differential subordination, J. Ineq. Pure and Appl. Math., **5**(2004), No. 3, 1-15.

[9] E. Strohh*ä*cker, Beitrage z*ü*r Theorie der schlichten Funktionen, Math. Z., **37**(1933), 356-380.

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On the convergence of some right circulant matrices

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Abstract In this paper, the conditions for the converegence of the matrices $RCIRC_n(\vec{d})$ and $RCIRC_n(\vec{g})$ were established.

Keywords Convergent matrix, right circulant matrix, spectral norm.

§1. Introduction

In [1] and [2] the followings were established about the spectral norm of $RCIRC_n(\vec{d})$ and $RCIRC_n(\vec{g})$:

1. $||RCIRC_n(\vec{d})||_2 = max\left\{|na + \frac{nd(n-1)}{2}|, \frac{|nd|}{2\sin\frac{\pi}{n}}\right\}$,

2. $||RCIRC_n(\vec{g})||_2 = max\left\{|\frac{a(1-r^n)}{1-r}|, \frac{|a(r^n-1)|}{\sqrt{r^2-2r\cos\frac{2\pi m}{n}+1}}\right\}$,

where

$$RCIRC_n(\vec{d}) = \begin{pmatrix} a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\ a+(n-1)d & ad & a+d & \cdots & a+(n-3)d & a+(n-2)d \\ a+(n-2)d & a+(n-1)d & a & \cdots & a+(n-4)d & a+(n-3)d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a+2d & a+3d & a+4d & \cdots & a & a+d \\ a+d & a+2d & a+3d & \cdots & a+(n-1)d & a \end{pmatrix},$$

and $$RCIRC_n(\vec{g}) = \begin{pmatrix} a & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\ ar^{n-1} & a & ar & \cdots & ar^{n-3} & ar^{n-2} \\ ar^{n-2} & ar^{n-1} & a & \cdots & ar^{n-4} & ar^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ar^2 & ar^3 & ar^4 & \cdots & a & ar \\ ar & ar^2 & ar^3 & \cdots & ar^{n-1} & a \end{pmatrix}.$$

Definition 1.1. A matrix C is said to be convergent if its spectral norm is less than 1, that is $||C||_2 < 1$.

§2. Main results

Theorem 2.1. $RCIRC_n(\vec{d})$ is convergent provided that $|2an + nd(n-1)| < 2$ or $d^2 < \frac{2\sin^2(\frac{\pi}{n})}{n^2}$.

Proof.

Case 1: $||RCIRC_n(\vec{d})||_2 = \left|na + \frac{nd(n-1)}{2}\right| < 1$, then,

$$|2an + nd(n-1)| < 2.$$

Case 2: $||RCIRC_n(\vec{d})||_2 = \frac{|nd|}{2\sin\frac{\pi}{n}} < 1$, then,

$$|nd| < 2\sin\left(\frac{\pi}{n}\right) \text{ and } d^2 < \frac{2\sin^2(\frac{\pi}{n})}{n^2}.$$

Theorem 2.2. $RCIRC_n(\vec{g})$ is convergent provided that $a^2 < (\frac{r-1}{r^n-1})^2$ or $a^2 < \frac{r^2-2r\cos(\frac{2\pi m}{n})+1}{(r^n-1)^2}$.

Proof.

Case 1: $||RCIRC_n(\vec{g})||_2 = |\frac{a(1-r^n)}{1-r}| < 1$, then,

$$a^2(1-r^n)^2 < (1-r)^2 \text{ and } a^2 < \left(\frac{1-r}{1-r^n}\right)^2.$$

Case 2: $||RCIRC_n(\vec{g})||_2 = \frac{|a(1-r^n)|}{\sqrt{r^2-2r\cos\frac{2\pi m}{n}+1}} < 1$, then,

$$a^2(1-r^n)^2 < r^2 - 2r\cos\frac{2\pi m}{n} + 1 \text{ and } a^2 < \frac{r^2-2r\cos\frac{2\pi m}{n}+1}{(1-r^n)^2}.$$

References

[1] M. Bahsi and S. Solak, On the circulant matrices with arithmetic sequence, International Journal of Contemporary Matematical Sciences, **5**(2010), No. 25, 1213-1222.

[2] A. C. F. Bueno, Right circulant matrices with geometric progression, International Journal of Applied Mathematical Research, **1**(2012), No. 4, 593-603.

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Two summation formulae of half argument linked with contiguous relation

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Abstract The main object of present paper is the establishment of two summation formulae associated with the contiguous relation and hypergeometric function.

Keywords Contiguous relation, recurrence relation, *Legendre*'s duplication formula.

2000 Mathematics Subject Classification: 33C05, 33C20, 33D15, 33D50, 33D60

§1. Introduction

Generalized Gaussian hypergeometric function of one variable :

$$ {}_AF_B\left[\begin{array}{ll} a_1, a_2, \cdots, a_A ; & \\ & z \\ b_1, b_2, \cdots, b_B ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}, \tag{1} $$

or

$$ {}_AF_B\left[\begin{array}{ll} (a_A) ; & \\ & z \\ (b_B) ; & \end{array}\right] \equiv {}_AF_B\left[\begin{array}{ll} (a_j)_{j=1}^{A} ; & \\ & z \\ (b_j)_{j=1}^{B} ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!}, \tag{2} $$

where the parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

Contiguous relation: [1]

$$ (a-b)\,(1-z)\,{}_2F_1\left[\begin{array}{ll} a, b; & \\ c; & z \end{array}\right] = (c-b)\,{}_2F_1\left[\begin{array}{ll} a, b-1; & \\ c; & z \end{array}\right] + (a-c)\,{}_2F_1\left[\begin{array}{ll} a-1, b; & \\ c; & z \end{array}\right]. \tag{3} $$

Recurrence relation :

$$ \Gamma(z+1) = z\,\Gamma(z). \tag{4} $$

Legendre's duplication formula :

$$ \sqrt{\pi}\,\Gamma(2z) = 2^{(2z-1)}\,\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right), \tag{5} $$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \frac{2^{(b-1)}\ \Gamma(\frac{b}{2})\ \Gamma(\frac{b+1}{2})}{\Gamma(b)}, \tag{6}$$

$$= \frac{2^{(a-1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\Gamma(a)}. \tag{7}$$

In the monograph of Prudnikov et al., a summation formula is given in the form: [3]

$$ {}_2F_1\left[\begin{matrix} a, \ b \ ; & \frac{1}{2} \\ \frac{a+b-1}{2} \ ; & 2 \end{matrix}\right] = \sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} + \frac{2\ \Gamma(\frac{a+b-1}{2})}{\Gamma(a)\Gamma(b)}\right]. \tag{8}$$

Now using *Legendre*'s duplication formula and recurrence relation for Gamma function, the above formula can be written in the form

$$ {}_2F_1\left[\begin{matrix} a, \ b \ ; & \frac{1}{2} \\ \frac{a+b-1}{2} \ ; & 2 \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})} + \frac{2^{(a-b+1)}\ \Gamma(\frac{a}{2})\ \Gamma(\frac{a+1}{2})}{\{\Gamma(a)\}^2} + \frac{\Gamma(\frac{b+2}{2})}{\Gamma(\frac{a+1}{2})}\right]. \tag{9}$$

It is noted that the above formula [3], i.e. equation (8) or (9) is not correct. The correct form of equation (8) or (9) is obtained by [2].

$$ {}_2F_1\left[\begin{matrix} a, \ b \ ; & \frac{1}{2} \\ \frac{a+b-1}{2} \ ; & 2 \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-1}{2})}{\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-1}{2})}\left\{\frac{(b+a-1)}{(a-1)}\right\} + \frac{2\ \Gamma(\frac{b+1}{2})}{\Gamma(\frac{a}{2})}\right]. \tag{10}$$

Involving the formula obtained by [2], we establish the main formulae.

§2. Main results of summation formulae

For all the results $a \neq b$,

$$ {}_2F_1\left[\begin{matrix} a, \ b \ ; & \frac{1}{2} \\ \frac{a+b-23}{2} \ ; & 2 \end{matrix}\right] = \frac{2^{(b-1)}\ \Gamma(\frac{a+b-23}{2})}{(a-b)\Gamma(b)}\left[\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-23}{2})}\right.$$

$$\times\left\{\frac{(316234143225a - 703416314160a^2 + 590546123298a^3 - 264300628944a^4 + 72578259391a^5)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}\right.$$

$$+\frac{(-13137458400a^6 + 1628301884a^7 - 140529312a^8 + 8439783a^9 - 345840a^{10} + 9218a^{11})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-144a^{12} + a^{13} - 316234143225b + 987903828090a^2b - 1002491094240a^3b)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(523055123685a^4b - 135562896000a^5b + 27768329500a^6b - 2975972160a^7b + 299768865a^8b)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-14044800a^9b+718410a^{10}b-12000a^{11}b+275a^{12}b+7034163314160b^2-987903828090ab^2)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(521997079830a^3b^2-287175957600a^4b^2+104854420780a^5b^2-15893250240a^6b^2)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(2529523900a^7b^2-149067600a^8b^2+12283150a^9b^2-242880a^{10}b^2+10350a^{11}b^2)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-5905461123298b^3+1002491094240ab^3-521997079830a^2b^3+82963937100a^4b^3)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-25551704640a^5b^3+6975270260a^6b^3-573638400a^7b^3+73053750a^8b^3-1821600a^9b^3)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(123970a^10b^3+2643006628944b^4-523055123685ab^4+287175957600a^2b^4-82963937100a^3b^4)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(4841754610a^5b^4-785551200a^6b^4+167502100a^7b^4-5768400a^8b^4+600875a^9b^4)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-725782593391b^5+135562896000ab^5-104854420780a^2b^5+25551704640a^3b^5)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-4841754610a^4b^5+107126180a^6b^5-7131840a^7b^5+1225785a^8b^5+131374584000b^6)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-27768329500ab^6+15893250240a^2b^6-6975270260a^3b^6+785551200a^4b^6-107126180a^5b^6)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(742900a^7b^6-16283018884b^7+2975972160ab^7-2529523900a^2b^7+573638400a^3b^7)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-167502100a^4b^7+7131840a^5b^7-742900a^6b^7+140529312b^8-299768865ab^8)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(149067600a^2b^8-73053750a^3b^8+5768400a^4b^8-1225785a^5b^8-8439783b^9+14044800ab^9)}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-12283150a^2b^9+1821600a^3b^9-600875a^4b^9+345840b^{10}-718410ab^{10}+242880a^2b^{10})}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$\left.+\frac{(-123970a^3b^{10}-9218b^{11}+12000ab^{11}-10350a^2b^{11}+144b^{12}-275ab^{12}-b^{13})}{\prod_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}\right\}$$

$$+\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-22}{2})}\left\{\frac{(-483585689160a+789891354792a^2-460744729944a^3+184587960184a^4)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}\right.$$

$$+\frac{(-36115854800a^5+6465269136a^6-547131312a^7+50602992a^8-1870440a^9+89672a^{10})}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(-1144a^{11}+24a^{12}+483585689160b-628259352120a^2b+596428543440a^3b)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(-195420950000a^4b+56438379200a^5b-6566394800a^6b+920232160a^7b-43386200a^8b)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(3256000a^9b-51480a^{10}b+2000a^{11}b-789891354792b^2+628259352120ab^2)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(-173665216800a^3b^2+110863118800a^4b^2-20734777840a^5b^2+4558206240a^6b^2)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(-290628000a^7b^2+32738200a^8b^2-657800a^9b^2+40480a^{10}b^2+460744729944b^3)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(-596428543440ab^3+173665216800a^2b^3-15168969200a^4b^3+7064396640a^5b^3)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(-704664800a^6b^3+122396800a^7b^3-3289000a^8b^3+303600a^9b^3-184587960184b^4)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(195420950000ab^4-110863118800a^2b^4+15168969200a^3b^4-461297200a^5b^4)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(165186000a^6b^4-6817200a^7b^4+961400a^8b^4+36115854800b^5-56438379200ab^5)}{\prod_{\zeta=1}^{11}\left\{a-2\zeta\right\}}$$

$$+\frac{(20734777840a^2b^5-7064396640a^3b^5+461297200a^4b^5-4160240a^6b^5+1188640a^7b^5)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-6465269136b^6+6566394800ab^6-4558206240a^2b^6+704664800a^3b^6-165186000a^4b^6)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(4160240a^5b^6+547131312b^7-920232160ab^7+290628000a^2b^7-122396800a^3b^7)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(6817200a^4b^7-1188640a^5b^7-50602992b^8+43386200ab^8-32738200a^2b^8+3289000a^3b^8)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-961400a^4b^8+1870440b^9-3256000ab^9+657800a^2b^9-303600a^3b^9-89672b^{10}+51480ab^{10})}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\left.\left.\frac{(-40480a^2b^{10}+1144b^{11}-2000ab^{11}-24b^{12})}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}\right\}\right]. \tag{11}$$

$${}_2F_1\left[\begin{matrix}a, & b & ; & \frac{1}{2}\\ \frac{a+b-24}{2} & ; & & \end{matrix}\right]=\frac{2^{(b-1)}\,\Gamma(\frac{a+b-24}{2})}{(a-b)\Gamma(b)}$$

$$\times\left[\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-23}{2})}\left\{\frac{(1961990553600a-4325828198400a^2+3874205859840a^3-1634441932800a^4)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}\right.\right.$$

$$+\frac{(512039040000a^5-83602361600a^6+12742301120a^7-946046400a^8+77391600a^9)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-2574000a^{10}+111540a^{11}-1300a^{12}+25a^{13}-1961990553600b+1281593180160ab)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(3094576496640a^2b-3825569710080a^3b+2245742361600a^4b-568140505600a^5b)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(132653041600a^6b-13220253440a^7b+1600835600a^8b-67152800a^9b+4493060a^{10}b)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-64480a^11b+2275a^12b+30442350182400b^2-4990865375232ab^2+1232718520320a^2b^2)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(1115851130880a^3b^2-814203974400a^4b^2+341856934080a^5b^2-51313920000a^6b^2)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(9319592640a^7b^2-518226800a^8b^2+50979500a^9b^2-920920a^{10}b^2+50830a^{11}b^2)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-1977916981248b^3+3503317499904ab^3-2219737221120a^2b^3+272783385600a^3b^3)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(136393348800a^4b^3-58313839360a^5b^3+18533032000a^6b^3-1521532800a^7b^3)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(221648700a^8b^3-5262400a^9b^3+427570a^{10}b^3+723975622656b^4-1485650361344ab^4)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(849188691200a^2b^4-304571115200a^3b^4+19959363200a^4b^4+6518851360a^5b^4)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-1509922400a^6b^4+378005000a^7b^4-13066300a^8b^4+1562275a^9b^4-168244950016b^5)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(317597785600ab^5-259454322880a^2b^5+65660179200a^3b^5-15636448800a^4b^5)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(523700800a^5b^5+122727080a^6b^5-11886400a^7b^5+2414425a^8b^5+26376979200b^6)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-56740812736ab^6+32671797760a^2b^6-15299636800a^3b^6+1786327200a^4b^6)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-308714280a^5b^6+4160240a^6b^6+742900a^7b^6-2879374784b^7+5281066752ab^7)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-4623802560a^2b^7+1051523200a^3b^7-329797000a^4b^7+14543360a^5b^7-1931540a^6b^7)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(221652288b^8-477823632ab^8+237437200a^2b^8-120706300a^3b^8+9538100a^4b^8)}{\prod\limits_{\delta=1}^{12}\left\{a-(2\delta-1)\right\}}$$

$$+\frac{(-2187185a^5b^8-11991408b^9+19974240ab^9-17803500a^2b^9+2631200a^3b^9-904475a^4b^9)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(446160b^{10}-932932ab^{10}+314600a^2b^{10}-164450a^3b^{10}-10868b^{11}+14144ab^{11})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-12350a^2b^{11}+156b^{12}-299ab^{12}-b^{13})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}\Bigg\}$$

$$+\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-24}{2})}\Bigg\{\frac{(1961990553600a-3044235018240a^2+1977916981248a^3-723975622656a^4)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(168244950016a^5-26376979200a^6+2879374784a^7-221652288a^8+11991408a^9)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-446160a^{10}+10868a^{11}-156a^{12}+a^{13}-1961990553600b-1281593180160ab)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(4990865375232a^2b-3503317499904a^3b+1485650361344a^4b-317597785600a^5b)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(56740812736a^6b-5281066752a^7b+477823632a^8b-19974240a^9b+932932a^{10}b)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-14144a^{11}b+299a^{12}b+4325828198400b^2-3094576496640ab^2-1232718520320a^2b^2)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(2219737221120a^3b^2-849188691200a^4b^2+259454322880a^5b^2-32671797760a^6b^2)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(4623802560a^7b^2-237437200a^8b^2+17803500a^9b^2-314600a^{10}b^2+12350a^{11}b^2)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-3874205859840b^3+3825569710080ab^3-1115851130880a^2b^3-272783385600a^3b^3)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(304571115200a^4b^3-65660179200a^5b^3+15299636800a^6b^3-1051523200a^7b^3)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(120706300a^8b^3 - 2631200a^9b^3 + 164450a^10b^3 + 1634441932800b^4 - 2245742361600ab^4)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(814203974400a^2b^4 - 136393348800a^3b^4 - 19959363200a^4b^4 + 15636448800a^5b^4)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(568140505600ab^5 - 341856934080a^2b^5 + 58313839360a^3b^5 - 6518851360a^4b^5)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-523700800a^5b^5 + 308714280a^6b^5 - 14543360a^7b^5 + 2187185a^8b^5 + 83602361600b^6)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-132653041600ab^6 + 51313920000a^2b^6 - 18533032000a^3b^6 + 1509922400a^4b^6)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-122727080a^5b^6 - 4160240a^6b^6 + 1931540a^7b^6 - 127423011120b^7 + 13220253440ab^7)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-9319592640a^2b^7 + 1521532800a^3b^7 - 378005000a^4b^7 + 11886400a^5b^7 - 742900a^6b^7)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(946046400b^8 - 1600835600ab^8 + 518226800a^2b^8 - 221648700a^3b^8 + 13066300a^4b^8)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(-2414425a^5b^8 - 77391600b^9 + 67152800ab^9 - 50979500a^2b^9 + 5262400a^3b^9 - 1562275a^4b^9)}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\frac{(2574000b^{10} - 4493060ab^{10} + 920920a^2b^{10} - 427570a^3b^{10} - 111540b^{11} + 64480ab^{11})}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}$$

$$+\left.\left.\frac{(-50830a^2b^{11} + 1300b^{12} - 2275ab^{12} - 25b^{13})}{\prod_{\vartheta=1}^{12}\{a-2\vartheta\}}\right\}\right]. \tag{12}$$

§3. Derivation of summation formulae

Derivation of (11): substituting $c = \frac{a+b-23}{2}$ and $z = \frac{1}{2}$ in equation (3), we get

$$\left(\frac{a-b}{2}\right) {}_2F_1\left[\begin{matrix} a, b; & \frac{1}{2} \\ \frac{a+b-23}{2}; & \end{matrix}\right] = \left(\frac{a-b-23}{2}\right) {}_2F_1\left[\begin{matrix} a, b-1; & \frac{1}{2} \\ \frac{a+b-23}{2}; & \end{matrix}\right]$$

$$+\left(\frac{a-b+23}{2}\right){}_2F_1\left[\begin{matrix} a-1,b; & \frac{1}{2} \\ \frac{a+b-23}{2}; & \frac{1}{2}\end{matrix}\right],$$

or

$$(a-b){}_2F_1\left[\begin{matrix} a,b; & \frac{1}{2} \\ \frac{a+b-23}{2}; & \frac{1}{2}\end{matrix}\right]=(a-b-23)\,{}_2F_1\left[\begin{matrix} a,b-1; & \frac{1}{2} \\ \frac{a+b-23}{2}; & \frac{1}{2}\end{matrix}\right]+(a-b+23)\,{}_2F_1\left[\begin{matrix} a-1,b; & \frac{1}{2} \\ \frac{a+b-23}{2}; & \frac{1}{2}\end{matrix}\right].$$

Now involving (10), we get

$$L.H.S=\frac{2^{(b-1)}\ \Gamma(\frac{a+b-23}{2})}{\Gamma(b)}\left[\frac{(a-b-23)(b-1)}{(a-b+1)}\ \frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-23}{2})}\left\{\frac{(-316234143225+373432860360a)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}\right.\right.$$

$$+\frac{(129106402182a^2-320632172520a^3+177781840313a^4-51711155760a^5+9260845396a^6)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-1085127120a^7+84910089a^8-4402200a^9+145222a^{10}-2760a^{11}+23a^{12})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(7034163141 60b-1139061201588ab+396896330784a^2b+152000003492a^3b)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-147359650912a^4b+49578268792a^5b-9037988736a^6b+1074415560a^7b-78535248a^8b)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(3929596a^9b-105248a^10b+1748a^11b-5905461232 98b^2+1069825103544ab^2)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-576865073266a^2b^2+81954887264a^3b^2+36378600348a^4b^2-16834239856a^5b^2)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(3616937772a^6b^2-391633696a^7b^2+30456646a^8b^2-1111176a^9b^2+31878a^{10}b^2)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(264300628944b^3-503451140140ab^3+312991442432a^2b^3-79652150448a^3b^3)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(4715754400a^4b^3+2713871384a^5b^3-611995776a^6b^3+85728912a^7b^3-4383984a^8b^3)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(211508a^9b^3-72578259391b^4+140570255696ab^4-93342239796a^2b^4+27793239376a^3b^4)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-3690175706a^4b^4+70724080a^5b^4+70100044a^6b^4-6299792a^7b^4+572033a^8b^4)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(13137458400b^5-25911980616ab^5+16865721152a^2b^5-5516113736a^3b^5+842326240a^4b^5)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-56995288a^5b^5+534888a^7b^5-16283018884b^6+3107909616ab^6-2154337780a^2b^6)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(618811872a^3b^6-112314244a^4b^6+7488432a^5b^6-208012a^6b^6+140529312b^7)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-275591448ab^7+160818944a^2b^7-55226864a^3b^7+6306784a^4b^7-653752a^5b^7-8439783b^8)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(14821224ab^8-10248018a^2b^8+2021976a^3b^8-389367a^4b^8+345840b^9-655556ab^9)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(267168a^2b^9-92092a^3b^9-9218b^{10}+12760ab^{10}-8602a^2b^{10}+144b^{11}-252ab^{11}-b^{12})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}\Bigg\}$$

$$+\frac{(a-b-23)}{(a-b+1)}\;\frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-22}{2})}\Bigg\{\frac{(-497334999735+240349776192a+283337378706a^2)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-301005003392a^3+120738023135a^4-28127876736a^5+4053197084a^6-397208064a^7)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(24985719a^8-1076416a^9+26290a^{10}-384a^{11}+a^{12}+1257240832632b-1100102048556ab)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(8888092648a^2b+296874862508a^3b-147610629200a^4b+35668641544a^5b-5306483504a^6b)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(497150104a^7b-31051944a^8b+1180388a^9b-27192a^{10}b+252a^{11}b-1245022762902b^2)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(1343501390896ab^2-389126369998a^2b^2-55506401216a^3b^2+62934811220a^4b^2)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-17261090784a^5b^2+2644880308a^6b^2-240208320a^7b^2+14146242a^8b^2-457424a^9b^2)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(8602a^{10}b^2+659273219368b^3-778752070180ab^3+316618638112a^2b^3-36277638352a^3b^3)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-8824358480a^4b^3+4191946696a^5b^3-630661472a^6b^3+61671280a^7b^3-2750616a^8b^3)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(92092a^9b^3-212974169753b^4+263201438400ab^4-119279262332a^2b^4+23952997824a^3b^4)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-825887174a^4b^4-370603968a^5b^4+96750052a^6b^4-6460608a^7b^4+389367a^8b^4)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(448294350056b^5-57198911416ab^5+26664410416a^2b^5-6095706008a^3b^5+634852624a^4b^5)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(7072408a^5b^5-4160240a^6b^5+653752a^7b^5-6609754996b^6+8045450784ab^6)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-3994074476a^2b^6+873356736a^3b^6-108374252a^4b^6+4992288a^5b^6+208012a^6b^6)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(6449137744b^7-843555912ab^7+353776800a^2b^7-90259728a^3b^7+8201616a^4b^7-534888a^5b^7)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-487781689b^8+51717248ab^8-26461270a^2b^8+3999424a^3b^8-572033a^4b^8+2151512b^9)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-2852828ab^9+795432a^2b^9-211508a^3b^9-84502b^{10}+60720ab^{10}-31878a^2b^{10})}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(1288b^{11}-1748ab^{11}-23b^{12})}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}\Bigg\}\Bigg]+\frac{2^{(b-1)}\ \Gamma(\frac{a+b-23}{2})}{\Gamma(b)}\Bigg[\frac{(a-b+23)}{(a-b-1)}\ \frac{\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a-22}{2})}$$

$$\times\Bigg\{\ \frac{(497334999735-1257240832632a+1245022762902a^2-659273219368a^3)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(212974169753a^4-44829435056a^5+6609754996a^6-644913744a^7+48781689a^8)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-2151512a^9+84502a^{10}-1288a^{11}+23a^{12}-240349776192b+1100102048556ab)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}+$$

$$+\frac{(-1343501390896a^2b+778752070180a^3b-263201438400a^4b+57198911416a^5b)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}+$$

$$+\frac{(-8045450784a^6b+843555912a^7b-51717248a^8b+2852828a^9b-60720a^{10}b+1748a^{11}b)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-283337378706b^2-8888092648ab^2+389126369998a^2b^2-316618638112a^3b^2)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(119279262332a^4b^2-26664410416a^5b^2+3994074476a^6b^2-353776800a^7b^2+26461270a^8b^2)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-795432a^9b^2+31878a^10b^2+301005003392b^3-296874862508ab^3+55506401216a^2b^3)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(36277638352a^3b^3-23952997824a^4b^3+6095706008a^5b^3-873356736a^6b^3+90259728a^7b^3)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-3999424a^8b^3+211508a^9b^3-120738023135b^4+147610629200ab^4-62934811220a^2b^4)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(8824358480a^3b^4+825887174a^4b^4-634852624a^5b^4+108374252a^6b^4-8201616a^7b^4)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(572033a^8b^4+28127876736b^5-35668641544ab^5+17261090784a^2b^5-4191946696a^3b^5)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(370603968a^4b^5-7072408a^5b^5-4992288a^6b^5+534888a^7b^5-40531197084b^6)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(5306483504ab^6-2644880308a^2b^6+630661472a^3b^6-96750052a^4b^6+4160240a^5b^6)}{\prod_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-208012a^6b^6+397208064b^7-497150104ab^7+240208320a^2b^7-61671280a^3b^7)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(6460608a^4b^7-653752a^5b^7-24985719b^8+31051944ab^8-14146242a^2b^8+2750616a^3b^8)}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(-389367a^4b^8+1076416b^9-1180388ab^9+457424a^2b^9-92092a^3b^9-26290b^{10})}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}$$

$$+\frac{(27192ab^{10}-8602a^2b^{10}+384b^{11}-252ab^{11}-b^{12})}{\prod\limits_{\zeta=1}^{11}\{a-2\zeta\}}\Bigg\}+\frac{(a-b+23)}{(a-b-1)}\frac{\Gamma(\frac{b}{2})}{\Gamma(\frac{a-23}{2})}$$

$$\times\Bigg\{\frac{(316234143225-703416314160a+590546123298a^2-264300628944a^3+72578259391a^4)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-13137458400a^5+1628301884a^6-140529312a^7+8439783a^8-345840a^9+9218a^{10})}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-144a^{11}+a^{12}-373432860360b+1139061201588ab-1069825103544a^2b)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(503451140140a^3b-140570255696a^4b+25911980616a^5b-3107909616a^6b+275591448a^7b)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-14821224a^8b+655556a^9b-12760a^{10}b+252a^{11}b-129106402182b^2-396896330784ab^2)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(576865073266a^2b^2-312991442432a^3b^2+93342239796a^4b^2-16865721152a^5b^2)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(2154337780a^6b^2-160818944a^7b^2+10248018a^8b^2-267168a^9b^2+8602a^{10}b^2)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(320632172520b^3-152000003492ab^3-81954887264a^2b^3+79652150448a^3b^3)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-27793239376a^4b^3+5516113736a^5b^3-618811872a^6b^3+55226864a^7b^3-2021976a^8b^3)}{\prod\limits_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(92092a^9b^3-177781840313b^4+147359650912ab^4-36378600348a^2b^4-4715754400a^3b^4)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(3690175706a^4b^4-842326240a^5b^4+112314244a^6b^4-6306784a^7b^4+389367a^8b^4)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(51711155760b^5-49578268792ab^5+16834239856a^2b^5-2713871384a^3b^5-70724080a^4b^5)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(56995288a^5b^5-7488432a^6b^5+653752a^7b^5-9260845396b^6+9037988736ab^6)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-3616937772a^2b^6+611995776a^3b^6-70100044a^4b^6+208012a^6b^6+1085127120b^7)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-1074415560ab^7+391633696a^2b^7-85728912a^3b^7+6299792a^4b^7-534888a^5b^7)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-849100089b^8+78535248ab^8-30456646a^2b^8+4383984a^3b^8-572033a^4b^8+4402200b^9)}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$+\frac{(-3929596ab^9+1111176a^2b^9-211508a^3b^9-145222b^10+105248ab^10-31878a^2b^{10})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}$$

$$\left.\left.+\frac{(2760b^{11}-1748ab^{11}-23b^{12})}{\prod_{\delta=1}^{12}\{a-(2\delta-1)\}}\right\}\right].$$

On simplification, we get the result (11). By applying same method we can prove the result (12).

References

[1] M. A. Abramowitz and Stegun, Irene, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards, 1970.

[2] Arora Asish, Singh Rahul and Salahuddin, Development of a family of summation formulae of half argument using Gauss and Bailey theorems, Journal of Rajasthan Academy of Physical Sciences, **7**(2008), 335-342.

[3] A. P. Prudnikov, Brychkov, A. Yu and O. I. Marichev, More special functions, Nauka, Moscow, Integrals and Series, **3**1986, translated from the Russian by G. G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo, Melbourne, 1990.

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νg-separation axioms

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Abstract In this paper we discuss new separation axioms using νg-open sets.

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§1. Introduction

Norman Levine introduced generalized closed sets, K. Balachandaran and P. Sundaram studied generalized continuous functions and generalized homeomorphism. V. K. Sharma studied generalized separation axioms. Following V. K. Sharma we are going to define a new variety of generalized axioms called νg-separation axioms and study their basic properties and interrelation with other type of generalized separation axioms. Throughout the paper a space X means a topological space (X,τ). For any subset A of X its complement, interior, closure, νg-interior, νg-closure are denoted respectively by the symbols A^c, A^0, $\overline{A}$, $\nu g(A)^0$ and $\nu g\overline{(A)}$.

§2. Preliminaries

Definition 2.1. $A \subset X$ is said to be

(i) Regular closed [46] [resp: α-closed [29]; pre-closed [28]; β-closed [1]] if $A = \overline{A^o}$ [resp: $(\overline{(A^o)})^o \subseteq A$; $\overline{(A^o)} \subseteq A$; $\overline{(\overline{A})^o} \subseteq A$] and Regular open [46] [resp: α-open [34]; pre-open [28]; β-open [1]] if $A = (\overline{A})^o$ [resp: $A \subseteq (\overline{(A^o)})^o$; $A \subseteq \overline{(A^o)}$; $A \subseteq \overline{(\overline{A})^o}$].

(ii) Semi open [35] [resp: ν-open] if there exists an open [resp: regular open] set $U \ni U \subseteq A \subseteq \overline{U}$ and semi closed [13] [resp: ν-closed] if its complement if semi open [ν-open].

(iii) g-closed [36] [resp: rg-closed [41]] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(iv) sg-closed [11,18,25] [resp: gs-closed [32]] if $s(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is semi-open{open} in X.

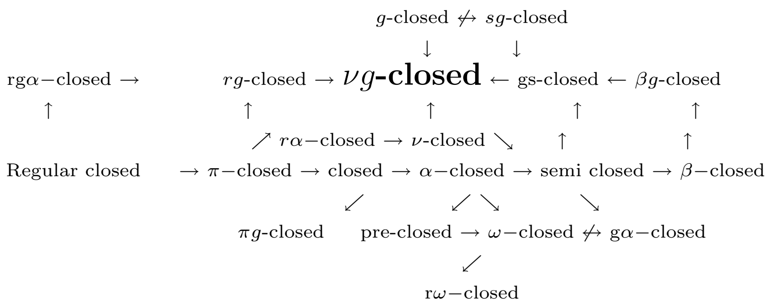
(v) pg-closed [resp: gp-closed; gpr-closed [23]] if $p(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is pre-open{open; regular-open} in X.

(vi) αg-closed [resp: $g\alpha$-closed [25]; rgα-closed [47]] if $\alpha(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is {α-open; rα-open} open in X.

(vii) νg-closed [6] if $\nu(\overline{A}) \subseteq U$ whenever $A \subseteq U$ and U is ν-open in X.

(viii) clopen [resp: nearly-clopen; ν-clopen; semi-clopen; g-clopen; rg-clopen; sg-clopen; νg-clopen] if it is both open [resp: regular-open; ν-open; semi-open; g-open; rg-open; sg-open; νg-open] and closed [resp: regular-closed; ν-closed; semi-closed; g-closed; rg-closed; sg-closed; νg-closed].

Note 2.1. From definition 2.1 we have the following interrelations among the closed sets.



Definition 2.2. A function $f: X \to Y$ is said to be [4,7]

(i) ν-continuous [resp: νg-continuous; ν-irresolute; νg-irresolute] if the inverse image of every open [resp: closed; ν-open; νg-closed] set is ν-open [resp: νg-closed; ν-open; νg-closed].

(ii) ν-open [resp: νg-open] if the image of open set is ν-open [resp: νg-open].

(iii) ν-homeomorphism [resp: νc-homeomorphism; νg-homeomorphism; νgc-homeomorphism] if f and f^{-1} are bijective ν-continuous [ν-irresolute; νg-continuous; νg-irresolute].

Definition 2.3. X is said to be $T_{\frac{1}{2}}$ [resp: $rT_{\frac{1}{2}}, \nu T_{\frac{1}{2}}, sT_{\frac{1}{2}}$] if every generalized [resp: regular generalzed, ν-generalized, semi-generalized] closed set is closed [resp: regular-closed, ν-closed, semi-closed].

Note 2.2. The class of regular open sets, ν-open sets, open sets, g-open sets, rg-open sets and νg-open sets are denoted by $RO(X)$, $\nu O(X)$, $\tau(X)$, $GO(X)$, $RGO(X)$ and $\nu GO(X)$ respectively. Clearly $RO(X) \subset \tau(X) \subset GO(X) \subset RGO(X) \subset \nu GO(X)$.

Note 2.3. For any subset A in X, $A \in \nu GO(X, x)$ means A is a νg-neighborhood of x.

Definition 2.4. X is said to be

(i) compact [resp: nearly compacat, ν-compact, semi-compact, g-compact, rg-compact, sg-compact, νg-compact] if every open [resp: regular-open, ν-open, semi-open, g-open, rg-open, sg-open, νg-open] cover has a finite subcover;

(ii) T_0 [resp: rT_0, νT_0 [5], sT_0, g_0 [35], rg_0 [11], sg_0 [34]] space if for each $x \neq y \in X$, $\exists U \in \tau(X)$ [resp: $RO(X)$; $\nu O(X)$; $SO(X)$; $GO(X)$; $RGO(X)$; $SGO(X)$] containing either x or y;

(iii) T_1 [resp: rT_1, νT_1 [5], sT_1, g_1 [35], rg_1 [11], sg_1 [34]] space if for each $x \neq y \in X$, $\exists U, V \in \tau(X)$ [resp: $RO(X)$; $\nu O(X)$; $SO(X)$; $GO(X)$; $RGO(X)$; $SGO(X)$] such that $x \in U - V$ and $y \in V - U$;

(iv) T_2 [resp: rT_2, νT_2 [5], sT_2, g_2 [35], rg_2 [11], sg_2 [34]] space if for each $x \neq y \in X$, $\exists U, V \in \tau(X)$ [resp: $RO(X)$; $\nu O(X)$; $SO(X)$; $GO(X)$; $RGO(X)$; $SGO(X)$] such that $x \in U$, $y \in V$ and $U \cap V = \phi$;

(v) C_0 [resp: $rC_0, \nu C_0$ [5], sC_0, gC_0 [7], rgC_0 [11], sgC_0 [10]] space if for each $x \neq y \in X$, $\exists U \in \tau(X)$ [resp: $RO(X)$; $\nu O(X)$; $SO(X)$; $GO(X)$; $RGO(X)$; $SGO(X)$] whose closure contains either x or y;

(vi) C_1 [resp: rC_1, νC_1 [5], sC_1, gC_1 [7], rgC_1 [11], sgC_1 [10]] space if for each $x \neq y \in$

$X,\ \exists U, V \in \tau(X)$ [resp: $RO(X);\ \nu O(X);\ SO(X);\ GO(X);\ RGO(X);\ SGO(X);$] such that $x \in \overline{U}$ and $y \in \overline{V}$;

(vii) C_2 [resp: $rC_2,\ \nu C_2$ [5], $sC_2,\ gC_2$ [7], rgC_2 [11], sgC_2 [10]] space if for each $x \neq y \in X,\ \exists U, V \in \tau(X)$ [resp: $RO(X);\ \nu O(X);\ SO(X);\ GO(X);\ RGO(X);\ SGO(X)$] such that $x \in \overline{U}$, $y \in \overline{V}$ and $U \cap V = \phi$;

(viii) D_0 [resp: $rD_0,\ \nu D_0$ [5], $sD_0,\ gD_0$ [7], rgD_0 [11], sgD_0 [10]] space if for each $x \neq y \in X,\ \exists U \in D(X)$ [resp: $RDO(X);\ \nu DO(X);\ SDO(X);\ GDO(X);\ RGDO(X);\ SGDO(X)$] containing either x or y;

(ix) D_1 [resp: $rD_1,\ \nu D_1$ [5], $sD_1,\ gD_1$ [7], rgD_1 [11], sgD_1 [10]] space if for each $x \neq y \in X,\ \exists U, V \in \tau(X)$ [resp: $RDO(X);\ \nu DO(X);\ SDO(X);\ GDO(X);\ RGDO(X);\ SGDO(X)$] $\ni x \in U - V$ and $y \in V - U$;

(x) D_2 [resp: $rD_2,\ \nu D_2$ [5], $sD_2,\ gD_2$ [7], rgD_2 [11], sgD_2 [10]] space if for each $x \neq y \in X,\ \exists U, V \in \tau(X)$ [resp: $RDO(X);\ \nu DO(X);\ SDO(X);\ GDO(X);\ RGDO(X);\ SGDO(X)$] $\ni x \in U - V;\ y \in V - U$ and $U \cap V = \phi$;

(xi) R_0 [resp: $rR_0,\ \nu R_0$ [5], $sR_0,\ gR_0$ [7], rgR_0 [11], sgR_0 [10]] space if for each $x \in X,\ \exists U \in \tau(X)$ [resp: $RO(X);\ \nu O(X);\ SO(X);\ GO(X);\ RGO(X);\ SGO(X)$]$\overline{\{x\}} \subseteq U$ [resp: $r\overline{\{x\}} \subseteq U;\ \nu\overline{\{x\}} \subseteq U;\ s\overline{\{x\}} \subseteq U;\ g\overline{\{x\}} \subseteq U;\ rg\overline{\{x\}} \subseteq U;\ sg\overline{\{x\}} \subseteq U$] whenever $x \in U \in \tau(X)$ [resp: $x \in U \in RGO(X);\ x \in U \in \nu O(X);\ x \in U \in SO(X);\ x \in U \in GO(X);\ x \in U \in RGO(X);\ x \in U \in SGO(X)$];

(xii) R_1 [resp: $rR_1,\ \nu R_1$ [5], $sR_1,\ gR_1$ [7], rgR_1 [11], sgR_1 [10]] space for $x, y \in X \ni \overline{\{x\}} \neq \overline{\{y\}}$ [resp:$\ni r\overline{\{x\}} \neq r\overline{\{y\}};\ \ni \nu\overline{\{x\}} \neq \nu\overline{\{y\}};\ \ni s\overline{\{x\}} \neq s\overline{\{y\}};\ \ni g\overline{\{x\}} \neq g\overline{\{y\}};\ \ni rg\overline{\{x\}} \neq rg\overline{\{y\}};\ \ni sg\overline{\{x\}} \neq sg\overline{\{y\}}$]$V \in \tau(X)$ [resp: $RO(X);\ \nu O(X);\ SO(X);\ GO(X);\ RGO(X);\ SGO(X)$], $\exists$ disjoint $U; V \in \tau(X) \ni \overline{\{x\}} \subseteq U$ [resp: $RO(X) \ni r\overline{\{x\}} \subseteq U;\ \nu O(X) \ni \nu\overline{\{x\}} \subseteq U;\ SO(X) \ni s\overline{\{x\}} \subseteq U;\ GO(X) \ni g\overline{\{x\}} \subseteq U;\ RGO(X) \ni rg\overline{\{x\}} \subseteq U;\ SGO(X) \ni sg\overline{\{x\}} \subseteq U$] and $\overline{\{y\}} \subseteq V$ [resp: $r\overline{\{y\}} \subseteq V;\ \nu\overline{\{y\}} \subseteq V;\ s\overline{\{y\}} \subseteq V;\ g\overline{\{y\}} \subseteq V;\ rg\overline{\{y\}} \subseteq V;\ sg\overline{\{y\}} \subseteq V$].

Theorem 2.1.

(i) If x is a νg-limit point of any $A \subset X$, then every νg-neighbourhood of x contains infinitely many distinct points.

(ii) Let $A \subseteq Y \subseteq X$ and Y is regularly open subspace of X then A is νg-open in X iff A is νg-open in $\tau_{/Y}$.

Theorem 2.2. If f is νg-continuous [resp: νg-irresolute{νg-homeomorphism}] and G is open [resp: νg-open [νg-closed]] set in Y, then $f^{-1}(G)$ is νg-open [resp: νg-open [νg-closed]] in X.

Theorem 2.3. Let Y and $\{X_\alpha : \alpha \in I\}$ be topological spaces. Let $f : Y \to \Pi X_\alpha$ be a function. If f is νg-continuous, then $\pi_\alpha \circ f : Y \to X_\alpha$ is νg-continuous.

Theorem 2.4. If Y is $\nu T_{\frac{1}{2}}$ and $\{X_\alpha : \alpha \in I\}$ be topological spaces. Let $f : Y \to \Pi X_\alpha$ be a function, then f is νg-continuous iff $\pi_\alpha \circ f : Y \to X_\alpha$ is νg-continuous.

Corollary 2.1. Let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f((x_\alpha)_{\alpha\in I}) = (f_\alpha(x_\alpha))_{\alpha\in I}$. If f is νg-continuous then each f_α is νg-continuous.

Corollary 2.2. For each α, let X_α be $\nu T_{\frac{1}{2}}$ and let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f((x_\alpha)_{\alpha\in I}) = (f_\alpha(x_\alpha))_{\alpha\in I}$, then f is νg-continuous iff each f_α is νg-continuous.

§3. νg_i spaces, $i = 0, 1, 2$

Definition 3.1. X is said to be

(i) a νg_0 space if for each pair of distinct points x, y of X, there exists a νg-open set G containing either x or y;

(ii) a νg_1 space if for each pair of distinct points x, y of X, there exists a νg-open set G containing x but not y and a νg-open set H containing y but not x;

(iii) a νg_2 space if for each pair of distinct points x, y of X, there exists disjoint νg-open sets G and H such that G containing x but not y and H containing y but not x.

Note 3.1.(i)

$$\begin{array}{ccccccc} rT_i & \Rightarrow & T_i & \Rightarrow & g_i & \Rightarrow & rg_i, \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \nu T_i & \Rightarrow & sT_i & \Rightarrow & sg_i & \Rightarrow & \nu g_i, \end{array}$$

i = 0, 1, 2. But the converse is not true in general.

(ii) X is $\nu g_2 \Rightarrow$ X is $\nu g_1 \Rightarrow$ X is νg_0.

Example 3.1. Let X = {a, b, c} and

(i) $\tau = \{\phi, \{a, c\}, X\}$ then X is νg_i but not $r - T_0$ and T_0, $i = 0, 1, 2$;

(ii)$\tau = \{\phi, \{a\}, \{a, c\}, X\}$ then X is not νg_i for $i = 0, 1, 2$.

Remark 3.1. If X is $\nu - T_{\frac{1}{2}}$ then $\nu - T_i$ and νg_i are one and the same for $i = 0, 1, 2$.

Proof. Since X is ν-$T_{\frac{1}{2}}$, every νg-closed set is ν-closed and so the proof is straightforward from the respective definitions.

Theorem 3.1.

(i) Every open subspace [resp: regular open] of νg_i space is νg_i for $i = 0, 1, 2$;

(ii) The product of νg_i spaces is again νg_i for $i = 0, 1, 2$.

Theorem 3.2.

(i) X is νg_0 iff $\forall x \in X, \exists U \in \nu GO(X)$ containing $x \ni$ the subspace U is νg_0.

(ii) X is νg_0 iff distinct points of X have disjoint νg-closures.

Theorem 3.3. The followings are equivalent:

(i) X is νg_1,

(ii) Each one point set is νg-closed,

(iii) Each subset of X is the intersection of all νg-open sets containing it,

(iv) For any $x \in X$, the intersection of all νg-open sets containing the point is the set $\{x\}$.

Theorem 3.4. If X is νg_1 then distinct points of X have disjoint νg-closures.

Theorem 3.5. Suppose x is a νg-limit point of a subset of A of a νg_1 space X. Then every neighborhood of x contains infinitely many distinct points of A.

Theorem 3.6. X is νg_2 iff the intersection of all νg-closed, νg-neighbourhoods of each point of the space is reduced to that point.

Proof. Let X be νg_2 and $x \in X$, then for each $y \neq x$ in X, $\exists U, V \in \nu GO(X) \ni x \in U$, $y \in V$ and $U \cap V = \phi$. Since $x \in U - V$, hence $X - V$ is a νg-closed, νg-neighbourhood of x to which y does not belong. Consequently, the intersection of all νg-closed, νg-neighbourhoods of x is reduced to $\{x\}$.

Conversely let $y \neq x$ in X, then by hypothesis there exists a νg-closed, νg-neighbourhood U of x such that $y \notin U$. Now $\exists G \in \nu GO(X) \ni x \in G \subset U$. Thus G and $X - U$ are disjoint νg-open sets containing x and y respectively. Hence X is νg_2.

Theorem 3.7. If to each point $x \in X$, there exist a νg-closed, νg-open subset of X containing x which is also a νg_2 subspace of X, then X is νg_2.

Proof. Let $x \in X, U$ a νg-closed, νg-open subset of X containing x and which is also a νg_2 subspace of X, then the intersection of all νg-closed, νg-neighbourhoods of x in U is reduced to x. U being νg-closed, νg-open, these are νg-closed, νg-neighbourhoods of x in X. Thus the intersection of all νg-closed, νg-neighbourhoods of x is reduced to $\{x\}$. Hence by theorem 3.6, X is νg_2.

Theorem 3.8. If X is νg_2, then the diagonal Δ in $X \times X$ is νg-closed.

Proof. Suppose $(x, y) \in X \times X - \Delta$. As $(x, y) \notin \Delta$ and $x \neq y$. Since X is νg_2, $\exists U,\ V \in \nu GO(X) \ni x \in U,\ y \in V$ and $U \cap V = \phi$. $U \cap V = \phi \Rightarrow (U \times V) \cap \Delta = \phi$ and therefore $(U \times V) \subset X \times X - \Delta$. Further $(x, y) \in (U \times V)$ and $(U \times V)$ is νg-open in $X \times X$ gives $X \times X - \Delta$ is νg-open. Hence Δ is νg-closed.

Corollary 3.1.

(i) In an T_1 [resp: rT_1; νT_1; sT_1; g_1; rg_1; sg_1] space, each singleton set is νg-closed;

(ii) if X is T_1 [resp: rT_1; νT_1; sT_1; g_1; rg_1; sg_1] then distinct points of X have disjoint νg-closures;

(iii) if X is T_2 [resp: rT_2; νT_2; sT_2; g_2; rg_2; sg_2] then the diagonal Δ in $X \times X$ is νg-closed.

Theorem 3.9. In νg_2-space, νg-limits of sequences, if exist, are unique.

Theorem 3.10. In a νg_2 space, a point and disjoint νg-compact subspace can be separated by disjoint νg-open sets.

Proof. Let X be a νg_2 space, $x \in X$ and C a νg-compact subspace of X not containing x. Let $y \in C$ then for $x \neq y$ in X, there exist disjoint νg-open neighborhoods G_x and H_y. Allowing this for each y in C, we obtain a class $\{H_y\}$ whose union covers C; and since C is νg-compact, some finite subclass, which we denote by $\{H_i, i = 1 \text{ to } n\}$ covers C. If G_i is νg-neighborhood of x corresponding to H_i, we put $G = \bigcup_{i=1}^{n} G_i$ and $H = \bigcap_{i=1}^{n} H_i$, satisfying the required properties.

Theorem 3.11. Every νg-compact subspace of a νg_2 space is νg-closed.

Proof. Let C be νg-compact subspace of a νg_2 space. If x be any point in C^c, by above theorem x has a νg-neighborhood $G \ni x \in G \subset \mathrm{C}^c$. This shows that C^c is the union of νg-open sets and therefore C^c is νg-open. Thus C is νg-closed.

Corollary 3.2.

(i) Show that in a T_2 [resp: rT_2; νT_2; sT_2; g_2; rg_2; sg_2] space, a point and disjoint compact [resp: nearly-compact; ν-compact; semi-compact; g-compact; rg-compact; sg-compact] subspace can be separated by disjoint νg-open sets;

(ii) every compact [resp: nearly-compact; ν-compact; semi-compact; g-compact; rg-compact; sg-compact] subspace of a T_2 [resp: rT_2; νT_2; sT_2; g_2; rg_2; sg_2] space is νg-closed.

Theorem 3.12. If f is injective, νg-irresolute and Y is νg_i then X is νg_i, $i = 0,\ 1,\ 2$.

Proof. Obvious from the definitions and so omitted.

Corollary 3.3.

(i) If f is injective, νg-continuous and Y is T_i then X is νg_i, $i = 0,\ 1,\ 2$;

(ii) if f is injective, r-irresolute [r-continuous] and Y is rT_i then X is νg_i, $i = 0,\ 1,\ 2$;

(iii) the property of being a space is νg_0 is a νg-topological property;

(iv) let f is a νgc-homeomorphism, then X is νg_i if Y is νg_i, $i = 0,\ 1,\ 2$.

Theorem 3.13. Let X be T_1 and $f : X \to Y$ be νg-closed subjection. Then X is νg_1.

Theorem 3.14. Every νg-irresolute map from a νg-compact space into a νg_2 space is νg-closed.

Proof. Suppose $f : X \to Y$ is νg-irresolute where X is νg-compact and Y is νg_2. Let $C \subset X$ be closed, then $C \subset X$ is νg-closed and hence C is νg-compact and so $f(C)$ is νg-compact. But then $f(C)$ is νg-closed in Y. Hence the image of any νg-closed set in X is νg-closed set in Y. Thus f is νg-closed.

Theorem 3.15. Any νg-irresolute bijection from a νg-compact space onto a νg_2 space is a νgc-homeomophism.

Proof. Let $f : X \to Y$ be a νg-irresolute bijection from a νg-compact space onto a νg_2 space. Let G be an νg-open subset of X. Then $X - G$ is νg-closed and hence $f(X - G)$ is νg-closed. Since f is bijective $f(X - G) = Y - f(G)$. Therefore $f(G)$ is νg-open in Y. This means that f is νg-open. Hence f is bijective νg-irresolute and νg-open. Thus f is νgc-homeomorphism.

Corollary 3.4. Any νg-continuous bijection from a νg-compact space onto a νg_2 space is a νg-homeomorphism.

Theorem 3.16. The followings are equivalent:

(i) X is νg_2;

(ii) for each pair $x \neq y \in X$, $\exists$ a νg-open, νg-closed set $V \ni x \in V$ and $y \notin V$;

(iii) for each pair $x \neq y \in X$, $\exists$ $f : X \to [0, 1]$ such that $f(x) = 0$, $f(C) = 1$ and f is νg-continuous.

Theorem 3.17. If $f : X \to Y$ is νg-irresolute and Y is νg_2 then

(i) the set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is νg-closed in $X \times X$;

(ii) $G(f)$, graph of f is νg-closed in $X \times Y$.

Proof. (i) Let $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow \exists$ disjoint $V_1, V_2 \in \nu GO(Y) \ni f(x_1) \in V_1$ and $f(x_2) \in V_2$, then by νg-irresoluteness of f, $f^{-1}(V_j) \in \nu GO(X, x_j)$ for each j. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in \nu GO(X \times X)$. Therefore $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A \Rightarrow X \times X - A$ is νg-open. Hence A is νg-closed.

(ii) Let $(x, y) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint νg-open sets V and $W \ni f(x) \in V$ and $y \in W$. Since f is νg-irresolute, $\exists U \in \nu GO(X) \ni x \in U$ and $f(U) \subset W$. Therefore we obtain $(x, y) \in U \times V \subset X \times Y - G(f)$. Hence $X \times Y - G(f)$ is νg-open. Hence $G(f)$ is νg-closed in $X \times Y$.

Theorem 3.18. If $f : X \to Y$ is νg-open and the set $A = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is closed in $X \times X$. Then Y is νg_2.

Theorem 3.19. Let Y and $\{X_\alpha : \alpha \in I\}$ be topological spaces. If $f : Y \to \Pi X_\alpha$ be a νg-continuous function and Y is $\nu T_{\frac{1}{2}}$, then ΠX_α and each X_α are νg_i, $i = 0,\ 1,\ 2$.

Problem 3.1. If Y be a νg_2 space and A be regular-open subspace of X. If $f : (A, \tau_{/A}) \to (Y, \sigma)$ is νg-irresolute. Does there exist any extension $F : (X, \tau) \to (Y, \sigma)$.

Theorem 3.20. Let X be an arbitrary space, R an equivalence relation in X and $p : X \to X/R$ the identification map. If $R \subset X \times X$ is νg-closed in $X \times X$ and p is νg-open map, then X/R is νg_2.

Proof. Let $p(x)$, $p(y)$ be distinct members of X/R. Since x and y are not related,

$R \in \nu GC(X \times X)$. There are $U,\ V \in \nu GO(X) \ni x \in U,\ y \in V$ and $U \times V \subset R^c$. Thus $p(U),\ p(V)$ are disjoint and also νg-open in X/R since p is νg-open.

Theorem 3.21. The following four properties are equivalent:

(i) X is νg_2;

(ii) let $x \in X$. For each $y \neq x$, $\exists U \in \nu GO(X) \ni x \in U$ and $y \notin \nu g\overline{(U)}$;

(iii) for each $x \in X$, $\cap\{\nu g\overline{(U)}/U \in \nu GO(X)$ and $x \in U\} = \{x\}$;

(iv) the diagonal $\Delta = \{(x,x)/x \in X\}$ is νg-closed in $X \times X$.

Proof. (i)$\Rightarrow$ (ii). Let $x \neq y \in X$, then there are disjoint νg-open sets $U,\ V \in \nu GO(X) \ni x \in U$ and $y \in V$. Clearly V^c is νg-closed, $\nu g\overline{(U)} \subset V^c, y \notin V^c$ and therefore $y \notin \nu g\overline{(U)}$.

(ii) $\Rightarrow$ (iii). If $y \neq x$, then $\exists U \in \nu GO(X),\ x \in U$ and $y \notin \nu g\overline{(U)}$. So $y \notin \cap\{\nu g\overline{(U)}/U \in \nu GO(X)$ and $x \in U\}$.

(iii) $\Rightarrow$ (iv). We prove Δ^c is νg-open. Let $(x,y) \notin \Delta$. Then $y \neq x$ and $\cap\{\nu g\overline{(U)}/U \in \nu GO(X)$ and $x \in U\} = \{x\}$, there is some $U \in \nu GO(X)$ with $x \in U$ and $y \notin \nu g\overline{(U)}$. Since $U \cap (\nu g\overline{(U)})^c = \phi$, $U \times (\nu g\overline{(U)})^c$ is a νg-open set such that $(x,y) \in U \times (\nu g\overline{(U)})^c \subset \Delta^c$.

(iv) $\Rightarrow$ (i). $y \neq x$, then $(x,y) \notin \Delta$ and thus $\exists U,\ V \in \nu GO(X) \ni (x,y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for the $U,\ V \in \nu GO(X)$ we have $x \in U,\ y \in V$ and $U \cap V = \phi$.

§4. νgg_3 and νgg_4 spaces

Definition 4.1. X is said to be

(i) a νg_3 space if for every νg-closed sets F and a point $x \notin F$, $\exists$ disjoint $U, V \in \nu O(X) \ni F \subseteq U,\ x \in V$;

(ii) a νgg_3 space if for every νg-closed sets F and a point $x \notin F$, $\exists$ disjoint $U,\ V \in \nu GO(X) \ni F \subseteq U,\ x \in V$;

(iii) a νg_4 space if for each pair of disjoint νg-closed sets F and H, $\exists$ disjoint $U,\ V \in \nu O(X) \ni F \subseteq U,\ H \subseteq V$;

(iv) a νgg_4 space if for each pair of disjoint νg-closed sets F and H, $\exists$ disjoint $U,\ V \in \nu gO(X) \ni F \subseteq U,\ H \subseteq V$.

Note 4.1.

(i) Every νT_3 space is νg_3 space but not conversely;

(ii) every νT_4 space is νg_4 space but not conversely;

(iii) every νg_3 space is νgg_3 space but not conversely;

(iv) every νg_4 space is νgg_4 space but not conversely.

Example 4.1. Let Y and Z be disjoint infinite sets and let $X = Y \cup Z$ and $\tau = \{\phi, Y, Z, X\}$. Clearly X is locally indiscrete and thus every ν-closed set is clopen, hence X is ν-T_3. If $\phi \neq A \subset X$ and $x \in Y - A$ then A is νg-closed, but A and x cannot be separated by disjoint ν-open sets, thus X is not νg_3. Similarly X is ν-T_4, but not νg_4.

Example 4.2. Let $X = \{a,b,c\}$ and

(i) $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ then X is νgg_i; νg_i and νT_i for $i = 3,\ 4$;

(ii) $\tau = \{\phi, \{a\}, X\}$ then X is νgg_i but not νg_i and νT_i for $i = 3,\ 4$.

Lemma 4.1. X is νg-regular iff X is ν-regular and $\nu - T_{\frac{1}{2}}$.

Proof. X is νg-regular, then obviously X is ν-regular. Let $A \subseteq X$ be νg-closed. For each $x \notin A$, $\exists V_x \in \nu O(X, x) \ni V_x \cap A = \phi$. If $V = \bigcup\{V_x : x \notin A\}$, then V is ν-open and $V = X - A$. Hence A is ν-closed implies X is $\nu - T_{\frac{1}{2}}$.

Theorem 4.1. If X is νg_3. Then for each $x \in X$ and each $U \in \nu O(X, x), \exists V \in \nu O(X, x) \ni \nu\overline{(A)} \subset U$.

Proof. Let $x \in U \in \nu O(X, x)$. Let $B = X - U$, then B is ν-closed and by νg-regularity of X, $\exists$ disjoint $V,\ W \in \nu O(X) \ni x \in V \& B \subseteq W$. Then $\nu\overline{(V)} \cap B = \phi \Rightarrow \nu\overline{(V)} \subseteq X - B$.

Theorem 4.2. The folowings are equivalent:

(i) X is νg_3;

(ii) $\forall x \in X$ and $\forall G \in \nu GO(X, x)$, $\exists U \in \nu O(X) \ni x \in U \subseteq \nu\overline{(U)} \subseteq G$;

(iii) $\forall F \in \nu GC(X)$, the intersection of all ν-closed ν-neighbourhoods of F is exactly F;

(iv) for every set A and $B \in \nu GO(X) \ni A \cap B \neq \phi$, $\exists G \in \nu O(X) \ni A \cap G \neq \phi$ and $\nu\overline{(G)} \subseteq B$;

(v) for $A \neq \phi$ and $B \in \nu GC(X)$ with $A \cap B = \phi$, $\exists$ disjoint $G,\ H \in \nu O(X) \ni A \subseteq G$ and $B \subseteq H$.

Theorem 4.3. If X is νgg_3. Then for each $x \in X$ and each $U \in \nu GO(X, x)$, $\exists V \in \nu GO(X, x) \ni \nu g\overline{(A)} \subset U$.

Proof. Let $x \in U \in \nu O(X, x)$. Let $B = X - U$, then B is νg-closed and by νg-regularity of X, $\exists$ disjoint $V,\ W \in \nu GO(X) \ni x \in V \& B \subseteq W$. Then $\nu g\overline{(V)} \cap B = \phi \Rightarrow \nu g\overline{(V)} \subseteq X - B$.

Corollary 4.1. If X is T_3 [resp: rT_3; νT_3; sT_3; g_3; rg_3; sg_3]. Then for each $x \in X$ and each νg-open neighborhood U of x there exists a νg-open neighborhood V of $x \ni \nu g\overline{(A)} \subset U$.

Theorem 4.4. If f is ν-closed, νg-irresolute bijection. Then X is νgg_3 iff Y is νgg_3.

Proof. Let F be closed set in X and $x \notin F$, then $f(x) \notin f(F)$ and $f(F)$ is νg-closed in Y. By νgg_3 of Y, $\exists V,\ W \in \nu GO(Y) \ni f(x) \in V$ and $f(F) \subseteq W$. Hence $x \in f^{-1}(V)$ and $F \subseteq f^{-1}(W)$, where $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint νg-open sets in X (by νg-irresoluteness of f). Hence X is νgg_3.

Conversely, X be νgg_3 and K any νg-closed in Y with $y \notin K$, then $f^{-1}(K)$ is νg-closed in $X \ni f^{-1}(y) \notin f^{-1}(K)$. By νgg_3 of X, $\exists$ disjoint $V,\ W \in \nu GO(X) \ni f^{-1}(y) \in V$ and $f^{-1}(K) \subseteq W$. Hence $y \in f(V)$ and $K \subseteq f(W) \ni f(V)$ and $f(W)$ are disjoint νg-open sets in X. Thus Y is νgg_3.

Theorem 4.5. X is νg-normal iff $\forall F \in \nu GC(X)$ and a νg-open set $G \in \nu GO(X, A)$, $\exists V \in \nu O(X) \ni F \subseteq V \subseteq \nu\overline{V} \subseteq G$.

Theorem 4.6. X is νg-normal iff $\forall$ disjoint $A,\ B \in \nu GC(X)$, $\exists$ disjoint $U,\ V \in \nu GO(X) \ni A \subseteq U$ and $B \subseteq V$.

Proof. Necessity: Follows from the fact that every ν-open set is νg-open.

Sufficiency: Let $A,\ B$ are disjoint νg-closed sets and $U,\ V$ are disjoint νg-open sets such that $A \subseteq U$ and $B \subseteq V$. Since U and V are νg-open sets, $A \subseteq U$ and $B \subseteq V \Rightarrow A \subseteq \nu(U)^o$ and $B \subseteq \nu(V)^o$. Hence $\nu(U)^o$ and $\nu(V)^o$ are disjoint ν-open sets satisfying the axiom of νg-normality.

Theorem 4.7. The followings are equivalent:

(i) X is ν-normal;

(ii) for any pair of disjoint closed sets A and B, $\exists$ disjoint $U,\ V \in \nu GO(X)$and $\ni A \subseteq U$ and $B \subseteq V$;

(iii) for every closed set A and an open $B \supset A$, $\exists U \in \nu GO(X) \ni A \subseteq U \subseteq \nu\overline{(U)} \subseteq B$;

(iv) for every closed set A and a νg-open $B \supset A$, $\exists U \in \nu O(X) \ni A \subseteq U \subseteq \nu\overline{(U)} \subseteq (B)^o$;

(v) for every νg-closed set A and every open $B \supset A$, $\exists U \in \nu O(X) \ni A \subseteq \nu\overline{(A)} \subseteq U \subseteq \nu\overline{(U)} \subseteq B$.

Proof. (i) $\Rightarrow$ (ii). Let A and B be two disjoint closed subsets of X. Since X is ν-normal, $\exists$ disjoint $U,\ V \in \nu O(X) \ni A \subseteq U$ and $B \subseteq V$. Since ν-open sets are νg-open sets, it follows that $U,\ V \in \nu GO(X) \ni A \subseteq U$ and $B \subseteq V$.

(ii) $\Rightarrow$ (iii). Let A be a closed subset of X and $B \in \tau(X) \ni A \subseteq B$. Then A and $X - B$ are disjoint closed subsets of X. Therefore, $\exists$ disjoint $U,\ V \in \nu GO(X),\ A \subseteq U$ and $X - B \subseteq V$. Thus $A \subseteq U \subseteq X - V \subseteq B$. Since B is open and $X - V$ is νg-closed, therefore $\nu\overline{(X - V)} \subseteq B$. Hence $A \subseteq U \subseteq \nu\overline{(U)} \subseteq B$.

(iii) $\Rightarrow$ (iv). Let A be a closed subset of X and $B \in \nu GO(X) \ni A \subseteq B$. Since B is νg-open and A is closed, therefore $A \subseteq B^o$. In view of theorem 2 of Maheswari and Prasad, there exists a ν-open set U such that $A \subseteq U \subseteq \nu\overline{(U)} \subseteq B^o$.

(iv) $\Rightarrow$ (v). Let A be any νg-closed subset of X and B be an open set such that $A \subseteq B$. $A \subseteq B \Rightarrow \overline{A} \subseteq B$. By theorem 2 of Maheswari and Prasad, $\exists U \in \nu O(X) \ni \overline{A} \subseteq U \subseteq \nu\overline{(U)} \subseteq B^o$.

(v) $\Rightarrow$ (i). Let A and B be disjoint closed subsets of X. Then A is νg-closed and $A \subseteq X - B$. Therefore, $\exists U \in \nu O(X) \ni A \subseteq \nu\overline{(A)} \subseteq U \subseteq \nu\overline{(U)} \subseteq B$. Thus $A \subseteq U,\ B \subseteq X - \nu\overline{(U)}$, which is ν-open and $U \cap (X - \nu\overline{(U)}) = \phi$. Hence X is ν-normal.

Theorem 4.8. The followings are equivalent:

(i) X is νg-normal;

(ii) for every $A \in \nu GC(X)$ and every νg-open set containing A, there exists a ν-clopen set V such that $A \subseteq V \subseteq U$.

Theorem 4.9. Let X be an almost normal space and $F \cap A = \phi$ where F is regularly closed and A is νg-closed, then there exists disjoint open sets U and V such that $F \subseteq U,\ B \subseteq V$.

Theorem 4.10. X is almost normal iff for every disjoint regular closed set F and a closed set A, there exists disjoint νg-open sets in X such that $F \subseteq U,\ B \subseteq V$.

Proof. Necessity: Follows from the fact that every open set is νg-open.

Sufficiency: Let $F,\ A$ be disjoint subsets of X such that F is regular closed and A is closed, there exists disjoint νg-open sets in X such that $F \subseteq U,\ B \subseteq V$. Hence $F \subseteq U^o,\ B \subseteq V^o$, where U^o and V^o are disjoint open sets. Hence X is almost regular.

Theorem 4.11. The followings are equivalent:

(i) X is almost normal;

(ii) for every regular closed set A and for every νg-open set B containing A, $\exists U \in \tau(X) \ni A \subseteq U \subseteq \overline{U} \subseteq B$;

(iii) for every νg-closed set A and for every regular-open set B containing A, $\exists U \in \tau(X) \ni A \subseteq U \subseteq \overline{U} \subseteq B$;

(iv) for every pair of disjoint regularly closed set A and νg-closed set B, $\exists U,\ V \in \tau(X) \ni \overline{U} \cap \overline{V} = \phi$.

§5. νg-R_i, spaces $i = 0, 1$

Definition 5.1. Let $x \in X$. Then

(i) νg-kernel of x is defined and denoted by $Ker_{\nu g}\{x\} = \cap\{U : U \in \nu GO(X)$ and $x \in U\}$;

(ii) $Ker_{\nu g}F = \cap\{U : U \in \nu GO(X)$ and $F \subset U\}$.

Lemma 5.1. Let $A \subset X$, then $Ker_{\nu g}\{A\} = \{x \in X : \nu g\overline{\{x\}} \cap A \neq \phi\}$.

Lemma 5.2. Let $x \in X$. Then $y \in Ker_{\nu g}\{x\}$ iff $x \in \nu g\overline{\{y\}}$.

Proof. Suppose that $y \notin Ker_{\nu g}\{x\}$. Then $\exists V \in \nu GO(X)$ containing $x \ni y \notin V$. Therefore we have $x \notin \nu g\overline{\{y\}}$. The proof of converse part can be done similarly.

Lemma 5.3. For any points $x \neq y \in X$, the followings are equivalent:

(1) $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\}$;

(2) $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$.

Proof. (1) $\Rightarrow$ (2). Let $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\}$, then $\exists z \in X \ni z \in Ker_{\nu g}\{x\}$ and $z \notin Ker_{\nu g}\{y\}$. From $z \in Ker_{\nu g}\{x\}$ it follows that $x \cap \nu g\overline{\{z\}} \neq \phi \Rightarrow x \in \nu g\overline{\{z\}}$. By $z \notin Ker_{\nu g}\{y\}$, we have $\{y\} \cap \nu g\overline{\{z\}} = \phi$. Since $x \in \nu g\overline{\{z\}}, \nu g\overline{\{x\}} \subset \nu g\overline{\{z\}}$ and $\{y\} \cap \nu g\overline{\{x\}} = \phi$. Therefore $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Now $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\} \Rightarrow \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$.

(2) $\Rightarrow$ (1). If $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Then $\exists z \in X \ni z \in \nu g\overline{\{x\}}$ and $z \notin \nu g\overline{\{y\}}$. Then $\exists$ a νg-open set containing z and therefore containing x but not y, namely, $y \notin Ker_{\nu g}\{x\}$. Hence $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\}$.

Definition 5.2. X is said to be

(i) νg-R_0 iff $\nu g\overline{\{x\}} \subseteq G$ whenever $x \in G \in \nu GO(X)$,

(ii) weakly νg-R_0 iff $\cap \nu g\overline{\{x\}} = \phi$,

(iii) νg-R_1 iff for $x,\ y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$ $\exists$ disjoint $U, V \in \nu GO(X) \ni \nu g\overline{\{x\}} \subseteq U$ and $\nu g\overline{\{y\}} \subseteq V$.

Example 5.1.

(i) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then X is weakly νg-R_0 and νg-R_i, $i = 0,\ 1$;

(ii) For $\tau = \{\phi, \{a\}, \{a, b, c\}, X\}$, then X is not weakly νg-R_0 and νg-R_i, $i = 0,\ 1$.

Remark 5.1.

(i) $rR_i \Rightarrow R_i \Rightarrow gR_i \Rightarrow rgR_i \Rightarrow \nu g$-$R_i$, $i = 0,\ 1$;

(ii) Every weakly-R_0 space is weakly νg-R_0.

Lemma 5.1. Every νg-R_0 space is weakly νg-R_0.

Converse of the above lemma is not true in general by the following examples.

Example 5.2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Clearly, X is weakly νg-R_0, since $\cap \nu g\overline{\{x\}} = \phi$. But it is not νg-R_0, for $\{a\} \subset X$ is νg-open and $\nu g\overline{\{a\}} = \{a, c\} \not\subset \{a\}$.

Theorem 5.1. Every νg-regular space X is ν-T_2 and νg-R_0.

Proof. Let X be νg-regular and let $x \neq y \in X$. By lemma 4.1, $\{x\}$ is either ν-open or ν-closed. If $\{x\}$ is ν-open, $\{x\}$ is νg-open and hence ν-clopen. Thus $\{x\}$ and $X - \{x\}$ are separating ν-open sets. Similar argument, for $\{x\}$ is ν-closed gives $\{x\}$ and $X - \{x\}$ are separating ν-closed sets. Thus X is ν-T_2 and νg-R_0.

Theorem 5.2. X is νg-R_0 iff given $x \neq y \in X; \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$.

Proof. Let X be νg-R_0 and let $x \neq y \in X$. Suppose U is a νg-open set containing x but not y, then $y \in \nu g\overline{\{y\}} \subset X - U$ and so $x \notin \nu g\overline{\{y\}}$. Hence $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$.

Conversely, let $x \neq y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}} \Rightarrow \nu g\overline{\{x\}} \subset X - \nu g\overline{\{y\}} = U$(say) a νg-open set in X. This is true for every $\nu g\overline{\{x\}}$. Thus $\cap \nu g\overline{\{x\}} \subseteq U$ where $x \in \nu g\overline{\{x\}} \subseteq U \in \nu GO(\tau)$, which in turn implies $\cap \nu g\overline{\{x\}} \subseteq U$ where $x \in U \in \nu GO(X)$. Hence X is νg-R_0.

Theorem 5.3. X is weakly νg-R_0 iff $Ker_{\nu g}\{x\} \neq X$ for any $x \in X$.

Proof. Let $x_0 \in X \ni ker_{\nu g}\{x_0\} = X$. This means that x_0 is not contained in any proper νg-open subset of X. Thus x_0 belongs to the νg-closure of every singleton set. Hence $x_0 \in \cap \nu g\overline{\{x\}}$, a contradiction.

Conversely, assume $Ker_{\nu g}\{x\} \neq X$ for any $x \in X$. If there is an $x_0 \in X \ni x_0 \in \cap\{\nu g\overline{\{x\}}\}$, then every νg-open set containing x_0 must contain every point of X. Therefore, the unique νg-open set containing x_0 is X. Hence $Ker_{\nu g}\{x_0\} = X$, which is a contradiction. Thus X is weakly νg-R_0.

Theorem 5.4. The following statements are equivalent:

(i) X is νg-R_0 space;

(ii) for each $x \in X, \nu g\overline{\{x\}} \subset Ker_{\nu g}\{x\}$;

(iii) for any νg-closed set F and a point $x \notin F$, $\exists U \in \nu GO(X) \ni x \notin U$ and $F \subset U$;

(iv) each νg-closed set F can be expressed as $F = \cap\{G : G$ is νg-open and $F \subset G\}$;

(v) each νg-open set G can be expressed as $G = \cup\{A : A$ is νg-closed and $A \subset G\}$;

(vi) for each νg-closed set F, $x \notin F$ implies $\nu g\overline{\{x\}} \cap F = \phi$.

Proof. (i) $\Rightarrow$ (ii). For any $x \in X$, we have $Ker_{\nu g}\{x\} = \cap\{U : U \in \nu GO(X)$ and $x \in U\}$. Since X is νg-R_0, each νg-open set containing x contains $\nu g\overline{\{x\}}$. Hence $\nu g\overline{\{x\}} \subset Ker_{\nu g}\{x\}$.

(ii) $\Rightarrow$ (iii). Let $x \notin F \in \nu GC(X)$. Then for any $y \in F$, $\nu g\overline{\{y\}} \subset F$ and so $x \notin \nu g\overline{\{y\}} \Rightarrow y \notin \nu g\overline{\{x\}}$ that is $\exists U_y \in \nu GO(X) \ni y \in U_y$ and $x \notin U_y$, $\forall y \in F$. Let $U = \cup\{U_y : U_y$ is νg-open,$y \in U_y$ and$x \notin U_y\}$. Then $U \in \nu GO(X) \ni x \notin U$ and $F \subset U$.

(iii) $\Rightarrow$(iv). Let $F \in \nu GC(X)$ and $N = \cap\{G : G \in \nu GO(X)$ and $F \subset G\}$. Then $F \subset N \rightarrow (1)$. Let $x \notin F$, then by (iii) $\exists G \in \nu GO(X) \ni x \notin G$ and $F \subset G$, hence $x \notin N$ which implies $x \in N \Rightarrow x \in F$. Hence $N \subset F \rightarrow (2)$.

Therefore from (1) and (2), each νg-closed set $F = \cap\{G : G$ is νg-open and $F \subset G\}$

(iv) $\Rightarrow$ (v). obvious.

(v) $\Rightarrow$ (vi). Let $x \notin F \in \nu GC(X)$. Then $X - F = G$ is a νg-open set containing x. Then by (v), G can be expressed as the union of νg-closed sets A contained in G, and so there is an $M \in \nu GC(X) \ni x \in M \subset G$; and hence $\nu g\overline{\{x\}} \subset G$ which implies $\nu g\overline{\{x\}} \cap F = \phi$.

(vi) $\Rightarrow$ (i). Let $x \in G \in \nu GO(X)$. Then $x \notin (X - G)$, which is a νg-closed set. Therefore by (vi) $\nu g\overline{\{x\}} \cap (X - G) = \phi$, which implies that $\nu g\overline{\{x\}} \subseteq G$. Thus X is νg-R_0 space.

Theorem 5.5. If f is νg-closed one-one function and if X is weakly νg-R_0, so is Y.

Theorem 5.6. If X is weakly νg-R_0, then for every space Y, $X \times Y$ is weakly νg-R_0.

Proof. $\cap \nu g\overline{\{(x,y)\}} \subseteq \cap\{\nu g\overline{\{x\}} \times \nu g\overline{\{y\}}\} = \cap[\nu g\overline{\{x\}}] \times [\nu g\overline{\{y\}}] \subseteq \phi \times Y = \phi$. Hence $X \times Y$ is νg-R_0.

Corollary 5.1.

(i) If X and Y are weakly νg-R_0, then $X \times Y$ is weakly νg-R_0;

(ii) if X and Y are (weakly-)R_0, then $X \times Y$ is weakly νg-R_0;

(iii) if $X$and Y are νg-R_0, then $X \times Y$ is weakly νg-R_0;

(iv) if X is νg-R_0 and Y are weakly R_0, then $X \times Y$ is weakly νg-R_0.

Theorem 5.7. X is νg-R_0 iff for any $x,\ y \in X, \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}} \Rightarrow \nu g\overline{\{x\}} \cap \nu g\overline{\{y\}} = \phi$.

Proof. Let X is νg-R_0 and $x,\ y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Then $\exists z \in \nu g\overline{\{x\}} \ni z \notin \nu g\overline{\{y\}} (or z \in \nu g\overline{\{y\}}) \ni z \notin \nu g\overline{\{x\}}$. There exists $V \in \nu GO(X) \ni y \notin V$ and $z \in V$, hence $x \in V$. Therefore, $x \notin \nu g\overline{\{y\}}$. Thus $x \in [G\overline{y}]^c \in \nu GO(X)$, which implies $\nu g\overline{\{x\}} \subset [\nu g\overline{\{y\}}]^c$ and $\nu g\overline{\{x\}} \cap \nu g\overline{\{y\}} = \phi$. The proof for otherwise is similar.

Sufficiency: Let $x \in V \in \nu GO(X)$. We show that $\nu g\overline{\{x\}} \subset V$. Let $y \notin V$, i.e. $y \in [V]^c$. Then $x \neq y$ and $x \notin \nu g\overline{\{y\}}$. Hence $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. By assumption, $\nu g\overline{\{x\}} \cap \nu g\overline{\{y\}} = \phi$. Hence $y \notin \nu g\overline{\{x\}}$. Therefore $\nu g\overline{\{x\}} \subset V$.

Theorem 5.8. X is νg-R_0 iff for any points $x, y \in X, Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\} \Rightarrow Ker_{\nu g}\{x\} \cap Ker_{\nu g}\{y\} = \phi$.

Proof. Suppose X is νg-R_0. Thus by lemma 5.3, for any $x, y \in X$ if $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\}$ then $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Assume that $z \in Ker_{\nu g}\{x\} \cap Ker_{\nu g}\{y\}$. By $z \in Ker_{\nu g}\{x\}$ and lemma 5.2, it follows that $x \in \nu g\overline{\{z\}}$. Since $x \in \nu g\overline{\{z\}}, \nu g\overline{\{x\}} = \nu g\overline{\{z\}}$. Similarly, we have $\nu g\overline{\{y\}} = \nu g\overline{\{z\}} = \nu g\overline{\{x\}}$. This is a contradiction. Therefore, we have $Ker_{\nu g}\{x\} \cap Ker_{\nu g}\{y\} = \phi$.

Conversely, let $x,\ y \in X,\ \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, then by lemma 5.3, $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\}$. Hence by hypothesis $Ker_{\nu g}\{x\} \cap Ker_{\nu g}\{y\} = \phi$ which implies $\nu g\overline{\{x\}} \cap \nu g\overline{\{y\}} = \phi$. Because $z \in \nu g\overline{\{x\}}$ implies that $x \in Ker_{\nu g}\{z\}$ and $Ker_{\nu g}\{x\} \cap Ker_{\nu g}\{z\} \neq \phi$. Therefore by theorem 5.7, X is a νg-R_0 space.

Theorem 5.9. The following properties are equivalent:

(1) X is a νg-R_0 space;

(2) For any $A \neq \phi$ and $G \in \nu GO(X, \tau) \ni A \cap G \neq \phi$, $\exists F \in \nu GC(X, \tau) \ni A \cap F \neq \phi$ and $F \subset G$.

Proof.

(1) $\Rightarrow$ (2). Let $A \neq \phi$ and $G \in \nu GO(X) \ni A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \nu GO(X),\ \nu g\overline{\{x\}} \subset G$. Set $F = \nu g\overline{\{x\}}$, then $F \in \nu GC(X),\ F \subset G$ and $A \cap F \neq \phi$;

(2) $\Rightarrow$ (1). Let $G \in \nu GO(X)$ and $x \in G$. By (2) $\nu g\overline{\{x\}} \subset G$. Hence X is νg-R_0.

Theorem 5.10. The following properties are equivalent:

(1) X is a νg-R_0 space;

(2) $x \in \nu g\overline{\{y\}}$ iff $y \in \nu g\overline{\{x\}}$, for any points x and y in X.

Proof. (1) $\Rightarrow$ (2). Assume X is νg-R_0. Let $x \in \nu g\overline{\{y\}}$ and D be any νg-open set such that $y \in D$. Now by hypothesis, $x \in D$. Therefore, every νg-open set which contain y contains x. Hence $y \in \nu g\overline{\{x\}}$.

(2) $\Rightarrow$ (1). Let $x \in U \in \nu GO(X)$. If $y \notin U$, then $x \notin \nu g\overline{\{y\}}$ and hence $y \notin \nu g\overline{\{x\}}$. This implies that $\nu g\overline{\{x\}} \subset U$. Hence X is νg-R_0.

Theorem 5.11. The following properties are equivalent:

(1) X is a νg-R_0 space;

(2) If F is νg-closed, then $F = Ker_{\nu g}(F)$;

(3) If F is νg-closed and $x \in F$, then $Ker_{\nu g}\{x\} \subseteq F$;

(4) If $x \in X$, then $Ker_{\nu g}\{x\} \subset \nu g\overline{\{x\}}$.

Proof. (1) $\Rightarrow$ (2). Let $x \notin F \in \nu GC(X) \Rightarrow (X - F) \in \nu GO(X, x)$. For X is νg-R_0, $\nu g\overline{(\{x\})} \subset (X - F)$. Thus $\nu g\overline{(\{x\})} \cap F = \phi$ and $x \notin Ker_{\nu g}(F)$. Hence $Ker_{\nu g}(F) = F$.

(2) $\Rightarrow$ (3). $A \subset B \Rightarrow Ker_{\nu g}(A) \subset Ker_{\nu g}(B)$. Therefore, by(2) $Ker_{\nu g}\{x\} \subset Ker_{\nu g}(F) = F$.

(3) $\Rightarrow$ (4). Since $x \in \nu g\overline{\{x\}}$ and $\nu g\overline{\{x\}}$ is νg-closed, by (3) $Ker_{\nu g}\{x\} \subset \nu g\overline{\{x\}}$.

(4) $\Rightarrow$ (1). Let $x \in \nu g\overline{\{y\}}$. Then by lemma 5.2, $y \in Ker_{\nu g}\{x\}$. Since $x \in \nu g\overline{\{x\}}$ and $\nu g\overline{\{x\}}$ is νg-closed, by (4) we obtain $y \in Ker_{\nu g}\{x\} \subseteq \nu g\overline{\{x\}}$. Therefore $x \in \nu g\overline{\{y\}}$ implies $y \in \nu g\overline{\{x\}}$.

The converse is obvious and X is νg-R_0.

Corollary 5.2. The following properties are equivalent:

(1) X is νg-R_0;

(2) $\nu g\overline{\{x\}} = Ker_{\nu g}\{x\} \forall x \in X$.

Proof. Straight forward from theorems 5.4 and 5.11.

Recall that a filterbase F is called νg-convergent to a point x in X, if for any νg-open set U of X containing x, there exists $B \in F$ such that $B \subset U$.

Lemma 5.4. Let x and y be any two points in X such that every net in X νg-converging to y νg-converges to x. Then $x \in \nu g\overline{\{y\}}$.

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $\nu g\overline{(\{y\})}$. Since $\{x_n\}_{n \in N} \nu g$-converges to y, then $\{x_n\}_{n \in N} \nu g$-converges to x and this implies that $x \in \nu g\overline{\{y\}}$.

Theorem 5.12. The following statements are equivalent:

(1) X is a νg-R_0 space;

(2) If $x,\ y \in X$, then $y \in \nu g\overline{\{x\}}$ iff every net in X νg-converging to y νg-converges to x.

Proof.

(1) $\Rightarrow$ (2). Let $x,\ y \in X \ni y \in \nu g\overline{\{x\}}$. Suppose that $\{x_\alpha\}_{\alpha \in \Lambda}$ is a net in $X \ni \{x_\alpha\}_{\alpha \in \Lambda}$ νg-converges to y. Since $y \in \nu g\overline{\{x\}}$, by theorem 5.7 we have $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$. Therefore $x \in \nu g\overline{\{y\}}$. This means that $\{x_\alpha\}_{\alpha \in \Lambda}$ νg-converges to x.

Conversely, let $x,\ y \in X$ such that every net in X νg-converging to y νg-converges to x. Then $x \in \nu g\overline{\{y\}}$ by theorem 5.4. By theorem 5.7, we have $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$. Therefore $y \in \nu g\overline{\{x\}}$.

(2) $\Rightarrow$ (1). Let $x,\ y \in X \ni \nu g\overline{\{x\}} \cap \nu g\overline{\{y\}} \neq \phi$. Let $z \in \nu g\overline{\{x\}} \cap\ \nu g\overline{\{y\}}$. So $\exists$ a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $\nu g\overline{\{x\}} \ni \{x_\alpha\}_{\alpha \in \Lambda}$ νg-converges to z. Since $z \in \nu g\overline{\{y\}}$, then $\{x_\alpha\}_{\alpha \in \Lambda}$ νg-converges to y. It follows that $y \in \nu g\overline{\{x\}}$. Similarly we obtain $x \in \nu g\overline{\{y\}}$. Therefore $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$. Hence, X is νg-R_0.

Theorem 5.13.

(i) Every subspace of νg-R_1 space is again νg-R_1;

(ii)Product of any two νg-R_1 spaces is again νg-R_1.

Theorem 5.14. X is νg-R_1 iff given $x \neq y \in X$, $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$.

Theorem 5.15. Every νg_2 space is νg-R_1.

The converse is not true. However, we have the following result.

Theorem 5.16. Every νg_1, νg-R_1 space is νg_2.

Proof. Let $x \neq y \in X$. Since X is νg_1, $\{x\}$ and $\{y\}$ are νg-closed sets $\ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Since X is νg-R_1, $\exists U,\ V \in \nu GO(X),\ \ni x \in U,\ y \in V$. Hence X is νg_2.

Corollary 6.1. X is νg_2 iff it is νg-R_1 and νg_1.

Theorem 5.17. The followings are equivalent

(i) X is νg-R_1;

(ii) $\cap\nu g\overline{\{x\}} = \{x\}$;

(iii) For every $x \in X$, intersection of all νg-neighborhoods of x is $\{x\}$.

Proof. (i) $\Rightarrow$ (ii). Let $y \neq x \in X \ni y \in \nu g\overline{\{x\}}$. since X is νg-R_1, $\exists U \in \nu GO(X) \ni y \in U$, $x \notin U$ or $x \in$U, $y \notin U$. In either case $y \notin \nu g\overline{\{x\}}$. Hence $\cap\nu g\overline{\{x\}} = \{x\}$.

(ii) $\Rightarrow$ (iii). If $y \neq x \in X$, then $x \notin \cap\nu g\overline{\{y\}}$, so there is a νg-open set containing x but not y. Therefore y does not belong to the intersection of all νg-neighborhoods of x. Hence intersection of all νg-neighborhoods of x is $\{x\}$.

(iii) $\Rightarrow$ (i). Let $x \neq y \in X$, by hypothesis, y does not belong to the intersection of all νg-neighborhoods of x and x does not belong to the intersection of all νg-neighborhoods of y, which implies $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Hence X is νg-R_1.

Theorem 5.18. The followings are equivalent:

(i) X is νg-R_1;

(ii) For each pair $x,\ y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, $\exists$ a νg-clopen set $V \ni x \in V$ and $y \notin V$;

(iii) For each pair $x,\ y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, $\exists f : X \to [0, 1] \ni f(x) = 0$ and $f(C) = 1$ and f is νg-continuous.

Proof.

(i) $\Rightarrow$ (ii). Let $x,\ y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, $\exists$ disjoint $U,\ W \in \nu GO(X) \ni \nu g\overline{\{x\}} \subset U$ and $\nu g\overline{\{y\}} \subset W$and $V = \nu g\overline{(U)}$ is νg-open and νg-closed such that $x \in V$ and $y \notin V$.

(ii) $\Rightarrow$ (iii). Let $x,\ y \in X$ such that $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, and let V be νg-open and νg-closed such that $x \in V$ and $y \notin V$. Then $f : X \to [0, 1]$ defined by $f(z) = 0$ if $z \in V$ and $f(z) = 1$ if $z \notin V$ satisfied the desired properties.

(iii) $\Rightarrow$ (i). Let $x,\ y \in X \ni \nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, let $f : X \to [0, 1]$ be νg-continuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$ are disjoint νg-open and νg-closed sets in X, such that $\nu g\overline{\{x\}} \subset U$ and $\nu g\overline{\{y\}} \subset V$.

Theorem 5.19. If X is νg-R_1, then X is νg-R_0.

Proof. Let $x \in U \in \nu GO(X)$. If $y \notin U$, then $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Hence, $\exists$ a νg-open $V_y \ni \nu g\overline{\{y\}} \subset V_y$ and $x \notin V_y$, $\Rightarrow y \notin \nu g\overline{\{x\}}$. Thus $\nu g\overline{\{x\}} \subset U$. Therefore X is νg-R_0.

Theorem 5.20. X is νg-R_1 iff for $x,\ y \in X$, $Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\}$, $\exists$ disjoint $U,\ V \in \nu GO(X) \ni \nu g\overline{\{x\}} \subset U$ and $\nu g\overline{\{y\}} \subset V$.

§6. νg-C_i and νg-D_i spaces, $i = 0, 1, 2$

Definition 6.1. X is said to be a

(i) νg-C_0 space if for each pair of distinct points $x,\ y$ of X there exists a νg-open set G whose closure contains either x or y.

(ii) νg-C_1 space if for each pair of distinct points $x,\ y$ of X there exists a νg-open set G whose closure containing x but not y and a νg-open set H whose closure containing y but not x.

(iii) νg-C_2 space if for each pair of distinct points $x,\ y$ of X there exists disjoint νg-open sets G and H such that G containing x but not y and H containing y but not x.

Note 6.1. νg-$C_2 \Rightarrow \nu g$-$C_1 \Rightarrow \nu g$-C_0 but converse need not be true in general.

Example 6.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then X is νg-C_i, $i = 0,\ 1,\ 2$.

Theorem 6.1.

(i) Every subspace of νg-C_i space is νg-C_i.

(ii) Every νg_i spaces is νg-C_i.

(iii) Product of νg-C_i spaces is νg-C_i.

Theorem 6.2. Let (X, τ) be any νg-C_i space and A be any non empty subset of X then A is νg-C_i iff $(A, \tau_{/A})$ is νg-C_i.

Theorem 6.3.

(i) If X is νg-C_1 then each singleton set is νg-closed.

(ii) In an νg-C_1 space disjoint points of X have disjoint νg-closures.

Definition 6.2. $A \subset X$ is called a νg Difference(Shortly νg-D-set) set if there are two U, $V \in \nu GO(X, \tau)$ such that $U \neq X$ and $A = U - V$.

Clearly every νg-open set U different from X is a νg-D-set if $A = U$ and $V = \phi$.

Definition 6.3. X is said to be a

(i) νg-D_0 if for any pair of distinct points x and y of X there exist a νg-D-set in X containing x but not y or a νg-D-set in X containing y but not x;

(ii) νg-D_1 if for any pair of distinct points x and y in X there exist a νg-D-set of X containing x but not y and a νg-D-set in X containing y but not x;

(iii) νg-D_2 if for any pair of distinct points x and y of X there exists disjoint νg-D-sets G and H in X containing x and y, respectively.

Example 6.2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is νg-D_i, $i = 0,\ 1,\ 2$.

Remark 6.1. (i) If X is $r - T_i$, then it is νg_i, $i = 0,\ 1,\ 2$ and converse is false;

(ii) If X is νg_i, then it is νg_{i-1}, $i = 1,\ 2$;

(iii) If X is νg_i, then it is νg-D_i, $i = 0,\ 1,\ 2$;

(iv) If X is νg-D_i, then it is νg-D_{i-1}, $i = 1,\ 2$.

Theorem 6.4. The following statements are true:

(i) X is νg-D_0 iff it is νg_0;

(ii) X is νg-D_1 iff it is νg-D_2.

Proof. (i) The sufficiency is stated in remark 6.1(iii). To prove necessity, let X be νg-D_0. Then for each $x \neq y \in X$, at least one of them, say x, belong to a νg-D-set G but $y \notin G$. Let $G = U_1 - U_2$ where $U_1 \neq X$ and $U_1,\ U_2 \in \nu GO(X)$. Then $x \in U_1$ and for $y \notin G$ we have two cases:

(a) $y \notin U_1$;

(b) $y \in U_1$ and $y \notin U_2$.

In case (a), $x \in U_1$ but $y \notin U_1$; In case (b), $y \in U_2$ but $x \notin U_2$. Hence X is νg_0.

(ii) Sufficiency. Remark 6.1(iv). Necessity. Suppose X is νg-D_1. Then for each $x \neq y \in X$, we have νg-D-sets $G_1,\ G_2 \ni x \in G_1;\ y \notin G_1;\ y \in G_2,\ x \notin G_2$. Let $G_1 = U_1 - U_2,\ G_2 = U_3 - U_4$. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1 - U_2$, it follows that $x \in U_1 - (U_2 \cup U_3)$ and by $y \in U_3 - U_4$ we have $y \in U_3 - (U_1 \cup U_4)$. Therefore $(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4) = \phi$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2,\ y \in U_2,\ (U_1 - U_2) \cap U_2 = \phi$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4,\ x \in U_4,\ (U_3 - U_4) \cap U_4 = \phi$.

Therefore X is νg-D_2.

Corollary 6.1. If X is νg-D_1, then it is νg_0.

Proof. Remark 6.1(iv) and theorem 6.2(i).

Definition 6.4. A point $x \in X$ which has X as the unique νg-neighborhood is called νgc.c point.

Theorem 6.5. For a νg_0 space X the followings are equivalent:

(1) X is νg-D_1;

(2) X has no νgc.c point.

Proof. (1) $\Rightarrow$ (2). Since X is νg-D_1, each point x of X is contained in a νg-D-set $O = U - V$ and thus in U. By definition $U \neq X$. This implies that x is not a νgc.c point.

(2) $\Rightarrow$ (1). If X is νg_0, then for each $x \neq y \in X$, at least one of them, x (say) has a νg-neighborhood U containing x and not y. Thus U which is different from X is a νg-D-set. If X has no νgc.c point, then y is not a νgc.c point. This means that there exists a νg-neighborhood V of y such that $V \neq X$. Thus $y \in (V - U)$ but not x and $V - U$ is a νg-D-set. Hence X is νg-D_1.

Corollary 6.2. A νg_0 space X is νg-D_1 iff there is a unique νgc.c point in X.

Proof. Only uniqueness is sufficient to prove. If x_0 and y_0 are two νgc.c points in X then since X is νg_0, at least one of x_0 and y_0 say x_0, has a νg-neighbourhood U such that $x_0 \in U$ and $y_0 \notin U$, hence $U \neq X$, x_0 is not a νgc.c point, a contradiction.

Remark 6.2. It is clear that an νg_0 space X is not νg-D_1 iff there is a unique νgc.c point in X. It is unique because if x and y are both νgc.c point in X, then at least one of them say x has a νg-neighborhood U containing x but not y. But this is a contradiction since $U \neq X$.

Definition 6.5. X is νg-symmetric if for x and y in X, $x \in \nu g\overline{\{y\}}$ implies $y \in \nu g\overline{\{x\}}$.

Theorem 6.6. X is νg-symmetric iff $\{x\}$ is νg-closed for each $x \in X$.

Proof. Assume that $x \in \nu g\overline{\{y\}}$ but $y \notin \nu g\overline{\{x\}}$. This means that $[\nu g\overline{\{x\}}]^c$ contains y. This implies that $\nu g\overline{\{y\}}$ is a subset of $[\nu g\overline{\{x\}}]^c$. Now $[\nu g\overline{\{x\}}]^c$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E \in \nu GO(X)$ but $\nu g\overline{\{x\}}$ is not a subset of E. This means that $\nu g\overline{\{x\}}$ and E^c are not disjoint. Let y belong to their intersection. Now we have $x \in \nu g\overline{\{y\}}$ which is a subset of E^c and $x \notin E$. But this is a contradiction.

Corollary 6.3. If X is a νg_1, then it is νg-symmetric.

Proof. Follows from theorem 3.3 and theorem 6.6.

Corollary 6.4. The following are equivalent:

(1) X is νg-symmetric and νg_0;

(2) X is νg_1.

Proof. By corollary 6.3 and remark 6.1 it suffices to prove only (1) $\rightarrow$ (2). Let $x \neq y$ and by νg_0, we may assume that $x \in G_1 \subset \{y\}^c$ for some $G_1 \in \nu GO(X)$. Then $x \notin \nu g\overline{\{y\}}$ and hence $y \notin \nu g\overline{\{x\}}$. There exists a $G_2 \in \nu GO(X)$ such that $y \in G_2 \subset \{x\}^c$ and X is a νg_1 space.

Theorem 6.7. For an νg-symmetric space X the following are equivalent:

(1) X is νg_0;

(2) X is νg-D_1;

(3) X is νg_1.

Proof. (1) $\Rightarrow$ (3) corollary 6.4 and (3) $\Rightarrow$ (2) $\Rightarrow$ (1) remark 6.1.

Theorem 6.8. If f is a νg-irresolute surjective function and E is a νg-D-set in Y, then the inverse image of E is a νg-D-set in X.

Proof. Let E be a νg-D-set in Y. Then there are νg-open sets $U_1,\ U_2 \in \nu GO(Y) \ni E = U_1 - U_2$ and $U_1 \neq Y$. By the νg-irresoluteness of $f, f^{-1}(U_1)$ and $f^{-1}(U_2)$ are νg-open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a νg-D-set.

Theorem 6.9. If Y is νg-D_1 and f is νg-irresolute and bijective, then X is νg-D_1.

Proof. Suppose that Y is a νg-D_1 space. Let $x \neq y \in X$. Since f is injective and Y is νg-D_1, there exist νg-D-sets G_z and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin G_z$ and $f(x) \notin G_y$. By theorem 6.8, $f^{-1}(G_z)$ and $f^{-1}(G_y)$ are νg-D-sets in X containing x and y respectively. Hence X is a νg-D_1 space.

Theorem 6.10. X is νg-D_1 iff for each $x \neq y \in X$, $\exists$ νg-irresolute surjective function f, where Y is a νg-D_1 space $\ni f(x) \neq f(y)$.

Proof. Necessity. For every $x \neq y \in X$, it suffices to take the identity function on X.

Sufficiency. Let $x \neq y \in X$. By hypothesis, $\exists$ a νg-irresolute, surjection f from X onto a νg-D_1 space $Y \ni f(x) \neq f(y)$. Therefore, $\exists$ disjoint νg-D-sets G_x and G_y in $Y \ni f(x) \in G_x$ and $f(y) \in G_y$. Since f is νg-irresolute and surjective, by theorem 6.8, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint νg-D-sets in X containing x and y respectively. Therefore X is νg-D_1 space.

Corollary 6.5. Let $\{X_\alpha/\alpha \in I\}$ be any family of topological spaces. If X_α is νg-D_1 for each $\alpha \in I$, then the product ΠX_α is νg-D_1.

Proof. Let (x_α) and (y_α) be any pair of distinct points in ΠX_α. Then there exists an index $\beta \in I \ni x_\beta \neq y_\beta$. The natural projection $P_\beta : \Pi X_\alpha \to X_\beta$ is almost continuous and almost open and $P_\beta((x_\alpha)) = P_\beta((y_\alpha))$. Since X_β is νg-D_1, ΠX_α is νg-D_1.

Conclusion. In this paper we defined new separation axioms using νg-open sets and studied their interrelations with other separation axioms.

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References

[1] D. Andrijevic, Semi-preopen sets, Mat. Vesnik , **38**(1986), 24-32.

[2] S. P. Arya and T. Nour, Characterizations of s-normal spaces, I. J. P. A. M., **21**(1990), No. 8, 717-719.

[3] S. N. Bairagya and S. P. Baisnab, On structure of Generalized open sets, Bull. Cal. Math. Soc., **79**(1987), 81-88.

[4] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi. Univ. (Math), **12**(1991), 05-13.

[5] S. Balasubramanian, C. Sandhy and P. A. S. Vyjayanthi, ν-T_i, ν-R_i and ν-C_i-Axioms-Scientia Magna, **4**(2008), No. 4, 86-103.

[6] S. Balasubramanian, ν-generalized closed sets, Bull. Kerala. Math. Soc., **6**(2009), No. 2, 81-92.

[7] S. Balasubramanian, Generalized separation axioms, Scientia Magna, **6**(2010), No. 4, 1-14.

[8] S. Balasubramanian νg-continuity-Proc, International seminar in new trends in Mathematics and its applications, Bharatha Matha College, Trikkakara, Kerala-2011.

[9] S. Balasubramanian, νg-boundary and νg-exterior operators, Acta Ciencia Indica, **XXXVIIM**(2011), No. 1, 11-18.

[10] S. Balasubramanian and C. Sandhya, sg-separation axioms, Indian Journal of Engineering, **1**(2012), No. 1, 46-54.

[11] S. Balasubramanian, C. Sandhya and M. D. S. Saikumar, rg-separation axioms, International Journal of Advanced Scientific and Technical Research, **5**(2012), No. 2, 146-161.

[12] P. Bhattachary and B. K. Lahiri, semi-generalized closed sets in topology, Indian J. Pure and Appl. Math., **29**(1987), No. 3, 375-382.

[13] N. Biswas, Characterization of semicontinuous mappings, Atti. Accad. Naz. Lience. Rend. Cl. Sci. Fis. Mat. Nat. ,**48**(1970), No. 8, 399-402.

[14] Chawalit Boonpok, Generalized continuous functions from any topological space into product, Naresuan University journal, **11**(2003), No. 2, 93-98.

[15] Chawalit Boonpok, Preservation theorems concering g-Hausdorf and rg-Hausdorff spaces, KKU. Sci. J., **31**(2003), No. 3, 138-140.

[16] R. Devi, K. Balachandran and H. Maki, semi-Generalized Homeomorphisms and Generalized semi-Homeomorphismin topological spaces, IJPAM, **26**(1995), No. 3, 271-284.

[17] J. Dontchev and T. Noiri, On properties of spaces defined in terms of semi-regular sets.

[18] J. Dontchev and H. Maki, On the behaviour of sg-closed sets.

[19] J. Dontchev and M. Ganster, More on sg-compact spaces.

[20] W. Dunham, $T_{\frac{1}{2}}$ spaces, Kyungpook Math. J., **17**(1977), 161-169.

[21] A. I. El-Maghrabi and A. A. Naset, Between semi-closed and GS-closed sets, J. Taibah. Uni. Sci., **2**(2009), 79-87.

[22] M. Ganster, S. Jafarai and G. B. Navalagi, On semi-g-regular and semi-g-normal spaces.

[23] Y. Gnanambal, Studies on gpr-closed sets and generalizations of locally closed sets, Bharathiar University, (1998).

[24] Jiling Cao, Sina Greenwood and Ivan Reilly, Generalized closed sets: A Unified Approach.

[25] Jiling Cao, M. Ganster and Ivan Reily, on sg-closed sets and $g\alpha-$closed sets.

[26] Jin Han Park, On s-normal spaces and some functions, IJPAM, **30**(1999), No. 6, 575-580.

[27] S. R. Malghan, Generalized closed maps, The J. Karnataka Univ., **XXVII**(1982), 82-88.

[28] A. S. Mashhour, M. E. Abd El-Monsef and I. A. Hasanein, On pretopological spaces, Bull. Math. Soc. Sci. R. S. R, **28**(1984), No. 76, 39-45.

[29] A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, On $\alpha-$continuous and $\alpha-$open mappings, Acta. Math. Hunga., **41**(1983), 213-218.

[30] Miguel Caldas and R. K. Saraj, A surve on semi$-T_{\frac{1}{2}}$ spaces, Pesquimat, **II**(1999), No. 1, 33-40.

[31] Miguel Caldas and R. K. Saraf, A Research on characterization of semi$-T_{\frac{1}{2}}$ spaces, Divulgenious Math., **8**(2000), No. 1, 43-50.

[32] G. B. Navalagi, Properties of gs-closed sets and sg-closed sets in topology.

[33] G. B. Navalagi, Semi-generalized separation in topology.

[34] O. Njastad, On some classes of nearly open sets, Pacific J. Math., **15**(1996), 961-970.

[35] Norman Levine, Semi open sets and semicontinuity in topological spaces, Amer. Math. Monthly., **70**(1963), 36-41.

[36] Norman Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo., **19**(1970), No. 2, 89-96.

[37] T. Noiri, semi-normal spaces and some functions, Acta Math Hungar, **65**(1994), No. 3, 305-311.

[38] T. Noiri, Mildly Normal spaces and some functions, Kyungpook Math. J., **36**(1996), 183-190.

[39] T. Noiri, A note on α-regular spaces, IJPAM, **30**(1999), No. 9, 889-891.

[40] T. Noiri and V. Popa, On G-regular spaces and some functions, Mem. Fac. Sci. Kochi. Univ.(Math), **20**(1999), 67-74.

[41] N. Palaniappan and K. Chandrasekhara rao, Regular Generalized closed sets, Kyungpook M. J., **33**(1993), No. 2, 211-219.

[42] V. K. Sharma, g-open sets and almost normality, Acta Ciencia Indica, **XXXIIIM**(2007), No. 3, 1249-1251.

[43] V. K. Sharma, sg-separation axioms, Acta Ciencia Indica, **XXXIIIM**(2007), No. 3, 1253-1259.

[44] V. K. Sharma, g-separation axioms, Acta Ciencia Indica, **XXXIIIM**(2007), No. 4, 1271-1276.

[45] P. Sundaram, H. Maki and K. Balachandran, Semi-generalized continuous maps and semi-$T_{\frac{1}{2}}$ spaces, Bull. Fukuoka Univ. of Edn., **40**(1991), No. III, 33-40.

[46] M. H. Stone, Applications of the theory of Boolean rings to general topology, TAMS, **41**(1937), 375-81.

[47] A. Vadivel, Studies on $rg\alpha$-closed sets in topological and bitopological spaces, Annamalai University, 2009.

[48] M. K. R. S. Veerakumar, concerning semi-$T_{\frac{1}{3}}$ spaces.

[49] M. K. R. S. Veerakumar, pre-semi-closed sets, Indian J. Math., **44**(2002), No. 2, 165-181.

[50] M. K. R. S. Veerakumar, Between closed sets and g-closed sets, Mem. Fac. Sci. Kochi. Univ., **21**(2000), 01-19.

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More on νg-separation axioms

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Abstract In this paper by using νg-open sets, I define almost νg-normality and mild νg-normality also we continue the study of further properties of νg-normality. We show that these three axioms are regular open hereditary. I also define the class of almost νg-irresolute mappings and show that νg-normality is invariant under almost νg-irresolute M-νg-open continuous surjection.

Keywords νg-open, semiopen, semipreopen, almost normal, midly normal, M-νg-closed, M-νg-open, rc-continuous.
2000 Mathematics Subject Classification: 54D15, 54D10

§1. Introduction

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C. E. Aull studied some separation axioms between T_1 and T_2 spaces, namely, S_1 and S_2. In 1982, S. P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G. B. Navlagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vyjayanthi studied ν-Normal Almost-ν-Normal, Mildly-ν-Normal and ν-US spaces. νg-open sets and νg-continuous mappings were introduced in 2009 and 2011 by S. Balasubramanian. The purpose of this paper is to examine the normality axioms, νg-US, νg-S_1 and νg-S_2 spaces. νg-convergence, sequentially νg-compact, sequentially νg-continuous maps, and sequentially sub νg-continuous maps are also introduced and studied in this paper. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote topological spaces on which no separation axioms are assumed explicitly stated.

§2. Preliminaries

Definition 2.1. A $\subset X$ is called
(i) closed [resp: Semi closed; ν-closed] if its complement if is open [resp: semi open; ν-open];
(ii) $r\alpha$-open [ν-open] if $U \in \alpha O(X)$ $[RO(X)]$ such that $U \subset A \subset \alpha\overline{(U)}$ $[U \subset A \subset \overline{(U)}]$;
(iii) semi-θ-open if it is the union of semi-regular sets and its complement is semi-θ-closed;

(iv) r-closed [α-closed; pre-closed; β-closed] if $A = \overline{(A^o)}[(\overline{(A^o)})^o \subseteq A; \overline{(A^o)} \subseteq A; \overline{((\overline{(A)})^o)} \subseteq A]$;

(v) g-closed [rg-closed; g''-closed; $\hat{g}$-closed] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open [gs-open; semi-open] in X;

(vi) sg-closed [gs-closed] if $s\overline{(A)} \subseteq U$ whenever $A \subseteq U$ and U is semi-open{open} in X;

(vii) pg-closed [gp-closed; gpr-closed] if $p\overline{(A)} \subseteq U$ whenever $A \subseteq U$ and U is pre-open [open; regular-open] in X;

(viii) αg-closed [$g\alpha$-closed; r$g\alpha$-closed; $r\alpha g$-closed; αgs-closed; $g\alpha$-closed] if $\alpha\overline{(A)} \subseteq U$ whenever $A \subseteq U$ and U is open [α-open; $r\alpha$-open; r-open; semi-open; gs-open] in X;

(ix) νg-closed if $\nu\overline{(A)} \subseteq U$ whenever $A \subseteq U$ and U is ν-open in X;

(x) The family of all νg-open sets of X containing point x is denoted by $\nu GO(X, x)$.

Definition 2.2. Let $A \subset X$. Then a point x is said to be a

(i) limit point of A if each open set containing x contains some point y of A such that $x \neq y$;

(ii) T_0-limit point [20] of A if each open set containing x contains some point y of A such that $cl\{x\} \neq cl\{y\}$, or equivalently, such that they are topologically distinct;

(iii) ν-T_0-limit point [11] of A if each open set containing x contains some point y of A such that $\nu cl\{x\} \neq \nu cl\{y\}$, or equivalently, such that they are topologically distinct.

Definition 2.3.[10] A function f is said to be

(i) almost-ν-irresolute if for each $x \in X$ and each ν-neighborhood V of $f(x)$, $\nu\overline{(f^{-1}(V))}$ is a ν-neighborhood of x;

(ii) sequentially ν-continuous at $x \in X$ if $f(x_n) \rightarrow^{\nu} f(x)$ whenever $< x_n > \rightarrow^{\nu} X$. If f is sequentially ν-continuous at all $x \in X$, then f is said to be sequentially ν-continuous;

(iii) sequentially nearly ν-continuous if for each point $x \in X$ and each sequence $< x_n > \rightarrow^{\nu} x \in X$, there exists a subsequence $< x_{nk} >$ of $< x_n >$ such that $< f(x_{nk}) > \rightarrow^{\nu} f(x)$.

(iv) sequentially sub-ν-continuous if for each $x \in X$ and each sequence $< x_n > \rightarrow^{\nu} x$ in X, there exists a subsequence $< x_{nk} >$ of $< x_n >$ and a point $y \in Y$ such that $< f(x_{nk}) > \rightarrow^{\nu} y$;

(v) sequentially ν-compact preserving if the image $f(K)$ of every sequentially ν-compact set K of X is sequentially ν-compact in Y.

§3. νg-T_0 limit point

Definition 3.1. In X, a point x is said to be a νg-T_0-limit point of A if each νg-open set containing x contains some point y of A such that $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, or equivalently, such that they are topologically distinct with respect to νg-open sets.

Example 3.1. Since regular open $\Rightarrow$ ν-open $\Rightarrow$ νg-open, then r-T_0-limit point $\Rightarrow$ ν-T_0-limit point $\Rightarrow$ νg-T_0-limit point.

Definition 3.2. A set A together with all its νg-T_0-limit points is denoted by T_0-$\nu g\overline{(A)}$.

Lemma 3.1. If x is a νg-T_0-limit point of a set A then x is νg-limit point of A.

Lemma 3.2 If X is νg-T_0-space then every νg-T_0-limit point and every νg-limit point are equivalent.

Corollary 3.1.

(i) If X is r-T_0-space then every νg-T_0-limit point and every νg-limit point are equivalent;

(ii)If X is ν-T_0-space then every νg-T_0-limit point and every νg-limit point are equivalent.

Theorem 3.1. For $x \neq y \in X$,

(i) x is a νg-T_0-limit point of $\{y\}$ iff $x \notin \nu g\overline{\{y\}}$ and $y \in \nu g\overline{\{x\}}$;

(ii) x is not a νg-T_0-limit point of $\{y\}$ iff either $x \in \nu g\overline{\{y\}}$ or $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$;

(iii) x is not a νg-T_0-limit point of $\{y\}$ iff either $x \in \nu g\overline{\{y\}}$ or $y \in \nu g\overline{\{x\}}$.

Corollary 3.2.

(i) If x is a νg-T_0-limit point of $\{y\}$, then y cannot be a νg-limit point of $\{x\}$;

(ii) If $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$, then neither x is a νg-T_0-limit point of $\{y\}$ nor y is a νg-T_0-limit point of $\{x\}$;

(iii) If a singleton set A has no νg-T_0-limit point in X, then $\nu g\overline{(A)} = \nu g\overline{\{x\}}$ for all $x \in \nu g\overline{(A)}$.

Lemma 3.3. In X, if x is a νg-limit point of a set A, then in each of the following cases, X becomes νg-T_0-limit point of $A(\{x\} \neq A)$.

(i) $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$ for $y \in A, x \neq y$;

(ii) $\nu g\overline{\{x\}} = \{x\}$;

(iii) X is a νg-T_0-space;

(iv) A-$\{x\}$ is νg-open.

Corollary 3.3. In X, if x is a limit point [resp: r-limit point; ν-limit point] of a set A, then in each of the following cases x becomes νg-T_0-limit point of A ($\{x\} \neq A$).

(i) $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$ for $y \in A, x \neq y$;

(ii) $\nu g\overline{\{x\}} = \{x\}$;

(iii) X is a νg-T_0-space;

(iv) A-$\{x\}$ is νg-open.

§4. νg-T_0 and νg-R_i axioms, $i = 0, 1$

In view of lemma 3.3(iii), νg-T_0-axiom implies the equivalence of the concept of limit point of a set with that of νg-T_0-limit point of the set. But for the converse, if $x \in \nu g\overline{\{y\}}$ then $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$ in general, but if x is a νg-T_0-limit point of $\{y\}$, then $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$.

Lemma 4.1. In a space X, a limit point x of {y} is a νg-T_0-limit point of $\{y\}$ iff $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$.

This lemma leads to characterize the equivalence of νg-T_0-limit point and νg-limit point of a set as the νg-T_0-axiom.

Theorem 4.1. The following conditions are equivalent:

(i) X is a νg-T_0 space;

(ii) Every νg-limit point of a set A is a νg-T_0-limit point of A;

(iii)Every r-limit point of a singleton set $\{x\}$ is a νg-T_0-limit point of $\{x\}$;

(iv) For any $x, y \in X$, $x \neq y$ if $x \in \nu g\overline{\{y\}}$, then x is a νg-T_0-limit point of $\{y\}$.

Note 4.1. In a νg-T_0-space X if every point of X is a r-limit point of X, then every point of X is νg-T_0-limit point of X. But a space X in which each point is a νg-T_0-limit point of X is not necessarily a νg-T_0-space.

Theorem 4.2. The following conditions are equivalent:

(i) X is a $\nu g - R_0$ space;

(ii) For any $x, y \in$X, if $x \in \nu g\overline{\{y\}}$, then x is not a νg-T_0-limit point of $\{y\}$;

(iii) A point νg-closure set has no νg-T_0-limit point in X;

(iv) A singleton set has no νg-T_0-limit point in X.

Since every r-R_0-space is νg-R_0-space, we have the following Corollary.

Corollary 4.1. The following conditions are equivalent:

(i) X is a $r - R_0$ space;

(ii) For any $x, y \in X$, if $x \in \nu g\overline{\{y\}}$, then X is not a νg-T_0-limit point of $\{y\}$;

(iii) A point νg-closure set has no νg-T_0-limit point in X;

(iv) A singleton set has no νg-T_0-limit point in X.

Since every ν-R_0-space is νg-R_0-space, we have the following Corollary.

Corollary 4.2. The following conditions are equivalent:

(i) X is a ν-R_0 space;

(ii) For any $x, y \in X$, if $x \in \nu g\overline{\{y\}}$, then X is not a νg-T_0-limit point of $\{y\}$;

(iii) A point νg-closure set has no νg-T_0-limit point in X;

(iv) A singleton set has no νg-T_0-limit point in X.

Theorem 4.3. In a νg-R_0 space X, a point x is νg-T_0-limit point of A iff every νg-open set containing x contains in finitely many points of A with each of which x is topologically distinct.

If νg-R_0 space is replaced by rR_0 space in the above theorem, we have the following corollaries.

Corollary 4.3. (a) In an rR_0-space X,

(i) If a point x is rT_0-limit point of a set then every νg-open set containing X contains infinitely many points of A with each of which X is topologically distinct;

(ii) If a point x is νg-T_0-limit point of a set then every νg-open set containing X contains infinitely many points of A with each of which X is topologically distinct.

(b) In an ν-R_0-space X,

(i) If a point x is rT_0-limit point of a set then every νg-open set containing x contains infinitely many points of A with each of which x is topologically distinct;

(ii) If a point x is νg-T_0-limit point of a set then every νg-open set containing x contains infinitely many points of A with each of which x is topologically distinct.

Theorem 4.4. X is νg-R_0 space iff a set A of the form $A = \cup \nu g\overline{\{x_i :_{i=1ton}\}}$, a finite union of point closure sets has no νg-T_0-limit point.

Corollary 4.4. (a) If X is rR_0 space and

(i) If $A = \cup \nu g\overline{\{x_i :_{i=1ton}\}}$, a finite union of point closure sets has no νg-T_0-limit point;

(ii) If $X = \cup \nu g\overline{\{x_i :_{i=1ton}\}}$, then X has no νg-T_0-limit point.

(b) If X is ν-R_0 space and

(i) If $A = \cup \nu g\overline{\{x_i :_{i=1ton}\}}$, a finite union of point closure sets has no νg-T_0-limit point;

(ii) If $X = \cup \nu g\overline{\{x_i :_{i=1ton}\}}$, then X has no νg-T_0-limit point.

Theorem 4.5. The following conditions are equivalent:

(i) X is νg-R_0-space;

(ii) For any X and a set A in X, x is a νg-T_0-limit point of A iff every νg-open set containing x contains infinitely many points of A with each of which X is topologically distinct.

Various characteristic properties of νg-T_0-limit points studied so far is enlisted in the following Theorem for a ready reference.

Theorem 4.6. In a νg-R_0-space, we have the following:

(i) A singleton set has no νg-T_0-limit point in X;

(ii) a finite set has no νg-T_0-limit point in X;

(iii) a point νg-closure has no set νg-T_0-limit point in X;

(iv) a finite union point νg-closure sets have no set νg-T_0-limit point in X;

(v) for $x, y \in X,\ x \in T_0 - \nu g\overline{\{y\}}$ iff $x = y$;

(vi) for any $x, y \in X,\ x \neq y$ iff neither X is νg-T_0-limit point of $\{y\}$ nor y is νg-T_0-limit point of $\{x\}$;

(vii) for any $x, y \in X,\ x \neq y$ iff $T_0 - \nu g\overline{\{x\}} \cap T_0 - \nu g\overline{\{y\}} = \phi$;

(viii) any point $x \in X$ is a νg-T_0-limit point of a set A in X iff every νg-open set containing X contains infinitely many points of A with each which X is topologically distinct.

Theorem 4.7. X is νg-R_1 iff for any νg-open set U in X and points x, y such that $x \in X - U, y \in U$, there exists a νg-open set V in X such that $y \in V \subset U, x \notin V$.

Lemma 4.2. In νg-R_1 space X, if x is a νg-T_0-limit point of X, then for any non empty νg-open set U, there exists a non empty νg-open set V such that $V \subset U, x \notin \nu g\overline{(V)}$.

Lemma 4.3. In a νg-regular space X, if X is a νg-T_0-limit point of X, then for any non empty νg-open set U, there exists a non empty νg-open set V such that $\nu g\overline{(V)} \subset U, x \notin \nu g\overline{(V)}$.

Corollary 4.5. In a regular space X,

(i) If X is a νg-T_0-limit point of X, then for any non empty νg-open set U, there exists a non empty νg-open set V such that $\nu g\overline{(V)} \subset U, x \notin \nu g\overline{(V)}$;

(ii) If x is a T_0-limit point of X, then for any non empty νg-open set U, there exists a non empty νg-open set V such that $\nu g\overline{(V)} \subset U, x \notin \nu g\overline{(V)}$.

Theorem 4.8. If X is a νg-compact νg-R_1-space, then X is a *Baire* space.

Proof. Let $\{A_n\}$ be a countable collection of νg-closed sets of X, each A_n having empty interior in X. Take A_1, since A_1 has empty interior, A_1 does not contain any νg-open set say U_0. Therefore we can choose a point $y \in U_0$ such that $y \notin A_1$. For X is νg-regular, and $y \in (X - A_1) \cap U_0$, a νg-open set, we can find a νg-open set U_1 in X such that $y \in U_1, \nu g\overline{(U_1)} \subset (X - A_1) \cap U_0$. Hence U_1 is a non empty νg-open set in X such that $\nu g\overline{(U_1)} \subset U_0$ and $\nu g\overline{(U_1)} \cap A_1 = \phi$. Continuing this process, in general, for given non empty νg-open set U_{n-1}, we can choose a point of U_{n-1} which is not in the νg-closed set A_n and a νg-open set U_n containing this point such that $\nu g\overline{(U_n)} \subset U_{n-1}$ and $\nu g\overline{(U_n)} \cap A_n = \phi$. Thus we get a sequence of nested non empty νg-closed sets which satisfies the finite intersection property. Therefore $\cap \nu g\overline{(U_n)} \neq \phi$. Then some $x \in \cap \nu g\overline{(U_n)}$ which in turn implies that $x \in U_{n-1}$ as $\nu g\overline{(U_n)} \subset U_{n-1}$ and $x \notin A_n$ for each n.

Corollary 4.6. If X is a compact νg-R_1-space, then X is a *Baire* space.

Corollary 4.7. Let X be a νg-compact νg-R_1-space. If $\{A_n\}$ is a countable collection of νg-closed sets in X, each A_n having non-empty νg-interior in X, then there is a point of X which is not in any of the A_n.

Corollary 4.8. Let X be a νg-compact R_1-space. If $\{A_n\}$ is a countable collection of νg-closed sets in X, each A_n having non-empty νg-interior in X, then there is a point of X which is not in any of the A_n.

Theorem 4.9. Let X be a non empty compact νg-R_1-space. If every point of X is a νg-T_0-limit point of X then X is uncountable.

Proof. Since X is non empty and every point is a νg-T_0-limit point of X, X must be infinite. If X is countable, we construct a sequence of νg-open sets $\{V_n\}$ in X as follows:

Let $X = V_1$, then for set x_1 is a νg-T_0-limit point of X, we can choose a non empty νg-open V_2 in X such that $V_2 \subset V_1$ and $x_1 \notin \nu g\overline{V_2}$. Next for x_2 and non empty νg-open set V_2, we can choose a non empty νg-open set V_3 in X such that $V_3 \subset V_2$ and $x_2 \notin \nu g\overline{V_3}$. Continuing this process for each x_n and a non empty νg-open set V_n, we can choose a non empty νg-open set V_{n+1} in X such that $V_n{+}_1 \subset V_n$ and $x_n \notin \nu g\overline{V_n{+}_1}$.

Now consider the nested sequence of νg-closed sets $\nu g\overline{V_1} \supset \nu g\overline{V_2} \supset \nu g\overline{V_3} \supset \cdots \supset \nu g\overline{V_n} \supset \cdots$, since X is νg-compact and $\{\nu g\overline{V_n}\}$ the sequence of νg-closed sets satisfy finite intersection property. By Cantors intersection theorem, there exists an x in X such that $x \in \nu g\overline{V_n}$. Further $x \in X$ and $x \in V_1$, which is not equal to any of the points of x. Hence X is uncountable.

Corollary 4.9. Let X be a non empty νg-compact νg-R_1-space. If every point of X is a νg-T_0-limit point of x, then X is uncountable.

§5. νg-T_0-identification spaces and νg-separation axioms

Definition 5.1. Let X be a topological space and let $\Re$ be the equivalence relation on X defined by $x\Re y$ iff $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$.

Problem 5.1. Show that $x\Re y$ iff $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$ is an equivalence relation.

Definition 5.1. The space $(X_0, Q(X_0))$ is called the νg-T_0-identification space of (X, τ), where X_0 is the set of equivalence classes of τ and $Q(X_0)$ is the decomposition topology on X_0.

Let $P_X : (X, \tau) \to (X_0, Q(X_0))$ denote the natural map.

Lemma 5.1. If $x \in X$ and $A \subset X$, then $x \in \nu g\overline{A}$ iff every νg-open set containing X intersects A.

Theorem 5.1. The natural map $P_X : (X, \tau) \to (X_0, Q(X_0))$ is closed, open and $P_X^{-1}(P_{X_{(O)}}) = O$ for all $O \in PO(X, \tau)$ and $(X_0, Q(X_0))$ is νg-T_0.

Proof. Let $O \in PO(X, \tau)$ and let $C \in P_X(O)$. Then there exists $x \in O$ such that $P_X(x) = C$. If $y \in C$, then $\nu g\overline{\{y\}} = \nu g\overline{\{x\}}$, which, implies $y \in O$. Since $\tau \subset PO(X, \tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U \in \tau$, which implies P_X is closed and open.

Let $G, H \in X_0$ such that $G \neq H$ and let $x \in G$ and $y \in H$. Then $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, which implies that $x \notin \nu g\overline{\{y\}}$ or $y \notin \nu g\overline{\{x\}}$, say $x \notin \nu g\overline{\{y\}}$. Since P_X is continuous and open, then $G \in A = P_X\{X - \nu g\overline{\{y\}}\} \notin PO(X_0, Q(X_0))$ and $H \notin A$.

Theorem 5.2. The followings are equivalent:

(i) X is $\nu g R_0$;

(ii) $X_0 = \{\nu g\overline{\{x\}} : x \in X\}$;

(iii) $(X_0, Q(X_0))$ is $\nu g T_1$.

Proof. (i) $\Rightarrow$ (ii). Let $C \in X_0$, and let $x \in C$. If $y \in C$, then $y \in \nu g\overline{\{y\}} = \nu g\overline{\{x\}}$, which implies $C \in \nu g\overline{\{x\}}$. If $y \in \nu g\overline{\{x\}}$, then $x \in \nu g\overline{\{y\}}$, since, otherwise, $x \in X - \nu g\overline{\{y\}} \in PO(X, \tau)$ which implies $\nu g\overline{\{x\}} \subset X - \nu g\overline{\{y\}}$, which is a contradiction. Thus, if $y \in \nu g\overline{\{x\}}$, then $x \in \nu g\overline{\{y\}}$, which implies $\nu g\overline{\{y\}} = \nu g\overline{\{x\}}$ and $y \in C$. Hence $X_0 = \{\nu g\overline{\{x\}} : x \in X\}$.

(ii) $\Rightarrow$ (iii). Let $A \neq B \in X_0$. Then there exists $x, y \in X$ such that $A = \nu g\overline{\{x\}}; B = \nu g\overline{\{y\}}$, and $\nu g\overline{\{x\}} \cap \nu g\overline{\{y\}} = \phi$. Then $A \in C = P_X(X - \nu g\overline{\{y\}}) \in PO(X_0, Q(X_0))$ and $B \notin C$. Thus $(X_0, Q(X_0))$ is νg-T_1.

(iii) $\Rightarrow$ (i). Let $x \in U \in \nu GO(X)$. Let $y \notin U$ and $C_x, C_y \in X_0$ containing x and y respectively. Then $x \notin \nu g\overline{\{y\}}$, which implies $C_x \neq C_y$ and there exists νg-open set A such that $C_x \in A$ and $C_y \notin A$. Since P_X is continuous and open, then $y \in B = P_X^{-1}(A) \in x \in \nu GO(X)$ and $x \notin B$, which implies $y \notin \nu g\overline{\{x\}}$. Thus $\nu g\overline{\{x\}} \subset U$. This is true for all $\nu g\overline{\{x\}}$ implies $\cap \nu g\overline{\{x\}} \subset U$. Hence X is νg-R_0

Theorem 5.3. X is $\nu g - R_1$ iff $(X_0, Q(X_0))$ is νg-T_2.

The proof is straight forward from using theorems 5.1 and 5.2 and is omitted.

§6. νg-open functions and νg-T_i spaces, $i = 0, 1, 2$

Theorem 6.1. X is νg-T_i, $i = 0, 1, 2$ iff there exists a νg-continuous, almost-open, 1-1 function $f : X$ into a νg-T_i space, $i = 0, 1, 2$, respectively.

Theorem 6.2. If $f : X \to Y$ is νg-continuous, νg-open, and $x, y \in X$ such that $\nu g\overline{\{x\}} = \nu g\overline{\{y\}}$, then $\nu g\overline{\{f(x)\}} = \nu g\overline{\{f(y)\}}$.

Proof. Suppose $\nu g\overline{\{f(x)\}} \neq \nu g\overline{\{f(y)\}}$. Then $f(x) \notin \nu g\overline{\{f(y)\}}$ or $f(y) \notin \nu g\overline{\{f(x)\}}$, say $f(x) \notin \nu g\overline{\{f(y)\}}$. Then $f(x) \in A = Y - \nu g\overline{\{f(y)\}} \in \nu GO(Y)$. If $B = Y - \overline{A}^0$, then $f(x) \notin B$, and $B \cap \nu g\overline{\{f(y)\}} \neq \phi$, which implies $f(y) \in B, y \in f^{-1}(B) \in \nu GO(X)$, and $x \notin f^{-1}(B)$ which is a contradiction. Thus $\overline{A^o} = Y$. Since $f(y) \notin A$, then $y \notin (f^{-1}(A))^o$. If $x \in \overline{(f^{-1}(A))^o}$, then $\{x\} \cup (f^{-1}(A))^o$ is νg-open containing x and not y, which is a contradiction. Hence $x \in U = X - \overline{(f^{-1}(A))^o}$ and $\phi \neq f(U) \in \nu GO(Y)$. Then $C = (f(U))^o \cap A^o = \phi$, for suppose not. Then $f^{-1}(C) \in \nu GO(X)$, which implies $f^{-1}(C) \subset \overline{(f^{-1}(C))^o} \subset \overline{(f^{-1}(A))^o}$, which is a contradiction. Hence $C = \phi$, which contradicts $\overline{A^o} = Y$.

Theorem 6.3. The followings are equivalent:

(i) X is νg-T_0;

(ii) Elements of X_0 are singleton sets;

(iii) There exists a νg-continuous, νg-open, 1-1 function $f : X \to Y$, where Y is νg-T_0.

Proof.(i) is equivalent to (ii) and (i) $\Rightarrow$ (iii) are straight forward and is omitted.

(iii) $\Rightarrow$ (i). Let $x, y \in X$ such that $f(x) \neq f(y)$, which implies $\nu g\overline{\{f(x)\}} \neq \nu g\overline{\{f(y)\}}$. Then by theorem 6.2, $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$. Hence X is νg-T_0.

Corollary 6.1. A space X is νg-T_i, $i = 1, 2$ iff X is νg-T_{i-1}, $i = 1, 2$, respectively, and there exists a νg-continuous, νg-open, 1-1 function $f : X$ into a νg-T_0 space.

Definition 6.1. $f : X \to Y$ is point-νg-closure 1-1 iff for $x, y \in X$ such that $\nu g\overline{\{x\}} \neq \nu g\overline{\{y\}}$, $\nu g\overline{\{f(x)\}} \neq \nu g\overline{\{f(y)\}}$.

Theorem 6.4. (i) If $f : X \to Y$ is point νg-closure 1-1 and X is νg-T_0, then f is 1-1;

(ii) If $f : X \to Y$, where X and Y are νg-T_0, then f is point νg-closure 1-1 iff f is 1-1.

Proof. Omitted.

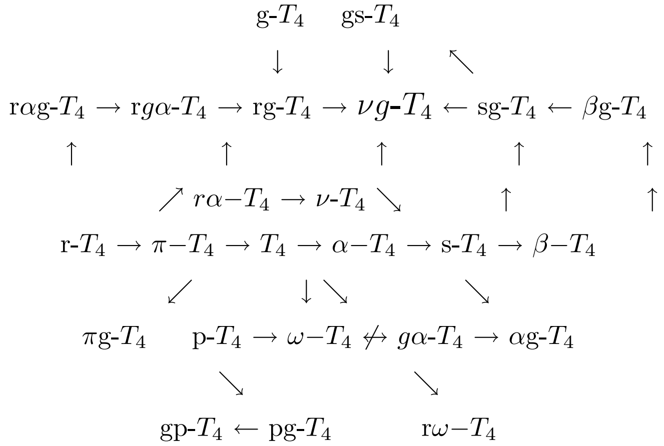
The following result can be obtained by combining results for νg-T_0-identification spaces, νg-induced functions and νg-T_i spaces, $i = 1, 2$.

Theorem 6.5. X is $\nu g - R_i$, $i = 0, 1$ iff there exists a νg-continuous, almost-open point νg-closure 1-1 function $f : X$ into a νg-R_i space, $i = 0, 1$, respectively.

§7. νg-normal, almost νg-normal and mildly νg-normal spaces

Definition 7.1. A space X is said to be νg-normal if for any pair of disjoint closed sets F_1 and F_2, there exist disjoint νg-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Note 7.1. From the above Definition we have the following implication diagram.



Example 7.1 Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then X is νg-normal.

Example 7.2 Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$. Then X is νg-normal and is not normal.

We have the following characterization of νg-normality.

Theorem 7.1. For a space X the followings are equivalent:

(a) X is νg-normal;

(b) For every pair of open sets U and V whose union is X, there exist νg-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$;

(c) For every closed set F and every open set G containing F, there exists a νg-open set U such that $F \subset U \subset \nu g\overline{(U)} \subset G$.

Proof. (a) $\Rightarrow$ (b). Let U and V be a pair of open sets in a νg-normal space X such that $X = U \cup V$. Then $X - U$, $X - V$ are disjoint closed sets. Since X is νg-normal there exist disjoint νg-open sets U_1 and V_1 such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $A = X - U_1, B = X - V_1$. Then A and B are νg-closed sets such that $A \subset U, B \subset V$ and $A \cup B = X$.

(b) $\Rightarrow$ (c). Let F be a closed set and G be an open set containing F. Then $X - F$ and G are open sets whose union is X. Then by (b), there exist νg-closed sets W_1 and W_2 such that $W_1 \subset X - F$ and $W_2 \subset G$ and $W_1 \cup W_2 = X$. Then $F \subset X - W_1, X - G \subset X - W_2$ and $(X - W_1) \cap (X - W_2) = \phi$. Let $U = X - W_1$ and $V = X - W_2$. Then U and V are disjoint

νg-open sets such that $F \subset U \subset X - V \subset G$. As $X - V$ is νg-closed set, we have $\nu g(\overline{U}) \subset X - V$ and $F \subset U \subset \nu g(\overline{U}) \subset G$.

(c) $\Rightarrow$ (a). Let F_1 and F_2 be any two disjoint closed sets of X. Put $G = X - F_2$, then $F_1 \cap G = \phi . F_1 \subset G$ where G is an open set. Then by (c), there exists a νg-open set U of X such that $F_1 \subset U \subset \nu g(\overline{U}) \subset G$. It follows that $F_2 \subset X - \nu g(\overline{U}) = V$, say, then V is νg-open and $U \cap V = \phi$. Hence F_1 and F_2 are separated by νg-open sets U and V. Therefore X is νg-normal.

Theorem 7.2. A regular open subspace of a νg-normal space is νg-normal.

Proof. Let Y be a regular open subspace of a νg-normal space X. Let A and B be disjoint closed subsets of Y. By νg-normality of X, there exist disjoint νg-open sets U and V in X such that $A \subset U$ and $B \subset V$, $U \cap Y$ and $V \cap Y$ are νg-open in Y such that $A \subset U \cap Y$ and $B \subset V \cap Y$. Hence Y is νg-normal.

Example 7.3. Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ is νg-normal and νg-regular.

Now, we define the following.

Definition 7.2. A function $f : X \to Y$ is said to be almost-νg-irresolute if for each $x \in X$ and each νg-neighborhood V of $f(x)$, $\nu g(\overline{f^{-1}(V)})$ is a νg-neighborhood of X.

Clearly every νg-irresolute map is almost νg-irresolute.

The Proof of the following lemma is straightforward and hence omitted.

Lemma 7.1. f is almost νg-irresolute iff $f^{-1}(V) \subset \nu g - int(vgcl(f^{-1}(V))))$ for every $V \in \nu GO(Y)$.

Now we prove the following.

Lemma 7.2. f is almost νg-irresolute iff $f(\nu g(\overline{U})) \subset \nu g(\overline{f(U)})$ for every $U \in \nu GO(X)$.

Proof. Let $U \in \nu GO(X)$. Suppose $y \notin \nu g(\overline{f(U)})$. Then there exists $V \in \nu GO(y)$ such that $V \cap f(U) = \phi$. Hence $f^{-1}(V) \cap U = \phi$. Since $U \in \nu GO(X)$, we have $\nu g(\nu g(\overline{f^{-1}(V)}))^o \cap \nu g(\overline{U}) = \phi$. Then by lemma 7.1, $f^{-1}(V) \cap \nu g(\overline{U}) = \phi$ and hence $V \cap f(\nu g(\overline{U})) = \phi$. This implies that $y \notin f(\nu g(\overline{U}))$.

Conversely, if $V \in \nu GO(Y)$, then $W = X - \nu g(\overline{f^{-1}(V)}) \in \nu GO(X)$. By hypothesis, $f(\nu g(\overline{W})) \subset \nu g(\overline{f(W)}))$ and hence $X - \nu g(\nu g(\overline{f^{-1}(V)}))^o = \nu g(\overline{W}) \subset f^{-1}(\nu g(\overline{f(W)})) \subset f^{-1}(\nu g[\overline{f(X - f^{-1}(V))}]) \subset f^{-1}[\nu g(\overline{Y - V})] = f^{-1}(Y - V) = X - f^{-1}(V)$. Therefore, $f^{-1}(V) \subset \nu g(\nu g(\overline{f^{-1}(V)}))^o$. By lemma 7.1, f is almost νg-irresolute.

Now we prove the following result on the invariance of νg-normality.

Theorem 7.3. If f is an M-νg-open continuous almost νg-irresolute function from a νg-normal space X onto a space Y, then Y is νg-normal.

Proof. Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is νg-normal, there exists a νg-open set U in X such that $f^{-1}(A) \subset U \subset \nu g(\overline{U}) \subset f^{-1}(B)$. Then $f(f^{-1}(A)) \subset f(U) \subset f(\nu g(\overline{U})) \subset f(f^{-1}(B))$. Since f is M-νg-open almost νg-irresolute surjection, we obtain $A \subset f(U) \subset \nu g(\overline{f(U)}) \subset B$. Then by theorem 7.1 Y is νg-normal.

Lemma 7.3. A mapping f is M-νg-closed if and only if for each subset B in Y and for each νg-open set U in X containing $f^{-1}(B)$, there exists a νg-open set V containing B such that $f^{-1}(V) \subset U$.

Now we prove the following.

Theorem 7.4. If f is an M-νg-closed continuous function from a νg-normal space onto a space Y, then Y is νg-normal.

Proof of the theorem is routine and hence omitted.

Now in view of lemma 2.2 [19] and lemma 7.3, we prove the following result.

Theorem 7.5. If f is an M-νg-closed map from a weakly Hausdorff νg-normal space X onto a space Y such that $f^{-1}(y)$ is S-closed relative to X for each $y \in Y$, then Y is νg-T_2.

Proof. Let y_1 and y_2 be any two distinct points of Y. Since X is weakly $Hausdorff$, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X by lemma 2.2 [19]. As X is νg-normal, there exist disjoint νg-open sets V_1 and V_2 such that $f^{-1}(y_i) \subset V_i$, for $i = 1, 2$. Since f is M-νg-closed, there exist νg-open sets U_i containing y_i such that $f^{-1}(U_i) \subset V_i$ for $i = 1, 2$. Then it follows that $U_1 \cap U_2 = \phi$. Hence Y is νg-T_2.

Theorem 7.6. For a space X we have the following:

(a) If X is normal then for any disjoint closed sets A and B, there exist disjoint νg-open sets U, V such that $A \subset U$ and $B \subset V$;

(b) if X is normal then for any closed set A and any open set V containing A, there exists an νg-open set U of X such that $A \subset U \subset \nu g\overline{(U)} \subset V$.

Definition 7.2. X is said to be almost νg-normal if for each closed set A and each regular closed set B such that $A \cap B = \phi$, there exist disjoint νg-open sets U and V such that $A \subset U$ and $B \subset V$.

Note 7.2. From the above definition we have the following implication diagram.

Al-g-T_4 Al-gs-T_4

$\downarrow$ $\downarrow$ $\nwarrow$

Al-$r\alpha g$-T_4 $\rightarrow$ Al-$rg\alpha$-T_4 $\rightarrow$ Al-rg-T_4 $\rightarrow$ **Al-νg-T_4** $\leftarrow$ Al-sg-T_4 $\leftarrow$ Al-βg-T_4

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$

$\nearrow$ Al-$r\alpha$-T_4 $\rightarrow$ Al-ν-T_4 $\searrow$ $\uparrow$ $\uparrow$

Al-r-T_4 $\rightarrow$ Al-π-T_4 $\rightarrow$ Al-T_4 $\rightarrow$ Al-α-T_4 $\rightarrow$ Al-s-T_4 $\rightarrow$ Al-β-T_4

$\swarrow$ $\downarrow$ $\searrow$ $\searrow$

Al-πg-T_4 Al-p-T_4 $\rightarrow$ Al-ω-T_4 $\not\leftrightarrow$ Al-$g\alpha$-T_4 $\rightarrow$ Al-αg-T_4

$\searrow$ $\searrow$

Al-gp-T_4 $\leftarrow$ Al-pg-T_4 Al-$r\omega-T_4$

Example 7.4. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then X is almost νg-normal and νg-normal.

Now, we have characterization of almost νg-normality in the following.

Theorem 7.7. For a space X the following statements are equivalent:

(a) X is almost νg-normal;

(b) for every pair of sets U and V, one of which is open and the other is regular open whose union is X, there exist νg-closed sets G and H such that $G \subset U, H \subset V$ and $G \cup H = X$;

(c) for every closed set A and every regular open set B containing A, there is a νg-open set V such that $A \subset V \subset \nu g\overline{(V)} \subset B$.

Proof. (a) $\Rightarrow$ (b). Let U be an open set and V be a regular open set in an almost νg-normal space X such that $U \cup V = X$. Then $(X - U)$ is closed set and $(X - V)$ is regular closed set with $(X - U) \cap (X - V) = \phi$. By almost νg-normality of X, there exist disjoint νg-open sets U_1 and V_1 such that $X - U \subset U_1$ and $X - V \subset V_1$. Let $G = X - U_1$ and $H = X - V_1$. Then G and H are νg-closed sets such that $G \subset U, H \subset V$ and $G \cup H = X$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious.

One can prove that almost νg-normality is also regular open hereditary.

Almost νg-normality does not imply almost νg-regularity in general. However, we observe that every almost νg-normal νg-R_0 space is almost νg-regular.

Next, we prove the following.

Theorem 7.8. Every almost regular, ν-compact space X is almost νg-normal.

Recall that $f : X \rightarrow Y$ is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost νg-normality in the following.

Theorem 7.9. If f is continuous M-νg-open rc-continuous and almost νg-irresolute surjection from an almost νg-normal space X onto a space Y, then Y is almost νg-normal.

Definition 7.3. A space X is said to be mildly νg-normal if for every pair of disjoint regular closed sets F_1 and F_2 of X, there exist disjoint νg-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Note 7.3. From the above Definition we have the following implication diagram.

Mild-g-T_4 Mild-gs-T_4

$\downarrow$ $\downarrow$ $\nwarrow$

Mild-rαg-T_4 $\rightarrow$ Mild-r$g\alpha$-T_4 $\rightarrow$ Mild-rg-T_4 $\rightarrow$ **Mild-νg-T_4** $\leftarrow$ Mild-sg-T_4 $\leftarrow$ Mild-βg-T_4

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$

$\nearrow$ Mild-$r\alpha-T_4$ $\rightarrow$ Mild-ν-T_4 $\searrow$ $\uparrow$ $\uparrow$

Mild-r-T_4 $\rightarrow$ Mild-$\pi-T_4$ $\rightarrow$ Mild-T_4 $\rightarrow$ Mild-$\alpha-T_4$ $\rightarrow$ Mild-s-T_4 $\rightarrow$ Mild-$\beta-T_4$

$\swarrow$ $\downarrow \searrow$ $\searrow$

Mild-πg-T_4 Mild-p-T_4 $\rightarrow$ Mild-$\omega-T_4$ $\not\leftrightarrow$ Mild-$g\alpha$-T_4 $\rightarrow$ Mild-αg-T_4

$\searrow$ $\searrow$

Mild-gp-T_4 $\leftarrow$ Mild-pg-T_4 Mild-r$\omega-T_4$

We have the following characterization of mild νg-normality.

Theorem 7.10. For a space X the following are equivalent.

(a) X is mildly νg-normal;

(b) for every pair of regular open sets U and V whose union is X, there exist νg-closed sets G and H such that $G \subset U, H \subset V$ and $G \cup H = X$;

(c) for any regular closed set A and every regular open set B containing A, there exists a νg-open set U such that $A \subset U \subset \nu g\overline{(U)} \subset B$;

(d) for every pair of disjoint regular closed sets, there exist νg-open sets U and V such that $A \subset U, B \subset V$ and $\nu g\overline{(U)} \cap \nu g\overline{(V)} = \phi$.

This theorem may be proved by using the arguments similar to those of theorem 7.7.

Also, we observe that mild νg-normality is regular open hereditary.

We define the following:

Definition 7.4. A space X is weakly νg-regular if for each point X and a regular open set U containing $\{x\}$, there is a νg-open set V such that $x \in V \subset \overline{V} \subset U$.

Theorem 7.11. If $f : X \to Y$ is an M-νg-open rc-continuous and almost νg-irresolute function from a mildly νg-normal space X onto a space Y, then Y is mildly νg-normal.

Proof. Let A be a regular closed set and B be a regular open set containing A. Then by rc-continuity of f, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since X is mildly νg-normal, there exists a νg-open set V such that $f^{-1}(A) \subset V \subset \nu g\overline{(V)} \subset f^{-1}(B)$ by theorem 7.10. As f is M-νg-open and almost νg-irresolute surjection, it follows that $f(V) \in \nu GO(Y)$ and $A \subset f(V) \subset \nu g\overline{(f(V))} \subset B$. Hence Y is mildly νg-normal.

Theorem 7.12. If $f : X \to Y$ is rc-continuous, M-νg-closed map from a mildly νg-normal space X onto a space Y, then Y is mildly νg-normal.

§8. νg-US spaces

Definition 8.1. A sequence $< x_n >$ is said to νg-converges to $x \in X$, written as $< x_n > \to^{\nu g} x$ if $< x_n >$ is eventually in every νg-open set containing x.

Clearly, if a sequence $< x_n > \to^{\nu} X$ of X, then $< x_n > \to^{\nu g}$ to X.

Definition 8.2. A space X is said to be νg-US if every sequence $< x_n >$ in X νg-converges to a unique point.

Theorem 8.1. Every νg-US space is νg-T_1.

Proof. Let X be νg-US space. Let $x \neq y \in$X. Consider the sequence $< x_n >$ where $x_n = x$ for every n. Cleary, $< x_n > \to^{\nu g} X$. Also, since $x \neq y$ and X is νg-US, $< x_n > \not\to^{\nu g} y$, i.e. there exists a νg-open set V containing y but not X. Similarly, if we consider the sequence $< y_n >$ where $y_n = y$ for all n, and proceeding as above we get a νg-open set U containing X but not y. Thus, the space X is νg-T_1.

Theorem 8.2. Every νg-T_2 space is νg-US.

Proof. Let X be νg-T_2 space and $< x_n >$ be a sequence in X. If $< x_n >$ νg-converge to two distinct points X and y. That is, $< x_n >$ is eventually in every νg-open set containing X and also in every νg-open set containing y. This is contradiction since X is νg-T_2 space. Hence the space X is νg-US.

Definition 8.3. A set F is sequentially νg-closed if every sequence in F νg-converges to a point in F.

Theorem 8.3. X is νg-US iff the diagonal set is a sequentially νg-closed subset of $X \times X$.

Proof. Let X be νg-US. Let $< x_n, x_n >$ be a sequence in Δ. Then $< x_n >$ is a sequence in X. As X is νg-US, $< x_n > \to^{\nu g} x$ for a unique $x \in X$, i.e. if $< x_n > \to^{\nu g} x$ and y. Thus, $x = y$. Hence Δ is sequentially νg-closed set.

Conversely, let Δ be sequentially νg-closed. Let a sequence $< x_n > \to^{\nu g} x$ and y. Hence $< x_n, x_n > \to^{\nu g} (x, y)$. Since Δ is sequentially νg-closed, $(x, y) \in \Delta$ which means that $x = y$ implies space X is νg-US.

Definition 8.4. A subset G of a space X is said to be sequentially νg-compact if every sequence in G has a subsequence which νg-converges to a point in G.

Theorem 8.4. In a νg-US space every sequentially νg-compact set is sequentially νg-closed.

Proof. Let Y be a sequentially νg-compact subset of νg-US space X. Let $< x_n >$ be a sequence in Y. Suppose that $< x_n >$ νg-converges to a point in $X - Y$. Let $< x_{np} >$ be subsequence of $< x_n >$ that νg-converges to a point $y \in Y$, since Y is sequentially νg-compact. Also, let a subsequence $< x_{np} >$ of $< x_n > \rightarrow^{\nu g} x \in X - Y$. Since $< x_{np} >$ is a sequence in the νg-US space X, $x = y$. Thus, Y is sequentially νg-closed set.

Next, we give a hereditary property of νg-US spaces.

Theorem 8.5. Every regular open subset of a νg-US space is νg-US.

Proof. Let X be a νg-US space and $Y \subset X$ be an regular open set. Let $< x_n >$ be a sequence in Y. Suppose that $< x_n > \rightarrow^{\nu g} x$ and y in Y. We shall prove that $< x_n > \rightarrow^{\nu g} x$ and y in X. Let U be any νg-open subset of X containing X and V be any νg-open set of X containing y. Then, $U \cap Y$ and $V \cap Y$ are νg-open sets in Y. Therefore, $< x_n >$ is eventually in $U \cap Y$ and $V \cap Y$ and so in U and V. Since X is νg-US, this implies that $x = y$. Hence the subspace Y is νg-US.

Theorem 8.6. A space X is νg-T_2 iff it is both νg-R_1 and νg-US.

Proof. Let X be νg-T_2 space. Then X is νg-R_1 and νg-US by theorem 8.2. Conversely, let X be both νg-R_1 and νg-US space. By theorem 8.1, X is both νg-T_1 and νg-R_1 and, it follows that space X is νg-T_2.

Definition 8.5. A point y is a νg-cluster point of sequence $< x_n >$ iff $< x_n >$ is frequently in every νg-open set containing X.

The set of all νg-cluster points of $< x_n >$ will be denoted by $\nu g\overline{(x_n)}$.

Definition 8.6. A point y is νg-side point of a sequence $< x_n >$ if y is a νg-cluster point of $< x_n >$ but no subsequence of $< x_n >$ νg-converges to y.

Now, we define the following.

Definition 8.7. A space X is said to be νg-S_1 if it is νg-US and every sequence $< x_n >$ νg-converges with subsequence of $< x_n >$ νg-side points.

Definition 8.8. A space X is said to be νg-S_2 if it is νg-US and every sequence $< x_n >$ in X νg-converges which has no νg-side point.

Lemma 8.1. Every $\nu g - S_2$ space is νg-S_1 and Every νg-S_1 space is νg-US.

Now using the notion of sequentially continuous functions, we define the notion of sequentially νg-continuous functions.

Definition 8.9. A function f is said to be sequentially νg-continuous at $x \in X$ if $f(x_n) \rightarrow^{\nu g} f(x)$ whenever $< x_n > \rightarrow^{\nu g} X$. If f is sequentially νg-continuous at all $x \in X$, then f is said to be sequentially νg-continuous.

Theorem 8.7. Let f and g be two sequentially νg-continuous functions. If Y is νg-US, then the set $A = \{x | f(x) = g(x)\}$ is sequentially νg-closed.

Proof. Let Y be νg-US and suppose that there is a sequence $< x_n >$ in A νg-converging to $x \in X$. Since f and g are sequentially νg-continuous functions, $f(x_n) \rightarrow^{\nu g} f(x)$ and $g(x_n) \rightarrow^{\nu g} g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, A is sequentially νg-closed.

Next, we prove the product theorem for νg-US spaces.

Theorem 8.8. Product of arbitrary family of νg-US spaces is νg-US.

Proof. Let $X = \Pi_{\lambda \in \Lambda} X_\lambda$ where X_λ is νg-US. Let a sequence $< x_n >$ in X νg-converges to $x(= x_\lambda)$ and $y(= y_\lambda)$. Then the sequence $< x_n >$ νg-converges to x_λ and y_λ for all $\lambda \in \Lambda$. For suppose there exists a $\mu \in \Lambda$ such that $< x_{n\mu} > \not\rightarrow^{\nu g} x_\mu$. Then there exists a τ_μ-νg-open set U containing x_μ such that $< x_{n\mu} >$ is not eventually in U . Consider the set $U = Pi_{\lambda \in \Lambda} X_\lambda \times U_\mu$. Then $U \in \nu GO(X, x)$. Also, $< x_n >$ is not eventually in U, which contradicts the fact that $< x_n >$ νg-converges to X. Thus we get $< x_{n\lambda} >$ νg-converges to x_λ and y_λ for all $\lambda \in \Lambda$. Since X is νg-US for each $\lambda \in \Lambda$. Thus $x = y$. Hence X is νg-US.

§9. Sequentially sub-νg-continuity

In this section we introduce and study the concepts of sequentially sub-νg-continuity, sequentially nearly νg-continuity and sequentially νg-compact preserving functions and study their relations and the property of νg-US spaces.

Definition 9.1. A function f is said to be sequentially nearly νg-continuous if for each $x \in X$ and each sequence $< x_n > \rightarrow^{\nu g} x \in X$, there exists a subsequence $< x_{nk} >$ of $< x_n >$ such that $< f(x_{nk}) > \rightarrow^{\nu g} f(x)$.

Definition 9.2. A function f is said to be sequentially sub-νg-continuous if for each $x \in X$ and each sequence $< x_n > \rightarrow^{\nu g} x \in X$, there exists a subsequence $< x_{nk} >$ of $< x_n >$ and a point $y \in Y$ such that $< f(x_{nk}) > \rightarrow^{\nu g} y$.

Definition 9.3. A function f is said to be sequentially νg-compact preserving if $f(K)$ is sequentially νg-compact in Y for every sequentially νg-compact set K of X.

Lemma 9.1. Every function f is sequentially sub-νg-continuous if Y is a sequentially νg-compact.

Proof. Let $< x_n >$ be a sequence in X νg-converging to a point x of X. Then $\{f(x_n)\}$ is a sequence in Y and as Y is sequentially νg-compact, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ νg-converging to a point $y \in Y$. Hence f is sequentially sub-νg-continuous.

Theorem 9.1. Every sequentially nearly νg-continuous function is sequentially νg-compact preserving.

Proof. Let f be sequentially nearly νg-continuous function and let K be any sequentially νg-compact subset of X. Let $< y_n >$ be any sequence in $f(K)$. Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $< x_n >$ is a sequence in the sequentially νg-compact set K, there exists a subsequence $< x_{nk} >$ of $< x_n > \rightarrow^{\nu g} x \in K$. By hypothesis, f is sequentially nearly νg-continuous and hence there exists a subsequence $< x_j >$ of $< x_{nk} >$ such that $f(x_j) \rightarrow^{\nu g} f(x)$. Thus, there exists a subsequence $< y_j >$ of $< y_n >$ νg-converging to $f(x) \in f(K)$. This shows that $f(K)$ is sequentially νg-compact set in Y.

Theorem 9.2. Every sequentially νg-compact preserving function is sequentially sub-νg-continuous.

Proof. Suppose f is a sequentially νg-compact preserving function. Let x be any point of X and $< x_n >$ any sequence in X νg-converging to x. We shall denote the set $\{x_n | n = 1, 2, 3, \cdots\}$ by A and $K = A \cup \{x\}$. Then K is sequentially νg-compact since $x_n \rightarrow^{\nu g} X$. By hypothesis, f is sequentially νg-compact preserving and hence $f(K)$ is a sequentially νg-compact set of Y. Since $\{f(x_n)\}$ is a sequence in $f(K)$, there exists a subsequence $\{f(x_{nk})\}$

of $\{f(x_n)\}$ νg-converging to a point $y \in f(K)$. This implies that f is sequentially sub-νg-continuous.

Theorem 9.3. A function $f : X \to Y$ is sequentially νg-compact preserving iff $f_{/K} : K \to f(K)$ is sequentially sub-νg-continuous for each sequentially νg-compact subset K of X.

Proof. Suppose f is a sequentially νg-compact preserving function. Then $f(K)$ is sequentially νg-compact set in Y for each sequentially νg-compact set K of X. Therefore, by lemma 9.1 above, $f_{/K} : K \to f(K)$ is sequentially νg-continuous function.

Conversely, let K be any sequentially νg-compact set of X. Let $< y_n >$ be any sequence in $f(K)$. Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $< x_n >$ is a sequence in the sequentially νg-compact set K, there exists a subsequence $< x_{nk} >$ of $< x_n >$ νg-converging to a point $x \in K$. By hypothesis, $f_{/K} : K \to f(K)$ is sequentially sub-νg-continuous and hence there exists a subsequence $< y_{nk} >$ of $< y_n >$ νg-converging to a point $y \in f(K)$. This implies that $f(K)$ is sequentially νg-compact set in Y. Thus, f is sequentially νg-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub-νg-continuous function to be sequentially νg-compact preserving.

Corollary 9.1. If f is sequentially sub-νg-continuous and $f(K)$ is sequentially νg-closed set in Y for each sequentially νg-compact set K of X, then f is sequentially νg-compact preserving function.

Proof. Omitted.

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References:

[1] S. P. Arya and M. P. Bhamini, A note on semi-US spaces, Ranchi Uni. Math. J., **13**(1982), 60-68.

[2] Ashish Kar and P. Bhattacharyya, Some weak separation axioms, Bull. Cal. Math. Soc., **82**(1990), 415-422.

[3] C. E. Aull, Sequences in topological spaces, Comm. Math. (1968), 329-36.

[4] S. Balasubramanian, C. Sandhya and P. Aruna Swathi Vyjayanthi, On ν-T_i, ν-R_i and ν-C_i axioms, Scientia Magna, **4**(2008), No. 4, 86-103.

[5] S. Balasubramanian, P. Aruna Swathi Vyjayanthi and C. Sandhya, ν-compact spaces, Scientia Magna, **5**(2009), No. 1, 78-82.

[6] S. Balasubramanian, νg-Closed sets, Bull. Kerala. Math. Association, **5**(2009), No. 2, 81-92.

[7] S. Balasubramanian and K. Rajasekhar, Pre T_0-identification spaces axioms, Jr. Tripura. Math. Soc., **11**(2009), 118-124.

[8] S. Balasubramanian, νg-Boundary and νg-Exterior operators, Acta Ciencia Indica, **XXXVII**(2011), No. 1, 11-18.

[9] S. Balasubramanian, νg-continuity, Proc. Internat. Seminar on Topology and Applied Mathematics, Kochi, Kerala, (2011).

[10] S. Balasubramanian and P. Aruna Swathi Vyjayanthi, On ν-separation axioms, Inter. J. Math. Archive., **2**(2011), No. 8, 1464-1473.

[11] S. Balasubramanian and P. Aruna Swathi Vyjayanthi, ν-T_0-limit point of a set and its implications; ν-T_0-identification spaces and ν-separation axioms, Reyona, **1**(2012), 151-160.

[12] S. Balasubramanian, νg-separation axioms, (Communicated).

[13] F. Cammaroto and T. Noiri ,Almost irresolute functions, Indian J. Pure and Appl. Math., **20**(1989), No. 5, 472-482.

[14] H. F. Cullen, Unique sequential limits, Boll. UMI, **20**(1965), 123-127.

[15] Charles Dorsett, semi-T_1, semi-T_2 and semi-R_1 spaces, Ann. Soc. Sci. Bruxelles, **92**(1978), 143-158.

[16] S. G. Crossley and S. K. Hildebrand, Semiclosure, Texas J. Sci., **22**(1970), 99-112.

[17] A. Davis, Indexed systems of neighbourhoods for general topological spaces, Amer. Math. Monthly., **68**(1961), 886-896.

[18] C. Dorsett, Semi-regularization spaces and the semiclosure operator, s-closed spaces and quasi-irresolute functions, Indian J. Pure and Appl. Math., **21**(1990), No. 5, 416-422.

[19] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. R. S. R., **27**(1983), No. 4, 311-315.

[20] K. K. Dube and B. L. namdeo, T_0-limit point of a set and its implications, J. Tripura. Math. Soc., **9**(2007), 85-96.

[21] M. Ganster and I. L. Reilly, On some strong forms of paracompactness, Q & A in general topology, **5**(1987), 303-310.

[22] M .Ganster and D. Andrijevic, On some questions concerning semipreopen sets, Jour. Inst. Math. Comp. Sci.(Math. Ser.), **1**(1988), No. 2, 65-75.

[23] G. L. Garg and D. Sivaraj, Presemiclosed mappings, Periodica Math. Hung., **19**(1988), No. 2, 97-106.

[24] D. S. Jankovic', A note on mappings of extremally disconnected spaces, Acta. Math. Hung., **46**(1985), No. 1-2, 83-92.

[25] S. R. Malghan and G. B. Navalagi, Almost-p-regular, p-completely regular and almost-p-completely regular spaces, Bull. Math. Soc. Sci. Math. R. S. R. Tome, **34**(1990), No. 4, 317-326.

[26] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt, **53**(1982), 47-56.

[27] A. S. Mashhour, M. E. Abd El-Monsef and I. A. Hasanein, On pretopological spaces, Bull. Math. Soc. Sci. R. S. R., **28**(1984), 39-45.

[28] A. S. Mashhour, M. E. Abd El-Monsef and I. A. Hasanein and T. Noiri, Strongly compact spaces, Delta J. Sci., **8**(1984), 30-46.

[29] S. N. Maheshwari and R. Prasad, Some new separation axioms, Ann. Soc. Sci., Bruxelles., **89**(1975), 395-402.

[30] G. B. Navalagi, Pre-neighbourhoods, The Mathematics Education, **XXXII**(1998), No. 4, 201-206.

[31] G. B. Navalagi, Further properties of preseparation axioms, (Unpublished)

[32] G. B. Navalagi, P-Normal, Almost-P-Normal, Mildly-P-Normal, Topology Atlas.

[33] G. B. Navalagi, Pre-US spaces, Topology Atlas.

[34] O. Njastad, On some classes of nearly open sets, Pacific J. Math., **15**(1965), 961-70.

[35] T. Noiri, Almost continuity and some separation axioms, Glasnik Mat., **9**(1974), 131-135.

[36] T. Noiri, Sequentially subcontinuous functions, Accad. Naz. Dei. Lincei. Estratto dei. Rendiconti. Della Classe di Sci. Fis. Mat. Nat., **LVIII**(1975), No. 3, 370-376.

[37] T. Noiri, On semi continuous mappings, Atti. Della Accad. Naz. Del Lincei. Rendi. Cl. Sci. Fis. Mat. Nat., **54**(1973), 210-217.

[38] Norman Levine, Semi open sets and semicontinuity in topological spaces, Amer. Math. Monthly., **70**(1963), 36-41.

[39] Paul and Bhattacharyya, On p-normal spaces, Soochow Jour. Math., **21**(1995), No. 3, 273-289.

[40] V. Popa, Characterization of H-almost continuous functions, Glasnik Math. Vol., **22**(1987), 157-161.

[41] V. K. Sharma, ν-open sets, Acta Ciencia Indica, **XXXII**(2006), No. 2, 685-688.

[42] V. K. Sharma, ν-axioms, Acta Ciencia Indica, **XXXIII M**(2007), No. 1, 1807-1810.

[43] G. Sina and I. L. Reilly, On feebly closed mappings, Indian J. Pure Appl. Math., **17**(1986), 1101-1105.

[44] M. K. Singal and S. P. Arya, On nearly compact spaces, Mat. Vesnik, **6**(1969), 3-16.

[45] M. K. Singal and S. P. Arya, On almost regular spaces, Glasnik Mat., **4**(1969), No. 1, 89-99.

[46] M. K. Singal and S. P. Arya, On almost normal and almost completely regular spaces, Glasnik Mat., **25**(1970), 141-152.

[47] M. K. Singal and A. R. Singal, Mildly normal spaces, Kyungpook Math. J., **13**(1973), No. 1, 27-31.

[48] D. Sivaraj, Semi open set characterizations of almost regular spaces, Glasnik Mat., **21**(1986), 437-440.

[49] T. Soundarajan, General topology and its applications to Modern Analysis and Algebra, Proc. Conf. Kanpur (1968), Acad. Progue., (1971), 301-306.

[50] T. Thompson , S-closed spaces, Proc. Amer. Math. Soc., **60**(1976), 335-338.

[51] A. Wilansky, Between T_1 and T_2, Amer. Math. Monthly., **74**(1967), 261-266.

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On intuitionistic fuzzy pre-β-irresolute functions

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Abstract In this paper the concept of intuitionistic fuzzy pre-β-irresolute functions are introduced and studied. Besides giving characterizations and properties of this function, preservation of some intuitionistic fuzzy topological structure under this function are also given. We also study relationship between this function with other existing functions.

Keywords Intuitionistic Fuzzy β-open set, intuitionistic fuzzy pre-α-irresolute, intuitionistic Fuzzy pre-β-irresolute.
2000 Mathematics Subject Classification: 54A40, 03E72

§1. Introduction

Ever since the introduction of Fuzzy sets by L. A. Zadeh [14], the Fuzzy concept has invaded almost all branches of mathematics. The concept of Fuzzy topological spaces was introduced and developed by C. L. Chang [2]. Atanassov [1] introduced the notion of intuitionistic fuzzy sets, Coker [4] introduced the intuitionistic fuzzy topological spaces. In this paper we have introduced the concept of intuitionistic Fuzzy pre-β-irresolute functions and studied their properties. Also we have given characterizations of intuitionistic fuzzy pre-β-irresolute functions. We also study relationship between this function with other existing functions.

§2. Preliminaries

Definition 2.1.[1] Let X be a nonempty fixed set and I the closed interval [0,1]. An intuitionistic fuzzy set (IFS) A is an object of the following form

$$A = \{< x, \mu_A(x), \nu_A(x) >; x \in X\},$$

where the mappings $\mu_A(x) : X \to I$ and $\nu_A(x) : X \to I$ denote the degree of membership(namely) $\mu_A(x)$ and the degree of nonmembership (namely) $\nu_A(x)$ for each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.

Definition 2.2.[1] Let A and B are IFS of the form $A = < x, \mu_A(x), \nu_A(x) >; x \in X$ and $B = < x, \mu_B(x), \nu_B(x) >; x \in X$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;

(ii) $\overline{A}$(or A^c) $= \{< x, \nu_A(x), \mu_A(x) >; x \in X\}$;

(iii) $A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >; x \in X\}$;

(iv) $A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >; x \in X\}$.

We will use the notation $A = \{< x, \mu_A, \nu_A >; x \in X\}$ instead of $A = \{< x, \mu_A(x), \nu_A(x) > ; x \in X\}$.

Definition 2.3.[4] $0_\sim = \{< x, 0, 1 >; x \in X\}$ and $1_\sim = \{< x, 1, 0 >; x \in X\}$.

Let α, $\beta \in$ [0,1] such that $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP) $p_{(\alpha,\beta)}$ is intuitionistic fuzzy set defined by $p_{(\alpha,\beta)}(x) = \begin{cases} (\alpha, \beta), \text{ if } x = p \text{ ;} \\ (0, 1), \text{ otherwise.} \end{cases}$

Definition 2.4.[10] Let $p_{(\alpha,\beta)}$ be an IFP in $IFTS$ X. An IFS A in X is called an intuitionistic Fuzzy neighborhood (IFN) of $p_{(\alpha,\beta)}$ if there exists an $IFOS$ B in X such that $p_{(\alpha,\beta)} \in B \subseteq A$.

Let X and Y are two non-empty sets and $f : (X, \tau) \to (Y, \sigma)$ be a function.

If $B = \{< y, \mu_B(y), \nu_B(y) >; y \in Y\}$ is an IFS in Y, then the pre-image of B under f is denoted and defined by $f^{-1}(B) = \{< x, f^{-1}(\mu_B(x)), f^{-1}(\nu_B(x)) >; x \in X\}$. Since $\mu_B(x)$, $\nu_B(x)$ are Fuzzy sets, we explain that $f^{-1}(\mu_B(x)) = \mu_B(x)(f(x))$, $f^{-1}(\nu_B(x)) = \nu_B(x)(f(x))$.

Definition 2.5.[4] An intuitionistic fuzzy topology (IFT) in Coker's sense on a nonempty set X is a family τ of intuitionistic fuzzy sets in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$;

(ii) $G_1 \cap G_2 \in \tau$, for any G_1, $G_2 \in \tau$;

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i; i \in J\} \subseteq \tau$.

In this paper by (X,τ) or simply by X we will denote the intuitionistic fuzzy topological space ($IFTS$). Each IFS which belongs to τ is called an intuitionistic fuzzy open set ($IFOS$) in X. The complement $\overline{A}$ of an $IFOS$ A in X is called an intuitionistic fuzzy closed set ($IFCS$) in X.

Definition 2.6.[4] Let (X,τ) be an $IFTS$ and $A = \{< x, \mu_A(x), \nu_A(x) >; x \in X\}$ be an IFS in X. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of A are defined by

(i) $cl(A) = \bigcap\{C : C$ is an $IFCS$ in X and $C \supseteq A\}$;

(ii) $int(A) = \bigcup\{D : D$ is an $IFOS$ in X and $D \subseteq A\}$.

It can be also shown that $cl(A)$ is an $IFCS$, $int(A)$ is an $IFOS$ in X and A is an $IFCS$ in X if and only if $cl(A) = A$; A is an $IFOS$ in X if and only $int(A) = A$.

Proposition 2.1.[4] Let (X,τ) be an $IFTS$ and A, B be $IFSs$ in X. Then the following properties hold:

(i) $cl\overline{A} = \overline{(int(A))}$, $int\overline{(A)} = \overline{(cl(A))}$;

(ii) $int(A) \subseteq A \subseteq cl(A)$.

Definition 2.7.[6] An IFS A in an $IFTS$ X is called an intuitionistic fuzzy pre open set ($IFPOS$) if $A \subseteq int(clA)$. The complement of an $IFPOS$ A in $IFTS$ X is called an intuitionistic fuzzy pre closed ($IFPCS$) in X.

Definition 2.8.[6] An IFS A in an $IFTS$ X is called an intuitionistic fuzzy α-open set ($IF\alpha OS$) if and only if $A \subseteq int(cl(intA))$. The complement of an $IF\alpha OS$ A in X is called intuitionistic fuzzy α-closed ($IF\alpha CS$) in X.

Definition 2.9.[6] An IFS A in an $IFTS$ X is called an intuitionistic fuzzy semi open set ($IFSOS$) if and only if $A \subseteq cl(int(A))$. The complement of an $IFSOS$ A in X is called intuitionistic fuzzy semi closed ($IFSCS$) in X.

Definition 2.10.[13] An IFS A in an $IFTS$ X is called an intuitionistic fuzzy β-open set ($IF\beta OS$) (otherwise called as intuitionistic Fuzzy semi pre-open set) if and only if $A \subseteq cl(int(clA))$. The complement of an $IF\beta OS$ A in X is called intuitionistic fuzzy β-closed($IF\beta CS$) in X.

Definition 2.11.[6,13] Let f be a mapping from an $IFTS$ X into an $IFTS$ Y. The mapping f is called:

(i) intuitionistic fuzzy continuous if and only if $f^{-1}(B)$ is an $IFOS$ in X, for each $IFOS$ B in Y;

(ii) intuitionistic fuzzy α-continuous if and only if $f^{-1}(B)$ is an $IF\alpha OS$ in X, for each $IFOS$ B in Y;

(iii) intuitionistic fuzzy pre continuous if and only if $f^{-1}(B)$ is an $IFPOS$ in X, for each $IFOS$ B in Y;

(iv) intuitionistic fuzzy semi contiunous if and only if $f^{-1}(B)$ is an $IFSOS$ in X, for each $IFOS$ B in Y;

(v) intuitionistic fuzzy β-contiunous if and only if $f^{-1}(B)$ is an $IF\beta OS$ in X, for each $IFOS$ B in Y.

Definition 2.12.[12] Let (X,τ) be an $IFTS$ and $A = \{< x, \mu_A(x), \nu_A(x) >; x \in X\}$ be an IFS in X. Then the intuitionistic fuzzy β-closure and intuitionistic fuzzy β-interior of A are defined by

(i) $\beta cl(A) = \bigcap\{C : C$ is an $IF\beta CS$ in X and $C \supseteq A\}$;

(ii) $\beta int(A) = \bigcup\{D : D$ is an $IF\beta OS$ in X and $D \subseteq A\}$.

Definition 2.13. A function $f : (X, \tau) \to (Y, \sigma)$ from a intuitionistic fuzzy topological space (X,τ) to another intuitionistic fuzzy topological space (Y,σ) is said to be intuitionistic fuzzy β-irresolute if $f^{-1}(B)$ is an $IF\beta OS$ in (X,τ) for each $IF\beta OS$ B in (Y,σ).

Definition 2.14.[11] A function $f : (X, \tau) \to (Y, \sigma)$ from a intuitionistic fuzzy topological space (X,τ) to another intuitionistic fuzzy topological space (Y,σ) is said to be intuitionistic fuzzy pre irresolute if $f^{-1}(B)$ is an $IFPOS$ in (X,τ) for each $IFPOS$ B in (Y,σ).

Definition 2.15.[11] A function $f : (X, \tau) \to (Y, \sigma)$ from a intuitionistic fuzzy topological space (X,τ) to another intuitionistic fuzzy topological space (Y,σ) is said to be intuitionistic fuzzy pre-α-irresolute if $f^{-1}(B)$ is an $IFPOS$ in (X,τ) for each $IF\alpha OS$ B in (Y,σ).

Definition 2.16.[4,9] Let X be an $IFTS$. A family of $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$ intuitionistic fuzzy open sets (intuitionistic fuzzy pre-open sets) in X satisfies the condition $1_\sim = \cup\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$ is called a intuitionistic fuzzy open (intuitionistic fuzzy pre-open) cover of X. A finite subfamily of a intuitionistic fuzzy open (intuitionistic fuzzy pre-open) cover $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$ of X which is also a intuitionistic fuzzy open (intuitionistic fuzzy pre-open) cover of X is called a finite subcover of $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >$

$; i \in J\}$.

Definition 2.17.[9] An $IFTS$ X is called intuitionistic fuzzy precompact (pre $Lindelof$) if each intuitionistic fuzzy pre-open cover of X has a finite (countable) subcover for X.

§3. Intuitionistic Fuzzy pre-β-irresolute functions

Definition 3.1. A function $f : (X, \tau) \to (Y, \sigma)$ from a intuitionistic fuzzy topological space (X,τ) to another intuitionistic fuzzy topological space (Y, σ) is said to be intuitionistic fuzzy pre-β-irresolute if $f^{-1}(B)$ is an $IFPOS$ in (X,τ) for each $IF\beta OS$ B in (Y,σ).

IF pre-α-irresolute $\nwarrow$ $\nearrow$ IF pre continuous

IF pre-β-irresolute

$\swarrow$ $\downarrow$ $\searrow$

IF pre irresolute IF β-irresolute IF β-continuous

Proposition 3.1. Every intuitionistic fuzzy pre-β-irresolute function is an intuitionistic fuzzy pre-α-irresolute function.

Proof. Follows from the definitions.

However, the converse of the above proposition 3.1 needs not to be true, as shown by the following example.

Example 3.1. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$ where

$$A = \{< x, (\tfrac{a}{0.6}, \tfrac{b}{0.3}), (\tfrac{a}{0.4}, \tfrac{b}{0.7}) >; x \in X\},$$
$$B = \{< y, (\tfrac{c}{0.4}, \tfrac{d}{0.3}), (\tfrac{c}{0.2}, \tfrac{d}{0.5}) >; y \in Y\},$$
$$C = \{< y, (\tfrac{c}{0.4}, \tfrac{d}{0.4}), (\tfrac{c}{0.6}, \tfrac{d}{0.5}) >; y \in Y\}.$$

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c, f(b) = d$. B ia an $IF\alpha OS$ in (Y,σ), since $B \subseteq int(cl(int(B))) = B$. $f^{-1}(B) = \{< x, (\frac{a}{0.4}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.5}) >;$ $x \in X\}$, and $int(clf^{-1}(B)) = 1_\sim$. Thus $f^{-1}(B) \subseteq int(clf^{-1}(B))$. Hence $f^{-1}(B)$ is $IFPOS$ in X, which implies f is IF pre-α-irresolute. C is an $IF\beta OS$ in (Y,σ) since $C \subseteq cl(int(cl(C)))$ $= 0^c_\sim$. $f^{-1}(C) = \{< x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.6}, \frac{b}{0.5}) >; x \in X\}$ and $int(clf^{-1}(C))) = 0_\sim$. So, $f^{-1}(C)$ $\nsubseteq int(clf^{-1}(C)))$. Thus $f^{-1}(C)$ is not $IFPOS$ in X. Hence f is not IF pre-β-irresolute.

Proposition 3.2. Every intuitionistic fuzzy pre-β-irresolute is an intuitionistic fuzzy pre continuous.

Proof. Follows from the definitions.

However the converse of the above proposition 3.2 needs not to be true, in general as shown by the following example.

Example 3.2. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau = \{0_\sim, 1_\sim, A\}$, $\sigma = \{0_\sim, 1_\sim, B\}$ where

$$A = \{< x, (\tfrac{a}{0.6}, \tfrac{b}{0.3}), (\tfrac{a}{0.4}, \tfrac{b}{0.7}) >; x \in X\},$$
$$B = \{< y, (\tfrac{c}{0.4}, \tfrac{d}{0.3}), (\tfrac{c}{0.2}, \tfrac{d}{0.5}) >; y \in Y\},$$
$$C = \{< y, (\tfrac{c}{0.4}, \tfrac{d}{0.4}), (\tfrac{c}{0.6}, \tfrac{d}{0.5}) >; y \in Y\}.$$

Define an intuitionistic fuzzy mapping f :$(X,\tau)\to(Y,\sigma)$ by $f(a) = c, f(b) = d$. B is an $IFOS$ in (Y, σ). $f^{-1}(B) = \{< x,(\frac{a}{0.4},\frac{b}{0.3}),(\frac{a}{0.2},\frac{b}{0.5})>;x \in X\}$ is an $IFPOS$ in (X,τ), since $int(clf^{-1}(B)) = 1_\sim$ and $f^{-1}(B) \subseteq int(clf^{-1}(B))$. Hence f is an IF pre continuous. By previous example, f is not IF pre-β-irresolute.

Proposition 3.3. Every intuitionistic fuzzy pre-β-irresolute is an intuitionistic fuzzy pre irresolute function.

Proof. Follows from the definitions.

However the converse of the above proposition 3.3 needs not be true, as shown by the following example.

Example 3.3. Let $X = \{a, b\}, Y = \{c, d\}$, $\tau=\{0_\sim, 1_\sim, A\}$, $\sigma=\{0_\sim, 1_\sim, B\}$ where

$$A = \{< x,(\frac{a}{0.4},\frac{b}{0.5}),(\frac{a}{0.4},\frac{b}{0.2})>;x \in X\},$$
$$B = \{< y,(\frac{c}{0.4},\frac{d}{0.5}),(\frac{c}{0.5},\frac{d}{0.5})>;y \in Y\},$$
$$C = \{< y,(\frac{c}{0.2},\frac{d}{0.2}),(\frac{c}{0.4},\frac{d}{0.5})>;y \in Y\}.$$

Define an intuitionistic fuzzy mapping f :$(X,\tau)\to(Y,\sigma)$ by $f(a) = c, f(b) = d$. B is an $IFOS$ in (Y, σ), and B is an $IFPOS$ in Y, since $B \subseteq int(cl(B)) = B$. $f^{-1}(B) = \{< x, (\frac{a}{0.4},\frac{b}{0.5}), (\frac{a}{0.5},\frac{b}{0.5})>;x \in X\}$, and $int(clf^{-1}(B)) = 1_\sim$. Thus $f^{-1}(B) \subseteq int(clf^{-1}(B))$. Hence $f^{-1}(B)$ is $IFPOS$ in X, which implies f is IF pre irresolute. C is an IFS in Y. $cl(int(cl(C))) = B^c$. Hence $C \subseteq cl(int(cl(C)))$. Thus C is $IF\beta OS$ in Y. $f^{-1}(C) = \{< x, (\frac{a}{0.2},\frac{b}{0.2}), (\frac{a}{0.4},\frac{b}{0.5})>;x \in X\}$ And $int(clf^{-1}(C)) = 0_\sim$. Since $f^{-1}(C) \nsubseteq int(clf^{-1}(C))$, $f^{-1}(C)$ is not an $IFPOS$ in X. Hence f is not IF pre-β-irresolute.

Proposition 3.4. Every intuitionistic fuzzy pre-β-irresolute is an intuitionistic fuzzy β-irresolute function.

Proof. Follows from the definitions.

However the converse of the above proposition 3.4 needs not be true, in general as shown by the following example.

Example 3.4. Let $X = \{a, b\}, Y = \{c, d\}$, $\tau=\{0_\sim, 1_\sim, A\}$, $\sigma=\{0_\sim, 1_\sim, B\}$ where

$$A = \{< x,(\frac{a}{0.3},\frac{b}{0.4}),(\frac{a}{0.6},\frac{b}{0.5})>; x \in X\},$$
$$B=\{< y,(\frac{c}{0.4},\frac{d}{0.5}),(\frac{c}{0.5},\frac{d}{0.5})>; y \in Y\}.$$

Define an intuitionistic fuzzy mapping f :$(X,\tau)\to(Y,\sigma)$ by $f(a) = d, f(b) = c$. B ia an $IF\beta OS$ in (Y,σ) since $B \subseteq cl(int(cl(B))) = B^c$, and $f^{-1}(B) = \{< x,(\frac{a}{0.5},\frac{b}{0.4}), (\frac{a}{0.5},\frac{b}{0.5})>;x \in X\}$, since $f^{-1}(B) \subseteq cl(int(clf^{-1}(B))) = A^c$. So $f^{-1}(B)$ is an $IF\beta OS$ in X. Thus f is $IF\beta$-irresolute. B is an $IF\beta OS$ in (Y,σ), and $int(clf^{-1}(B)) = A$. So $f^{-1}(B) \nsubseteq int(clf^{-1}(B))$. Hence $f^{-1}(B)$ is not $IFPOS$ in X . So f is not IF pre-β-irresolute function.

Proposition 3.5. Every intuitionistic fuzzy pre-β-irresolute function is an intuitionistic fuzzy β-continuous function.

Proof. Follows from the definitions.

However, the converse of the above proposition 3.5 needs not to be true, as shown by the following example.

Example 3.5. Let $X = \{a, b\}$, $Y = \{c, d\}$, $\tau=\{0_\sim, 1_\sim,A\}$, $\sigma=\{0_\sim, 1_\sim, B\}$ where

$$A = \{< x,(\frac{a}{0.6},\frac{b}{0.3}),(\frac{a}{0.4},\frac{b}{0.7})>; x \in X\},$$

$B = \{< y, (\frac{c}{0.4}, \frac{d}{0.3}), (\frac{c}{0.2}, \frac{d}{0.5})>; y \in Y\}$,
$C = \{< y, (\frac{c}{0.4}, \frac{d}{0.4}), (\frac{c}{0.6}, \frac{d}{0.5})>; y \in Y\}$.

Define an intuitionistic fuzzy mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = c, f(b) = d$. B ia an $IFOS$ in (Y,σ). $f^{-1}(B) = \{< x, (\frac{a}{0.4}, \frac{b}{0.3}), (\frac{a}{0.2}, \frac{b}{0.5})>; x \in X\}$, and $cl(int(clf^{-1}(B))) = 1_\sim$. Thus $f^{-1}(B) \subseteq cl(int(clf^{-1}(B)))$. Hence $f^{-1}(B)$ is $IF\beta OS$ in X, which implies f is IF β-continuous function. C ia an $IF\beta OS$ in (Y,σ) since $C \subseteq cl(int(cl(C))) = 0^c_\sim$. $f^{-1}(C) = \{< x, (\frac{a}{0.4}, \frac{b}{0.4}), (\frac{a}{0.6}, \frac{b}{0.5})>; x \in X\}$ and $int(clf^{-1}(C))) = 0_\sim$. So, $f^{-1}(C) \nsubseteq int(clf^{-1}(C)))$. Thus $f^{-1}(C)$ is not $IFPOS$ in X. Hence f is not IF pre-β-irresolute.

Theorem 3.1. *IF f is a function from an $IFTS$ (X,τ) to another $IFTS(Y,\sigma)$ then the followings are equivalent:*

(a) f is IF pre-β-irresolute,

(b) $f^{-1}(B) \subseteq int(clf^{-1}(B))$ for every $IF\beta OS$ B in Y,

(c) $f^{-1}(C)$ is IF pre closed in X for every $IF\beta$-closed set C in Y,

(d) $cl(intf^{-1}(D)) \subseteq f^{-1}(\beta cl(D))$ for every IFS D of Y,

(e) $f(cl(int(E))) \subseteq \beta cl(f(E))$ for every IFS E of X.

Proof. (a) $\Rightarrow$ (b). Let B be $IF\beta OS$ in Y. By (a), $f^{-1}(B)$ is IF pre open in X. $\Rightarrow$ $f^{-1}(B) \subseteq int(clf^{-1}(B))$. Hence (a) $\Rightarrow$ (b) is proved.

(b)$\Rightarrow$(c). Let C be any $IF\beta CS$ in Y, then $\overline{C}$ be $IF\beta OS$ in Y. By (b), $f^{-1}(\overline{C}) \subseteq int(clf^{-1}(\overline{C}))$. But $\overline{f^{-1}(C)} \subseteq int(cl(\overline{f^{-1}(C)})= int(\overline{intf^{-1}(C)}) = \overline{cl(intf^{-1}(C)}$ which implies $\overline{f^{-1}(C)} \subseteq \overline{cl(intf^{-1}(C)}$. Thus $cl(intf^{-1}(C)) \subseteq f^{-1}(C)$. Hence $f^{-1}(C)$ is IF pre closed in X. Hence (b)$\Rightarrow$(c) is proved.

(c)$\Rightarrow$(d). Let D be IFS in Y, then $\beta cl(D)$ is a IF β-closed in Y. $\Rightarrow f^{-1}(\beta cl(D))$ is IF pre closed in X. Then $cl(intf^{-1}(\beta cl(D))) \subseteq f^{-1}(\beta cl(D))$. Thus we have $cl(intf^{-1}(D)) \subseteq f^{-1}(\beta cl(D))$. Hence (c)$\Rightarrow$(d) is proved.

(d)$\Rightarrow$(e). Let E be an IFS in X. $cl(int(E)) \subseteq cl(int(f^{-1}(f(E)))) \subseteq cl(int(f^{-1}(\beta cl(f(E)))$ $)) \subseteq f^{-1}(\beta cl(f(E)))$, then, $cl(int(E)) \subseteq f^{-1}(\beta cl(f(E))$. We get $f(cl(int(E))) \subseteq \beta cl(f(E))$. Hence (d)$\Rightarrow$(e) is proved.

(e)$\Rightarrow$(a). Let B be $IF\beta$-open set in Y. Then $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ is an IFS in X. By (e), $f(cl((int(f^{-1}(\overline{B})))) \subseteq \beta cl(f(f^{-1}(\overline{B}))) \subseteq \beta cl(\overline{B}) = \overline{\beta int B} = \overline{B}$.

Thus

$$f(cl(int(f^{-1}(\overline{B}))) \subseteq \overline{B}. \quad (1)$$

Consider,

$$\overline{int(cl(f^{-1}(B)))} = cl(\overline{cl(f^{-1}(B))}) = cl(int(\overline{(f^{-1}(B))})) = cl(int((f^{-1}(\overline{B}))$$

$$\subseteq (f^{-1}(f(cl(int((f^{-1}(\overline{B}))). \quad (2)$$

By (1), (2), $\overline{int(cl(f^{-1}(B)))} \subseteq f^{-1}(f(cl(int((f^{-1}(\overline{B}))))) \subseteq f^{-1}(\overline{B})) = \overline{f^{-1}(B)}$. Thus, $f^{-1}(B) \subseteq int(cl(f^{-1}(B))$. Hence $f^{-1}(B)$ is an IF pre open in X. Therefore f is IF pre-β-irresolute. Hence (e)$\Rightarrow$(a) is proved.

§4. Properties of intuitionistic fuzzy pre-β-irresolute functions

The following four lemmas are given here for convenience of the reader.

Lemma 4.1.[4] Let $f : X \to Y$ be a mapping, and A_α be a family of IF sets of Y. Then

(a) $f^{-1}(\bigcup A_\alpha) = \bigcup f^{-1}(A_\alpha)$,

(b) $f^{-1}(\bigcap A_\alpha) = \bigcap f^{-1}(A_\alpha)$.

Lemma 4.2.[7] Let $f : X_i \to Y_i$ be a mapping and A, B are $IFSs$ of Y_1 and Y_2 respectively then $(f_1 \times f_2)^{-1}(A \times B) = {f_1}^{-1}(A) \times {f_2}^{-1}(B)$.

Lemma 4.3.[7] Let $g : X \to X \times Y$ be a graph of a mapping $f : (X,\tau) \to (Y,\sigma)$. If A and B are $IFSs$ of X and Y respectivly, then $g^{-1}(1_\sim \times B) = (1_\sim \cap f^{-1}(B))$.

Lemma 4.4.[7] Let X and Y be intuitionistic fuzzy topological spaces, then (X,τ) is product related to (Y,σ) if for any IFS C in X, D in Y whenever $\overline{A} \not\supseteq C$, $\overline{B} \not\supseteq D$ implies $\overline{A} \times 1_\sim \bigcup 1_\sim \times \overline{B} \supseteq C \times D$ there exists $A_1 \in \tau$, $B_1 \in \sigma$ such that $\overline{A_1} \supseteq C$ and $\overline{B_1} \supseteq D$ and $\overline{A_1} \times 1_\sim \bigcup 1_\sim \times \overline{B_1} = \overline{A} \times 1_\sim \bigcup 1_\sim \times \overline{B}$.

Lemma 4.5. Let X and Y be intuitionistic fuzzy topological spaces such that X is product related to Y. Then the product $A \times B$ of $IF\beta OS$ A in X and a $IF\beta OS$ B in Y is a $IF\beta OS$ in Fuzzy product spaces $X \times Y$.

Theorem 4.1. Let $f : X \to Y$ be a function and assume that X is product related to Y. If the graph $g : X \to X \times Y$ of f is IF pre-β-irresolute, so is f.

Proof. Let B be $IF\beta OS$ in Y. Then by lemma 4.3 $f^{-1}(B) = 1_\sim \cap f^{-1}(B) = g^{-1}(1_\sim \times B)$.

Now $1_\sim \times B$ is a $IF\beta OS$ in $X \times Y$. Since g is IF pre-β-irresolute then $g^{-1}(1_\sim \times B)$ is IF pre open in X. Hence $f^{-1}(B)$ is IF pre open in X. Thus f is IF pre-β-irresolute.

Theorem 4.2. If a function $f : X \to \Pi Y_i$ is a IF pre-β-irresolute, then $P_i \circ f : X \to Y_i$ is IF pre-β-irresolute, where P_i is the projection of ΠY_i onto $Y-i$.

Proof. Let B_i be any $IF\beta OS$ of Y_i. Since P_i is IF continuous and $IFOS$, it is $IF\beta OS$. Now $P_i : \Pi\, Y_i \to Y_i$; ${P_i}^{-1}(B_i)$ is $IF\beta OS$ in $\Pi\, Y_i$. Therefore, P_i is IF β-irresolute function. Now $(P_i \circ f)^{-1}(B_i) = f^{-1}\ ({P_i}^{-1}(B_i))$, since f is IF pre-β-irresolute and ${P_i}^{-1}(B_i)$ is $IF\beta OS$, $f^{-1}\ ({P_i}^{-1}(B_i))$ is $IFPOS$ in X. Hence $(P_i \circ f)$ is IF pre-β-irresolute.

Theorem 4.3. If $f_i : X_i \to Y_i, (i = 1, 2)$ are IF pre-β-irresolute and X_1 is product related to X_2 then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is IF pre-β-irresolute.

Proof. Let $C = \bigcup(A_i \times B_i)$ where A_i and B_i are $IF\beta$ open sets of Y_1 and Y_2 respectively. Since Y_1 is product related to Y_2 , by previous lemma 4.5, that $C = \bigcup(A_i \times B_i)$ is $IF\beta$-open of $Y_1 \times Y_2$. Then by lemma 4.1 and 4.2 we have $(f_1 \times f_2)^{-1}(C) = (f_1 \times f_2)^{-1} \bigcup(A_i \times B_i) = \bigcup ({f_1}^{-1}(A_i) \times {f_2}^{-1}(B_i))$. Since f_1 and f_2 are IF pre-β-irresolute, $(f_1 \times f_2)^{-1}(C)$ is an $IFPOS$ in $X_1 \times X_2$ and hence $f_1 \times f_2$ is IF pre-β-irresolute function.

Definition 4.1.[5] Let (X,τ) be any $IFTS$ and let A be any IFS in X. Then A is called IF dense set if $clA = 1_\sim$ and A is called nowhere IF dense set if $int(cl(A)) = 0_\sim$.

Theorem 4.4. If a function $f : (X, \tau) \to (Y, \sigma)$ is IF pre-β-irresolute, then $f^{-1}(A)$ is IF pre closed in X for any nowhere IF dense set A of Y.

Proof. Let A be any nowhere IF dense set in Y. Then $int(clA) = 0_\sim$. Now, $\overline{int(clA)} = 1_\sim \Longrightarrow cl(\overline{cl(A)}) = 1_\sim$ which implies $cl(int(\overline{A})) = 1_\sim$.

Since $int1_\sim = 1_\sim$, $cl(int(\overline{A})) \subseteq cl(int(cl\overline{A}))$. So $\overline{A} \subseteq 1_\sim = cl(int(\overline{A}) \subseteq cl(int(cl\overline{A})$. Hence $\overline{A} \subseteq cl(int(cl\overline{A})))$. Then $\overline{A}$ is $IF\beta OS$ in Y. Since f is IF pre-β-irresolute, $f^{-1}(\overline{A})$ is IF pre open set in X. Hence $f^{-1}(A)$ is $IFPCS$ in X.

Theorem 4.5. A mapping $f : X \to Y$ from an $IFTS$ X into an $IFTS$ Y is IF pre-β-irresolute if and only if for each IFP $p_{(\alpha,\beta)}$ in X and $IF\beta OS$ B in Y such that $f(p_{(\alpha,\beta)}) \in B$, there exists an $IFPOS$ A in X such that $p_{(\alpha,\beta)} \in A$ and $f(A) \subseteq B$.

Proof. Let f be any IF pre-β-irresolute mapping, $p_{(\alpha,\beta)}$ be an IFP in X and B be any $IF\beta OS$ in Y such that $f(p_{(\alpha,\beta)}) \in B$. Then $p_{(\alpha,\beta)} \in f^{-1}(B)$. Let $A = f^{-1}(B)$. Then A is an $IFPOS$ in X which containing IFP $p_{(\alpha,\beta)}$ and $f(A) = f(f^{-1}(B)) \subseteq B$.

Conversely, let B be an $IF\beta OS$ in Y and $p_{(\alpha,\beta)}$ be IFP in X such that $p_{(\alpha,\beta)} \in f^{-1}(B)$. According to assumption there exists $IFPOS$ A in X such that $p_{(\alpha,\beta)} \in A$ and $f(A) \subseteq B$.

Hence $p_{(\alpha,\beta)} \in A \subseteq f^{-1}(B)$. We have $p_{(\alpha,\beta)} \in A \subseteq int(clA) \subseteq int(clf^{-1}(B))$. Therefore, $f^{-1}(B) \subseteq int(clf^{-1}(B))$. So f is IF pre-β-irresolute mapping.

Theorem 4.6. A mapping $f : X \to Y$ from an $IFTS$ X into an $IFTS$ Y is IF pre-β-irresolute if and only if for each IFP $p_{(\alpha,\beta)}$ in X and $IF\beta OS$ B in Y such that $f(p_{(\alpha,\beta)}) \in B$, $cl(f^{-1}(B)$ is IFN of IFP $p_{(\alpha,\beta)}$ in X.

Proof. Let f be any IF pre-β-irresolute mapping, $p_{(\alpha,\beta)}$ be an IFP in X and B be any $IF\beta OS$ in Y such that $f(p_{(\alpha,\beta)}) \in B$. Then $p_{(\alpha,\beta)} \in f^{-1}(B) \subseteq int(cl(f^{-1}(B))) \subseteq cl(f^{-1}(B))$.

Hence $cl(f^{-1}(B))$ is IFN of $p_{(\alpha,\beta)}$ in X.

Conversely, let B be any $IF\beta OS$ in Y and $p_{(\alpha,\beta)}$ be IFP in X such that $f(p_{(\alpha,\beta)}) \in B$. Then $p_{(\alpha,\beta)} \subseteq f^{-1}(B)$. According to assumption $cl(f^{-1}(B))$ is IFN of IFP $p_{(\alpha,\beta)}$ in X. So $p_{(\alpha,\beta)} \in int(cl(f^{-1}(B)))$. So, $f^{-1}(B) \subseteq int(cl(f^{-1}(B)))$. Hence $f^{-1}(B)$ is $IFPOS$ in X. Therefore f is IF pre-β-irresolute.

Theorem 4.7. The followings hold for functions f: $X \to Y$ and $g : Y \to Z$.

(i)If f is IF pre-β-irresolute and g is IF β-irresolute then $g \circ f$ is IF pre-β-irresolute.

(ii)If f is IF pre-β-irresolute and g is IF β-continuous then $g \circ f$ is IF pre continuous.

(iii)If f is IF pre irresolute and g is IF pre-β-irresolute then $g \circ f$ is IF pre-β-irresolute.

Proof. (i) Let B be an $IF\beta OS$ in Z. Since g is IF β-irresolute, $g^{-1}(B)$ is an $IF\beta OS$ in Y. Now $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Since f is IF pre-β-irresolute, $f^{-1}(g^{-1}(B))$ is $IFPOS$ in X. Hence $g \circ f$ is IF pre-β-irresolute.

(ii) Let B be $IFOS$ in Z. Since g is IF β-continuous, $g^{-1}(B)$ is an $IF\beta OS$ in Y. Now $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Since f is IF pre-β-irresolute, $f^{-1}(g^{-1}(B))$ is $IFPOS$ in X which implies $g \circ f$ is IF pre continuous.

(iii) Let B be an $IF\beta OS$ in Z. Since g is IF pre-β-irresolute, $g^{-1}(B)$ is an $IFPOS$ in Y. Now $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Since f is IF pre irresolute, $f^{-1}(g^{-1}(B))$ is $IFPOS$ in X. Hence $g \circ f$ is IF pre-β-irresolute.

§5. Preservation of some intuitionistic fuzzy topological structure

Definition 5.1. An $IFTS$ (X, τ) is $IF\beta$-disconnected if there exists intuitionistic fuzzy β-open sets A, B in X, $A \neq 0_\sim, B \neq 0_\sim$ such that $A \cup B = 1_\sim$ and $A \cap B = 0_\sim$. If X is not

$IF\beta$-disconnected then it is said to be $IF\beta$-connected.

Proposition 5.1. Let $f : (X, \tau) \to (Y, \sigma)$ be a IF pre-β-irresolute surjection, $(X,\ \tau)$ is an $IF\beta$-connected, then (Y, σ) is $IF\ \beta$-connected.

Proof. Assume that (Y,σ) is not $IF\ \beta$-connected then there exists nonempty intuitionistic fuzzy β-open sets A and B in (Y,σ) such that $A \cup B = 1_\sim$ and $A \cap B = 0_\sim$. Since f is IF pre-β-irresolute mapping, $C = f^{-1}(A) \neq 0_\sim$ and $D = f^{-1}(B) \neq 0_\sim$ which are intuitionistic fuzzy pre-open sets in X and hence intuitionistic Fuzzy β-open sets in X. And $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(1_\sim) = 1_\sim$ which implies $C \cup D = 1_\sim$.

$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(0_\sim) = 0_\sim$ which implies $C \cap D = 0_\sim$. Thus X is $IF\beta$-disconnected, which is a contradiction to our hypothesis. Hence Y is $IF\beta$-connected.

Definition 5.2. Let X be an $IFTS$. A family of $\{< x,\ \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$ intuitionistic fuzzy β-open sets in X satisfies the condition $1_\sim = \cup \{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$ is called a $IF\beta$-cover of X. A finite subfamily of a $IF\beta$-open cover $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$ of X which is also a $IF\beta$-open cover of X is called a finite subcover of $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J\}$.

Definition 5.3. A space X is called Intuitionistic Fuzzy β-compact ($Lindelof$) if every intuitionistic fuzzy β-open cover of X has a finite (countable) subcover.

Theorem 5.1. Every surjective Intuitionistic Fuzzy pre-β-irresolute image of a intuitionistic fuzzy precompact space is intuitionistic fuzzy β-compact.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be intuitionistic fuzzy pre-β-irresolute mapping of a intuitionistic fuzzy precompact space (X,τ) onto a space $\to (Y,\sigma)$. Let $\{G_i(i \in I)\}$ be any intuitionistic fuzzy-β-open cover of (Y,σ). Then $\{f^{-1}(G_i)(i \in I)\}$ is a intuitionistic fuzzy pre open cover of X. Since X is intuitionistic fuzzy precompact, there exists a finite subfamily $\{f^{-1}(G_{i_j})(j = 1, 2, \cdots, n)\}$ of $\{f^{-1}(G_i)(i \in I)\}$ which covers X. It follows that $\{G_{i_j}(j = 1, 2, \cdots, n)\}$ is a finite subfamily of $\{G_i(i \in I)\}$ which covers Y. Hence Y is intuitionistic fuzzy β-compact.

References

[1] K. T. Atanassov, Intuitionistic Fuzzy sets, Fuzzy Sets and Systems, **20**(1986), 87-96.

[2] C. L. Chang, Fuzzy topological spaces, J. Math. Anal.Appl., **24**(1968), 182-190.

[3] G. Balasubramanian, On fuzzy β-Compact spaces and fuzzy β-extremally disconneced spaces, Kybernetika, **33**(1997), 271-277.

[4] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, **88**(1997), 81-89.

[5] R. Dhavaseelan, E. Roja and M. K. Uma, Intuitionistic Fuzzy resolvable and intuitionistic fuzzy irresolvable spaces, Scientia Magna, **7**(2011), 59-67.

[6] H. Gurcay and D. Coker, On Fuzzy continuity in intuitionistic Fuzzy topological spaces, J. Fuzzy. Math., **5**(1997), 365-378.

[7] I. M. Hanafy, Completely continuous functions in intuitionistic fuzzy topological spaces, Czechoslovak Math. J., **53**(2003), No. 4, 793-803.

[8] J. K. Joen, Y. B. Jun and J. H. Park, Intuitionistic Fuzzy alpha-continuity and intuitionistic fuzzy precontinuity, Int. J. Math. Math. Sci., **19**(2005), 3091-3101.

[9] B. Krsteska and S. Abbas, Intuitionistic fuzzy strong preccompactness in Cokers sense, Mathematica Moravica, **11**(2007), 59-67.

[10] S. J. Lee and E. P. Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean. Math. Soc., **37**(2000), No. 1, 63-76.

[11] V. Seenivasan and R. Renuka, Intuitionistic Fuzzy pre-α-irresolute functions, Far East J. of Mathematical sciences, **72**(2013), No. 2, 251-267.

[12] S. S. Thakur and R. Chaturvedi, Regular generalized closed sets in intuitionistic fuzzy topological spaces, Universitatea Din Bacau, Studii Si Cercetari Stiintifice, Seria: Matematica, **16**(2006), 257-272.

[13] Young Bae Jun and Seok-Zun Song, Intuitionistic fuzzy semi-pre open sets and intuitionistic fuzzy semi-pre continuous mappings, J. Appl. Math and computing, **19**(2005), 467-474.

[14] L. A. Zadeh, Fuzzy Sets, Information and control, **8**(1965), 338-353.

[15] Zhen-Guo Xu and Fu-Gui Shi, Some weakly mappings on Intuitionistic fuzzy topological spaces, Tamkang Journal of Mathematics, **39**(2008), No. 1, 25-32.

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On some trigonometric and hyperbolic functions evaluated on circulant matrices

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Abstract In this paper, we investigate the outcome when the trigonometric functions $\sin x$, $\cos x$, e^x and the hyperbolic functions $\sinh x$ and $\cosh x$ are evaluated on circulant matrices.

Keywords circulant matrix, trigonometric functions, hyperbolic functions.
2000 Mathematics Subject Classification: 54D15, 54D10

§1. Introduction

Given any sequence of numbers, $c_0,\ c_1,\ \cdots, c_{n-1}$, we can form circulant matrices. From [2], circulant matrices have four types: the right circulant, the left circulant, the skew-right circulant and the skew-left circulant and take the following forms, respectively:

$$RCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}, \tag{1}$$

$$LCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \end{pmatrix}, \tag{2}$$

$$SRCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ -c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ -c_{n-2} & -c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_2 & -c_3 & -c_4 & \cdots & c_0 & c_1 \\ -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} & c_0 \end{pmatrix}, \tag{3}$$

$$SLCIRC_n(\vec{c}) = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & -c_0 \\ c_2 & c_3 & c_4 & \cdots & -c_0 & -c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-1} & -c_0 & \cdots & -c_{n-4} & -c_{n-3} \\ c_{n-1} & -c_0 & -c_1 & \cdots & -c_{n-3} & -c_{n-2} \end{pmatrix}, \tag{4}$$

In each matrix, $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$ is called the circulant vector.

The said matrices have significant applications in different fields. These fields include physics, image processing, probability and statistics, number theory, geometry, numerical solutions of ordinary and partial differential equations [2], frequency analysis, signal processing, digital encoding, graph theory [4], and time-series analysis [3].

§2. Preliminaries

In this section, we shall use $diag(c_0, c_1, \cdots, c_{n-1})$ to denote the diagonal matrix

$$\begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_{n-1} \end{pmatrix},$$

and $adiag(c_0, c_1, \cdots, c_{n-1})$ to denote the anti diagonal matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & c_0 \\ 0 & 0 & 0 & \cdots & c_1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_{n-2} & 0 & \cdots & 0 & 0 \\ c_{n-1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that these matrices are not necessarily circulant.

Now, we define the Fourier matrix to establish the relationship of the circulant matrices.

Definition 2.1. The unitary matrix F_n given by

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-2} & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-2)} & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{n-2} & \omega^{2(n-2)} & \cdots & \omega^{(n-2)(n-2)} & \omega^{(n-1)(n-2)} \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-2)(n-1)} & \omega^{(n-1)(n-1)} \end{pmatrix}, \quad (5)$$

where $\omega = e^{2i\pi/n}$ is called the Fourier matrix.

Here are the equations that show the relationship between the circulant matrices:

$$RCIRC_n(\vec{c}) = F_n D F_n^{-1}, \quad (6)$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$ and $D = diag(d_0, d_1, \cdots, d_{n-1})$.

$$LCIRC_n(\vec{c}) = \Pi RCIRC_n(\vec{c}), \quad (7)$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = diag(d_0, d_1, \cdots, d_{n-1})$, $\Pi = \begin{pmatrix} 1 & \mathbb{O}_1 \\ \mathbb{O}_1^T & \tilde{I}_{n-1} \end{pmatrix}$ (an $n * n$ matrix), $\tilde{I}_{n-1} = adiag\,(1, 1, \cdots, 1, 1)$ (an $(n-1) * (n-1)$ matrix), $\mathbb{O}_1 = (0, 0, \cdots, 0, 0)$ (an $(n-1) * 1$ matrix).

$$SRCIRC_n(\vec{c}) = \Delta F_n D F_n^{-1} \Delta^{-1}, \quad (8)$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = diag(d_0, d_1, \cdots, d_{n-1})$, $\Delta = diag\left(1, \theta, \cdots, \theta^{n-1}\right)$, and $\theta = e^{i\pi/n}$.

$$SLCIRC_n(\vec{c}) = \Sigma SRCIRC_n(\vec{c}), \quad (9)$$

where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = diag(d_0, d_1, \cdots, d_{n-1})$, $\Delta = diag\left(1, \theta, \cdots, \theta^{n-1}\right)$, $\Sigma = \begin{pmatrix} 1 & \mathbb{O}_1 \\ \mathbb{O}_1^T & -\tilde{I}_{n-1} \end{pmatrix}$ (an $n * n$ matrix), $\tilde{I}_{n-1} = adiag\,(1, 1, \cdots, 1, 1)$ (an $(n-1) * (n-1)$ matrix), $\mathbb{O}_1 = (0, 0, \cdots, 0, 0)$.

The matrices Π and Σ are orthogonal meaning $\Pi = \Pi^T = \Pi^{-1}$ and $\Sigma = \Sigma^T = \Sigma^{-1}$. In case of right multiplication with these matrices, we have the following results from [1]:

1. $RCIRC_n(\vec{c})\Pi = LCIRC_n(\vec{\gamma})$ where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$ and $\vec{\gamma} = (c_0, c_{n-1}, c_{n-2}, \cdots, c_2, c_1)$,

2. $SRCIRC_n(\vec{c})\Sigma = SLCIRC_n(\vec{\rho})$ where $\vec{c} = (c_0, c_1, \cdots, c_{n-1})$ and $\vec{\rho} = (c_0, -c_{n-1}, -c_{n-2}, \cdots, -c_2, -c_1)$.

§2. Preliminary results

For the rest of the paper we shall use the following notations:

1. The set of complex right circulant matrices

$$RCIRC_n(\mathbb{C}) = \{RCIRC_n(\vec{c}) |\ \vec{c} \in \mathbb{C}^n\},$$

2. The set of complex left circulant matrices

$$LCIRC_n(\mathbb{C}) = \{LCIRC_n(\vec{c}) |\ \vec{c} \in \mathbb{C}^n\},$$

3. The set of complex skew-right circulant matrices

$$SRCIRC_n(\mathbb{C}) = \{SRCIRC_n(\vec{c}) |\ \vec{c} \in \mathbb{C}^n\},$$

4. The set of complex skew-left circulant matrices

$$SLCIRC_n(\mathbb{C}) = \{SLCIRC_n(\vec{c}) |\ \vec{c} \in \mathbb{C}^n\},$$

5. $e^x = E[x]$,

6. $\sin x = S[x]$,

7. $\cos x = C[x]$,

8. $\sinh x = Sh[x]$,

9. $\cosh x = Ch[x]$.

From [2], it has been shown that:

1. the sum of circulant matrices of the same type is a circulant matrix of the same type,

2. the product of right circulant matrices is a right circulant matrix,

3. the product of skew-right circulant matrices is a skew-right circulant matrix.

For the other products, we have the following lemmas which will be used to prove our results:

Lemma 2.1. The product of two left circulant matrices is a right circulant matrix.

Proof.

$$\begin{aligned}
& LCIRC_n(\vec{a})LCIRC_n(\vec{b}) \\
= \ & \Pi RCIRC_n(\vec{a})\Pi RCIRC_n(\vec{b}) \\
= \ & \Pi LCIRC_n(\vec{\alpha})RCIRC_n(\vec{b}) \\
= \ & RCIRC_n(\vec{\alpha})RCIRC_n(\vec{b}) \in RCIRC_n(\mathbb{C}).
\end{aligned}$$

Lemma 2.2. The product of a left circulant and a right circulant is a left circulant matrix.

Proof.
Case 1

$$\begin{aligned}&RCIRC_n(\vec{a})LCIRC_n(\vec{b})\\=\ &RCIRC_n(\vec{a})\Pi RCIRC_n(\vec{b})\\=\ &LCIRC_n(\vec{\alpha})RCIRC_n(\vec{b})\\=\ &\Pi RCIRC_n(\vec{\alpha})RCIRC_n(\vec{b})\in LCIRC_n(\mathbb{C}).\end{aligned}$$

Case 2

$$\begin{aligned}&LCIRC_n(\vec{b})RCIRC_n(\vec{a})\\=\ &\Pi RCIRC_n(\vec{b})RCIRC_n(\vec{b})\in LCIRC_n(\mathbb{C}).\end{aligned}$$

Lemma 2.3. The product of two skew-left circulant matrices is a skew-right circulant matrix.

Proof.

$$\begin{aligned}&SLCIRC_n(\vec{a})SLCIRC_n(\vec{b})\\=\ &\Sigma SRCIRC_n(\vec{a})\Sigma SRCIRC_n(\vec{b})\\=\ &\Sigma SLCIRC_n(\vec{\alpha})SRCIRC_n(\vec{b})\\=\ &SRCIRC_n(\vec{\alpha})SRCIRC_n(\vec{b})\in SRCIRC_n(\mathbb{C}).\end{aligned}$$

Lemma 2.4. The product of a skew-left circulant and a skew-right circulant is a skew-left circulant matrix.

Proof.
Case 1

$$\begin{aligned}&SRCIRC_n(\vec{a})SLCIRC_n(\vec{b})\\=\ &SRCIRC_n(\vec{a})\Sigma SRCIRC_n(\vec{b})\\=\ &SLCIRC_n(\vec{\alpha})SRCIRC_n(\vec{b})\\=\ &\Sigma SRCIRC_n(\vec{\alpha})SRCIRC_n(\vec{b})\in SLCIRC_n(\mathbb{C}).\end{aligned}$$

Case 2

$$\begin{aligned}&SLCIRC_n(\vec{b})SRCIRC_n(\vec{a})\\=\ &\Sigma SRCIRC_n(\vec{b})SRCIRC_n(\vec{a})\in SLCIRC_n(\mathbb{C}).\end{aligned}$$

Lemma 2.5.

$$\text{If k is odd, } LCIRC_n^k(\vec{a})\in LCIRC_n^k(\mathbb{C});$$
$$\text{If k is even, } LCIRC_n^k(\vec{a})\in RCIRC_n^k(\mathbb{C}).$$

Lemma 2.6.

$$\text{If k is odd, } SLCIRC_n^k(\vec{a})\in SLCIRC_n^k(\mathbb{C});$$
$$\text{If k is even, } SLCIRC_n^k(\vec{a})\in SRCIRC_n^k(\mathbb{C}).$$

§3. Main results

Theorem 3.1. $E\left[RCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$\begin{aligned} E\left[RCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[RCIRC_n^k(\vec{c})\right]}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[F_n D F_n^{-1}\right]^k}{k!} \\ &= F_n \left[\sum_{k=0}^{+\infty} \frac{D^k}{k!}\right] F_n^{-1}. \end{aligned}$$

It takes the form of Eq. 6, so it is a right circulant matrix.

Theorem 3.2. $E\left[LCIRC_n(\vec{c})\right]$ is a sum of right circulant matrix and a left circulant matrix.

Proof.

$$\begin{aligned} E\left[LCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[LCIRC_n^k(\vec{c})\right]}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[LCIRC_n^{2k}(\vec{c})\right]}{(2k)!} + \sum_{k=0}^{+\infty} \frac{\left[LCIRC_n^{2k+1}(\vec{c})\right]}{(2k+1)!}. \end{aligned}$$

The first summand is a right circulant matrix because the powers of $LCIRC_n(\vec{c})$ are even while the second summand is a left circulant matrix because the powers of $LCIRC_(\vec{c})$ are odd.

Theorem 3.3. $E\left[SRCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$\begin{aligned} E\left[SRCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[SRCIRC_n^k(\vec{c})\right]}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[\Delta F_n D F_n^{-1} \Delta^{-1}\right]^k}{k!} \\ &= \Delta F_n \left[\sum_{k=0}^{+\infty} \frac{D^k}{k!}\right] F_n^{-1} \Delta^{-1}. \end{aligned}$$

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

Theorem 3.4. $E\left[SLCIRC_n(\vec{c})\right]$ is a sum of skew-right circulant matrix and a skew-left circulant matrix.

Proof.

$$\begin{aligned} E\left[SLCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[SLCIRC_n^k(\vec{c})\right]}{k!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[SLCIRC_n^{2k}(\vec{c})\right]}{(2k)!} + \sum_{k=0}^{+\infty} \frac{\left[SLCIRC_n^{2k+1}(\vec{c})\right]}{(2k+1)!}. \end{aligned}$$

The first summand is a skew-right circulant matrix because the powers of $SLCIRC_n(\vec{c})$ are even while the second summand is a skew-left circulant matrix because the powers of $SLCIRC_(\vec{c})$ are odd.

Theorem 3.5. $S\left[RCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$\begin{aligned} S\left[RCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty}(-1)^k \frac{\left[RCIRC_n^{2k+1}\right]}{(2k+1)!} \\ &= \sum_{k=0}^{+\infty}(-1)^k \frac{\left[F_n D F_n^{-1}\right]^{2k+1}}{(2k+1)!} \\ &= F_n \left[\sum_{k=0}^{+\infty}(-1)^k \frac{D^{2k+1}}{(2k+1)!}\right] F_n^{-1}. \end{aligned}$$

It takes the form of Eq. 6, so it is a right circulant matrix.

Theorem 3.6. $S\left[LCIRC_n(\vec{c})\right]$ is a left circulant matrix.

Proof.

$$S\left[LCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty}(-1)^k \frac{\left[LCIRC_n^{2k+1}\right]}{(2k+1)!}.$$

Since the powers of $LCIRC_n(vecc)$ are odd, it is a left circulant matrix.

Theorem 3.7. $S\left[SRCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$\begin{aligned} S\left[SRCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty}(-1)^k \frac{\left[SRCIRC_n^{2k+1}\right]}{(2k+1)!} \\ &= \sum_{k=0}^{+\infty}(-1)^k \frac{\left[\Delta F_n D F_n^{-1}\Delta^{-1}\right]^{2k+1}}{(2k+1)!} \\ &= \Delta F_n \left[\sum_{k=0}^{+\infty}(-1)^k \frac{D^{2k+1}}{(2k+1)!}\right] F_n^{-1}\Delta^{-1}. \end{aligned}$$

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

Theorem 3.8. $S\left[SLCIRC_n(\vec{c})\right]$ is a skew-left circulant matrix.

Proof.

$$S\left[SLCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty}(-1)^k \frac{\left[SLCIRC_n^{2k+1}\right]}{(2k+1)!}.$$

Since the powers of $SLCIRC_n(vecc)$ are odd, it is a left circulant matrix.

Theorem 3.9. $C\left[RCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$\begin{aligned} C\left[RCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty}(-1)^k\frac{RCIRC_n^{2k}(\vec{c})}{(2k)!} \\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{\left[F_nDF_n^{-1}\right]^{2k}}{(2k)!} \\ &= F_n\left[\sum_{k=0}^{+\infty}(-1)^k\frac{D^{2k}}{(2k)!}\right]F_n^{-1}. \end{aligned}$$

It takes the form Eq. 6, so it is a right circulant matrix.

Theorem 3.10. $C\left[LCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$C\left[LCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty}(-1)^k\frac{LCIRC_n^{2k}(\vec{c})}{(2k)!}.$$

Since the powers of $LCIRC_n(\vec{c})$ are even, it is a right circulant matrix.

Theorem 3.11. $C\left[SRCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$\begin{aligned} C\left[SRCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty}(-1)^k\frac{SRCIRC_n^{2k}(\vec{c})}{(2k)!} \\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{\left[\Delta F_nDF_n^{-1}\Delta^{-1}\right]^{2k}}{(2k)!} \\ &= \Delta F_n\left[\sum_{k=0}^{+\infty}(-1)^k\frac{D^{2k}}{(2k)!}\right]F_n^{-1}\Delta^{-1}. \end{aligned}$$

It takes the form Eq. 8, so it is a right circulant matrix.

Theorem 3.12. $C\left[SLCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$C\left[SLCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty}(-1)^k\frac{SLCIRC_n^{2k}(\vec{c})}{(2k)!}.$$

Since the powers of $SLCIRC_n(\vec{c})$ are even, it is a skew-right circulant matrix.

Theorem 3.13. $Sh\left[RCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$\begin{aligned} Sh\left[RCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty}\frac{\left[RCIRC_n^{2k+1}(\vec{c})\right]}{(2k+1)!} \\ &= \sum_{k=0}^{+\infty}\frac{\left[F_nDF_n^{-1}\right]^{2k+1}}{(2k+1)!} \\ &= F_n\left[\sum_{k=0}^{+\infty}\frac{D^{2k+1}}{(2k+1)!}\right]F_n^{-1}. \end{aligned}$$

It takes the form of Eq. 6, so it is a right circulant matrix.

Theorem 3.14. $Sh\left[LCIRC_n(\vec{c})\right]$ is a left circulant matrix.

Proof.

$$Sh\left[LCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty} \frac{\left[LCIRC_n^{2k+1}(\vec{c})\right]}{(2k+1)!}.$$

Because the powers of $LCIRC_n(\vec{c})$ are odd, it is a left circulant matrix.

Theorem 3.15. $Sh\left[SRCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$\begin{aligned} Sh\left[SRCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[SRCIRC_n^{2k+1}(\vec{c})\right]}{(2k+1)!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[\Delta F_n D F_n^{-1} \Delta^{-1}\right]^{2k+1}}{(2k+1)!} \\ &= \Delta F_n \left[\sum_{k=0}^{+\infty} \frac{D^{2k+1}}{(2k+1)!}\right] F_n^{-1} \Delta^{-1}. \end{aligned}$$

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

Theorem 3.16. $Sh\left[SLCIRC_n(\vec{c})\right]$ is a skew-left circulant matrix.

Proof.

$$Sh\left[SLCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty} \frac{\left[SLCIRC_n^{2k+1}(\vec{c})\right]}{(2k+1)!}.$$

Because the powers of $SLCIRC_n(\vec{c})$ are odd, it is a skew-left circulant matrix.

Theorem 3.17. $Ch\left[RCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$\begin{aligned} Ch\left[RCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[RCIRC_n^{2k}\right]}{(2k)!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[F_n D F_n^{-1}\right]^{2k}}{(2k)!} \\ &= F_n \left[\sum_{k=0}^{+\infty} \frac{D^{2k}}{(2k)!}\right] F_n^{-1}. \end{aligned}$$

It takes the form of Eq. 6, so it is a right circulant matrix.

Theorem 3.18. $Ch\left[LCIRC_n(\vec{c})\right]$ is a right circulant matrix.

Proof.

$$Ch\left[LCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty} \frac{\left[LCIRC_n^{2k}\right]}{(2k)!}.$$

Because the powers of $LCIRC_n(\vec{c})$ are even, it is a right circulant matrix.

Theorem 3.19. $Ch\left[SRCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$\begin{aligned} Ch\left[SRCIRC_n(\vec{c})\right] &= \sum_{k=0}^{+\infty} \frac{\left[SRCIRC_n^{2k}\right]}{(2k)!} \\ &= \sum_{k=0}^{+\infty} \frac{\left[\Delta F_n D F_n^{-1} \Delta^{-1}\right]^{2k}}{(2k)!} \\ &= \Delta F_n \left[\sum_{k=0}^{+\infty} \frac{D^{2k}}{(2k)!}\right] F_n^{-1} \Delta^{-1}. \end{aligned}$$

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

Theorem 3.20. $Ch\left[SLCIRC_n(\vec{c})\right]$ is a skew-right circulant matrix.

Proof.

$$Ch\left[SLCIRC_n(\vec{c})\right] = \sum_{k=0}^{+\infty} \frac{\left[SLCIRC_n^{2k}\right]}{(2k)!}.$$

Because the powers of $SLCIRC_n(\vec{c})$ are even, it is a skew-right circulant matrix.

§4. Conclusion

Right and skew-right circulant matrices remains invariant on their type when evaluated on trigonometric and hyperbolic functions that are used in this paper. On the other hand, the left and skew-left circulant matrices change their type depending on the parity of the trigonometric and hyperbolic function.

References

[1] A. Bueno, Operations and inversions on circulant matrices, masters thesis, University of the Philippines Baguio.

[2] H. Karner, J. Schneid and C. Ueberhuber, Spectral decomposition of real circulant matrices, Linear Algebra and Its Applications, **367**(2003), 301-311.

[3] D. Pollock, circulant matrices and time-Series analysis, Queen Mary, University of London, August 2001.

[4] A. Wyn-Jones, Circulants, Carlisle, Pennsylvania.

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Generalization of an intuitionistic fuzzy $\mathscr{G}_{str}$ open sets in an intuitionistic fuzzy grill structure spaces

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Abstract The purpose of this paper is to introduce the concepts of an intuitionistic fuzzy grill, intuitionistic fuzzy $\mathscr{G}$ structure space, intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set and intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set. The concepts of an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$T_i$ space, $i = 0,\ 1,\ 2$ and intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-*co*-closed graphs are defined. Some interesting properties are established.

Keywords Intuitionistic fuzzy grill, intuitionistic fuzzy $\mathscr{G}$ structure space, intuitionistic fuz -zy $\delta_{\mathscr{G}_{str}}$ set and intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$, ($\alpha_{\mathscr{G}_{str}}$, semi$_{\mathscr{G}_{str}}$, pre$_{\mathscr{G}_{str}}$, regular$_{\mathscr{G}_{str}}$ and $\beta_{\mathscr{G}_{str}}$) open set, intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ exterior, intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$T_i$ space, $i = 0,\ 1,\ 2$ and intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-*co*-closed graphs.
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§1. Introduction and preliminaries

The concept of fuzzy sets was introduced by Zadeh [7] and later Atanassov [1] generalized the idea to intuitionistic fuzzy sets. On the other hand, Coker [2] introduced the notions of an intuitionistic fuzzy topological spaces, intuitionistic fuzzy continuity and some other related concepts. The concept of an intuitionistic fuzzy α-closed set was introduced by H. Gurcay and D. Coker [5]. The concept of fuzzy grill was introduced by Sumita Das, M. N. Mukherjee [6]. Erdal Ekici [4] studied slightly precontinuous functions, separation axioms and pre-*co*-closed graphs in fuzzy topological space. In this paper, the concepts of an intuitionistic fuzzy grill, intuitionistic fuzzy $\mathscr{G}$ structure space, intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set and intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set are introduced. Some interesting properties of separation axioms in intuitionistic fuzzy grill structure space with intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$continuous function are established.

§2. Preliminaries

Definition 2.1.[1] Let X be a nonempty fixed set and I be the closed interval $[0, 1]$. An intuitionistic fuzzy set (IFS) A is an object of the following form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in$

$X\}$, where the mappings $\mu_A : X \longrightarrow I$ and $\gamma_A : X \longrightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously, every fuzzy set A on a nonempty set X is an IFS of the following form, $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 2.2.[1] Let X be a nonempty set and the IFSs A and B in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,

(ii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$.

Definition 2.3.[1] The IFSs $0_\sim$ and $1_\sim$ are defined by $0_\sim$=$\{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim$=$\{\langle x, 1, 0\rangle : x \in X\}$.

Definition 2.4.[2] An intuitionistic fuzzy topology (IFT) in Coker's sense on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$,

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \tau$.

In this case the ordered pair (X, τ) is called an intuitionistic fuzzy topological space ($IFTS$) on X and each IFS in τ is called an intuitionistic fuzzy open set ($IFOS$). The complement $\overline{A}$ of an $IFOS$ A in X is called an intuitionistic fuzzy closed set ($IFCS$) in X.

Definition 2.5.[2] Let A be an IFS in $IFTS$ X. Then $int(A) = \bigcup\{G \mid G$ is an $IFOS$ in X and $G \subseteq A\}$ is called an intuitionistic fuzzy interior of A; $clA = \bigcap\{G \mid G$ is an $IFCS$ in X and $G \supseteq A\}$ is called an intuitionistic fuzzy closure of A.

Proposition 2.6.[1] For any IFS A in (X, τ) we have

(i) $cl(\overline{A}) = \overline{int(A)}$,

(ii) $int(\overline{A}) = \overline{cl(A)}$.

Corollary 2.1.[2] Let $A, A_i (i \in J)$ be IFSs in $X, B, B_j (j \in K)$ IFSs in Y and $f : X \to Y$ a function. Then

(i) $A \subseteq f^{-1}(f(A))$ (If f is injective, then $A = f^{-1}(f(A))$),

(ii) $f(f^{-1}(B)) \subseteq B$ (If f is surjective, then $f(f^{-1}(B)) = B$),

(iii) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$,

(iv) $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$,

(v) $f^{-1}(1_\sim) = 1_\sim$,

(vi) $f^{-1}(0_\sim) = 0_\sim$,

(vii) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Definition 2.7.[3] Let X be a nonempty set and $x \in X$ a fixed element in X. If $r \in I_0$, $s \in I_1$ are fixed real numbers such that $r + s \leq 1$, then the IFS $x_{r,s} = \langle x, x_r, 1 - x_{1-s}\rangle$ is called an intuitionistic fuzzy point (IFP) in X, where r denotes the degree of membership of $x_{r,s}$, s denotes the degree of nonmembership of $x_{r,s}$ and $x \in X$ the support of $x_{r,s}$. The $IFPx_{r,s}$ is contained in the IFS $A(x_{r,s} \in A)$ if and only if $r < \mu_A(x), s > \gamma_A(x)$.

Definition 2.8.[4] An IFSs A and B are said to be quasi coincident with the the $IFSA$, denoted by AqB if and only if there exists an element $x \in X$ such that $\mu_A(x) > \gamma_B(x)$ or $\gamma_A(x) < \mu_B(x)$. If A is not quasi coincident with B, denoted $A\widetilde{q}B$.

Definition 2.9.[5] Let A be an IFS of an $IFTS$ X. Then A is called an intuitionistic fuzzy α-open set ($IF\alpha OS$) if $A \subseteq int(cl(int(A)))$. The complement of an intuitionistic fuzzy α-open set is called an intuitionistic fuzzy α-closed set ($IF\alpha CS$).

§3. Intuitionistic fuzzy operators with respect to an intuitionistic fuzzy grills

Definition 3.1. Let ζ^X be the collection of all intuitionistic fuzzy sets in X. A collection $\mathscr{S} \subseteq \zeta^X$ is said be an intuitionistic fuzzy stack on X if $A \subseteq B$ and $A \in \mathscr{S}$ then $B \in \mathscr{S}$.

Definition 3.2. Let ζ^X be the collection of all intuitionistic fuzzy sets in X. An intuitionistic fuzzy grill $\mathscr{G}$ on X is an intuitionistic fuzzy stack on X if $\mathscr{G}$ satisfies the following conditions:

(i) $0_\sim \notin \mathscr{G}$,

(ii) If $A, B \in \zeta^X$ and $A \cup B \in \mathscr{G}$, then $A \in \mathscr{G}$ or $B \in \mathscr{G}$.

Definition 3.3. Let (X, T) be an intuitionistic fuzzy topological space and let ζ^X be the collection of all intuitionistic fuzzy sets in X. Let $\mathscr{G}$ be an intuitionistic fuzzy grill on X. A function $\Phi_{\mathscr{G}} : \zeta^X \to \zeta^X$ is defined by

$$\Phi_{\mathscr{G}}(A) = \cup\{IFint(\overline{A}) \cap \overline{U} \mid AqU, A \cap U \in \mathscr{G}, U \in T\},$$

for each $A \in \zeta^X$. The function $\Phi_{\mathscr{G}}$ is an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathscr{G}$ and an intuitionistic fuzzy topology T.

Remark 3.1. Let (X, T) be an intuitionistic fuzzy topological space. Let $\Phi_{\mathscr{G}}$ be an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathscr{G}$ and an intuitionistic fuzzy topology T. Then

(i) $\Phi_{\mathscr{G}}(0_\sim) = 0_\sim = \Phi_{\mathscr{G}}(1_\sim)$,

(ii) If $A, B \in \zeta^X$ and $A \subseteq B$, then $\Phi_{\mathscr{G}}(A) \subseteq \Phi_{\mathscr{G}}(B)$.

Proof. The proof of (i) and (ii) are follows from the definition of $\Phi_{\mathscr{G}}$.

Definition 3.4. Let (X, T) be an intuitionistic fuzzy topological space and let $\mathscr{G}$ be an intuitionistic fuzzy grill on X. Let $\Phi_{\mathscr{G}}$ be an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathscr{G}$ and an intuitionistic fuzzy topology T. A function $\Psi_{\mathscr{G}} : T \to \zeta^X$ is defined by $\Psi_{\mathscr{G}}(A) = A \cup IFint(\overline{\Phi_{\mathscr{G}}})$ for each $A \in T$. The function $\Psi_{\mathscr{G}}$ is an intuitionistic fuzzy operator associated with $\Phi_{\mathscr{G}}$.

Definition 3.5. Let $(X,\ T)$ be an intuitionistic fuzzy topological space and let $\mathscr{G}$ be an intuitionistic fuzzy grill on X. Let $\Psi_{\mathscr{G}}$ be an intuitionistic fuzzy operator associated with $\Phi_{\mathscr{G}}$. A collection $\mathscr{G}_{str} = \{A \mid \Psi_{\mathscr{G}}(A) = A\} \cup \{1_\sim\}$ is said to be an intuitionistic fuzzy $\mathscr{G}$ structure on X. Then $(X, \mathscr{G}_{str})$ is said to be an intuitionistic fuzzy $\mathscr{G}$ structure space. Every member of $\mathscr{G}_{str}$ is an intuitionistic fuzzy $\mathscr{G}_{str}$ open set(in short, $IF\mathscr{G}_{str}OS$) and the complement of an intuitionistic fuzzy $\mathscr{G}_{str}$ open set is an intuitionistic fuzzy $\mathscr{G}_{str}$ closed set (in short, $IF\mathscr{G}_{str}CS$).

Definition 3.6. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space and let $A \in \zeta^X$. Then

(i) the intuitionistic fuzzy $\mathscr{G}_{str}$ closure of A is denoted and defined by

$$IF_{\mathscr{G}_{str}}cl(A) = \cap\{B \in \zeta^X \mid B \supseteq A \text{ and } \overline{B} \in \mathscr{G}_{str}\},$$

(ii) the intuitionistic fuzzy $\mathscr{G}_{str}$ interior of A is denoted and defined by

$$IF_{\mathscr{G}_{str}}int(A) = \cup\{B \in \zeta^X \mid B \subseteq A \text{ and } B \in \mathscr{G}_{str}\}.$$

Remark 3.1. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. For any $A,\ B \in \zeta^X$,

(i) $IF_{\mathscr{G}_{str}}cl(A) = A$ if and only if A is an intuitionistic fuzzy $\mathscr{G}_{str}$ closed set,

(ii) $IF_{\mathscr{G}_{str}}int(A) = A$ if and only if A is an intuitionistic fuzzy $\mathscr{G}_{str}$ open set,

(iii) $IF_{\mathscr{G}_{str}}int(A) \subseteq A \subseteq IF_{\mathscr{G}_{str}}cl(A)$,

(iv) $IF_{\mathscr{G}_{str}}int(1_\sim) = 1_\sim = IF_{\mathscr{G}_{str}}cl(1_\sim)$ and $IF_{\mathscr{G}_{str}}int(0_\sim) = 0_\sim = IF_{\mathscr{G}_{str}}cl(0_\sim)$,

(v) $IF_{\mathscr{G}_{str}}int(\overline{A}) = \overline{IF_{\mathscr{G}_{str}}cl(A)}$ and $IF_{\mathscr{G}_{str}}cl(\overline{A}) = \overline{IF_{\mathscr{G}_{str}}int(A)}$.

§4. Properties of an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set and $\alpha_{\mathscr{G}_{str}}$ open set in an intuitionistic fuzzy $\mathscr{G}$ structure spaces

Definition 4.1. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space and let $A \in \zeta^X$. Then A is said be an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set (in short, $IF\delta_{\mathscr{G}_{str}}S$) if

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A)).$$

Proposition 4.1. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. Then

(i) The complement of an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set,

(ii) Finite union of intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ sets is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set,

(iii) Finite intersection of intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ sets is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set,

(iv) Every intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ sets is an intuitionistic fuzzy $\mathscr{G}_{str}$ open set.

Proof. (i) Let A be an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set. Then

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A)).$$

Taking complement on both sides, we have

$$\overline{IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A))} \supseteq \overline{IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A))},$$

$$IF_{\mathscr{G}_{str}}cl(\overline{IF_{\mathscr{G}_{str}}cl(A)}) \supseteq IF_{\mathscr{G}_{str}}int(\overline{IF_{\mathscr{G}_{str}}int(A)}),$$

$$IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}cl(\overline{A})) \supseteq IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}int(\overline{A})).$$

Hence $\overline{A}$ is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set.

(ii) Let A and B be any two intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ sets. Then

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A)), \tag{1}$$

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(B)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(B)). \quad (2)$$

Now, taking union of 3.1 and 3.2, we have

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) \cup IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(B))$$

$$\subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A)) \cup IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(B)),$$

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A) \cup IF_{\mathscr{G}_{str}}cl(B)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A) \cup IF_{\mathscr{G}_{str}}int(B)),$$

$$IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A \cup B)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A \cup B)).$$

Hence $A \cup B$ is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set.

(iii) The proof is obvious by taking complement of (ii).

(iv) It is obvious.

Note 4.1. In general, arbitrary union of an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set needs not to be intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set.

Definition 4.2. Let $(X, \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space and let $A \in \zeta^X$. Then A is said be an

(i) intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ open set (in short, $IFS_{\mathscr{G}_{str}}OS$) if $A \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A))$. The complement of an intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ open set is said to be an intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ closed set (in short, $IFS_{\mathscr{G}_{str}}CS$).

(ii) intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$ open set (in short, $IFP_{\mathscr{G}_{str}}OS$) if $A \subseteq IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A))$. The complement of an intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$ open set is said to be an intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$ closed set (in short, $IFP_{\mathscr{G}_{str}}CS$).

(iii) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set (in short, $IF\alpha_{\mathscr{G}_{str}}OS$) if

$$A \subseteq IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A))).$$

The complement of an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set is said to be an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ closed set (in short, $IF\alpha_{\mathscr{G}_{str}}CS$).

(iv) intuitionistic fuzzy $\beta_{\mathscr{G}_{str}}$ open set (in short, $IF\beta_{\mathscr{G}_{str}}OS$) if

$$A \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A))).$$

The complement of an intuitionistic fuzzy $\beta_{\mathscr{G}_{str}}$ open set is said to be an intuitionistic fuzzy $\beta_{\mathscr{G}_{str}}$ closed set (in short, $IF\beta_{\mathscr{G}_{str}}CS$).

(v) intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set (in short, $IFR_{\mathscr{G}_{str}}OS$) if

$$A = IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)).$$

The complement of an intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set is said to be an intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ closed set (in short, $IFR_{\mathscr{G}_{str}}CS$).

Note 4.2. The family of all intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$(resp. pre$_{\mathscr{G}_{str}}$, $\alpha_{\mathscr{G}_{str}}$, $\beta_{\mathscr{G}_{str}}$ and regular$_{\mathscr{G}_{str}}$) open sets are denoted by $IFS_{\mathscr{G}_{str}}O(X)$ (resp. $IFP_{\mathscr{G}_{str}}O(X)$, $IF\alpha_{\mathscr{G}_{str}}O(X)$, $IF\beta_{\mathscr{G}_{str}}O(X)$ and $IFR_{\mathscr{G}_{str}}O(X)$).

Definition 4.3. Let $(X, \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space and let $A \in \zeta^X$. Then

(i) the intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ closure of A is denoted and defined by

$$IF_{\alpha_{\mathscr{G}_{str}}}cl(A) = \cap\{B \in \zeta^X \mid B \supseteq A \text{ and } \overline{B} \in IF\alpha_{\mathscr{G}_{str}}O(X)\};$$

(ii) the intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ interior of A is denoted and defined by

$$IF_{\alpha_{\mathscr{G}_{str}}}int(A) = \cup\{B \in \zeta^X \mid B \subseteq A \text{ and } B \in IF\alpha_{\mathscr{G}_{str}}O(X)\}.$$

Remark 4.1. Let $(X, \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. For any $A, B \in \zeta^X$,

(i) $IF_{\alpha_{\mathscr{G}_{str}}}cl(A) = A$ if and only if A is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ closed set,

(ii) $IF_{\alpha_{\mathscr{G}_{str}}}int(A) = A$ if and only if A is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set,

(iii) $IF_{\alpha_{\mathscr{G}_{str}}}int(A) \subseteq A \subseteq IF_{\alpha_{\mathscr{G}_{str}}}cl(A)$,

(iv) $IF_{\alpha_{\mathscr{G}_{str}}}int(1_\sim) = 1_\sim = IF_{\alpha_{\mathscr{G}_{str}}}cl(1_\sim)$ and $IF_{\alpha_{\mathscr{G}_{str}}}int(0_\sim) = 0_\sim = IF_{\alpha_{\mathscr{G}_{str}}}cl(0_\sim)$,

(v) $IF_{\alpha_{\mathscr{G}_{str}}}int(\overline{A}) = \overline{IF_{\alpha_{\mathscr{G}_{str}}}cl(A)}$ and $IF_{\alpha_{\mathscr{G}_{str}}}cl(\overline{A}) = \overline{IF_{\alpha_{\mathscr{G}_{str}}}int(A)}$.

Proposition 4.2. Let $(X, \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. Then

(i) Every intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set is an intuitionistic fuzzy $\mathscr{G}_{str}$ open set;

(ii) Every intuitionistic fuzzy intuitionistic fuzzy $\mathscr{G}_{str}$ open set is an intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ (resp. pre$_{\mathscr{G}_{str}}$, $\alpha_{\mathscr{G}_{str}}$ and $\beta_{\mathscr{G}_{str}}$)open set.

Proof. It is obvious.

Remark 4.2. The converse of the proposition 4.2 needs not to be true as shown in example 4.1.

Example 4.1. Let $X = \{a, b\}$ be a nonempty set. Let $A = \langle x, (\frac{a}{0.3}, \frac{b}{0.5}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle, B = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4})\rangle, C = \langle x, (\frac{a}{0.2}, \frac{b}{0.2}), (\frac{a}{0.7}, \frac{b}{0.5})\rangle$ and $E = \langle x, (\frac{a}{0.7}, \frac{b}{0.8}), (\frac{a}{0.2}, \frac{b}{0.2})\rangle$ be intuitionistic fuzzy sets on X. The family $T = \{0_\sim, 1_\sim, A, B, C, D\}$ is an intuitionistic fuzzy topology on X and the family $\mathscr{G} = \{G \in \zeta^X \mid 0.2 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.8\}$ is an intuitionistic fuzzy grill on X. Then the family $\mathscr{G}_{str} = \{0_\sim, 1_\sim, A\}$ is an intuitionistic fuzzy $\mathscr{G}$ structure on X. Therefore, $(X, \mathscr{G}_{str})$ is an intuitionistic fuzzy $\mathscr{G}$ structure space. Now,

(i) $F = \langle x, (\frac{a}{0.3}, \frac{b}{0.5}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$ is an intuitionistic fuzzy $\mathscr{G}_{str}$ open set but needs not to be an intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set in X.

(ii) $H = \langle x, (\frac{a}{0.3}, \frac{b}{0.6}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$(resp. semi$_{\mathscr{G}_{str}}$) open set but needs not to be an intuitionistic fuzzy $\mathscr{G}_{str}$ open set in X.

(iii) $K = \langle x, (\frac{a}{0.3}, \frac{b}{0.7}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle$ is an intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$(resp. $\beta_{\mathscr{G}_{str}}$) open set but needs not to be an intuitionistic fuzzy $\mathscr{G}_{str}$ open set in X.

Proposition 4.3. Every intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set.

Proof. Let A be an intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set. Then $IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) = A \subseteq IF_{\mathscr{G}_{str}}cl(A)$. Since every intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set is an intuitionistic fuzzy $\mathscr{G}_{str}$ open set, $IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A))$. Hence A is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set.

Remark 4.3. The converse of the proposition 4.3 needs not to be true as shown in example 4.2.

Example 4.2. Let $X = \{a, b\}$ be a nonempty set. Let $A = \langle x, (\frac{a}{0.3}, \frac{b}{0.5}), (\frac{a}{0.2}, \frac{b}{0.3})\rangle, B = \langle x, (\frac{a}{0.3}, \frac{b}{0.4}), (\frac{a}{0.2}, \frac{b}{0.4})\rangle, C = \langle x, (\frac{a}{0.2}, \frac{b}{0.2}), (\frac{a}{0.7}, \frac{b}{0.5})\rangle$ and $E = \langle x, (\frac{a}{0.7}, \frac{b}{0.8}), (\frac{a}{0.2}, \frac{b}{0.2})\rangle$ be intuitionistic fuzzy sets on X. The family $T = \{0_\sim, 1_\sim, A, B, C, D\}$ is an intuitionistic fuzzy topology on X and the family $\mathscr{G} = \{G \in \zeta^X \mid 0.2 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.8\}$ is an intuitionistic fuzzy grill on X. Then the family $\mathscr{G}_{str} = \{0_\sim, 1_\sim, A\}$ is an intuitionistic fuzzy $\mathscr{G}$ structure on X. Therefore, $(X,\ \mathscr{G}_{str})$ is an intuitionistic fuzzy $\mathscr{G}$ structure space. Now, $F = \langle x, (\frac{a}{0.3}, \frac{b}{0.5}), (\frac{a}{0.2}, \frac{b}{0.2})\rangle$ is an intuitionistic fuzzy $\delta_{\mathscr{G}_{str}}$ set but needs not to be an intuitionistic fuzzy regular$_{\mathscr{G}_{str}}$ open set in X.

Proposition 4.4. Every intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set is an intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ open set.

Proof. Let A be an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set. Then

$$A \subseteq IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A))) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(A)).$$

Hence A is an intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ open set.

Remark 4.4. The converse of the proposition 4.4 needs not to be true as shown in example 4.3.

Proposition 4.5. Every intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$ open set is an intuitionistic fuzzy $\beta_{\mathscr{G}_{str}}$ open set.

Proof. Let A be an intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$ set. Then

$$A \subseteq IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A)) \subseteq IF_{\mathscr{G}_{str}}cl(IF_{\mathscr{G}_{str}}int(IF_{\mathscr{G}_{str}}cl(A))).$$

Hence A is an intuitionistic fuzzy $\beta_{\mathscr{G}_{str}}$ open set.

Remark 4.5. The converse of the proposition 4.5 needs not to be true as shown in example 4.3.

Example 4.3. Let $X = \{a,\ b\}$ be a nonempty set. Let

$A_1 = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.7}, \frac{b}{0.7})\rangle$, $A_2 = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}), (\frac{a}{0.2}, \frac{b}{0.4})\rangle$, $A_3 = \langle x, (\frac{a}{0.7}, \frac{b}{0.6}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$,

$A_4 = \langle x, (\frac{a}{0.1}, \frac{b}{0.8}), (\frac{a}{0.9}, \frac{b}{0.2})\rangle$, $A_5 = \langle x, (\frac{a}{0.1}, \frac{b}{0.3}), (\frac{a}{0.9}, \frac{b}{0.7})\rangle$, $A_6 = \langle x, (\frac{a}{0.7}, \frac{b}{0.8}), (\frac{a}{0.3}, \frac{b}{0.2})\rangle$,

$A_7 = \langle x, (\frac{a}{0.7}, \frac{b}{0.6}), (\frac{a}{0.2}, \frac{b}{0.4})\rangle$, $A_8 = \langle x, (\frac{a}{0.7}, \frac{b}{0.8}), (\frac{a}{0.2}, \frac{b}{0.2})\rangle$, $A_9 = \langle x, (\frac{a}{0.4}, \frac{b}{0.8}), (\frac{a}{0.2}, \frac{b}{0.2})\rangle$,

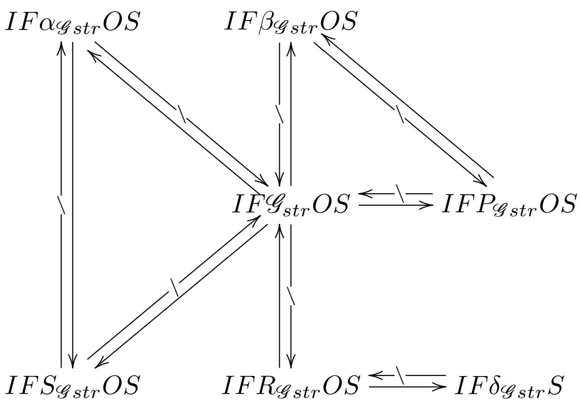
$A_{10} = \langle x, (\frac{a}{0.2}, \frac{b}{0.8}), (\frac{a}{0.7}, \frac{b}{0.2})\rangle$, $A_{11} = \langle x, (\frac{a}{0.1}, \frac{b}{0.6}), (\frac{a}{0.9}, \frac{b}{0.4})\rangle$, $A_{12} = \langle x, (\frac{a}{0.4}, \frac{b}{0.8}), (\frac{a}{0.3}, \frac{b}{0.2})\rangle$ and $A_{13} = \langle x, (\frac{a}{0.4}, \frac{b}{0.6}), (\frac{a}{0.3}, \frac{b}{0.4})\rangle$ be intuitionistic fuzzy sets on X.

The family $T = \{0_\sim, 1_\sim, A_i, i = 1, 2, \cdots, 13\}$ is an intuitionistic fuzzy topology on X and the family $\mathscr{G} = \{G \in \zeta^X \mid 0.1 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.9\}$ is an intuitionistic fuzzy grill on X. Then the family $\mathscr{G}_{str} = \{0_\sim, 1_\sim, A_1, A_2, A_3, A_7\}$ is an intuitionistic fuzzy $\mathscr{G}$ structure on X. Therefore, $(X,\ \mathscr{G}_{str})$ is an intuitionistic fuzzy $\mathscr{G}$ structure space. Now,

(i) $B = \langle x, (\frac{a}{0.7}, \frac{b}{0.7}), (\frac{a}{0.3}, \frac{b}{0.3})\rangle$ is an intuitionistic fuzzy semi$_{\mathscr{G}_{str}}$ open set but needs not to be an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set in X.

(ii) $C = \langle x, (\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.4}, \frac{b}{0.6})\rangle$ is an intuitionistic fuzzy $\beta_{\mathscr{G}_{str}}$ open set but needs not to be an intuitionistic fuzzy pre$_{\mathscr{G}_{str}}$ open set in X.

Remark 4.6. From the diagram, the following implications hold:



§5. Separation axioms in an intuitionistic fuzzy $\mathscr{G}$ structure space

Definition 5.1. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. Then an intuitionistic fuzzy set A is said to be an intuitionistic fuzzy $\mathscr{G}_{str}$ (resp. $\alpha_{\mathscr{G}_{str}}$) clopen set if and only if it is both intuitionistic fuzzy $\mathscr{G}_{str}$ open (resp. $\alpha_{\mathscr{G}_{str}}$) and intuitionistic fuzzy $\mathscr{G}_{str}$ closed (resp. $\alpha_{\mathscr{G}_{str}}$).

Definition 5.2. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. Then an intuitionistic fuzzy set A is said to be an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ exterior of A if $IFExt_{\alpha_{\mathscr{G}_{str}}}(A) = IF_{\alpha_{\mathscr{G}_{str}}}int(\overline{A})$.

Remark 5.1. Let $(X,\ \mathscr{G}_{str})$ be an intuitionistic fuzzy $\mathscr{G}$ structure space. Then

(i) If A is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ clopen set, then $IFExt_{\alpha_{\mathscr{G}_{str}}}(A) = A = \overline{A}$;

(ii) If $A \subseteq B$ then $IFExt_{\alpha_{\mathscr{G}_{str}}}(A) \supseteq IFExt_{\alpha_{\mathscr{G}_{str}}}(B)$;

(iii) $IFExt_{\alpha_{\mathscr{G}_{str}}}(1_{\sim}) = 0_{\sim}$ and $IFExt_{\alpha_{\mathscr{G}_{str}}}(0_{\sim}) = 1_{\sim}$.

Definition 5.3. Let $(X, \mathscr{G}_{1str})$ and $(Y, \mathscr{G}_{2str})$ be any two intuitionistic fuzzy $\mathscr{G}$ structure spaces. Let $f : (X, \mathscr{G}_{1str}) \to (Y, \mathscr{G}_{2str})$ be an intuitionistic fuzzy function. Then

(i) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy $\mathscr{G}_{2str}$ open set B in Y containing $f(x_{r,s})$, there exists an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set A in X containing $x_{r,s}$ such that $f(A) \subseteq B$;

(ii) intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function if for each intuitionistic fuzzy point $x_{r,s}$ in X and each intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set B in Y containing $f(x_{r,s})$, there exists an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ clopen set A in X containing $x_{r,s}$ such that $f(IFExt_{\alpha_{\mathscr{G}_{1str}}}(A)) \subseteq B$;

(iii) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open function if for each intuitionistic fuzzy $\mathscr{G}_{1str}$ open set A in X, $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}$ open set A in Y.

Proposition 5.1. Let $(X,\ \mathscr{G}_{1str})$ and $(Y,\ \mathscr{G}_{2str})$ be any two intuitionistic fuzzy $\mathscr{G}$ structure spaces. Let $f : (X,\ \mathscr{G}_{1str}) \to (Y,\ \mathscr{G}_{2str})$ be an intuitionistic fuzzy function. Then the followings are equivalent:

(i) f is intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function;

(ii) $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set in X, for each intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set A in Y;

(iii) $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ closed set in X, for each intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set A in Y;

(iv) $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ clopen set in X, for each intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set A in Y.

Proof. (i) $\Rightarrow$ (ii). Let B be an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set in Y and let $x_{r,s} \in f^{-1}(B)$. Since $f(x_{r,s}) \in B$, by (i), there exists an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ clopen set A in X containing $x_{r,s}$ such that

$$f(IFExt_{\alpha_{\mathscr{G}_{1str}}}(A)) = f(IF_{\alpha_{\mathscr{G}_{1str}}}int(\overline{A})) \subseteq B,$$

$$IF_{\alpha_{\mathscr{G}_{1str}}}int(\overline{A}) \subseteq f^{-1}(B).$$

This implies that

$$f^{-1}(B) = \cup_{x_{r,s}\in IF_{\alpha_{\mathscr{G}_{1str}}}int(\overline{A})} IF_{\alpha_{\mathscr{G}_{1str}}}int(\overline{A}).$$

Thus $f^{-1}(B)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set in X.

(ii) $\Rightarrow$ (iii). Let B be an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set in Y. Then $\overline{B}$ is an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set in Y. Thus, $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$. Since $f^{-1}(B)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set in X, $f^{-1}(\overline{B})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ closed set in X.

(iii) $\Rightarrow$ (iv). The proof is easily.

(iv) $\Rightarrow$ (v). Let B be an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set in Y containing $f(x_{r,s})$. By (iv), $f^{-1}(B)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ clopen set in X. If we take $A = f^{-1}(B)$, then $f(A) \subseteq B$. By remark 5.1, $f(IFExt_{\alpha_{\mathscr{G}_{1str}}}(A)) \subseteq B$.

Remark 5.2. Let $(X,\ \mathscr{G}_{1str})$ and $(Y,\ \mathscr{G}_{2str})$ be any two intuitionistic fuzzy $\mathscr{G}$ structure spaces. If $f : (X,\ \mathscr{G}_{1str}) \to (Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ exterior set connected function, then f is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ continuous function.

Definition 5.4. An intuitionistic fuzzy $\mathscr{G}$ structure space $(X,\ \mathscr{G}_{str})$ is said to be an

(i) intuitionistic fuzzy $\mathscr{G}_{str}clo$-T_0 if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exists an intuitionistic fuzzy $\mathscr{G}_{str}$ clopen set A of X containing one intuitionistic fuzzy point $x_{r,s}$ but not $y_{m,n}$;

(ii) intuitionistic fuzzy $\mathscr{G}_{str}clo$-T_1 if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exist intuitionistic fuzzy $\mathscr{G}_{str}$ clopen sets A and B containing $x_{r,s}$ and $y_{m,n}$ respectively such that $y_{m,n} \notin A$ and $x_{r,s} \notin B$;

(iii) intuitionistic fuzzy $\mathscr{G}_{str}clo$-T_2 if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exist intuitionistic fuzzy $\mathscr{G}_{str}$ clopen sets A and B containing $x_{r,s}$ and $y_{m,n}$ respectively such that $A \cap B = 0_{\sim}$;

(iv) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}clo$-regular if for each intuitionistic fuzzy $\mathscr{G}_{str}$ clopen set A and an intuitionistic fuzzy point $x_{r,s} \notin A$, there exist disjoint intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open sets B and C such that $A \subseteq B$ and $x_{r,s} \in C$;

(v) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}clo$-normal if for each pair of disjoint intuitionistic fuzzy $\mathscr{G}_{str}$ clopen sets A and B in X, there exist disjoint intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open sets C and D such that $A \subseteq C$ and $B \subseteq D$.

Definition 5.5. An intuitionistic fuzzy $\mathscr{G}$ structure space $(X, \mathscr{G}_{str})$ is said to be an an

(i) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$T_0$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exists an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open set A of X containing one intuitionistic fuzzy point $x_{r,s}$ but not $y_{m,n}$;

(ii) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$T_1$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exist intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open sets A and B containing $x_{r,s}$ and $y_{m,n}$ respectively such that $y_{m,n} \notin A$ and $x_{r,s} \notin B$;

(iii) intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$T_2$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exist an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open sets A and B containing $x_{r,s}$ and $y_{m,n}$ respectively such that $A \cap B = 0_\sim$;

(iv) intuitionistic fuzzy strongly $\alpha_{\mathscr{G}_{str}}$-regular if for each intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ closed set A and an intuitionitic fuzzy point $x_{r,s} \notin A$, there exist disjoint intuitionistic fuzzy $\mathscr{G}_{str}$ open sets B and C such that $A \subseteq B$ and $x_{r,s} \in C$;

(v) intuitionistic fuzzy strongly $\alpha_{\mathscr{G}_{str}}$-normal if for each pair of disjoint intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ closed sets A and B in X, there exist disjoint intuitionistic fuzzy $\mathscr{G}_{str}$ open sets C and D such that $A \subseteq C$ and $B \subseteq D$.

Proposition 5.2. Let $(X, \mathscr{G}_{1str})$ and $(Y, \mathscr{G}_{2str})$ be any two intuitionistic fuzzy $\mathscr{G}$ structure spaces. Let $f : (X, \mathscr{G}_{1str}) \to (Y, \mathscr{G}_{2str})$ be an injective, intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function.

(i) If $(Y, \mathscr{G}_{2str})$ is intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_0, then $(X, \mathscr{G}_{1str})$ is intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_0$;

(ii) if $(Y, \mathscr{G}_{2str})$ is intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_1, then $(X, \mathscr{G}_{1str})$ is intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_1$;

(iii) if $(Y, \mathscr{G}_{2str})$ is intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_2, then $(X, \mathscr{G}_{1str})$ is intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_2$.

Proof. (i) Suppose that $(Y, \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_0 space. For any distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exists an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set A in Y such that $f(x_{r,s}) \in A$ and $f(y_{r,s}) \notin A$. Since f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set in X such that $x_{r,s} \in f^{-1}(A)$. This implies that $(X, \mathscr{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_0$ space.

(ii) Suppose that $(Y, \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_1 space. For any distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exist an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen sets A and B in Y such that $f(x_{r,s}) \in A$, $f(x_{r,s}) \notin B$, $f(y_{r,s}) \notin A$ and $f(y_{r,s}) \in B$. Since f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, $f^{-1}(A)$ and $f^{-1}(B)$ are intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open sets in X respectively such that $x_{r,s} \in f^{-1}(A)$, $x_{r,s} \notin f^{-1}(B)$, $y_{r,s} \notin f^{-1}(A)$ and $y_{r,s} \in f^{-1}(B)$. This implies that $(X, \mathscr{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_1$ space.

(iii) Suppose that $(Y, \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_2 space. For any distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in X, there exist intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen sets A and B in Y such that $f(x_{r,s}) \in A$, $f(x_{r,s}) \notin B$, $f(y_{r,s}) \notin A$, $f(y_{r,s}) \in B$ and $A \cap B = 0_\sim$. Since f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, $f^{-1}(A)$ and $f^{-1}(B)$ are intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open sets in X, containing $x_{r,s}$ and $y_{r,s}$ respectively such that

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(0_\sim) = 0_\sim.$$

This implies that $(X,\ \mathscr{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_2$ space.

Proposition 5.3. Let $(X,\ \mathscr{G}_{1str})$ and $(Y,\ \mathscr{G}_{2str})$ be any two intuitionistic fuzzy $\mathscr{G}$ structure spaces. Let $f:(X,\ \mathscr{G}_{1str})\to(Y,\ \mathscr{G}_{2str})$ be an intuitionistic fuzzy function.

(i) If f is an injective, intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open function and intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function from intuitionistic fuzzy strongly $\alpha_{\mathscr{G}_{1str}}$-regular space $(X,\ \mathscr{G}_{1str})$ onto an intuitionistic fuzzy $\mathscr{G}$ structure space $(Y,\ \mathscr{G}_{2str})$, then $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}clo$-regular space;

(ii) if f is an injective, intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open function and intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function from intuitionistic fuzzy strongly $\alpha_{\mathscr{G}_{1str}}$-normal space $(X,\ \mathscr{G}_{1str})$ onto an intuitionistic fuzzy $\mathscr{G}$ structure space $(Y,\mathscr{G}_{2str})$, then $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}clo$-normal space.

Proof. (i) Let A be an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set in Y such that $y_{m,n}\notin A$. Take $y_{m,n}=f(x_{r,s})$. Since f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, $B=f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ closed sets in X such that $x_{r,s}\notin B$. Since $(X,\ \mathscr{G}_{1str})$ is an intuitionistic fuzzy strongly $\alpha_{\mathscr{G}_{1str}}$-regular space, there exist disjoint intuitionistic fuzzy $\mathscr{G}_{1str}$ open sets C and D such that $B\subseteq C$ and $x_{r,s}\in D$. Since f is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open function, we have $A=f(B)\subseteq f(C)$ and $y_{m,n}=f(x_{r,s})\in f(D)$ such that $f(C)$ and $f(D)$ are disjoint intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}$ open sets in Y. This implies that $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}clo$-regular space.

(ii) Let A_1 and A_2 be disjoint intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen sets in Y. Since f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, $f^{-1}(A_1)$ and $f^{-1}(A_2)$ are intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ closed sets in X. Take $B=f^{-1}(A_1)$ and $C=f^{-1}(A_2)$. This implies that $B\cap C=0_\sim$. Since $(X,\ \mathscr{G}_{1str})$ is an intuitionistic fuzzy strongly $\alpha_{\mathscr{G}_{1str}}$-normal space, there exist disjoint intuitionistic fuzzy $\mathscr{G}_{1str}$ open sets D_1 and D_2 such that $B\subseteq D_1$ and $C\subseteq D_2$. Since f is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ open function, we have $A_1=f(B)\subseteq f(D_1)$ and $A_2=f(C)\subseteq f(D_2)$ such that $f(D_1)$ and $f(D_2)$ are disjoint intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}$ open sets in Y. This implies that $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}clo$-normal space.

Definition 5.6. Let $(X,\ \mathscr{G}_{1str})$ and $(Y,\ \mathscr{G}_{2str})$ be any two intuitionistic fuzzy $\mathscr{G}$ structure spaces. Let $f:X\to Y$ be an intuitionistic fuzzy function. An intuitionistic fuzzy graph $G(f)=\{(x_{r,s},f(x_{r,s}):x_{r,s}\in\zeta^X\}\subseteq\zeta^X\times\zeta^Y$ of an intuitionistic fuzzy function f is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$co$-closed graph if and only if for each $(x_{r,s},y_{m,n})\in\zeta^X\times\zeta^Y\setminus G(f)$, there exist an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set A in X containing $x_{r,s}$ and an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set B in Y containing $y_{m,n}$ such that $f(A)\cap B=0_\sim$.

Proposition 5.4. Let $(X,\mathscr{G}_{1str})$, $(Y,\ \mathscr{G}_{2str})$ and $(X\times Y,\ \mathscr{G}_{str})$ be any three intuitionistic fuzzy $\mathscr{G}$ structure spaces. Let $f:(X,\ \mathscr{G}_{1str})\to(Y,\ \mathscr{G}_{2str})$ be an intuitionistic fuzzy function.

(i) If f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function and $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}$-$T_2$ space, then $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$co$-closed graph in $\zeta^X\times\zeta^Y$;

(ii) if f is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ continuous function and $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy intuitionistic fuzzy $\mathscr{G}_{2str}clo$-T_1 space, then $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$co$-closed graph in $\zeta^X\times\zeta^Y$;

(iii) If f is an intuitionistic fuzzy injective function has an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-$co$-

closed graph in $\zeta^X \times \zeta^Y$. Then $(X,\ \mathscr{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_1$ space.

Proof. (i) Let $(x_{r,s}, y_{m,n}) \in \zeta^X \times \zeta^Y \setminus G(f)$ and $f(x_{r,s}) \notin y_{m,n}$. Since $(Y,\ \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}$-$T_2$ space, there exist intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen sets A and B in Y containing $f(x_{r,s})$ and $y_{m,n}$ respectively such that $A \cap B = 0_\sim$. Since f is an intuitionistic fuzzy $E_{\alpha_{\mathscr{G}_{str}}}$ continuous function, there exists an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set C in X containing $x_{r,s}$ such that $f(Ext_{\alpha_{\mathscr{G}_{1str}}}(C)) = f(C) \subseteq A$ and $f(C) \cap A = 0_\sim$. This implies that $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-*co*-closed graph in $\zeta^X \times \zeta^Y$.

(ii) Let $(x_{r,s}, y_{m,n}) \in \zeta^X \times \zeta^Y \setminus G(f)$ and $f(x_{r,s}) \notin y_{m,n}$. Since $(Y, \mathscr{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{2str}}$-$T_2$ space, there exists an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set A in Y such that $f(x_{r,s}) \in A$ and $f(x_{r,s}) \notin A$. Since f is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$ continuous function, there exists an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set B in X containing $x_{r,s}$ such that $f(B) \subseteq A$. Therefore, $f(B) \cap \overline{A} = 0_\sim$ and $\overline{A}$ is an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen in Y containing $y_{m,n}$. This implies that $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-*co*-closed set in $\zeta^X \times \zeta^Y$.

(iii) Let $x_{r,s}$ and $y_{m,n}$ be any two intuitionistic fuzzy points in X. Then $(x_{r,s}, f(x_{r,s}) \in \zeta^X \times \zeta^Y \setminus G(f)$. Since $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{str}}$-*co*-closed graph in $\zeta^X \times \zeta^Y$, there exists an intuitionitic fuzzy $\alpha_{\mathscr{G}_{1str}}$ open set A in X and an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set B in Y such that $x_{r,s} \in A, f(y_{m,n}) \in B$ and $f(A) \cap B = 0_\sim$. Since f is an intuitionistic fuzzy injective function, $A \cap f^{-1}(B) = 0_\sim$ and $f^{-1}(B)$ is an intuitionistic fuzzy $\mathscr{G}_{2str}$ clopen set in Y containing $y_{r,s}$. and $y_{m,n} \notin A$. Thus $(X,\ \mathscr{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathscr{G}_{1str}}$-$T_1$ space.

References

[1] K. T. Atanassov, Intuitionistic fuzzy Sets, Fuzzy Sets and Systems, **20**(1986), 87-96.

[2] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy sets and systems, **88**(1997), No. 1, 81-89.

[3] D. Coker and M. Demirci, On intuitionistic fuzzy points, Notes IFS, **1**(1995), No. 2, 79-84.

[4] Erdal Ekici, Generalization of some fuzzy functions, Bulletin of the Institute of Mathematics, **33**(2005), No. 3, 277-289.

[5] H. Gurcay and D. Coker, On fuzzy continuity in intuitiionistic fuzzy topological spaces, J. fuzzy. Math., **5**(1997), 365-378.

[6] Sumita Das, M. N. Mukherjee, fuzzy generalized closed sets via fuzzy grills, Filomat, **26**(2012), No. 3, 563-571.

[7] L. A. Zadeh, Fuzzy sets, Information and Control, **9**(1965), 338-353.

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