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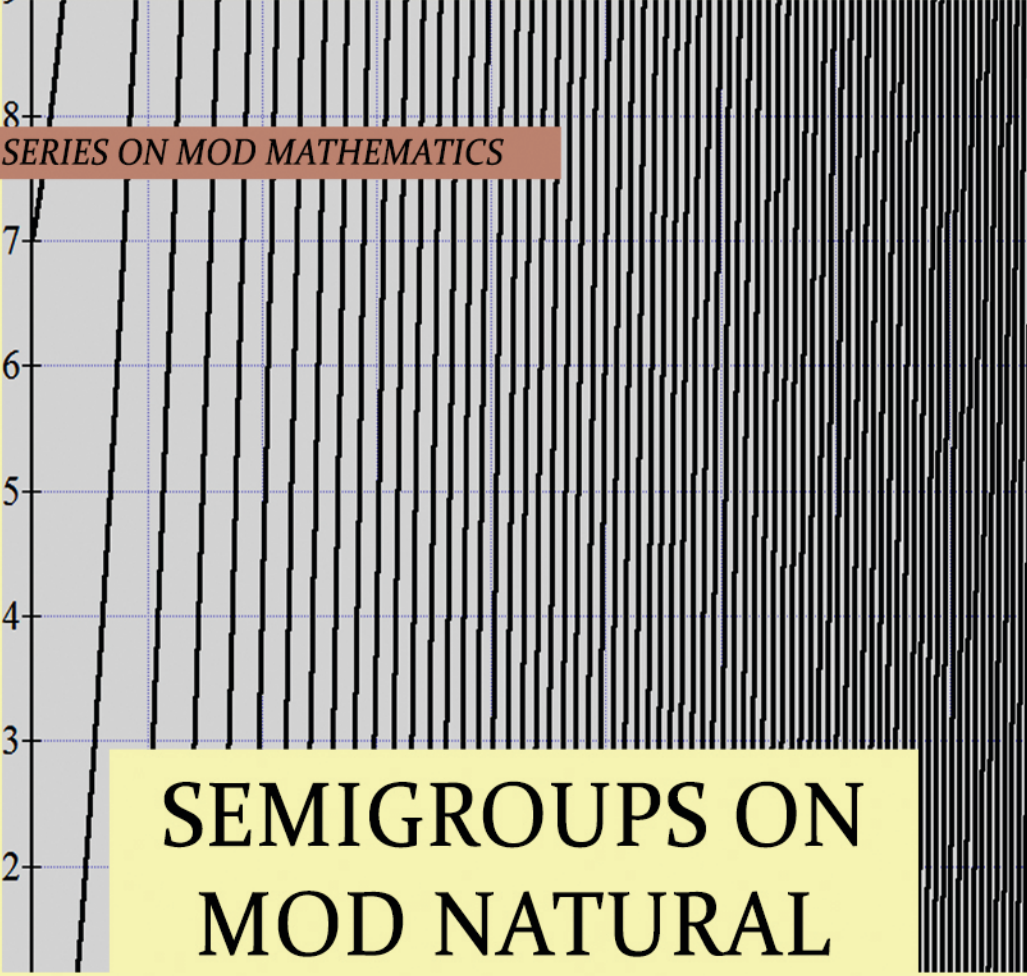

SEMIGROUPS ON MOD NATURAL NEUTROSOPHIC ELEMENTS

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PREFACE

In this book the notion of semigroups under + is constructed using the Mod natural neutrosophic integers or MOD natural neutrosophic-neutrosophic numbers or mod natural neutrosophic finite complex modulo integer or MOD natural neutrosophic dual number integers or MOD natural neutrosophic special dual like number or MOD natural neutrosophic special quasi dual numbers are analysed in a systematic way.

All these semigroups under + have an idempotent subsemigroup under +. This is the first time we are able to give a class of idempotent subsemigroups under + by taking only those MOD natural neutrosophic elements of Z_n^I or $C^I(Z_n)$ or $\langle Z_n \cup g\rangle_I$. However all these semigroups are S-semigroups.

Secondly on these above mentioned six sets, the operation of $\times$ is defined and under $\times$ also these sets form only a semigroup.

Several important properties about them are analysed.

For the MOD natural neutrosophic numbers please refer [24]. For natural product refer [19]. For S-semigroups and the related properties refer [5]. Several open conjectures are proposed.

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Chapter One

SEMIGROUPS ON MOD NATURAL NEUTROSOPHIC ELEMENTS UNDER +

In this chapter we for the first time introduce finite semigroups under "+". Such semigroups are very rare in mathematical literature.

First we will illustrate this situation by some examples.

Recall from [24] the MOD natural neutrosophic numbers

$Z_n^I = \{Z_n,\ I_0^n, I_t^n / t$ is a zero divisor or a non unit of $Z_n\}$. Depending on n; Z_n will have more zero divisors and idempotents.

Depending on these zero divisors and idempotents we get the natural neutrosophic numbers and MOD neutrosophic numbers.

Example 1.1: Let $Z_2^I = \{0, 1, I_0^2\}$ is the natural neutrosophic MOD integers. $o(Z_2^I) = 3$ and $Z_2 \subseteq Z_2^I$.

Example 1.2: Let $Z_3^I = \{0, 1, I_o^3, 2\}$ is the natural neutrosophic MOD integers $o(Z_3^I) = 4$, $Z_3 \subseteq Z_3^I$.

Example 1.3: Let $Z_5^I = \{0, 1, 2, 3, 4, I_0^5\}$ be the MOD natural neutrosophic numbers; $o(Z_5^I) = 6$.

Example 1.4: Let $Z_{23}^I = \{0, 1, 2, 3, \ldots, 22, I_0^{23}\}$ be the MOD natural neutrosophic numbers, $o(Z_{23}^I) = 24$.

Example 1.5: Let $Z_4^I = \{0, 1, 2, 3, I_0^4, I_2^4\}$ be the MOD natural neutrosophic numbers, $o(Z_4^I) = 6$.

Example 1.6: Let $Z_{12}^I = \{0, 1, 2, \ldots, 11, I_0^{12}, I_2^{12}, I_3^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_{10}^{12}, I_9^{12}\}$ be the MOD natural neutrosophic numbers $o(Z_{12}^I) = 20$.

In view of all these we have the following result.

THEOREM 1.1: *Let $Z_P^I = \{0, 1, 2, \ldots, p-1, I_0^P\}$ be the natural neutrosophic modulo integers; p a prime; $o(Z_P^I) = p + 1$.*

Proof is direct and hence left as an exercise to the reader.

Clearly $Z_{10}^I = \{0, 1, 1, \ldots, 9, I_0^{10}, I_2^{10}, I_4^{10}, I_6^{10}, I_8^{10}, I_5^{10}\}$ be the MOD natural neutrosophic numbers. $o(Z_{10}^I) = 16$.

We define + on Z_{10}^I as follows.

$$I_0^{10} + I_0^{10} = I_0^{10}, \quad I_2^{10} + I_2^{10} = I_2^{10}, \quad I_4^{10} + I_4^{10} = I_4^{10}, \; I_6^{10} + I_6^{10} = I_6^{10},$$

$$I_8^{10} + I_8^{10} = I_8^{10}, \quad I_5^{10} + I_5^{10} = I_5^{10}$$

$$a + I_t^{10} = I_t^{10} + a \text{ for all } a \in Z_{10} \text{ and}$$

$$I_t^{10} \in \{I_0^{10}, I_2^{10}, I_4^{10}, I_6^{10}, I_8^{10}, I_5^{10}\}.$$

So we define $I_0^{10} + I_2^{10}, \ldots, I_5^{10} + I_8^{10}$, $I_0^{10} + I_2^{10} + I_4^{10}$, $I_0^{10} + I_2^{10} + I_4^{10} + I_6^{10}$, $I_2^{10} + I_6^{10} + I_8^{10}$, $I_0^{10} + I_4^{10} + I_6^{10} + I_8^{10} + I_5^{10}$ and so on.

Example 1.7: Let $\langle Z_3^I, +\rangle$ = {0, 1, 2, I_0^3, 1 + I_0^3, 2 + I_0^3} be the MOD natural neutrosophic semigroup under +.

Clearly $I_0^3 + I_0^3 = I_0^3$, 0 + $I_0^3 = I_0^3$ + 0 serves as the additive identity.

Since $I_0^3 + I_0^3 = I_0^3$ we see Z_3^I under + is only a semigroup and o($\langle Z_3^I, +\rangle$) = 6.

Example 1.8: Let $\langle Z_4^I, +\rangle$ = {0, 1, 2, 3, I_0^4, I_2^4, 1 + I_0^4, 2 + I_0^4, 3 + I_0^4, 1 + I_2^4, 2 + I_2^4, 3 + I_2^4, $I_0^4 + I_2^4$, 1 + I_0^4 + I_2^4, 2 + I_0^4 + I_2^4, 3 + I_0^4 + I_2^4} be the semigroup of MOD natural neutrosophic numbers under +.

For I_0^4 + I_0^4 = I_0^4 and $I_2^4 + I_2^4 = I_2^4$.

o($\langle Z_4^I, +\rangle$) = 16.

Example 1.9: Let
$\langle Z_5^I, +\rangle$ = {0, 1, 2, 3, 4, I_0^5, 1 + I_0^5, 2 + I_0^5, 3 + I_0^5, 4 + I_0^5} be the MOD natural neutrosophic semigroup under +.

Infact $\langle Z_5^I, +\rangle$ is a monoid, as 0 acts as the additive identity.

o($\langle Z_5^I$, 0, +$\rangle$) = 10.

Example 1.10: Let $\langle Z_6^I, +\rangle$ = {0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_4^6, I_3^6, 1 + I_0^6, 1 + I_2^6, 1 + I_3^6, 1 + I_4^6, 2 + I_0^6, 2 + I_3^6, 2 + I_4^6, 2 + I_2^6, 3 + I_0^6, 3 + I_2^6, 3 + I_4^6, 3 + $I_3^6$4 + I_0^6, 4 + I_2^6, 4 + I_4^6, 4 + I_3^6 , 5 + I_0^6, 5 + I_2^6, 5 + I_3^6, 5 + I_4^6, I_0^6 + I_2^6, I_0^6 + I_3^6, I_3^6 + I_4^6, I_0^6 + I_2^6 + I_4^6, I_0^6 +

$I_2^6 + I_3^6$, $I_0^6 + I_3^6 + I_4^6$, $I_2^6 + I_3^6 + I_4^6$, $I_0^6 + I_3^6 + I_4^6 + I_2^6$, and so on} be the MOD natural neutrosophic semigroup under +.

$$o(\langle Z_6^I, +\rangle) = 96.$$

Example 1.11: Let $\langle Z_8^I, +\rangle = \{0, 1, 2, , \ldots, 7, I_0^8, I_2^8, I_4^8, I_6^8, a + I_0^8, a + I_2^8, a + I_4^8, a + I_6^8, a + I_0^8 + I_2^8, a + I_0^8 + I_4^8, a + I_0^8 + I_6^8, a + I_2^8 + I_4^8, a + I_2^8 + I_6^8, a + I_4^8 + I_6^8, a + I_0^8 + I_2^8 + I_4^8, a + I_0^8 + I_2^8 + I_6^8, a + I_0^8 + I_4^8 + I_6^8, a + I_2^8 + I_4^8 + I_6^8, a + I_0^8 + I_2^8 + I_4^8 + I_6^8$ / $a \in Z_8\}$ be a MOD natural neutrosophic monoid;

$$o(\langle Z_8^I, +\rangle) = 128.$$

Example 1.12: Let $\langle Z_{12}^I, +\rangle = \{Z_{12}, I_0^{12}, I_2^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_{10}^{12}, I_3^{12}, I_9^{12}, a_1 + I_0^{12}, \ldots, a_1 + I_0^{12}, a + I_0^{12} + I_2^{12}, \ldots, a + I_3^{12} + I_9^{12}, a + I_0^{12} + I_2^{12} + I_3^{12}$ and so on $a + I_0^{12} + I_2^{12} + I_4^{12} + I_3^{12} + I_6^{12} + I_8^{12} + I_9^{12} + I_{10}^{12}$ / $a \in Z_{12} \setminus \{0\}\}$ be the MOD natural neutrosophic semigroup of mod integers.

$o(\langle Z_{12}^I, +\rangle = 3072$ consider the elements of $\langle Z_{12}^I, +\rangle$;

we see $P = \{I_0^{12}, I_2^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_{10}^{12}, I_3^{12}, I_9^{12}\}$ are all idempotent neutrosophic elements of Z_{12}^I under +.

In fact they form a semigroup under '+'.

Clearly the semigroup P generated under + is not a monoid. $\langle P, +\rangle = \{I_0^{12}, I_2^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_{10}^{12}, I_3^{12}, I_9^{12}, I_0^{12} + I_2^{12}, \ldots, I_9^{12} + I_{10}^{12}, I_0^{12} + I_2^{12} + I_3^{12}, \ldots, I_8^{12} + I_9^{12} + I_{10}^{12}, I_0^{12} + I_2^{12} + I_3^{12} + I_4^{12}, \ldots, I_6^{12} + I_8^{12} + I_9^{12} + I_{10}^{12}, \ldots, I_0^{12} + I_2^{12} + I_3^{12} + I_4^{12} + I_8^{12} + I_6^{12} + I_{10}^{12}, \ldots, I_0^{12} + I_2^{12} + I_3^{12} + I_4^{12} + I_6^{12} + I_8^{12} + I_{10}^{12} + I_9^{12}\}$.

Now $\langle P, +\rangle = 255$. $\langle P, +\rangle$ is a subsemigroup of order 255.

Infact this is an example of an idempotent semigroup under +.

Clearly o(⟨P, +⟩) $\not|$ 3072.

We can by using the natural neutrosophic elements of mod integers obtain finite idempotent semigroups under +; which is non abstract and realistic.

Thus associated with each Z_n (n a non prime) we can have a natural neutrosophic collection of elements which forms an idempotent subsemigroup under +.

Apart from set theoretic union or intersection getting such elements is an impossibility to the best of our knowledge that too mainly under addition.

However we have such concepts under product.

So by this method of defining natural neutrosophic numbers using division arrive at natural neutrosophic elements which are idempotents under addition.

Next we give one more example.

Example 1.13: Let Z_{32} be the modulo integers. Using division as an operation on Z_{32} we obtain the following natural neutrosophic elements.

N = { I_0^{32}, I_2^{32}, I_4^{32}, I_6^{32}, I_8^{32}, I_{10}^{32}, I_{12}^{32}, I_{14}^{32}, I_{16}^{32}, I_{18}^{32}, I_{20}^{32}, I_{22}^{32}, I_{24}^{32}, I_{26}^{32}, I_{28}^{32}, I_{30}^{32} }.

Clearly {⟨N, +⟩} is an idempotent semigroup and not a monoid will also be known as pure neutrosophic natural MOD semigroup.

o(⟨N, +⟩ = 16 + ${}_{16}C_2$ + ${}_{16}C_2$ + … + ${}_{16}C_{16}$.

In view of all these we can have the following result.

THEOREM 1.2: *Let Z_n be the mod integers; n a composite number. If n has t number of elements in Z_n which are not units then associated with Z_n we have t natural neutrosophic numbers denoted by N and $o\langle N, +\rangle = (t + {}_tC_2 + {}_tC_3 + \ldots + {}_tC_t)$.*

Further $\langle N, +\rangle$ is an idempotent semigroup under +.

Proof is direct and hence left as an exercise to the reader.

We proceed onto supply some more examples of them.

Example 1.14: Let Z_{24} be the modulo integers. The natural neutrosophic elements got using Z_{24} by division are $\{I_0^{24}, I_2^{24}, I_3^{24}, I_4^{24}, I_6^{24}, I_8^{24}, I_9^{24}, I_{10}^{24}, I_{12}^{24}, I_{14}^{24}, I_{16}^{24}, I_{18}^{24}, I_{20}^{24}, I_{15}^{24}, I_{22}^{24}, I_{21}^{24}\} = N$.

N under + is an idempotent semigroup also known as the natural neutrosophic semigroup.

$$o(\langle N, +\rangle) = \{16 + {}_{16}C_2 + \ldots + {}_{16}C_{16}\}.$$

Example 1.15: Let Z_{26} be the modulo integers. Let N be the collection of all natural neutrosophic elements got by division.

$N = \{I_0^{26}, I_2^{26}, I_4^{26}, I_6^{26}, I_8^{26}, I_{10}^{26}, I_{12}^{26}, I_{14}^{26}, I_{18}^{26}, I_{20}^{26}, I_{22}^{26}, I_{16}^{26}, I_{24}^{26}, I_{13}^{26}\}$ generates an idempotent semigroup under '+'.

Clearly $o(\langle N, +\rangle) = (14 + {}_{14}C_2 + \ldots + {}_{14}C_{13} + {}_{14}C_{14})$.

Let $P_1 = \{I_0^{26}, I_{10}^{26}\} \subseteq N$ is an idempotent subsemigroup of order 3 given by $\langle P_1, +\rangle = \{I_0^{26}, I_{10}^{26}, I_0^{26} + I_{10}^{26}\}$ (as $I_0^{26} + I_0^{26} = I_0^{26}$, $I_{10}^{26} + I_{10}^{26} = I_{10}^{26}$) and order $\langle P_1, +\rangle$ is 3.

Let $P_2 = \{I_{13}^{26}, I_4^{26}, I_{16}^{26}\} \subseteq N$.

$\langle P_2, + \rangle = \{ I_{13}^{26}, I_4^{26}, I_{16}^{26}, I_4^{26}+I_{13}^{26}, I_4^{26}+I_{16}^{26}, I_{16}^{26}+I_{13}^{26}, I_4^{26}+I_{13}^{26}+I_{16}^{26} \}$ is an idempotent semigroup under + of order 7.

Let $P_3 = \{ I_4^{26}, I_8^{26}, I_{13}^{26}, I_{16}^{26} \} \subseteq N$,

$\langle P_3, + \rangle = \{ I_4^{26}, I_8^{26}, I_{13}^{26}, I_{16}^{26}, I_4^{26}+I_8^{26}, I_4^{26}+I_{13}^{26}, I_4^{26}+I_{16}^{26}, I_8^{26}+I_{13}^{26}, I_8^{26}+I_{16}^{26}, I_{16}^{26}+I_{13}^{26}, I_4^{26}+I_8^{26}+I_{13}^{26}, I_4^{26}+I_8^{26}+I_{16}^{26}, I_8^{26}+I_{13}^{26}+I_{16}^{26}, I_4^{26}+I_{13}^{26}+I_{16}^{26}, I_4^{26}+I_8^{26}+I_{13}^{26}+I_{16}^{26} \}$ is a semigroup under + which is of order 15 which is the natural neutrosophic idempotent semigroup under +.

Suppose $P_4 = \{ I_{24}^{26}, I_{13}^{26}, I_8^{26}, I_{20}^{26}, I_{22}^{26} \} \subseteq N$ be a subset of N.

Now P_4 will generate an idempotent subsemigroup under '+'; given by $\langle P_4, + \rangle = \{ I_{24}^{26}, I_{13}^{26}, I_8^{26}, I_{20}^{26}, I_{22}^{26}, I_{24}^{26}+I_{13}^{26}, I_{24}^{26}+I_8^{26}, I_{24}^{26}+I_{22}^{26}$ and so on $I_{24}^{26}+I_{13}^{26} + I_8^{26}+I_{20}^{26}+I_{22}^{26} \}$ is an idempotent subsemigroup of order 31.

Thus we have several subsemigroups which are idempotent natural neutrosophic semigroups under + all of which are of finite order.

Example 1.16: Let Z_{14} be the mod integers. The natural neutrosophic number associated with Z_{14} are

$\{ I_0^{14}, I_2^{14}, I_4^{14}, I_6^{14}, I_8^{14}, I_{10}^{14}, I_{12}^{14}, I_7^{14} \} = P$.

$o(P) = 14$ and $o(\langle P, + \rangle) = 14 + {}_{14}C_2 + \ldots + {}_{14}C_{13} + {}_{14}C_{14}$.

Example 1.17: Let Z_{15} be the modulo integers 15. The natural neutrosophic elements associated with Z_{15} are

$P = \{ I_0^{15}, I_3^{15}, I_6^{15}, I_9^{15}, I_{12}^{15}, I_5^{15}, I_{10}^{15} \}$; $o(P) = 7$.

We see $\langle P, + \rangle$ will generate an idempotent natural neutrosophic semigroup under +.

$o\langle P, + \rangle = 7 + {}_7C_2 + \ldots + {}_7C_6 + {}_7C_7$.

Example 1.18: Let Z_{22} be the modulo integers. The natural neutrosophic elements associated with Z_{22} are

$P = \{ I_0^{22}, I_2^{22}, I_4^{22}, I_6^{22}, I_8^{22}, I_{10}^{22}, I_{12}^{22}, I_{14}^{22}, I_{16}^{22}, I_{18}^{22}, I_{20}^{22}, I_{11}^{22} \}$

generates an idempotent semigroup under + of order

${}_{12}C_1 + {}_{12}C_2 + \ldots + {}_{12}C_{11} + {}_{12}C_{12}$.

In view of all these we have the following theorem.

THEOREM 1.3: *Let Z_{2p}, p a prime be the modulo integers.*

i. The number of natural neutrosophic elements associated with Z_{2p} is $(p + 1)$.

ii. The order of the largest MOD natural neutrosophic idempotent semigroup under + is ${}_{p+1}C_1 + {}_{p+1}C_2 + \ldots + {}_{p+1}C_p + {}_{p+1}C_{p+1}$.

Proof is direct and hence left as an exercise to the reader.

Example 1.19: Let Z_{21} be the modulo integers. The natural neutrosophic elements associated with Z_{21} is

$\{ I_0^{21}, I_3^{21}, I_6^{21}, I_9^{21}, I_{12}^{21}, I_{15}^{21}, I_{18}^{21}, I_7^{21}, I_{14}^{21} \} = P$.

Clearly order of P is $9 = (7 + 2)$.

$\langle P, + \rangle$ is the natural neutrosophic semigroup of order ${}_9C_1 + {}_9C_2 + {}_9C_3 + \ldots + {}_9C_9$.

Example 1.20: Let Z_{33} be the modulo integers. The natural neutrosophic elements associated with Z_{33} are

$P = \{ I_0^{33}, I_3^{33}, I_6^{33}, I_9^{33}, I_{12}^{33}, I_{15}^{33}, I_{18}^{33}, I_{21}^{33}, I_{24}^{33}, I_{27}^{33}, I_{30}^{33}, I_{11}^{33}, I_{22}^{33} \}$.

Clearly order of P is $13 = (11 + 2)$,

P generate the MOD natural neutrosophic idempotent semigroup under + of order ${}_{13}C_1 + {}_{13}C_2 + \ldots + {}_{13}C_{13}$.

In view of this we have the following theorem.

THEOREM 1.4: *Let Z_{3p} (p a prime) be the modulo integer.*

i. *The natural neutrosophic numbers associated with Z_{3p} is p + 2.*

ii. *Clearly the order of the natural neutrosophic idempotent semigroup generated by these natural neutrosophic numbers is* $_{p+2}C_1 + {}_{p+2}C_2 + \ldots + {}_{p+2}C_{p+1} + {}_{p+2}C_{p+2}$.

Proof follows from simple number theoretic techniques.

Example 1.21: Let Z_{35} be the modulo integers. The MOD natural neutrosophic elements associated with Z_{35} are

$$P = \{ I_0^{35}, I_5^{35}, I_{10}^{35}, I_{15}^{35}, I_{20}^{35}, I_{25}^{35}, I_{30}^{35}, I_7^{35}, I_{14}^{35}, I_{21}^{35}, I_{28}^{35} \}.$$

The order of P is 11. The natural neutrosophic idempotent semigroup generated by P is of order

$$_{11}C_1 + {}_{11}C_2 + \ldots + {}_{11}C_{10} + {}_{11}C_{11}.$$

Example 1.22: Let Z_{65} be the modulo integers. The natural neutrosophic elements of Z_{65} got by the operation of division is

$$P = \{ I_0^{65}, I_5^{65}, I_{10}^{65}, \ldots, I_{60}^{65}, I_{13}^{65}, I_{26}^{65}, I_{39}^{65}, I_{52}^{65} \}.$$

Clearly o(P) = 13 + 4 = 17
= 5 + 12.

Example 1.23: Let Z_{143} be the modulo integers. The natural neutrosophic elements associated with Z_{143} is

$$P = \{ I_0^{143}, I_{11}^{143}, I_{22}^{143}, \ldots, I_{132}^{143}, I_{13}^{143}, I_{26}^{143}, I_{39}^{143}, I_{52}^{143}, I_{65}^{143}, \ldots, I_{130}^{143} \}$$

and order of P is 23 = 11 + 12 = 13 + 10.

P under + will generate an idempotent natural neutrosophic semigroup of order

$$_{23}C_1 + {}_{23}C_2 + \ldots + {}_{23}C_{22} + {}_{23}C_{23}.$$

In view of all these we have the following theorem.

THEOREM 1.5: *Let Z_{pq} (p and q two distinct primes) be the modulo integers. P = { I_0^{pq}, I_p^{pq}, I_{2p}^{pq}, … $I_{(p-1)q}^{pq}$ } be the collection of all natural neutrosophic elements associated with Z_{pq} by operation of division in Z_{pq}.*

i. $o(P) = p + q - 1 = q + p - 1$.

ii. *⟨P, + ⟩ generates an idempotent natural neutrosophic semigroup of order* ${}_{p+q-1}C_1 + {}_{p+q-1}C_2 + \ldots + {}_{p+q-1}C_{p+q-2} + {}_{p+q-1}C_{p+q-1}$.

Proof can be obtained by simple number theoretic methods.

Example 1.24: Let Z_{30} be the modulo integers. The natural neutrosophic elements got from Z_{30} by division is

P = { I_0^{30}, I_2^{30}, I_4^{30}, I_6^{30}, I_8^{30}, I_{10}^{30}, I_{12}^{30}, I_{14}^{30}, I_{16}^{30}, I_{26}^{30}, I_{18}^{30}, I_{20}^{30}, I_{22}^{30}, I_{24}^{30}, I_{28}^{30}, I_3^{30}, I_5^{30}, I_{25}^{30}, I_9^{30}, I_{15}^{30}, I_{21}^{30}, I_{27}^{30} } order of P is 22.

So the order of the idempotent semigroup generated by P is ${}_{22}C_1 + {}_{22}C_2 + \ldots + {}_{22}C_{21} + {}_{22}C_{22}$.

Example 1.25: Let Z_{105} be the modulo integers.

P = { I_0^{105}, I_3^{105}, I_6^{105},…, I_{102}^{105}, I_5^{105}, I_{10}^{105}, …, I_{100}^{105}, I_7^{105}, I_{14}^{105}, …, I_{98}^{105} } be the natural neutrosophic numbers got by the operation of division on Z_{105}.

Clearly o(P) = 57.

$o\langle P, +\rangle = {}_{57}C_1 + {}_{57}C_2 + \ldots, {}_{57}C_{56} + {}_{57}C_{57}$.

However at this stage we are not in a position to arrive at the general formula if = pqr where p, q and r are 3 distinct primes.

Example 1.26: Let $Z_{110} = Z_{2.5.11}$ be the mod integers.

Let P = {0, 2, 4, 6, 8, 10, 12, …, 98 100, … 108, 5, 15, 25, 35, 45, …, 105, 11, 33, 77, 99} be the collection of all natural neutrosophic numbers associated with Z_{110} under division.

$$o(P) = 70 \quad = \frac{110}{2} + 5 + 10 = 55 + 5 + 10$$

$$= 55 + 41 + 11$$
$$= qr + q + r - 1$$

where p = 2.

Clearly P will generate an idempotent natural neutrosophic semigroup of order

$${}_{70}C_1 + {}_{70}C_2 + \ldots + {}_{70}C_{69} + {}_{70}C_{70}.$$

Example 1.27: Let Z_{182} be the modulo integers. Let P denote the collection of all natural neutrosophic elements associated with division on Z_{182} gives

$$P = \{ I_0^{182}, I_2^{182}, I_4^{182}, \ldots, I_{180}^{182}, I_7^{182}, I_{21}^{182}, I_{35}^{187} \ldots, I_{175}^{182}, I_{13}^{182}, I_{39}^{182}, I_{65}^{182}, \ldots, I_{169}^{182} \}$$

$$\text{Clearly } o(P) = 91 + 7 + 12$$
$$= 91 + 6 + 13$$
$$= 110.$$

Let $\langle P, +\rangle$ generate the idempotent natural neutrosophic semigroup and $o(\langle P, +\rangle = {}_{110}C_1 + {}_{110}C_2 + \ldots + {}_{110}C_{109} + {}_{110}C_{110}$.

Just at this juncture we propose the following conjecture.

Conjecture 1.1: Let Z_n, $n = p_1\, p_2 \ldots p_t$ be t distinct primes of the modulo integer n.

(i) Find the number of natural neutrosophic elements got by the operation division on Z_n.

(ii) Find the order of the idempotent natural neutrosophic semigroup.

We will give examples of Z_n; n = $p_1^2 qr$ and so on.

Example 1.28: Let Z_{12} be the modulo integers.

Let P = { I_0^{12}, I_2^{12}, I_3^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_9^{12}, I_{10}^{12} } be the collection of all natural neutrosophic numbers associated with Z_{24}; o(P) = 8.

Example 1.29: Let $Z_{24} = Z_{2^3 3}$ be the modulo integers.

P = { I_0^{24}, I_2^{24}, I_4^{24} I_6^{24}, I_8^{24}, I_{10}^{24}, I_{12}^{24}, I_{14}^{24}, I_{16}^{24}, I_{18}^{24}, I_{20}^{24}, I_3^{24}, I_{22}^{24}, I_9^{24}, I_{15}^{24}, I_{21}^{24} } be the collection of all natural neutrosophic number associated with Z_{24}.

Example 1.30: Let $Z_{48} = Z_{2^4.3}$ be the modulo integers.

P denote the collection of all natural neutrosophic number got by introducing the operation division on Z_{48}.

P = { I_0^{48}, I_2^{48}, I_4^{48}, …, I_{46}^{48}, I_3^{48}, I_9^{48}, I_{15}^{48}, I_{21}^{48}, I_{27}^{48}, I_{33}^{48}, I_{39}^{48}, I_{45}^{48} } be the collection of natural neutrosophic number associated with Z_{48}.

o(P) = $2^4.2 = 2^5 = 32$.

Example 1.31: Let Z_{20} be the modulo integers. The natural neutrosophic elements associated with Z_{20} be

P = { I_0^{20}, I_2^{20}, I_4^{20}, I_6^{20}, I_8^{20}, I_{10}^{20}, I_{12}^{20}, I_{14}^{20}, I_{16}^{20}, I_{18}^{20}, I_5^{20}, I_{15}^{20} }; o(P) = 12.

Example 1.32 : Let Z_{80} be the modulo integer

P = { I_0^{80} , I_2^{80}, …, I_{76}^{80}, I_{78}^{80}, I_5^{80}, I_{15}^{80}, I_{25}^{80}, I_{35}^{80}, I_{45}^{80}, I_{55}^{80}, I_{65}^{80}, I_{75}^{80} } denote the collection of all natural neutrosophic elements associated with $Z_{80} = Z_{16.5} = Z_{2^4.5}$;

$$o(P) = 48 = 2^4.3.$$

In view of all these we have the following theorem.

THEOREM 1.6: *Let Z_{2^n5} be the modulo integers. P denote the collection of all natural neutrosophic elements associated with Z_{2^n5} obtained by the operation of division;*

$$o(P) = 2^n.3.$$

Proof can be obtained by simple number theoretic techniques.

THEOREM 1.7: *Let Z_{2^n3} be the modulo integers. P be the natural neutrosophic elements associated with Z_{2^n3}; then*

$$o(P) = 2^{n+1}.$$

Proof is direct by exploiting number theoretic techniques.

Example 1.33: Let Z_{28} be the modulo integers. The natural neutrosophic elements associated with Z_{28} is

P = { I_0^{28}, I_2^{28}, I_4^{28}, I_6^{28}, I_8^{28}, I_{10}^{28}, I_{12}^{28}, I_{14}^{28}, I_{16}^{28}, I_{18}^{28}, I_{20}^{28}, I_{22}^{28}, I_{24}^{28}, I_{26}^{28}, I_7^{28}, I_{21}^{28} }.

$$o(P) = 16 = 2^2 \times (7 - 3).$$

Example 1.34: Let Z_{56} be the modulo integers.

Let P denote the collection of all natural neutrosophic elements associated with Z_{56}.

$$P = \{ I_0^{56}, I_2^{56}, \ldots, I_{54}^{56},\ I_7^{56}, I_{21}^{56}, I_{35}^{56}, I_{49}^{56} \}$$

$$o(P) = 32 = 2^3 \times (7 - 3).$$

In view of this we have the following theorem.

THEOREM 1.8: *Let Z_{2^n7} be the modulo integers; P be the collection of all natural neutrosophic elements associated with Z_{2^n7}; then*

$$o(P) = 2^{n+2} = 2^n (7 - 3).$$

Proof is direct by exploiting simple number theoretic techniques.

Example 1.35: Let Z_{176} be the modulo integers.

P be the collection of all natural neutrosophic elements associated with Z_{176}.

$P = \{ I_0^{176},\ I_2^{176},\ I_4^{176},\ \ldots,\ I_{174}^{176},\ I_{11}^{176},\ I_{33}^{176},\ I_{55}^{176},\ I_{77}^{176},\ I_{99}^{176},\ I_{121}^{176},\ I_{143}^{176},\ I_{165}^{176} \}$.

$$\begin{aligned} o(P) &= 96 \\ &= 2^5 \times 3 \\ &= 2^4 (11 - 5). \end{aligned}$$

Example 1.36: Let $Z_{352} = Z_{2^511}$ be the modulo integers.

Let $P = \{ I_0^{352},\ I_2^{352},\ \ldots,\ I_{350}^{352},\ I_{11}^{352},\ I_{33}^{352},\ I_{55}^{352},\ I_{77}^{352},\ I_{99}^{352},\ I_{121}^{352},\ I_{143}^{352},\ I_{165}^{352},\ I_{187}^{352},\ I_{209}^{352},\ I_{231}^{352},\ I_{253}^{352},\ I_{275}^{352},\ I_{297}^{352},\ I_{319}^{352},\ I_{341}^{352} \}$ be the natural neutrosophic elements associated with Z_{352}.

$$o(P) = 192 = 2^6 \times 3 = 2^5 (11 - 5).$$

In view of this we have the following theorem.

THEOREM 1.9: *Let $Z_{2^n 11}$ be the modulo integers P be the collection of all natural neutrosophic elements associated with $Z_{2^n 11}$ by introducing the operation of division on $Z_{2^n 11}$.*

$$o(P) = 2^{n+1} \times 3 = 2^n (11 - 5).$$

Proof is direct and hence left as an exercise to the reader.

Example 1.37: Let $Z_{208} = Z_{2^4 \times 13}$ be the modulo integers. Let P be the natural neutrosophic elements associated with Z_{208} through the operation of division.

$$P = \{ I_0^{208},\ I_2^{208},\ I_4^{208},\ \ldots,\ I_{206}^{208},\ I_{13}^{208},\ I_{39}^{208},\ \ldots.,\ I_{195}^{208} \};$$

$$\begin{aligned} o(P) &= 2^4 \times 7 \\ &= 2^4 \times (1 - 6) \\ &= 112. \end{aligned}$$

From this we can generalize to get the following theorem.

THEOREM 1.10: *Let $Z_{2^n \times 13}$ the modulo integer P be the collection of all natural neutrosophic elements obtained by the operation of division on $Z_{2^n \times 13}$.*

$$\textit{Clearly } o(P) = 2^n \times 7$$
$$= 2^n \times (13 - 6).$$

Proof is direct by using simple number theoretic techniques.

Example 1.38: Let Z_{136} be the modulo integers. Let P denote the natural neutrosophic elements associated $Z_{136} = Z_{8 \times 17}$.

$$P = \{ I_0^{136},\ I_2^{136},\ \ldots,\ I_{134}^{136},\ I_{17}^{136},\ I_{51}^{136},\ I_{85}^{136},\ I_{119}^{136} \}.$$

$$\begin{aligned} o(P) &= 72 = 2^3 \times 9 \\ &= 2^3 \times (17 - 8). \end{aligned}$$

Example 1.39: Let $Z_{272} = Z_{2^n \times 17}$ be the modulo integers.

$P = \{ I_0^{272}, I_2^{272}, \ldots, I_{270}^{272}, I_{17}^{272}, I_{51}^{272}, I_{85}^{272}, I_{119}^{272}, I_{153}^{272}, I_{187}^{272}, I_{221}^{272}, I_{255}^{272} \}$ be the natural neutrosophic elements associated with Z_{272}.

$$o(P) = 2^4 \times 9 = 2^4 \times (17 - 8).$$

Thus in view of this we have the following theorem.

THEOREM 1.11: *Let $Z_{2^n \times 17}$ be the modulo integers.*

P = {Collection of all natural neutrosophic elements of associated with $Z_{2^n \times 17}$ got by binary operation division on $Z_{2^n \times 17}$}.

Then $o(P) = 2^n \times (17 - 8) = 2^n \times 9$.

The proof is direct and hence left as an exercise to the reader.

Example 1.40: Let $Z_{608} = Z_{2^5 \times 19}$ be the modulo integer P be the collection of all natural neutrosophic elements associated with Z_{608}.

Then $o(P) = 2^5 \times 10 = 2^5 \times (19 - 9)$.

So it is left as an exercise to prove if $Z_{2^5 \times 19}$ be the modulo integers.

P the associated natural neutrosophic elements of $Z_{2^5 \times 19}$.

$$\text{Then } o(P) = 2^n \times 10 = 2^n \times (19 - 9).$$

Example 1.41: Let $Z_{2^4 \times 23}$ be the modulo integer P be the associated natural neutrosophic elements of Z_{368}.

$$\begin{aligned} o(P) &= 2^4 \times 12 \\ &= 2^6 \times 3 \\ &= 2^4 \times (23 - 11). \end{aligned}$$

Thus if p is any prime and $Z_{2^n \times p}$ be the modulo integer.

Let P be the collection of all associated natural neutrosophic elements of $Z_{2^n \times p}$ by performing the operation of division on $Z_{2^n \times p}$.

Find the order of p.

This is also left as a open conjecture.

Further it is left as a open problem to find the order of P if $m = 2^n \times p \times q$ of Z_m where p and q two distinct odd primes.

However we see in case of Z_m, $M = 2^3 \times 3 \times 5$ the

$$o(P) = 2^3 \times (pq - 4) = 2^3 \times 11.$$

Infact we can generate this for $Z_m = Z_{2^n \times p \times q}$ where p = 3 and q = 5 as follows.

If P is the usual natural neutrosophic elements associated with $Z_{2^n \times 3 \times 5}$ we see $o(P) = 2^n \times (pq - 5)$.

Let $Z_m = Z_{2^n \times p \times q \times r}$ p, q, r are all odd primes.

Find the number of natural neutrosophic elements associated with $Z_{2^n \times p \times q \times r}$.

However we will study the problem in case of $Z_{840} = Z_{2^3 \times 3 \times 5 \times 7}$.

P be the natural neutrosophic elements associated with Z_{840}.

The reader is left with the task of finding the o(P).

Example 1.42: Let $Z_{420} = Z_{2^2.3.5.7}$ be the modulo integers.

Let P be the natural neutrosophic elements associated with Z_{420}. o(P) = 324.

The reader is expected to verify and obtain a general formula.

So finding a general formula for Z_n, $n = 2^m$ p q r (p, q, r odd primes) happens to be a challenging problem.

As the operation is addition we see there are many idempotents under +.

We see the natural neutrosophic semigroup under + associated with Z_6 is as follows.

$\langle Z_6^I, +\rangle = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_4^6, I_3^6, 1+I_0^6, 2 + I_0^6, 3 + I_0^6, 4 + I_0^6, 5 + I_0^6, 1 + I_2^6, 2 + I_2^6, 3 + I_2^6, 4 + I_2^6, 5 + I_2^6, \ldots, 5 + I_0^6 + I_2^6 + I_3^6 + I_4^6\}$.

We see all $x \in \langle Z_6^I, +\rangle$ in general does not yield $6x = 0$.

So we cannot define $\langle Z_6^I, +\rangle$ to be the modulo integers 6.

Consider $I_2^6 + I_2^6 + I_2^6 + I_2^6 + I_2^6 + I_2^6 = I_2^6 \neq 0$.

In fact this collection easily proves one can have elements under addition which gives idempotents.

This will be defined as the natural neutrosophic semigroup of modulo integers.

Next we proceed onto study the division operation on $C(Z_n)$ and the MOD natural neutrosophic complex numbers associated with it.

However the study in this direction can be had in [11].

We give examples of MOD natural neutrosophic complex numbers.

Example 1.43: $\langle C^I(Z_2), +\rangle = \{0, 1, I_0^c, i_F, I_{1+i_F}^c, 1 + i_F, 1 + I_0^c, i_F + I_0^c, 1 + i_F + I_0^c, 1 + I_{1+i_F}^c, 1 + i_F + I_{1+i_F}^c, i_F + I_{1+i_F}^c, I_0^c + I_{1+i_F}^c, 1 + I_0^c + I_{1+i_F}^c, i_F + I_0^c + I_{1+i_F}^c, 1 + i_F + I_{1+i_F}^c + I_0^c\}$.

$$o\langle C^I(Z_2), +\rangle = 16.$$

Note $I_0^c + I_0^c = I_0^c$, $I_{1+i_F}^c + I_{1+i_F}^c = I_{1+i_F}^2$ and so on.

This has subsemigroups of different orders.

$P_1 = \{0, I_0^c, I_{1+i_F}^c, I_0^c + I_{1+i_F}^c\}$ is a subsemigroup which is an idempotent semigroup.

$P_2 = \{I_0^c, I_{1+i_F}^c, I_0^c + I_{1+i_F}^c\}$ is also an idempotent subsemigroup..

Clearly Lagrange's theorem is not true [5].

However weak Lagrange's theorem is true for semigroups under + all of which are finite order.

Infact finding order of these $\langle C^I(Z_n), +\rangle$ is a challenging problem.

Example 1.44: Let $\langle C^I(Z_3), +\rangle = \{0, 1, 2, i_F, 2i_F, 1 + i_F, 1 + 2i_F, 2 + i_F, 2 + 2i_F, I_0^c, 1 + I_0^c, 2 + I_0^c, i_F + I_0^c, 2i_F + I_0^c, 1 + i_f + I_0^c, 1 + 2i_F + I_0^c, 2 + i_F + I_0^c, 2 + 2i_F + I_0^c\}$ be the natural neutrosophic complex number semigroup.

This is not a group as it has idempotents with respect to addition.

Example 1.45: Let $\langle C^I\text{-} (Z_6), +\rangle = \{0, 1, 2, 3, 4, 5, i_F, 3i_F, 4i_F, 5i_F, 1 + i_F, 2 + i_F, \ldots, 5 + i_F, 1 + 2i_F, 2 + 2i_F, \ldots, 5 + 2i_F, \ldots, 5 + 5i_F, I_0^c, I_2^c, I_3^c, I_4^c, I_{2i_F}^c, I_{3i_F}^c, I_{4i_F}^c, I_{2+2i_F}^c, I_{2+3i_F}^c, I_{2+4i_F}^c, I_{3+3i_F}^c, I_{3+4i_F}^c, I_{4+2i_F}^c, I_{3+2i_F}^c$ and so on, $I_0^c + I_2^c, \ldots, I_0^c + I_{2i_F}^c + I_2^c, \ldots, a + I_0^c + I_2^c + I_3^c + I_4^c + I_{2i_F}^c + I_{4i_F}^c + I_{3i_F}^c + \ldots + I_{4+4i_F}^c\}$ be the natural neutrosophic complex modulo integer semigroup.

Finding the order of $\langle C^I(Z_n), +\rangle$ happens to be a challenging problem.

Example 1.46: Let $\langle C^I(Z_5), +\rangle = \{0, 1, 2, 3, 4, i_F, 2i_F, 3i_F, 4i_F, 1 + i_F, 2 + i_F, 3 + 4i_F, 4 + i_F, 3 + i_F, 4 + 4i_F, I_0^c, I_{3+i_F}^c, \ldots, \}$ be the natural neutrosophic complex modulo integer semigroup under +.

Find order of $\langle C^I(Z_5), +\rangle$.

Find all natural neutrosophic elements of $C^I(Z_5)$.

It is left as a open conjecture for finding the number of neutrosophic elements of $C^I(Z_5)$.

It is still a open conjecture.

Conjecture 1.2: Let $C^I(Z_n)$ be the collection of all natural neutrosophic complex modulo integers Find the number of natural neutrosophic complex modulo integers associated with $C^I(Z_n)$.

Example 1.47: Let $C^I(Z_{10}) = \{0, 1, 2, \ldots, 9, I_0^c, I_2^c, I_4^c, I_6^c, I_8^c, I_{2i_F}^c, I_5^c, I_{5i_F}^c, I_{4i_F}^c, I_{6i_F}^c, I_{8i_F}^c, I_{2+2i_F}^c, I_{2+4i_F}^c, I_{4+2i_F}^c, \ldots, I_{8+8i_F}^c\}$ be the collection of natural neutrosophic complex modulo integer associated with $C(Z_{10})$.

If $P = \{I_0^c, I_2^c, I_4^c, I_6^c, I_8^c, I_5^c, I_{2i_F}^c, I_{4i_F}^c, I_{6i_F}^c, I_{8i_F}^c, I_{5i_F}^c, I_{2+2i_F}^c, I_{2+4i_F}^c, \ldots, I_{8i_F+8}^c, \ldots\}$ be the collection of all natural neutrosophic complex modulo integers.

(i) Find order of P.
(ii) Find order of $\langle P, + \rangle$.
(iii) Find order of $\langle C^I(Z_{10}), + \rangle$.
(iv) Does there exist any relation between the 3 orders in a, b and c?

Example 1.48: Let $\langle C^I(Z_{13}), + \rangle$ be a natural neutrosophic complex modulo integers semigroup.

The reader is expected to find the number of natural neutrosophic complex modulo numbers.

This has subsemigroups which are idempotent subsemigroups under +.

Finding even natural neutrosophic elements in $C^I(Z_n)$ happens to be a challenging problem.

Almost all results carried out for $\langle Z_n^I, + \rangle$ can be done for $\langle C^I(Z_{13}), + \rangle$.

This task is left as an exercise to the reader.

Next we proceed onto study the $\langle Z \cup I \rangle_I$; we will illustrate this situation by some examples.

Example 1.49: Let $\{\langle Z_5 \cup I\rangle_I, +\} = \{0, 1, 2, 0, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I, I_0^I, I_I^I, I_{2I}^I, I_{1+2I}^I, \ldots\} = S$ be the MOD natural neutrosophic-neutrosophic modulo integer semigroup.

It is a difficult problem to find the order of S.

Even for small Z_n the determination of natural neutrosophic-neutrosophic elements is a challenging one.

Example 1.50: Let $G = \{\{\langle Z_4 \cup I\rangle_I, +\} = \{0, 1, 2, 3, I, 2I, 3I, 1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I, I_0^I, I_2^I, I_{2I}^I, I_I^I, I_{3I}^I$, and so on} be the natural neutrosophic-neutrosophic semigroup.

What is the order of G?

Find all natural neutrosophic-neutrosophic elements of G.

Example 1.51: Let $S = \{\langle Z_{10} \cup I\rangle_I, +\} = \{0, 1, 2, 3, \ldots, 9, I, 2I, 3I, \ldots, 1 + 9I, 2 + I, \ldots, 2 + 9I, 9 + 9I, I_0^I, I_2^I, I_4^I, I_5^I, I_6^I, I_8^I, I_I^I, I_{2I}^I, I_{4I}^I, I_{5I}^I, I_{6I}^I, I_{8I}^I, I_{2+2I}^I, I_{2+4I}^I, I_{2+6I}^I, I_{8+8I}^I$ and so on} be the natural neutrosophic neutrosophic semigroup under +.

Find all subsemigroups of S which are idempotent subsemigroups of S.

It is also left as an exercise to the reader to find the cardinality of S.

$I_{2+4I}^I + I_{2+4I}^I = I_{2+4I}^I$, $I_{5I+2}^I + I_{5I+2}^I = I_{5I+2}^I$ and so on are all idempotents under +.

$I_{2+8I}^I + I_{2+8I}^I = I_{2+8I}^I$ is also an idempotent of S.

Example 1.52: Let $S = \langle Z_{19} \cup I\rangle_I, +\} = \{0, 1, 2, \ldots, 18, I, 2I, \ldots, 18I, 1 + I, \ldots, 1 + 18I, \ldots, 18 + 18I, I_0^I, I_I^I, I_{2I}^I, \ldots$, and so on} be the MOD natural neutrosophic-neutrosophic semigroup under +.

The reader is left to find the order of S.

Find the number of subsemigroups of S which are idempotent semigroups.

The main observation is that if Z_n is a field, then the order of $\langle Z_n \cup I\rangle_I$, +} is smaller in comparison with Z_n; where n is a composite number.

Finding number of elements in case Z_n is a ring happens to be a very difficult problem.

All properties of $\{\langle Z_n \cup I\rangle_I, +\}$ can be studied analogously as in case of $\langle Z_n^I, +\rangle$.

Now we study the MOD natural neutrosophic dual number semigroup by some examples. For more refer [12, 24].

Example 1.53: Let S = $\{\langle Z_9 \cup g\rangle_I, +\}$ be the natural neutrosophic dual number semigroup.

S = {0, 1, 2, …, 8, g, 2g, …, 18g, 1 + g, 1 + 2g, …, 1 + 8g, 2 + g, 2 + 2g, …, 8 + 8g, I_0^g, I_3^g, I_6^g, I_g^g, I_{2g}^g, …, I_{8g}^g, I_{3+3g}^g, I_{6+6g}^g and so on}.

We see $I_{2g}^I + I_{2g}^g = I_{2g}^g$, $I_{3+3g}^I + I_{3+3g}^g = I_{3+3g}^g$ and so on are all idempotents of S.

Example 1.54: Let S ={$\langle Z_2 \cup g\rangle$, +} = {0, 1, g, 1 + g, I_0^g, I_g^g, 1 + I_0^g, g+ I_0^g, 1 + g + I_0^g, g + I_g^g, 1 + I_g^g, 1= g + I_g^g, I_0^g + I_g^g, 1+ I_0^g + I_g^g, g + I_0^g + I_g^g, 1 + g + I_0^g + I_g^g} be the natural neutrosophic dual number semigroup.

P = { I_0^g, I_g^g } generates a subsemigroup of order there under + and each element in { I_0^g, I_g^g, $I_0^g + I_g^g$ } is an idempotent semigroup under +, as $I_0^g + I_0^g = I_0^g$, $I_g^g + I_g^g = I_g^g$ and

$(I_0^g + I_g^g) + (I_0^g + I_g^g) = I_0^g + I_g^g$, hence the claim o(S) = 16.

Example 1.55: Let S = {⟨$Z_4 \cup g$⟩, +} = {0, 1, 2, 3, g, 2g, 3g, 1 + g, 1 + 2g, …, 3 + 3g, I_0^g, I_g^g, I_{2g}^g, … I_{2+2g}^g, I_{2+g}^g and so on} be the natural neutrosophic dual number semigroup under +.

$I_0^g + I_0^g = I_0^g$, $I_{2+2g}^g + I_{2+2g}^g = I_{2+2g}^g$ are idempotents.

Finding order of S happens to be a challenging problem.

Example 1.56: Let S = {⟨$Z_4 \cup g$⟩, +} = {0, 1, 2, 3, 4, g, 2g, 3g, 4g, 1 + g, 1 + 2g, 1 + 3g, 1 + 4g, 2 + g, 2 + 2g, 2 + 3g, 2 + 4g, 3 + g, 3 + 2g, 3 + 3g, 3 + 4g, 4 + g, 4 + 2g, 4 + 3g, 4 + 4g, I_0^g, I_g^g, I_{2g}^g, I_{4g}^g, I_{1+g}^g and so on} be the natural neutrosophic dual number semigroup under +.

Clearly S has subsemigroups which are idempotents under +.

In fact if P = { I_0^g, I_g^g, I_{2g}^g, …, I_{4g}^g and so on} ⊆ S; finding order of P happens to be a challenging problem.

However {⟨P, +⟩}, is an idempotent semigroup under +.

Thus this study yields an infinite class of idempotent semigroups under +.

The study of natural neutrosophic dual number semigroups happens to be a matter of routine so left as an exercise to the reader.

However several interesting problems and some open conjectures are proposed at the end of this chapter for the reader.

Next we proceed onto study the MOD natural neutrosophic special dual like number semigroups under + by some examples.

Example 1.57: Let G ={⟨Z_3 ∪ h⟩, +} be the natural neutrosophic special dual like number semigroup under + operation.

G = {0, 1, 2, h, 2h, 1 + h, 2 + h, 2h + 2, 1 + 2, I_0^h, I_h^h, I_{2h}^h, I_{1+2h}^h, I_{2+h}^h, $I + I_0^h$, 2 + I_0^h, h + I_0^h, … } is a semigroup under +.

Clearly P = { I_0^h, I_h^h, I_{1+h}^h, I_{2h}^h, I_{2+h}^h } generates an idempotent semigroup of natural neutrosophic elements under +.

Clearly ⟨P, +⟩ ⊆ G; o(P) = 5.

Finding order of G happens to be a difficult problem in case n is large which is used in the construction of Z_n.

Example 1.58: Let S = {⟨Z_6 ∪ h⟩$_I$, +} be the natural neutrosophic special dual like number semigroup.

Clearly B = {⟨ I_0^h, I_3^h, I_{3h}^h I_{4h}^h, I_{2h}^h, I_2^h, I_{3+3h}^h, +⟩ is a subsemigroup which is an idempotent semigroup.

It is left as an exercise for the reader to find all idempotent subsemigroups of S.

Example 1.59: Let S = {⟨Z_7 ∪h⟩, +} be the natural neutrosophic special dual like number semigroup.

All results derived in case of natural neutrosophic semigroup $\langle Z_n^I, +\rangle$ can be derived in case of $\{\langle Z_n \cup h\rangle_I, +\}$ with appropriate modifications.

Next we proceed onto study the notion of natural neutrosophic special quasi dual number semigroups under + by some examples.

Example 1.60: Let S = $\{\langle Z_3 \cup k\rangle, +\}$ be the natural neutrosophic special quasi dual number semigroup under +.

S = $\{0, 1, 2, k, 2k, 1 + k, 1 + 2k, 2 + k, 2 + 2k, I_0^k, I_k^k, I_{2k}^k, I_{1+k}^k, I_{2k+2}^k, 1 + I_0^k, 2 + I_0^k, k + I_0^k, 2k + I_0^k, 1 + k + I_0^k, 1 + 2k + I_0^k, 2 + k + I_0^k, 2 + 2k + I_0^k$ and so on$\}$.

There are subsemigroups which are idempotent semigroups under +.

We see as in case of $\langle Z_n^I, +\rangle$ we can develop the notion for $\{Z_n \cup k\rangle_I, +\}$.

This is left as an exercise to the reader.

Now we develop the natural neutrosophic semigroup of matrices and polynomials under +. For we need all these notions in the construction of semivector spaces over the semirings which are not Smarandache semirings. So in this we study only semimodules over natural neutrosophic modulo integers semigroups of finite order. Thus we develop the notion of additive semigroups of natural neutrosophic elements.

We will illustrate this situation by some examples.

Example 1.61: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} / a_i \in Z_{12}^I = \{0, 1, 2, \ldots, 11, I_0^{12}, I_2^{12}, I_3^{12},\right.$$

I_4^{12}, I_6^{12}, I_8^{12}, I_9^{12}, I_{10}^{12}, a + I_t^{12}; a ∈ Z_{12}; t = {0, 2, 3, 4, 6, 8, 9, 10}; 1 ≤ i ≤ 6, +} be the natural neutrosophic matrix semigroup under +.

There are several matrices in M which under '+' are idempotent subsemigroups of M.

$$x = \begin{bmatrix} I_2^{12} & I_4^{12} & I_6^{12} \\ I_0^{12} & I_8^{12} & I_9^{12} \end{bmatrix} \in M \text{ is such that } x + x = x.$$

Thus x is an additive idempotent of M.

Further o(M) < ∞.

Thus we have a finite natural neutrosophic modulo integer matrix semigroup under +.

Example 1.62: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \text{ where } a_i \in Z_7^I;\ 1 \leq i \leq 10 \right\}$$

be the natural neutrosophic modulo integer matrix semigroup under +. o(T) is finite.

T is only a semigroup and not a group as there are x ∈ T which are such that x + x = x.

Example 1.63: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \mid a_o \in \left\langle Z_{40}^{I} \cup I\right\rangle_I;\ 1 \leq i \leq 9, +\right\}$$

be the natural neutrosophic- neutrosophic modulo integer matrix semigroup under +.

W has subsemigroups which are idempotent subsemigroups. Infact W is a monoid.

$$P_1 = \left\{\begin{bmatrix} I_{10}^{I} & I_{16}^{I} & I_{12}^{I} \\ 0 & I_{10}^{I} & I_{14}^{I} \\ I_4^{I} & I_6^{I} & I_2^{I} \end{bmatrix}, +\right\}$$

is a subsemigroup of order one and P is an idempotent semigroup.

$$P_2 = \left\{\begin{bmatrix} I_1^{I} & I_{10}^{I} & 0 \\ 0 & I_4^{I} & I_6^{I} \\ I_8^{I} & 0 & I_{12}^{I} \end{bmatrix}, \begin{bmatrix} I_4^{I} & I_6^{I} & 0 \\ 0 & I_8^{I} & I_6^{I} \\ I_2^{I} & I_0^{I} & I_{12}^{I} \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} I_4^{I} + I_2^{I} & I_{10}^{I} + I_6^{I} & 0 \\ 0 & I_4^{I} + I_8^{I} & I_6^{I} \\ I_8^{I} + I_2^{I} & I_0^{I} & I_{12}^{I} \end{bmatrix}, +\right\}$$

is a subsemigroup which is the idempotent subsemigroup of W.

Infact o(W) is finite.

There are several idempotent subsemigroups.

Example 1.64: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \Big/ a_i \in \langle Z_{10} \cup I \rangle_I,\ 1 \le i \le 10, + \right\}$$

be the real natural neutrosophic column matrix semigroup under addition. $o(M) < \infty$.

M has several subsemigroups some of which are idempotent semigroups under +.

Example 1.65: Let $P = \{(a_1, a_2, a_3) / a_i \in Z^I_{49},\ 1 \le i \le 3, +\}$ be the real natural neutrosophic row matrix semigroup under +.

P has idempotent subsemigroups of finite order.

THEOREM 1.12: *Let S = {Collection of all $m \times n$ matrices with entries from the real natural neutrosophic semigroup $\langle Z^I_s, +\rangle$; $2 \le s < \infty$, +} be the real matrix natural neutrosophic semigroup under +.*

i) $o(S) < \infty$ and is a commutative monoid.
ii) S has several subsemigroups which are idempotent subsemigroups under +.

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto give examples of natural neutrosophic complex modulo integer matrix semigroups under +.

Example 1.66: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \;/\; a_i \in \langle C^I(Z_8), +\rangle, 1 \leq i \leq 6, + \right\}$$

be the natural neutrosophic finite complex modulo integer matrix semigroup under +.

V is a monoid of finite order.

$$M = \left\{ \begin{bmatrix} 0 \\ I_0^c \\ I_4^c \\ I_2^c + I_6^c \\ I_0^c + I_4^c \end{bmatrix} \right\}$$

is an idempotent subsemigroup of order one under +.

$$x = \begin{bmatrix} I_0^c \\ I_4^c \\ I_2^c \\ I_6^c \\ I_4^c \\ I_2^c \end{bmatrix} \in V \text{ is an idempotent of V as } x + x = x.$$

Infact $\langle x, +\rangle$ is an idempotent subsemigroup of order one.

$$T = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2^c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_0^c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_6^c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^c + I_2^c \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^c + I_6^c \\ 0 \\ 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_2^c + I_6^c \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^c + I_2^c + I_6^c \\ 0 \\ 0 \end{bmatrix} \right\}$$

is an idempotent subsemigroup of order 7 of V.

There are several such idempotent subsemigroups in V.

The reader is left with the task of finding the order of V.

Example 1.67: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle/ a_i \in \langle C^I(Z_{12}), + \rangle; 1 \leq i \leq 8, + \right\}$$

be the natural neutrosophic complex modulo integer matrix semigroup under +.

$$o(W) < \infty.$$

$$x = \begin{bmatrix} 0 & I_2^c \\ I_4^c & 0 \\ 0 & I_8^c \\ I_3^c & 0 \end{bmatrix} \in W \text{ is such that } x + x = x \text{ is a nontrivial}$$

idempotent of W .

Infact {x} is an idempotent subsemigroup of W.

$$P_1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, x \right\} \subseteq W \text{ is a submonoid of W of order 2.}$$

$$P_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_0^c & I_2^c \\ 0 & 0 \\ 0 & 0 \\ I_4^c & I_6^c \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I_2^c \\ 0 & 0 \\ 0 & 0 \\ 0 & I_6^c \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_0^c & I_2^c \\ 0 & 0 \\ 0 & 0 \\ 0 & I_6^c \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \subseteq W$$

is a again an idempotent subsemigroup of order 4 of W.

Infact W has many such idempotent subsemigroup.

Example 1.68: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} / a_i \in \langle C^I(Z_{17}), +\rangle, 1 \leq i \leq 16, +\right\}$$

be the natural neutrosophic finite complex modulo integer semigroup (monoid) under +.

This has several idempotent subsemigroups and $o(S) < \infty$.

All results related to real natural neutrosophic matrix semigroup under + are true in case of natural neutrosophic matrix complex modulo integer semigroup.

Infact of the collection of real natural neutrosophic matrix semigroup of a fixed order is always a subsemigroup (proper subset) of the collection of all natural neutrosophic finite complex modulo integer semigroups.

Next we just study the natural neutrosophic-neutrosophic matrix semigroups under + by some illustrative examples.

Example 1.69: Let $\{\langle Z_6 \cup I\rangle_I, +\} = P$ be the natural neutrosophic-neutrosophic semigroup under +.

Let $M = \{(a_1, a_2, \ldots, a_9) / a_i \in P, 1 \leq i \leq 9, +\}$ be the natural neutrosophic-neutrosophic matrix semigroup under +.

$o(M) < \infty$, M is a commutative monoid of finite order.

$N_1 = \{ I_0^I, 0, 0, \ldots, I_2^I), +\}$ is an idempotent subsemigroup of order one.

$N_2 = \{(0, 0, \ldots, 0), (I_0^I, \ldots, 0), (I_2^I, 0, \ldots, 0), (I_4^I, 0, \ldots, 0), (I_2^I + I_4^I, 0, \ldots, 0), (I_0^I + I_2^I, 0, \ldots, 0), (I_0^I + I_4^I, 0, \ldots, 0),$

$(I_0^I + I_2^I + I_4^I, 0, \ldots, 0), +\} \subseteq M$ is an idempotent subsemigroup of order 8.

Infact M has several idempotent subsemigroups some of which are monoids and some are not monodies.

Example 1.70: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle/ a_i \in \{\langle Z_8 \cup I\rangle_I, + \rangle\}, 1 \leq i \leq 7, + \right\}$$

be the natural neutrosophic-neutrosophic matrix semigroup of finite order.

This has several idempotents as well as S has subsemigroups which are idempotent subsemigroups.

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2^I \\ I_0^I \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} I_2^I \\ I_0^I + I_4^I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_0^I + I_6^I \\ I_0^I \\ I_4^I + I_2^I \end{bmatrix}$$

are some of the idempotents of S under +.

Infact each idempotent of S generates a subsemigroup which is also an idempotent subsemigroup of order one.

Example 1.71: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle/ a_i \in \langle\langle Z_{24} \cup I\rangle_I;\ 1 \leq i \leq 24, +\right\}$$

be the natural neutrosophic-neutrosophic matrix semigroup S has several idempotents and also has idempotent subsemigroup.

$$x_1 = \begin{bmatrix} I_2^I & I_4^I & I_6^I & I_{22}^I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I_{16}^I & I_0^I & I_3^I & I_9^I \\ I_{10}^I & I_{75}^I & I_{20}^I & I_4^I \\ 0 & 0 & 0 & 0 \end{bmatrix} 0 \in S \text{ is such that } x_1 + x_1 = x_1,$$

that is an idempotent of S. S has several idempotents.

Infact the collection of all idempotents of s forms a subsemigroup also known as the idempotent subsemigroup.

Interested reader can find all the idempotents of S.

Since this study is also a matter of routine we leave it as an exercise for the reader to study all the properties associated with S.

Next we proceed onto study the matrix of natural neutrosophic dual number semigroup under + by some examples.

Example 1.72: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \mid a_i \in \{\langle Z_{10} \cup g\rangle_I, +\}, 1 \leq i \leq 12, + \right\}$$

be the natural neutrosophic dual number matrix semigroup.

Clearly B is of finite order and is a commutative monoid under +.

Let $x_1 = \begin{bmatrix} I_5^g & I_0^g & 0 & 0 \\ 0 & 0 & I_2^g & I_4^g \\ I_8^g & I_4^g & 0 & 0 \end{bmatrix} \in B$ is such that $x + x_1 = x_1$.

$$P_1 = \{x_1, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\} \subseteq B$$

is a monoid which is an idempotent subsemigroup of order two.

$$P_2 = \{ \begin{bmatrix} I_4^g & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2^g \\ 0 & I_8^g & 0 & 0 \end{bmatrix}, \begin{bmatrix} I_2^g & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2^g \\ 0 & I_8^g & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} I_4^g + I_2^g & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2^g \\ 0 & I_8^g & 0 & 0 \end{bmatrix}\} \subseteq B$$

is an idempotent subsemigroup of B of order three.

Clearly P_2 is not a monoid only an idempotent subsemigroup.

$$P_3 = \{\begin{bmatrix} I_0^g & I_2^g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & I_8^g & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & I_2^g & 0 & I_6^9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_2^g & I_8^g & I_6^8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} I_0^g & I_2^g & I_8^g & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} I_0^g & I_2^g & 0 & I_6^g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} I_0^g & I_2^g & I_8^g & I_6^8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \subseteq B$$

is an idempotent subsemigroup of order 8.

This B has many idempotent subsemigroups.

Example 1.73: Let

$$S = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle/ a_i \in \{\langle Z_{18} \cup g\rangle_I, +\}, 1 \leq i \leq 5, +\right\}$$

be the natural neutrosophic matrix dual number commutative monoid of finite order.

This S has several idempotents as well as idempotent subsemigroups.

$$y_1 = \begin{bmatrix} I_0^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_2 = \begin{bmatrix} I_2^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_3 \begin{bmatrix} I_3^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_4 = \begin{bmatrix} I_4^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_5 = \begin{bmatrix} I_6^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$y_6 = \begin{bmatrix} I_8^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_7 = \begin{bmatrix} I_9^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_8 = \begin{bmatrix} I_{10}^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_9 = \begin{bmatrix} I_{12}^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_{10} = \begin{bmatrix} I_{14}^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$y_{11} = \begin{bmatrix} I_{16}^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_{12} = \begin{bmatrix} I_0^g + I_2^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_{13} = \begin{bmatrix} I_0^g + I_2^g + I_3^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$y_{14} = \begin{bmatrix} I_{16}^g + I_{14}^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_{15} = \begin{bmatrix} I_0^g + I_2^g + I_3^g + I_4^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, y_{16} = \begin{bmatrix} 0 \\ I_0^g \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$y_{14} = \begin{bmatrix} I_0^g + I_2^g + I_3^g + I_4^g + I_6^g \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and so on are some of the idempotent matrices of S.

Study of idempotent submonoids and idempotent subsemigroups happens to be an interesting exercise.

All properties associated with natural neutrosophic matrix dual number semigroups can be derived / studied as in case of other semigroups.

Next we will describe the natural neutrosophic special dual like numbers semigroups by some examples.

Example 1.74: Let M = $\{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) / a_i \in \{\langle Z_{12} \cup h\rangle_I, +\}$ $h^2 = h$; $1 \leq i \leq 7$; $+\}$ be the natural neutrosophic special dual like number row matrix semigroup under +.

Clearly M is of finite order and is a monoid. M has several idempotents and idempotent subsemigroups.

$$a_1 = (0, I_0^h, I_h^h, I_{5h}^h, I_2^h, 0, 0),$$
$$a_2 = (0, 0, 0, 0, 0, I_{7h}^h, 0),$$
$$a_3 = (I_2^h + I_{2+2h}^h, 0, 0, 0, 0, 0, 0).$$
$$a_4 = (0, 0, 0, 0, 0, 0, I_0^h + I_{4h}^h + I_{8+4hh}^h)$$

and so on are some of the idempotents of M.

$P_1 = \{(0, 0, 0, 0, I_0^h + I_2^h, 0, 0), (0\ 0\ 0\ 0, I_0^h, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, I_2^h, 0, 0)\} \subseteq M$ is an idempotent subsemigroup of M infact an idempotent monoid.

$P_2 = \{(I_4^h, 0, 0, 0\ 0\ 0\ 0), (I_8^h, 0, 0, 0, 0, 0, 0), (I_3^h, 0, 0, 0, 0, 0, 0), (I_3^h + I_8^h, 0, 0, 0, 0, 0, 0), (I_3^h + I_4^h, 0, 0, 0, 0, 0, 0), (I_4^h + I_8^h, 0, 0, 0, 0, 0, 0), (I_3^h + I_4^h + I_8^h, 0, 0, 0, 0, 0, 0)\}$ is an idempotent subsemigroup of order seven. Clearly P_2 is not a monoid.

$P_3 = \{(0, 0, 0, I_0^h, 0, 0, 0), (0, 0, 0, I_2^h, 0, 0, 0), (0, 0, 0, I_{10}^h, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0), (0, 0, 0, I_8^h, 0, 0, 0), (0, 0, 0, I_6^h, 0, 0, 0), (0, 0, 0, I_0^h + I_2^h, 0, 0, 0), (0, 0, 0, I_0^h + I_{10}^h, 0, 0, 0, 0), (0, 0, 0, I_0^h + I_8^h, 0, 0, 0), (0, 0, 0, I_0^h + I_6^h, 0, 0, 0), (0, 0, 0, I_2^h + I_{10}^h, 0, 0, 0), (0, 0, 0, I_2^h + I_8^h, 0, 0, 0), (0, 0, 0, I_2^h + I_6^h, 0, 0, 0), (0, 0, 0\ I_{10}^h + I_8^h, 0, 0, 0), (0, 0, 0, I_{10}^h + I_6^h, 0, 0, 0), (0, 0, 0, I_6^h + I_8^h, 0, 0, 0), (0, 0, 0, I_0^h + I_2^h + I_{10}^h, 0\ 0\ 0), (0, 0, 0, I_{10}^h + I_2^h + I_8^h, 0, 0, 0), (0, 0, 0, I_0^h + I_2^h + I_6^h, 0, 0, 0), (0, 0, 0, I_0^h + I_8^h + I_{10}^h, 0, 0, 0), (0, 0, 0, I_0^h + I_8^h + I_6^h, 0, 0, 0), (0, 0, 0, I_2^h + I_8^h + I_{10}^h, 0, 0, 0), (0, 0, 0, I_2^h + I_8^h + I_6^h, 0, 0, 0), (0, 0, 0, I_2^h + I_{10}^h + I_6^h, 0, 0, 0), (0, 0, 0, I_8^h + I_{10}^h + I_6^h, 0, 0, 0), (0, 0, 0, I_0^h + I_{10}^h + I_6^h, 0, 0, 0), (0, 0, 0, I_0^h + I_2^h + I_8^h + I_{10}^h, 0, 0, 0), (0, 0, 0, I_0^h + I_2^h + I_8^h + I_6^h, 0, 0, 0), (0, 0, 0, I_0^h + I_2^h + I_6^h + I_{10}^h, 0, 0, 0), (0, 0, 0\ I_0^h + I_6^h + I_8^h + I_{10}^h, 0, 0, 0), (0, 0, 0, I_6^h + I_2^h + I_{10}^h + I_8^h, 0, 0, 0), (0, 0, 0, I_0^h + I_6^h + I_2^h + I_8^h + I_{10}^h, 0, 0, 0)\}$ is an idempotent subsemigroup.

Clearly $o(P_3) = 32$.

This is the way we can get seven at idempotent subsemigroups.

Example 1.75: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \Bigg/ a_i \in \{\langle Z_{48} \cup h\rangle_I, +\}, 1 \leq i \leq 15, + \right\}$$

be the natural neutrosophic special dual like number matrix semigroup under +.

M is a commutative monoid of finite order. M has several idempotents.

$$x_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{24}^h \end{bmatrix} \text{ is such that } x_1 + x_1 = x_1.$$

$$x_2 = \begin{bmatrix} 0 & 0 & I_{12}^h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_0^h & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M \text{ is such that } x_2 + x_2 = x_2 \text{ and } x_2$$

is an idempotent.

$$x_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_{10}^h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M \text{ is an idempotent of M as } x_3 + x_3 = x_3.$$

$$x_4 = \begin{bmatrix} 0 & I_4^h + I_{10}^h + I_8^h + I_9^h & 0 \\ 0 & I_2^h + I_{40}^h & 0 \\ 0 & I_3^h + I_6^h + I_{12}^h & 0 \\ 0 & I_{14}^h + I_{42}^h + I_0^h & 0 \\ 0 & I_{22}^h + I_{24}^h & 0 \end{bmatrix} \in M \text{ is an}$$

idempotent as $x_4 + x_4 = x_4$.

Infact M has several idempotents. M also has idempotent subsemigroup of order 1, order 2, order 3, order 4 and so on.

We will give one or two idempotent subsemigroups of M.

$$P_1 = \left\{ \begin{bmatrix} I_0^h 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_4^h \end{bmatrix}, \begin{bmatrix} I_2^h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_4^h \end{bmatrix}, \begin{bmatrix} I_0^h + I_2^h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_4^h \end{bmatrix} \right\} \subseteq M$$

is an idempotent subsemigroup of order three.

$$P_2 = \left\{ \begin{bmatrix} 0 & I_2^h + I_4^h + I_8^h + I_{10}^h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_3^h + I_6^h + I_0^h & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

is an idempotent subsemigroup of order one.

All properties associated with natural neutrosophic special dual like number matrix semigroups is a matter of routine so left as exercise to the reader.

Next we proceed onto study by examples the natural neutrosophic special quasi dual number matrix semigroup.

Example 1.76: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle/ a_i \in \{\langle Z_6 \cup k\rangle_I, +\}\ 1 \leq i \leq 6, +\right\}$$

be the natural neutrosophic special quasi dual number column matrix semigroup under +.

Clearly $o(P) < \infty$.

P has idempotents as well as idempotent subsemigroups.

Take $y_1 = \begin{bmatrix} I^k_{3+3k} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in P$ is such an idempotent matrix.

Let $y_2 = \begin{bmatrix} I^k_{2k} \\ I^k_{3k} \\ I^k_{0} \\ 0 \\ 0 \\ 0 \end{bmatrix} \in P.$

$y_2 + y_2 = y_2$ so y_2 is also an idempotent of P.

Let $y_3 = \begin{bmatrix} I^k_2 + I^k_{3k} \\ I^k_0 + I^k_{2k+3} \\ I^k_{2k} + I^k_4 \\ I^k_{2+4k} + I^k_0 \\ I^k_0 + I^k_{3+2k} \\ I^k_k + I^k_{2k} \end{bmatrix} \in P$, we see $y_3 + y_3 = y_3$ so y_3

is an idempotent of P.

Infact P has many more idempotents.

Consider

$$B_1 = \left\{ \begin{bmatrix} I_0^k \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq P$$

is an idempotent subsemigroup of order one.

Consider

$$B_2 = \left\{ \begin{bmatrix} I_2^k \\ I_3^k + I_{4k}^k \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} I_2^k \\ I_3^k \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq P$$

is an idempotent subsemigroup of order 2.

$$B_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ I_3^k + I_4^k \\ 0 \\ 0 \\ I_2^k \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_3^k \\ 0 \\ 0 \\ I_2^k \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ I_4^k \\ 0 \\ 0 \\ I_2^k \end{bmatrix} \right\} \subseteq P$$

is an idempotent subsemigroup of order 3.

$$B_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^k + I_{4k}^k \\ I_{3k}^k + I_{4+2k}^k \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^k \\ I_{3k}^k + I_{4+2k}^k \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_{4k}^k \\ I_{3k}^k + I_{4+2k}^k \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^k + I_{4k}^k \\ I_{3k}^k \end{bmatrix} \right\} \subseteq P$$

is an idempotent subsemigroup of order four.

Infact we can get all ordered idempotent subsemigroups of P.

Example 1.77: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle/ a_i \in \{\langle Z_{20} \cup k \rangle_I, +\}, 1 \leq i \leq 16, + \right\}$$

be the natural neutrosophic special quasi dual number semigroup under +.

$$\text{Let } x_1 = \begin{bmatrix} I_0^k + I_2^k & 0 & 0 & 0 \\ 0 & I_{4k}^k & 0 & 0 \\ 0 & 0 & 0 & I_{4+5k}^k \\ I_{5k+2}^k & 0 & I_{10k}^k & 0 \end{bmatrix} \in W, x_1 + x_1 = x_1$$

is an idempotent of W.

$$x_2 = \begin{bmatrix} 0 & 0 & 0 & I_8^k + I_{12k}^k \\ 0 & 0 & 0 & I_{14k+6}^k \\ I_{6k}^k & 0 & 0 & 0 \\ 0 & I_{12k+6}^k & 0 & 0 \end{bmatrix} \in \text{W is such that } x_2 + x_2 = x_2$$

is an idempotent of W.

$$\text{Let } S_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_2^k & I_{4k}^k \end{bmatrix} \right\}$$

is an idempotent subsemigroup of W.

$$S_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_2^k + I_{2k}^k & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{2k}^k & 0 \end{bmatrix} \right\} \subseteq W$$

is an idempotent subsemigroup of order two.

$$S_3 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{2k}^k + I_0^k \\ 0 & 0 & 0 & I_k^k + I_{4k}^k \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_0^k \\ 0 & 0 & 0 & I_k^k + I_{4k}^k \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{2k}^k \\ 0 & 0 & 0 & I_k^k + I_{4k}^k \end{bmatrix} \right\} \subseteq W$$

is an idempotent subsemigroup of order 3.

Study of properties associated with natural neutrosophic special quasi dual number semigroups happens to be a matter of routine so it is left for the reader as an exercise.

Next we study the notion of finite polynomials semigroups under + using natural neutrosophic elements by some examples.

Example 1.78: Let

$$P[x]_5 = \left\{ \sum_{i=0}^{5} a_i x^i \;/\; a_i \in P = \{ Z_8^I, +\} \right.$$

all polynomials of degree less than or equal to 5 with coefficients from $\{ Z_8^I, +\}$.

$P[x]_5$ is defined as the natural neutrosophic real polynomial semigroup under +.

Clearly $o(P[x]_5) < \infty$, that is $P[x]_5$ is of finite order.

$P[x]_5$ has idempotents.

For consider $y = I_4^8 \; x^3 \in P[x]_5$, clearly y + y = 5.

Let $y_1 = I_6^8 x^2 + I_0^8 x + I_2^8 \in P[x]_5$.

Clearly $y_1 + y_1 = y_1$ is an idempotent of $P[x]_5$.

$P[x]_5$ has several idempotent $P[x]_5$ also has idempotent subsemigroup as well as subsemigroups which are not idempotents.

Consider $A_1 = \{\langle 3x^3 + x + 1\rangle, +\} \subseteq P[x]_5$ is a subsemigroup which is not an idempotent semigroup.

$A_2 = \{ I_2^6 x^3 + I_4^8,\ I_6^8 x^3 + I_2^8,\ (I_2^8 + I_6^8)x^3 + I_4^8 + I_2^8 \}$ is an idempotent subsemigroup of $P[x]_4$.

$A_3 = \{ I_4^8 x^2 + I_2^8 \} \subseteq W$ is an idempotent subsemigroup of order one.

$A_4 = \{ I_2^8 x^2, (I_2^8 + I_6^8)\ x^2 + I_0^8 \} \subseteq P[x]_5$ is an idempotent subsemigroup of order two.

We can get subsemigroups which are idempotent or otherwise.

Example 1.79: Let

$$B[x]_8 = \left\{ \sum_{i=0}^{5} a_i x^i \,/\, a_i \in B = \{ Z_{12}^I, +\}; 0 \leq i \leq 8, +\right\}$$

be the natural neutrosophic real coefficient polynomial semigroup of finite order under +.

$B[x]_8$ has idempotents, subsemigroups and idempotent subsemigroups.

Let $P_1 = \{\langle 2x^3 + 4x + 1,\ 3x + 2\rangle\} \subseteq B[x]_8$ generates a subsemigroup of finite order which is not an idempotent subsemigroup.

$P_2 = \{ I_4^{12} x^3 + (I_3^{12} + I_6^{12})\ x^2 + I_6^{12} + I_0^{12} + I_8^{12} \}$ is an idempotent subsemigroup of order one.

$P_3 = \{ I_3^{12} x^3 + I_4^{12} x + I_6^{12} + I_0^{12},\ I_4^{12} x,\ I_3^{12} x^3 + I_6^{12},\ I_{4x}^{12} + I_3^{12} x^3 + I_6^{12} \}$ is an idempotent subsemigroup of order four.

We can find idempotent subsemigroups of finite order.

However finding the order of $B[x]_8$ happens to be a challenging problem.

Example 1.80: Let

$$B[x]_{10} = \left\{ \sum_{i=0}^{10} a_i x^i \,/\, a_i \in B = \{ Z_7^I, +\}; + ; 0 \leq i \leq 10 \right\}$$

be the natural neutrosophic real coefficient semigroup of polynomials of finite order under +.

Find order of $B[x]_{10}$ is a challenging one for it depends on $\langle Z_n^I, +\rangle$ also.

Next we proceed onto give examples of natural neutrosophic-neutrosophic coefficient polynomial semigroups of finite order under +.

Example 1.81: Let

$$G[x]_3 = \left\{ \sum_{i=0}^{3} a_i x^i \,/\, a_i \in G = \{Z_6 \cup I\}_I; +\} ; 0 \leq i \leq 3, + \right\}$$

be the natural neutrosophic-neutrosophic coefficient polynomial semigroup of finite order $G[x]_3$ has idempotents and idempotents subsemigroups.

Take $a = I_0^I x^3 + (I_2^I + I_{3I}^I)x^2 + (I_{3I+3}^I + I_{2I+4}^I)\, x + I_{4I}^I \in G[x]_3$.

Clearly a + a = a is an idempotent of $G[x]_3$.

$G[x]_3$ has several such idempotents. However $o(G[x]_3 < \infty$.

Now $P_1 = \{ I_{4I+2}^I x^3 + I_{2I}^I \}$ is an idempotent of subsemigroup of finite order.

There are several such idempotent subsemigroups.

Example 1.82: Let

$$S[x]_7 = \left\{\sum_{i=0}^{7} a_i x^i \,/\, a_i \in S = \{\langle Z_{10} \cup I\rangle_I, +\} \,;\, 0 \leq i \leq 7, +\right\}$$

be the natural neutrosophic-neutrosophic coefficient polynomial semigroup under +.

Clearly $S[x]_7$ is a commutative monoid of finite order.

$S[x]_7$ has several subsemigroup which are not idempotent. There are also idempotent subsemigroup.

Further $S[x]_7$ has idempotents.

$$Z_{10}[x]_7 = \left\{\sum_{i=0}^{7} a_i x^i \,/\, a_i \in Z_{10}\right\}$$

is a subsemigroup of finite order which is not idempotent.

$$(\langle Z_{10} \cup I\rangle)\,[x] = \left\{\sum_{i=0}^{7} a_i x^i \,/\, a_i \in \langle Z_{10} \cup I\rangle = \{a + bI \,/\, a, b \in Z_{10}^I\},\right.$$

$$\left. 0 \leq i \leq 7, +\right\}$$

is again a subsemigroup of finite order which is not an idempotent subsemigroup.

$P_1 = \{ I_5^I x^3 + (I_6^I + I_0^I)x^2 + (I_2^I + I_{2+4I}^I + I_{5I}^I),$ $I_2^I, I_{2+4I}^I, I_2^I + I_{2+4I}^I, I_5^I x^3,$ $I_2^I +$ $I_5^I x^3;$ $I_2^I + I_{2+4I}^I,$ $I_{2+4I}^I + I_5^I$ $x^3, +\} \subseteq S[x]_7$ is an idempotent subsemigroup of order 9.

We have several such idempotent subsemigroups of finite order.

It is left as an exercise for the reader to find the order of $S[x]_7$

a = $I^I_{5+4I}x^5 + (I^I_{2I} + I^I_{6I})\,x^2 + I^I_0 + I^I_8$ in S[x]$_7$ is an idempotent element of S[x]$_7$.

Infact {a} is also an idempotent subsemigroup of order 1.

Let b = $I^I_0 + I^I_8\,x + I^I_{6I}\,x^2 \in$ S[x]$_7$. Clearly b + b = b so b is an idempotent of S[x]$_7$.

Example 1.83: Let

$$P[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in P = \langle Z_{19} \cup I\rangle_I +\};\ 0 \le i \le 9, +\right\}$$

be the natural neutrosophic neutrosophic coefficient polynomial semigroup of finite order.

a = $(I^I_I + I^I_0)x^8 + I^I_{3I} \in$ P[x]$_9$ as an idempotent as well as the idempotent subsemigroup of order one.

The reader is left with the task of finding 3 subsemigroups and 3 idempotent subsemigroups of P[x]$_9$.

Next we proceed on to give examples of finite natural neutrosophic complex modulo integer coefficient polynomial semigroup.

Example 1.84: Let

$$B[x]_{12} = \left\{\sum_{i=0}^{12} a_i x^i \,/\, a_i \in B = \{\langle C^I(Z_4), +\rangle\} = B,\ 0 \le i \le 12, +\right\}$$

be the natural neutrosophic complex modulo integer coefficient polynomial semigroup.

Clearly B[x]$_{12}$ is a finite commutative monoid under +.

Further $B[x]_{12}$ is a S-semigroup.

Consider a = $I^c_{2+2i_F} x^8 + (I^c_0 + I^c_2) \in B[x]_{12}$.

Clearly a is an idempotent and $\langle a \rangle$ is an idempotent subsemigroup $B[x]_{12}$ of order 1.

$P_1Z_{14}[x]_{12} \subseteq B[x]_{12}$ is a subsemigroup of $B[x]_{12}$ which is again a subsemigroup of finite order.

$P_2 = C(Z_4)\ [x]_{12} \subseteq B[x]_{12}$ is also a subsemigroup of $B[x]_{12}$ of finite order.

Infact finding idempotent subsemigroup of $B[x]_{12}$ is left as an exercise to the reader.

Example 1.85: Let

$$S[x]_4 = \left\{ \sum_{i=0}^{4} a_i x^i \ / \ a_i \in S = \{\langle C^I(Z_7), +\rangle\} = S; 0 \leq i \leq 4, +\right\}$$

be the complex modulo integer natural neutrosophic coefficient polynomial semigroup $o(S[x]_4) < \infty$ and is infact a commutative monoid.

$a = I^I_{6I} + I^I_0 x + I^I_{2I} x^3 \in S[x]_5$ is such that a + a = a and infact a is an idempotent of $S[x]_4$ and a generates an idempotent subsemigroup of order 1.

Next we proceed onto provide examples of natural neutrosophic dual number coefficient polynomial semigroup under +.

Example 1.86: Let

$$D[x]_9 = \left\{ \sum_{i=0}^{9} a_i x^i \ / \ a_i \in D = \{\langle Z_{12} \cup g\rangle_I\ g^2 = 0, +\}; 0 \leq i \leq 9, +\right\}$$

be the natural neutrosophic dual number coefficient polynomial semigroup.

$a = I^g_{10+2g}\ x^6 + (I^g_{2g} + I^g_{4g+3})x^3 + I^g_{4g+4} + I^g_{2g+6} + I^g_{8+9g} \in D[x]_9$ is an idempotent as a + a = a and a generates an idempotent subsemigroup of order one

$P_1 = \{ I^g_{4g}\, x^3 + I^g_{2g}\, x + I^g_0 + + I^g_{6g},\ I^g_{2g}\, x,\ I^g_0 + I^g_{6g},\ I^g_{2g}\, x + I^g_0 + I^g_{6g} \} \subseteq D[x]_9$ is an idempotent subsemigroup of order four.

Reader is left with the task of finding the order of $D[x]_9$.

Find atleast 4 subsemigroups which are not idempotent subsemigroups.

Example 1.87: Let

$$F[x]_6 = \left\{ \sum_{i=0}^{6} a_i x^i \,/\, a_i \in F = \{\langle Z_{11} \cup g\rangle, +\}\ g^2 = 0; 0 \leq i \leq 6; +\right\}$$

be the natural neutrosophic dual number coefficient polynomial semigroup of finite order.

$a = I^g_{3g}\, x^6 + I^g_{4g}\, x^4 + I^g_0\, x^2 + (I^g_{4g} + I^g_{8g} + I^g_{10g}) \in F[x]_6$, a + a = a so a is an idempotent of $F[x]_6$. $\langle a\rangle$ also generates an idempotent subsemigroup of under 1.

The reader is left with the task of finding the order of $F[x]_6$ and 3 idempotent subsemigroups of $F[x]_6$.

All these semigroups are also S-semigroups.

Next we give examples of natural neutrosophic special dual like number coefficient polynomial semigroups of finite order.

Example 1.88: Let

$$G[x]_{23} = \left\{ \sum_{i=0}^{3} a_i x^i \,/\, a_i \in G = \{\langle Z_{14} \cup h\rangle_I, +\} = G; 0 \leq i \leq 23, +\right\}$$

be the natural neutrosophic special dual like number coefficient polynomial semigroup.

Clearly $G[x]_{23}$ is a finite commutative monoid.

$p = I_7^h + I_{2+6h}^g x \in G[x]_{23}$ is such that $p + p = p$.

$p_1 = \{(I_0^h + I_{7h}^h)x^{20} + (I_{4h+2}^h + I_{8h}^h)x^{10} + I_{2h+10}^h + I_{12h+4}^h + I_{10+12h}^h, (I_{4h+2}^h + I_{8h}^h)x^{10}, I_{12h+4}^h, I_{4h+2}^h + I_{8h}^h + I_{12h+4}^h\} \subseteq G[x]_{23}$ is an idempotent subsemigroup of order four.

Thus $G[x]_{23}$ has several idempotents and idempotents subsemigroup as well as subsemigroups.

$Z_{14}[x]_{23} \subseteq G[x]_{23}$ is a subsemigroup which is not an idempotent subsemigroup ($\langle Z_{14} \cup h\rangle[x]_{23} \subseteq G[x]_{23}$ is again a subsemigroup which is not an idempotent subsemigroup of $G[x]_{23}$.

The reader is left with the task of finding the order of $G[x]_{23}$.

Infact both $Z_{14}[x]_{23}$ and $(\langle Z_{14} \cup h\rangle)[x]_{23}$ are subsemigroups which are groups of $G[x]_{23}$.

Example 1.89: Let

$$W[x]_{12} = \left\{ \sum_{i=0}^{3} a_i x^i \,/\, a_i \in W = \{\langle Z_{10} \cup h\rangle_I, +\}, h^2 = h, \right.$$

$$\left. 0 \leq i \leq 12, +\right\}$$

be the natural neutrosophic special dual like number coefficient polynomials semigroup under +.

$Z_{10}[x]_{12}$ is a finite subsemigroup of $W[x]_{12}$ which is a group under +

$p = I_5^h x^8 + (I_2^h + I_{4h}^h + I_{8h+6}^h)x^6 + (I_5^h + I_5^h)x^3 + I_2^h + I_{6h}^h \in W[x]_{12}$ is an idempotent of $W[x]_{12}$ as p + p = p.

$P = \{ I_{8h}^h x^7,\ I_{4h}^h x^5 + I_{2h}^h x^3\ I_{8+8h}^h + I_h^h + I_0^h x,\ I_{8h}^h x^7 + I_{4h}^h x^5 + I_{2h}^h x^3,\ I_{8h}^h x^7 + I_{8+8h}^h + I_h^h + I_0^h x,\ I_{4h}^h x^5 + I_{2h}^h x^3 + I_{8+8h}^h + I_h^h + I_0^h x,\ I_{8h}^h x^7 + I_{4h}^h x^5 + I_{2h}^h x^3 + I_0^h x + I_h^h + I_{8+8h}^h \} \subseteq W[x]_{12}$ is an idempotent subsemigroup of order 7.

Thus $W[x]_{12}$ has subsemigroups, groups, idempotent subsemigroups and idempotents.

The reader is left with the task of finding all idempotents of $W[x]_{12}$.

Next we proceed onto give examples of natural neutrosophic special quasi dual number coefficient polynomial semigroups of finite order under +.

Example 1.90: Let

$$S[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \,/\, a_i \in S = \{\langle Z_{16} \cup k\rangle_I, k^2 = 15k, +\},\quad 0 \le i \le 6, +\right\}$$

be the natural neutrosophic special quasi dual number coefficient polynomial semigroups of finite order under +.

$Z_{16}[x]_6$ is a group under +. $(\langle Z_{16} \cup k\rangle)[x]_6$ is again a group under +. $S[x]_6$ has several idempotents.

The reader is expected to find the number of idempotents in $S[x]_6$.

Example 1.91: Let

$$B[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in B = \{\langle Z_9 \cup k\rangle_I, k^2 = 8k, +\}, 0 \leq i \leq 9, +\right\}$$

be the polynomial with coefficients from the natural neutrosophic special quasi dual number semigroup B.

$B[x]_9$ under + is a finite semigroup.

$B[x]_9$ has substructures under + which are groups of finite order.

Infact $B[x]_9$ has subsemigroups which are idempotent subsemigroups so are not groups under +.

Infact $B[x]_9$ has several idempotents.

The reader is left with the task of finding order of $B[x]_9$.

We can prove all natural neutrosophic semigroups real or complex or neutrosophic or dual number or special dual like number or special quasi dual number all them contain the basic modulo integers Z_n as a subset which is always a group under +.

So all these semigroups are Smarandache semigroups.

We have in this chapter seen natural neutrosophic semigroups of finite order of all types and obtained several interesting properties about them.

Infact all these semigroups are Smarandache semigroups.

Further we see all these semigroups cannot be groups as the contain elements which are idempotents under +.

This is the first time to the best of knowledge of authors which gives infinite number of finite semigroups under + which has idempotents.

In the following we suggest problems some of which are open conjectures taken as research problems.

Problems

1. Enumerate all special features associated with Z_n^I; $2 \leq n < \infty$.

2. Let $Z_4^I = \{0, 1, 2, 3, I_0^4, I_2^4\}$ be the MOD natural neutrosophic elements associated in Z_4.

   i) Find $o(\langle Z_4^I, +\rangle)$.
   ii) Prove $\langle P, +\rangle = \{I_0^4, I_2^4, I_2^4 + I_0^4, +\}$ is an idempotent semigroup.
   iii) Find all subsemigroups of $\langle Z_4^I, +\rangle$.

3. Let $G = \langle Z_{10}^I, +\rangle$ be the MOD natural neutrosophic semigroup.

   i) Find o(G).
   ii) If P = {(collection of all natural neutrosophic elements of Z_{10} got by division operation on Z_{10}} find o(P).
   iii) Find o(⟨P, +⟩ as an idempotent semigroup.
   iv) Prove ⟨P, +⟩ is an idempotent semigroup and not a monoid of G.
   v) Find all idempotents of G.
   vi) Prove G is a S-semigroup.
   vii) Find at least 3 subsemigroups which are not idempotent subsemigroups of G.

4. Let $P = \{\langle Z_{13}^I, +\rangle\}$ be the MOD real natural neutrosophic semigroup under +.

Study questions (i) to (vii) of problem (3) for this P.

5. Let B = $\{\langle Z^I_{248}, +\rangle\}$ be the MOD real natural neutrosophic semigroup under +.

   i) Study questions (i) to (vii) of problem 3 for this B.
   ii) Obtain any other special feature associated with this B.

6. Let T = $\{\langle Z^I_{23}, +\rangle\}$ be the MOD natural neutrosophic semigroup under +.

   i) Study questions (i) to (vii) of problem 3 for this T.
   ii) Show if in Z^I_p, p is a prime then $o(\langle Z^I_p, +\rangle)$ is not very large.
   iii) Find $o(\langle Z^I_p, +\rangle)$; when p is a prime.

7. Obtain all special features about the real natural neutrosophic semigroup under + in Z_n; n = $p_1^{t_1} p_2^{t_2}, \ldots, p_s^{t_s}$, p_i's are distinct primes and $t_i > 1$; $i \leq i \leq s$.

   i) Prove if s is very large then $o(\langle Z^I_n, +\rangle)$ is large.

8. Let S = $\{C_I(Z_{10}), +\}$ be the MOD natural neutrosophic finite complex modulo integer semigroup under +.

   i) Study questions (i) to (vii) of problem (3) for this S.
   ii) Prove $\langle Z^I_{10}, +\rangle \subseteq$ S and $\langle Z^I_{10}, +\rangle$ is always a subsemigroup of S.
   iii) Prove S is a S-semigroup.
   iv) Can there be a situation in which both S and $\langle Z^I_{10}, +\rangle$ contain same number of natural neutrosophic elements?

9. Let $S_1 = \{\langle C^I(Z_{11}), +\rangle\}$ be the natural neutrosophic complex modulo integer semigroup under +.

   i) Study questions (i) to (vii) of problem (3) for this S_1.
   ii) Compare this S_1 with S of problem 8.
   iii) Which semigroup has higher cardinality S or S_1? Justify your claim.

10. Let $S_2 = \{\langle C^I(Z_p), +\rangle\}$, $S_1 = \{C^I(Z_{p+1}), +\}$, $S = \{\langle C^I(Z_{p-1}), +\rangle\}$ be three natural neutrosophic finite complex modulo integer semigroup under + (p a prime).

    i) Study questions (i) to (vii) of problem (3) for these S_1, S_2 and S.
    ii) Show
       a) $o(S_2) > o(S)$.
       b) $o(S_2) > o(S_1)$.
       c) Will $o(S) > o(S)$.
       Justify your claim in case of (c).
    iii) Hence or otherwise show if Z_n, n a prime then $o(\langle C^I(Z_n), +\rangle)$ is smaller when instead on n, n–1 and n + 1 are used where both n + 1 and n – 1 are not primes.

11. Study problem (ii) for $C^I(Z_{12})$, $C^I(Z_{13})$ and $C^I(Z_{14})$ under the operation + are semigroups generated by $C^I(Z_{12})$ and so on.

    a) Show $o(\langle C^I(Z_{12}), +\rangle) > o(\langle C^I(Z_{14}), +\rangle)$.
    b) What is $o(\langle C^I(Z_{13}), +\rangle)$?

12. Study problem (11) for $\langle C^I(Z_{22}), +\rangle = S$, $S_1 = \langle C^I(Z_{23}), +\rangle$ and $S_2 = \langle C^I(Z_{24}), +\rangle$.

    Prove $o(S_2) > o(S)$ and $o(S_2) > o(S)$.

13. Let $B_1 = \langle Z^I_{42}, +\rangle$, $B_2 = \langle Z^I_{43}, +\rangle$ and $B_3 = \langle Z^I_{44}, +\rangle$ be real natural neutrosophic semigroup.

i) Is $o(B_2) < o(B_1)$?
ii) Is $o(B_1) > o(B_3)$?
iii) Is $o(B_3) > o(B_2)$?

14. Let S = $\{\langle Z_{10}^{I} \cup I\rangle, +\}$ be the natural neutrosophic-neutrosophic semigroup under +.

a) Find o(S).
b) Study questions (i) to (viii) of problem (3) for this S.
c) If R = $\langle Z_{10}^{I}, +\rangle$ and T = $\langle C^{I}(Z_{10}), +\rangle$ then compare the order of T, R and S.
d) Prove R $\not\subseteq$ S for and Z_n.

15. Obtain all special features associated with natural neutrosophic-neutrosophic semigroup under +.

16. Can we say $o(\langle\langle Z_n \cup I\rangle_I, +\rangle) > o(\langle C^{I}(Z_n), +\rangle)$?

17. Let $S_1 = \{\langle\langle Z_{9^2} \cup I\rangle_I, +\rangle$ and $S_2 = \{\langle\langle Z_{625} \cup I\rangle_I, +\rangle$ be any two natural neutrosophic-neutrosophic semigroup.

Study the common features enjoyed by S_1 and S_2.

18. Obtain all special features associated with $\{\langle\langle Z_n \cup I\rangle_I, +\rangle\}$ the natural neutrosophic semigroup under +.

i) n a prime.
ii) n = 2p p a prime.
iii) $n = p_1 \ldots p_t$; $p_1, p_2, \ldots, p_t$ are distinct primes.

19. Let B = $\{\langle\langle Z_{480} \cup I\rangle_I, +\rangle\}$ be the natural neutrosophic-neutrosophic additive semigroup.

i) Find the number idempotents in B.
ii) Find the number of idempotent subsemigroups of B.
iii) Prove B is a S-semigroup.

iv) Find all subsemigroups of B which are not idempotent subsemigroups.
v) Find all idempotent neutrosophic elements of B.
vi) Find all subgroups of B under +.

20. Let $S = \{\langle Z_{12} \cup g\rangle_I, +\rangle\}$ be the natural neutrosophic dual number semigroup.

a) Study questions (i) to (vii) of problem (3) for this S.
b) Compare S with $S_1 = \{\langle Z_{12} \cup I\rangle_I, +\rangle\}$, and $S_2 = \langle C^I(Z_{12}), +\rangle$.

Obtain any other special and distinct features associated with natural neutrosophic dual number semigroup under +.

21. Let $S = \{\langle Z_{19} \cup g\rangle_I, +\}$ be the natural neutrosophic dual number semigroup.

a) Study questions (i) to (vii) of problem (3) for this S.
b) Compare o(S) with $o(\langle Z_{19} \cup I\rangle_I, +)$ and $o(\langle C^I(Z_{19}) +\rangle)$.

22. Let $S[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \;/\; a_i \in \{ I_{20}^I, I\} = S; 0 \leq i \leq 10, +\right\}$

be the real natural neutrosophic polynomial coefficient semigroup under +.

i) Find order of $S[x]_{10}$.
ii) How many elements in $S[x]_{10}$ are idempotents?
iii) Find all idempotent subsemigroups of $S[x]_{10}$.
iv) Find all subsemigroups of $S[x]_{10}$ which are not idempotent subsemigroups.

23. Let $S_1 = \{\langle Z_{24} \cup g\rangle_I, +\rangle$, $S_2 = \{\langle Z_{24} \cup I\rangle_I, +\rangle\}$ and $S_3 = \{C^I(Z_{24}), +\}$ be three natural neutrosophic semigroups of different types.

i) Compare S_1, S_2 and S_3 among themselves.
ii) Which has highest number of natural neutrosophic elements associated with it?

24. Let $S = \{\langle Z_{16} \cup h\rangle_I ; h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup under +.

i) Study questions (i) to (vii) of problem (3) for this S
ii) Compare S with $B = \{\langle Z_{16} \cup g\rangle_I, g^2 = 0, +\}$.
iii) Compare S with $D = \{\langle Z_{16} \cup I\rangle_I, I^2 = I, +\}$.
iv) Compare S with $P = \{C^I (Z_{16}), +\}$.

25. Let $M = \{\langle Z_{43} \cup h\rangle_I, h^2 = h, +\}$ be the natural neutrosophic special dual like number semigroup under +.

i) Study questions (i) to (vii) of problem (3) for this M
ii) Compare M with $N = \{C^I(Z_{43})\ i_F^2 = 42, +\}$. Find o(M) and o(N).
iii) Let $S = \{\langle Z_{43} \cup g\rangle_I, g^2 = 0, +\}$ be the natural neutrosophic dual number semigroup. Is o(S) > o(M)?
iv) Let $T = \{\langle Z_{43} \cup h\rangle, I^2 = I, +\rangle\}$ be the natural neutrosophic-neutrosophic semigroup.
Is o(T) = o(M)?

26. Obtain any other special feature associated with $M = \{\langle Z_m \cup h\rangle_I, h^2 = h, +\}$ the natural neutrosophic special dual like number semigroup.

27. Let $W = \{\langle Z_{90} \cup k\rangle, k^2 = 89k, +\}$ be the natural neutrosophic special quasi dual number semigroup under +.

i) Study questions (i) to (vii) of problem (3) for this W.
ii) If $V = \{\langle Z_{90} \cup h\rangle_I, h^2 = h, +\}$. Compare V and W.

28. Let $B = \{\langle Z_{19} \cup k\rangle_I, k^2 = 18k, +\}$ be the natural neutrosophic special quasi dual number semigroup.

i) Study questions (i) to (vii) of problem (3) for this B.
ii) Compare W of problem 27 with this V.
iii) Compare V with $S = \{C^I (Z_{19}), +\}$.

29. Let $L = \{\langle Z_n \cup k\rangle_I,\ k^2 = (n-1)k,\ +\}$ be the natural neutrosophic special quasi dual number semigroup under +.

i) Study questions (i) to (vii) problem (3) for this L.
   a) Study L when n is a prime.
   b) $n = p^t$; p a prime $t > 1$.
   c) n a large composite number with many divisors.

30. Let $B = \{(a_1, a_2, \ldots, a_6) / a_i \in \langle Z^I_{48}, +\rangle\ 1 \leq i \leq 6, +\}$ be the natural neutrosophic real row matrix semigroup under +.

i) Find o(B).
ii) Study questions (i) to (vii) of problem (3) for this B.
iii) Find all real natural neutrosophic matrices of B.
iv) Prove B is a S-semigroup.
v) Find the number of subgroups in B.

31. Let $D = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} / a_i \in \langle Z^I_{47}, +\rangle; 1 \leq i \leq 18, +\right\}$

be the real natural neutrosophic matrix semigroup.

i) Study questions (i) to (v) of problem (30) for this D.

ii) If Z_{47} is replaced by Z_{48} compare the semigroups.

32. Let T = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}\right.$ / $\langle a_i \in C^I(43), +\rangle$, $1 \le i \le 8\}$ be the natural neutrosophic complex modulo integer matrix semigroup.

i) Study questions (i) to (v) of problem (30) for this T.

ii) If $C^I(43)$ is replaced by Z_{43}^I, compare those two semigroups.

33. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right.$ / $a_i \in \langle C^I(Z_{48}), +\rangle$; $1 \le i \le 16, +\}$ be the natural neutrosophic complex modulo integer matrix semigroup under +.

a) Study questions (i) to (v) of problem (30) for this V
b) If Z_{48} is replaced by Z_{47} compare the two semigroups.

34. Let R = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} / \langle Z_{23} \cup I\rangle_I, +\}; 1 \leq i \leq 21, +\right\}$

be the natural neutrosophic neutrosophic matrix semigroup.

i) Study questions (i) to (v) of problem (30) for this R.
ii) Compare R with the neutrosophic matrix semigroup if Z_{23} is replaced by Z_{24}.
iii) Find all the natural neutrosophic-neutrosophic matrices of R.
iv) Find the number of idempotent subsemigroups of R.

35. Enumerate all the special and distinct features associated with natural neutrosophic-neutrosophic matrix semigroups under +.

36. Let S = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} / a_i \in \{(\langle Z_{248} \cup g\rangle_I, +) \; 1 \leq i \leq 12, +\right\}$

be the natural neutrosophic dual number matrix semigroup under +.

i) Study questions (i) to (v) of problem (30) for this S.
ii) Find all matrix subgroups of S under +.
iii) Find the order of the largest idempotent subsemigroup of S.

iv) Find the number of idempotent subsemigroups of order one of S.

37. Let $G = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \middle/ a_i \in \{\langle Z_{47} \cup g\rangle_I, +\};\ 1 \leq i \leq 25, + \right\}$

be the natural neutrosophic dual number matrix semigroup under +.

i) Study questions (i) to (iv) of problem 36 for this G.
ii) Find the number of idempotent subsemigroup of order two.
iii) Find the number of groups of G.

38. Obtain any other special features associated with the natural neutrosophic dual number matrix semigroup.

39. Let $A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \middle/ a_i \in \{\langle Z_{12} \cup h\rangle_I, h^2 = h, +\};\ 1 \leq i \leq 28, + \right\}$

be the natural neutrosophic special dual like number matrix semigroup under +.

i) Study questions (i) to (iv) of problem (36) for this A.
ii) Obtain all special features associated with A.

40. Let $B = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix} \middle/ a_i \in \{\langle Z_{41} \cup h\rangle_I, h^2 = h, +\}; 1 \leq i \leq 15, + \right\}$

be the natural neutrosophic special dual like number matrix semigroup.

i) Find all subgroups of B.
ii) Find all idempotent subsemigroups of B.
iii) Find all the idempotents of B.
iv) Study questions (i) to (iv) of problem (36) for this B.

41. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \middle/ a_i \in \{\langle Z_{96} \cup k\rangle_I, k^2 = 95k, +\}, 1 \leq i \leq 24, + \right\}$

be the natural neutrosophic special quasi dual number matrix semigroup under +.

i) Find all idempotent matrices of W.
ii) Find all subgroups of W.

iii) Find all idempotents subsemigroups of W.
iv) Find all idempotent subsemigroup of order 6.

42. Study questions (i) to (iv) in case when $\langle Z_{96} \cup k\rangle_I$ is replaced by $\langle Z_{47} \cup k\rangle_I$ in problem 41.

43. Let M = {m × n matrices with entries by $\langle Z_s \cup k\rangle_I$, $k^2 = (s-1)k$; $2 \leq s < \infty$, +} be the natural neutrosophic special quasi dual number matrix semigroup under +}.

Obtain all special features enjoyed by M.

44. Let $S[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \,/\, a_i \in S = \langle Z_{40}^I, +\rangle; 0 \leq i \leq 8, +\right\}$ be

the real natural neutrosophic coefficient polynomial semigroup under +.

Study (i) to (iv) of problem (22) for this $S[x]_8$.

45. Let $T[x]_{12} = \left\{\sum_{i=0}^{12} a_i x^i \,/\, a_i \in T = \langle Z_{13}^I, +\rangle; 0 \leq i \leq 12, +\right\}$

be the real natural neutrosophic coefficient polynomial semigroup under +.

Study questions (i) to (iv) of problem (22) for this T.

46. Obtain all special features associated with real natural neutrosophic coefficient polynomial semigroups under +.

47. Let $B[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \,/\, a_i \in S = \langle C^I(Z_4), +\rangle; 0 \leq i \leq 6, +\right\}$

be the natural neutrosophic complex modulo integer coefficient polynomial semigroup under +.

i) Find all idempotents of $B[x]_6$ under +.
ii) Find all subsets which are subgroup under +.
iii) Find all idempotents subsemigroups of $B[x]_6$.
iv) Find all subsemigroups which are not idempotent subsemigroups.
v) Find all idempotent subsemigroups of order 3.

48. Let $P[x]_7 = \left\{\sum_{i=0}^{7} a_i x^i \,/\, a_i \in P = \langle C^I(Z_{12}), +\rangle; 0 \leq i \leq 7, +\right\}$

be the natural neutrosophic complex modulo integer coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (47) for this $P[x]_7$.

49. Obtain all special features associated with natural neutrosophic complex modulo integer polynomial coefficient semigroup under +.

50. Let $W[x]_4 = \left\{\sum_{i=0}^{4} a_i x^i \,/\, a_i \in W = \langle\langle Z_{17} \cup I\rangle_I + \rangle; I^2 = I; 0 \leq i \leq 4, +\right\}$

be the natural neutrosophic neutrosophic coefficient polynomial semigroup under +.

i) Study questions (i) to (v) for this $W[x]_4$.
ii) Study questions if W is replaced by $\{\langle Z_{20} \cup I\rangle_I, +\}$

51. Let $M[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in M = \{\langle Z_{484} \cup I\rangle_I + \rangle; 0 \leq i \leq 9, +\right\}$

be the natural neutrosophic-neutrosophic coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (47) for this $M[x]_9$.

52. Obtain all special features associated with

$$B[x]_n = \left\{ \sum_{i=0}^{n} a_i x^i \,/\, a_i \in B = \{\langle Z_m \cup I\rangle_I + ;\, I^2 = I,\ 2 \leq m < \infty, +\},\ 0 \leq i \leq n, +\right\}$$

be the natural neutrosophic-neutrosophic coefficient polynomial semigroup under +.

53. Let $M[x]_3 = \left\{ \sum_{i=0}^{3} a_i x^i \,/\, a_i \in M = \{\langle Z_3 \cup g\rangle_I + g^2 = 0, +\};\ 0 \leq i \leq 3, +\right\}$

be the natural neutrosophic dual number coefficient polynomial semigroup under +.

i) Study questions (i) to (v) problem (47) for this $M[x]_3$.
ii) Study questions if M is replaced by $\langle Z_{17} \cup g\rangle_I$.

54. Let $P[x]_2 = \left\{ \sum_{i=0}^{12} a_i x^i \,/\, a_i \in P = \{\langle Z_{19} \cup h\rangle_I + h^2 = h, +\},\ 0 \leq i \leq 12, +\right\}$

be the natural neutrosophic special quasi dual like number coefficient polynomial semigroup under +.

i) Study questions (i) to (v) of problem (47) for this $P[x]_{12}$.
ii) Study when P is replaced by $\langle Z_{48} \cup h\rangle_I + h^2 = h$.

55. Let $M[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \,/\, a_i \in M = \{\langle Z_{27} \cup k\rangle_I + k^2 = 26k, +\}\ 0 \le i \le 8, +\right\}$

be the natural neutrosophic special quasi dual number coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (47) of this $M[x]_8$.

56. Let $N[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \,/\, a_i \in N = \{\langle Z_{29} \cup k\rangle_I + k^2 = 28k, +\}\ 0 \le i \le 10, +\right\}$

be the natural neutrosophic special quasi dual number coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (47) for this $N[x]_{10}$.

57. Study all special features associated with $M[x]_n$, natural neutrosophic real coefficient polynomial semigroup under +.

58. Let $M[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in P = \{\langle C^I Z_2)\ , +\rangle, 0 \le i \le 9, +\right\}$

be the natural neutrosophic complex module integer coefficient polynomial semigroup under +.

i) Find the number of elements of $M[x]_9$.
ii) Prove $M[x]_9$ is a S-semigroup.
iii) Find the number of idempotent of $M[x]_9$.
iv) Find all subsets of $M[x]_9$ which are subgroups under +.
v) Find all idempotent subsemigroups of order 5.
vi) Find the largest idempotent subsemigroup of $M[x]_9$

59. Let $B[x]_7 = \left\{ \sum_{i=0}^{7} a_i x^i / a_i \in B = \langle C^I(Z_{14}), +\rangle, 0 \leq i \leq 7, + \right\}$

be the natural neutrosophic complex modular coefficient polynomials semigroup under +.

Study questions (i) to (vi) of problem (58) for this $B[x]_7$.

60. Let $C[x]_7 = \left\{ \sum_{i=0}^{7} a_i x^i / a_i \in P = \{\langle C^I Z_{24}), +\rangle, 0 \leq i \leq 7, + \right\}$

be the natural neutrosophic complex modulo integer coefficient polynomials, semigroup under +.

Study questions (i) to (vi) of problem (58) for this $C[x]_7$.

61. Let $P[x]_3 = \left\{ \sum_{i=0}^{3} a_i x^i / a_i \in P = \{ Z_{18}^I, +\}; 0 \leq i \leq 3, + \right\}$

be the real natural neutrosophic semigroup under +.

i) Study $P[x]_3$ if P is replaced by $\{\langle Z_{18} \cup g\rangle_I, +\}$.
ii) Study $P[x]_3$ if P is replaced by $\{\langle Z_{18} \cup I\rangle_I, +\}$.
iii) Study $P[x]_3$ if P is replaced by $\{C^I (Z_{18}), +\}$.
iv) If P is replaced by $\{\langle Z_{18} \cup k\rangle; k^2 = 17k, +\}$, study problem $P[x]_3$.

62. Find all special features associated with idempotent subsemigroup of $P = \{ Z_{18}^I, +\}$.

63. Let $M = \{C^I(Z_{48}), +\}$ be the natural neutrosophic complex modulo integer semigroup under +.

i) Find o(M).
ii) Find the number idempotent in M.

iii) Find the largest idempotent subsemigroup of M.
iv) Find all subsemigroups of M which are not idempotent subsemigroup.
v) Will subsemigroups which are not idempotent semigroup be subgroups under +?

64. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle/ a_i \in \{\langle Z_{18} \cup I\rangle_I; +\}; 1 \leq i \leq 10, + \right\}$

be the natural neutrosophic neutrosophic matrix semigroup under +.

Study questions (i) to (v) of problem (63) for this M.

65. Let $P[x]_{20} = \left\{ \sum_{i=0}^{20} a_i x^i \middle/ a_i \in \{\langle Z_{24} \cup g\rangle;\ g^2 = 0,\ +\} = P; \right.$

$\left. 0 \leq i \leq 20, + \right\}$

be the natural neutrosophic dual number coefficient polynomial semigroup under +.

Study questions (i) to (v) of problem (63) for this $P[x]_{20}$ with appropriate changes.

Chapter Two

SEMIGROUPS UNDER $\times$ ON NATURAL NEUTROSOPHIC ELEMENTS

In this chapter for the first time authors study the semigroup structure on the natural neutrosophic sets Z_n^I, $C^I(Z_n)$, $\langle Z_n \cup I\rangle_I$, $\langle Z_n \cup g\rangle_I$ ($g^2 = 0$) and so on.

We will first illustrate this situation by some examples.

For more about natural neutrosophic elements of Z_n, $C(Z_n)$ etc refer [24].

Example 2.1: Let $\{Z_4^I, \times\} = \{0, 1, 2, 3, I_0^4, I_2^4, \times\}$ be a semigroup $I_0^4 \times I_2^4 = I_0^4$ and $I_2^4 \times I_2^4 = I_0^4$ and $I_0^4 \times I_0^4 = I_0^4$. $2I_6^4 = I_0^4$, $2I_2^4 = I_2^4$, $3I_2^4 = I_2^4$, $3I_0^4 = I_0^4$, $1I_2^4 = I_2^4$, $0I_0^4 = I_0^4$, $1I^4 = I_0^4$, $0I_2^4 = I_2^4$.

$\{Z_4^I, \times\}$ is called natural neutrosophic semigroup of order 6 in this case.

Example 2.2: Let $\{Z_5^I, \times\} = \{0, 1, 2, 3, 4, I_0^5, \times\}$ be a natural neutrosophic semigroup of order six.

Example 2.3: Let $\{Z_{19}^{I}, \times\} = \{0, 1, 2, \ldots, 18, I_0^{19}, \times\}$ be again a natural neutrosophic semigroup of order 20.

In view of all these we have the following theorem.

THEOREM 2.1: *$\{Z_p^I, \times\}$, p a prime is the natural neutrosophic semigroup of order p + 1.*

Proof follows from the simple fact that if p is a prime then Z_p has only one natural neutrosophic element given by I_0^p. So $\{Z_p^I, \times\}$ is of order p + 1 given by $\{Z_p^I, \times\} = \{0, 1, 2, \ldots, p-1, I_0^p\}$.

However this is not the same situation when p is a composite number or if '+', addition operation is performed on Z_p^I.

We will illustrate this situation by an example or two.

Example 2.4: Let $\{Z_6^I, \times\} = \{0, 1, 2, 3, 4, 5, I_0^6, I_2^6, I_4^6, I_3^6, \times\}$ is the natural neutrosophic semigroup of order 10.

$I_2^6 \times I_3^6 = I_0^6$ so a natural neutrosophic zero divisor.

$I_3^6 \times I_3^6 = I_3^6$ so I_3^6 is a natural neutrosophic idempotent.

$I_4^6 \times I_3^6 = I_0^6$ which is again a natural neutrosophic zero divisor.

$I_4^6 \times I_4^6 = I_4^6$ is a natural neutrosophic idempotent.

$I_4^6 \times I_2^6 = I_2^6$ gives neither a zero divisor not an idempotent.

We see we do not have in this example natural neutrosophic nilpotent.

$I_2^6 \times I_2^6 = I_4^6$ and so on.

Thus this is the way the product operation is performed on Z_6^I.

Example 2.5: Let $\{Z_8^I, \times\}$ = {0, 1, 2, …, 7, $I_0^8, I_2^8, I_4^8, I_6^8$, ×} be a natural neutrosophic semigroup of order 12.

Clearly $I_4^8 \times I_4^8 = I_0^8$ is a natural neutrosophic nilpotent of order two.

$I_2^8 \times I_2^8 \times I_2^8 = I_0^8$ is also a natural neutrosophic nilpotent element of order three.

$I_6^8 \times I_6^8 \times I_6^8 = I_0^8$ is a natural neutrosophic nilpotent element of order three.

However Z_8^I has no natural neutrosophic idempotent.

Example 2.6: Let $\{Z_{16}^I, \times\}$ = {0, 1, 2, …., 14, 15, I_{10}^{16}, I_6^{16} I_2^{16}, I_4^6, I_8^6, $I_{10}^6, I_{12}^6, I_{14}^6$, ×} be a natural neutrosophic semigroup of order 24.

$I_2^{16} \times I_2^{16} \times I_2^{16} \times I_2^{16} = I_0^{16}$ is a natural neutrosophic nilpotent of order four.

$I_4^4 \times I_4^{16} = I_0^{16}$ is a natural neutrosophic nilpotent of order two $I_6^{16} \times I_6^{16} \times I_6^{16} \times I_6^{16} = I_0^{16}$ is a natural neutrosophic nilpotent of order four.

$I_8^{16} \times I_8^{16} = I_0^{16}$ is a natural neutrosophic nilpotent of order two $I_{10}^{16} \times I_{10}^{16} \times I_{10}^{16} \times I_{10}^{16} = I_{10}^{16}$ is a natural neutrosophic nilpotent of order four.

$I_{12}^{16} \times I_{12}^{16} = I_{0}^{16}$ is a natural neutrosophic nilpotent of order two $I_{14}^{16} \times I_{14}^{16} \times I_{14}^{16} \times I_{14}^{16} = I_{0}^{16}$ is a natural neutrosophic nilpotent of order two.

Thus $\{Z_{16}^{I}, \times\}$ is a natural neutrosophic semigroup which has no neutrosophic idempotents but has natural nilpotents of order two and four.

Infact Z_{16}^{I} has natural neutrosophic zero divisors.

$I_{2}^{16} \times I_{8}^{16} = I_{0}^{16}$ $\qquad$ $I_{4}^{16} \times I_{12}^{16} = I_{0}^{16}$

$I_{4}^{16} \times I_{8}^{16} = I_{0}^{16}$ $\qquad$ $I_{14}^{16} \times I_{8}^{16} = I_{0}^{16}$

$I_{6}^{16} \times I_{8}^{16} = I_{0}^{16}$

and $I_{10}^{16} \times I_{8}^{16} = I_{0}^{16}$ are the natural neutrosophic zero divisors.

In view of this we have the following result the proof of which is left as an exercise to the reader.

THEOREM 2.2: *$\{Z_{2^n}^{I}, \times\} = \{0, 1, 2, \ldots, 2^n - 1, I_{0}^{2^n}, I_{2}^{2^n}, \ldots, I_{2^{n}-1}^{2^n}, \times\}$ is a natural neutrosophic semigroup of order $2^n + 2^{n-1}$.*

Proof is direct and hence left as an exercise to the reader.

Example 2.7: Let $\{Z_{9}^{I}, \times\} = \{0, 1, 2, \ldots, 8, I_{0}^{9}, I_{3}^{9}\ I_{6}^{9}, \times\}$ be the natural neutrosophic semigroup of order 12.

$I_{3}^{9} \times I_{3}^{9} = I_{0}^{9}$ is a natural neutrosophic nilpotent of order two.

$I_{6}^{9} \times I_{6}^{9} = I_{0}^{9}$ is a natural neutrosophic nilpotent of order two.

$I_{6}^{9} \times I_{6}^{9} = I_{0}^{9}$ is a natural neutrosophic nilpotent of order two.

$I_3^9 \times I_6^9 = I_0^9$ is again a natural neutrosophic zero divisor of Z_9^I.

Example 2.8: Let $\{Z_{81}^I, \times\}$ = {0, 1, 2, …, 80, I_0^{81}, I_3^{81}, I_9^{81}, I_{12}^{81}, I_{15}^{81}, I_{18}^{81}, I_{21}^{81}, I_{24}^{81}, I_{27}^{81}, I_{30}^{81}, I_{33}^{81}, $I_{36}^{81}, I_{39}^{81}, I_{42}^{81}$, $I_{45}^{81}, I_{48}^{81}, I_{51}^{81}$, $I_{54}^{81}, I_{57}^{81}, I_{60}^{81}$, $I_{63}^{81}, I_{66}^{81}, I_{69}^{81}$, $I_{72}^{81}, I_{75}^{81}, I_{78}^{81}$, ×} be a natural neutrosophic semigroup of order 108.

In view of this we have the following two results.

THEOREM 2.3: *$\{Z_{3^n}^I, \times\}$ = {0, 1, 2, …, $3^n - 1$, $I_0^{3^n}, I_3^{3^n}$, $I_6^{3^n}$, …, $I_{3^{n-1}}^{3^n}$, ×} be the natural neutrosophic semigroup of order $3^n + 3^{n-1}$.*

Proof is direct and hence left as an exercise to the reader.

Now we prove the following theorem.

THEOREM 2.4: *$\{Z_{p^n}^I$, ×; (p a prime} = {0, 1, 2, …, $p^n - 1$, $I_0^{p^n}, I_p^{p^n}$, $I_{2p}^{p^n}$, …, $I_{p^{n-1}}^{p^n}$, ×} is the natural neutrosophic semigroup of order $p^n + p^{n-1}$.*

Proof involves only simple number theoretic techniques so left as an exercise to the reader.

Example 2.9: Let
$\{Z_{12}^I, \times\} = \{0, 1, 2, \ldots, 11, I_0^{12}, I_2^{12}, I_3^{12}, I_6^{12}, I_4^{12}, I_8^{12}, I_{10}^{12}, I_9^{12}, \times\}$ be the natural neutrosophic semigroup of order 20.

$I_2^{12} \times I_6^{12} = I_0^{12}$ is a neutrosophic zero divisor.

$I_6^{12} \times I_6^{12} = I_0^{12}$ is a natural neutrosophic nilpotent element of order two

$I_3^{12} \times I_4^{12} = I_0^{12}$ is a natural neutrosophic zero divisor.

$I_4^{12} \times I_4^{12} = I_4^{12}$ is a natural neutrosophic idempotent.

$I_8^{12} \times I_3^{12} = I_0^{12}$ is a natural neutrosophic zero divisor.

$I_9^{12} \times I_9^{12} = I_9^{12}$ is a natural neutrosophic idempotent.

$I_{10}^{12} \times I_6^{12} = I_0^{12}$ is a natural neutrosophic zero divisor.

Thus $\{Z_{12}^I, \times\}$ has natural neutrosophic zero divisor, natural neutrosophic nilpotent of order two and natural neutrosophic idempotents.

Example 2.10: Let $\{Z_{15}^I, \times\} = \{0, 1, 2, \ldots, 14, I_0^{15}, I_3^{15}, I_6^{15}, I_9^{15}, I_{12}^{15}, I_5^{15}, I_{10}^{15}, \times\}$ be the natural neutrosophic semigroup of order 22.

$I_3^{15} \times I_5^{15} = I_0^{15} = I_3^{15} \times I_{10}^{15}$ is a natural neutrosophic zero divisor of Z_{15}^I.

$I_6^{15} \times I_5^{15} = I_0^{15} = I_6^{15} \times I_{10}^{15}$ is a natural neutrosophic zero divisor of Z_{15}^I.

$I_6^{15} \times I_6^{15} = I_6^{15}$ and $I_{10}^{15} \times I_{10}^{15} = I_{10}^{15}$ are natural neutrosophic idempotents of Z_{15}^I.

$I_9^{15} \times I_5^{15} = I_0^{15} = I_9^{15} \times I_{10}^{15}$ is again a natural neutrosophic zero divisor of Z_{15}^I

$I_{12}^{15} \times I_5^{15} = I_{12}^{15} \times I_{10}^{15} = I_0^5$ is also a natural neutrosophic zero divisor.

Thus Z_{15}^{I} also has natural neutrosophic idempotents neutrosophic zero divisors and has no natural neutrosophic nilpotent.

Example 2.11: Let $\{Z_{14}^{I}, \times\} = \{0, 1, 2, \ldots, 13, I_0^{14}, I_2^{14}, I_4^{14}, I_6^{14}, I_8^{14}, I_{10}^{14}, I_{12}^{14}, I_7^{14}, \times\}$ be the natural neutrosophic semigroup of order twenty two.

$$\begin{aligned} \text{Clearly } I_2^{14} \times I_7^{14} &= I_0^{14} \\ &= I_4^{14} \times I_7^{14} \\ &= I_6^{14} \times I_7^{14} \\ &= I_8^{14} \times I_7^{14} = I_{10}^{14} \times I_7^{14} \\ &= I_{12}^{14} \times I_7^{14} = I_0^{14}. \end{aligned}$$

Thus $I_t^{14} \times I_7^{14} = I_0^{14}$ for all t = 2, 4, 6, 8, 10, 12, 0. $I_7^{14} \times I_7^{14} = I_7^{14}$ is a natural neutrosophic idempotent.

$$I_2^{14} \times I_2^{14} = I_4^{14} = I_2^{14},\ I_6^{14} \times I_6^{14} = I_8^{14}, I_8^{14} \times I_8^{14} = I_{12}^{14}$$

$$I_{10}^{14} \times I_{10}^{14} = I_6^{14} \text{ and so on.}$$

Infact Z_{14}^{I} has only one idempotent and zero divisors has no nilpotents.

Example 2.12: Let
$\{Z_{26}^{I}, \times\} = \{0, 1, 2, \ldots, 25, I_0^{26}, I_2^{20}, I_4^{26}, I_6^{26}, I_8^{26}, I_{10}^{26}, I_{22}^{26}, I_{24}^{26}, I_{13}^{26}\}$ be the natural neutrosophic semigroup of order 40.

In view of all these the following results can be proved.

THEOREM 2.5: *Let $\{Z_{2p}^{I}, \times;$ p a prime$\}$ be the natural neutrosophic semigroup.*

i) $o(Z_{2p}^{I}) = 2p + (p + 1)$.

ii) $\{Z^I_{2p}, \times\}$ has only one idempotent.

iii) $\{Z^I_{2p}, \times\}$ has no nilpotents.

iv) $\{Z^I_{2p}, \times\}$ has zero divisor of the form.

$$I_p^{2p} \times I_t^{2p} = I_0^{2p} \text{ for all } t;\ t \neq p.$$

Proof. Proof of (i) can be got by simple number theoretic techniques.

$I_p^{2p} \in Z^I_{2p}$ is such that $I_p^{2p} \times I_p^{2p} = I_p^{2p}$ the only idempotent for all other natural neutrosophic elements I_t^{2p} are such that t = 0 or t is even. Hence (ii) is true.

It is impossible to find $I_t^{2p} \times I_t^{2p} = I_0^{2p}$ for no product of two even numbers can contribute to 2p (p a prime). Hence (iii) is true.

Clearly t × p = 0 (mod 2p) as all t are even in $I_t^{2p} \in Z^I_{2p}$ barring p. Hence (iv) is true.

THEOREM 2.6: *Let $\{Z^I_{p^n}, \times\}$ be the natural neutrosophic semigroup. $Z^I_{p^n}$ has no nontrivial neutrosophic idempotents. The trivial natural neutrosophic idempotent being $I_0^{p^n}$.*

Proof is direct and hence left as an exercise to the reader.

Thus we give more results in this direction by examples.

Example 2.12: Let $\{Z^I_{20}, \times\}$ = {0, 1, 2, …, 19, I_0^{20}, I_2^{20},…, I_{18}^{20}, I_5^{20}, I_{15}^{20}, ×} be the natural neutrosophic semigroup of order 32.

Thus has natural neutrosophic nilpotent of order two given by $I_{10}^{20} \times I_{10}^{20} = I_0^{20}$, $I_5^{20} \times I_5^{20} = I_5^{20}$ is a natural neutrosophic idempotent.

Further there are several natural neutrosophic zero divisors.

$I_t^{20} \times I_{10}^{20} = I_0^{20}$ t ≠ 5, 15 and for all other t = {0, 2, 4, …, 18}.

Example 2.14: Let
Z_{24}^I = {0, 1, 2, …, 23, $I_p^{24}, I_2^{24}, I_4^{24}$, I_6^{24}, …, I_{22}^{24}, I_3^{24}, $I_{21}^{24}, I_9^{24}, I_{15}^{24}$, ×} be the natural neutrosophic semigroup of order 40.

$I_{12}^{24} \times I_{12}^{24} = I_0^{24}$ is a natural neutrosophic nilpotent of order two.

$I_{16}^{24} \times I_{16}^{24} = I_{16}^{24}$ is a natural neutrosophic idempotent.

Infact there are several zero divisors $I_{12}^{24} \times I_{10}^{12} = I_0^{12}$.

Example 2.15: Let
{ Z_{28}^I, ×} = {0, 2, 4, …., 26, I_0^{28}, I_2^{28}, I_4^{28}, …, I_{26}^{28}, I_7^{28}, I_{21}^{28},×} be the natural neutrosophic semigroup of order 44.

Clearly $I_{14}^{28} \times I_{14}^{28} = I_0^{28}$, $I_{14}^{28} \times I_t^{28} = I_0^{28}$; t = 2, 4, …, 26.

Thus Z_{28}^I has both nilpotents of order two and natural neutrosophic zero divisors.

$I_4^{28} \times I_{21}^{28} = I_0^{28}$; however $I_{21}^{28} \times I_{21}^{28} = I_{21}^{28}$ is a natural neutrosophic idempotent of Z_{28}^I.

In view of this we have the following theorem.

THEOREM 2.7: *Let {Z_{4p}^I; ×, p a prime} be the natural neutrosophic semigroup.*

i) Order of Z_{4p}^I is 4p + 2p + 2.

ii) Z_{4p}^I has I_{3p}^{4p} to be an idempotent.

iii) Z_{4p}^{I} *has many natural neutrosophic zero divisors.*

iv) Z_{4p}^{I} *has* I_{3p}^{4p} *to be the natural neutrosophic nilpotent of order two.*

Proof is direct and hence left as an exercise to the reader.

Example 2.16: Let $\{Z_{30}^{I}, \times\}$ be the natural neutrosophic semigroup.

$\{Z_{30}^{I}, \times\} = \{0, 1, 2, \ldots, 29, I_0^{30}, I_2^{30}, I_4^{30}, \ldots., I_{28}^{30}, I_3^{30}, I_9^{30}, I_{15}^{30}, I_{21}^{30}, I_{27}^{30}, I_5^{30}, I_{25}^{30}, \times\}$ be the natural semigroup of order 52.

$I_{15}^{30} \times I_{15}^{30} = I_{15}^{30}$ is a natural neutrosophic idempotent.

$I_{10}^{30} \times I_{10}^{30} = I_{10}^{30}$ is again a natural neutrosophic idempotent.

$I_{6}^{30} \times I_{6}^{30} = I_{6}^{30}$ is a natural neutrosophic idempotent.

$I_{16}^{30} \times I_{16}^{30} = I_{16}^{30}$ is a natural neutrosophic idempotent.

$I_{25}^{30} \times I_{25}^{30} = I_{25}^{30}$ is again an idempotent.

Infact Z_{30}^{I} has several natural neutrosophic zero divisors but no natural neutrosophic nilpotents of order two.

In view of this we propose the following open problem.

Problem 2.1: Let $\{Z_{p_1, p_2, \ldots, p_t}^{I}; \times\}$ be the natural neutrosophic semigroup (p_i's are distinct primes).

Can $Z_{p_1, p_2, \ldots, p_t}^{I}$ have nontrivial natural neutrosophic nilpotent?

Example 2.17: Let $\{Z^I_{42}, \times\}$ = $\{0, 1, 2, \ldots, 40, 41, I^{42}_0, I^{42}_2, \ldots, I^{42}_{40}, I^{42}_3, I^{42}_9, I^{42}_{15}, I^{42}_{21}, I^{42}_{27}, I^{42}_{33}, I^{42}_{39}, I^{42}_7, I^{42}_{35}, \times\}$ be the natural neutrosophic semigroup of order 72.

$I^{42}_7 \times I^{42}_7 = I^{42}_7$ is a natural neutrosophic idempotent.

$I^{42}_{22} \times I^{42}_{22} = I^{42}_{22}$ is also a natural neutrosophic idempotent.

$I^{42}_{28} \times I^{42}_{28} = I^{42}_{28}$ is again a natural neutrosophic idempotent.

$I^{42}_{36} \times I^{42}_{36} = I^{42}_{36}$ is a natural neutrosophic idempotent.

$I^{42}_{15} \times I^{42}_{15} = I^{42}_{15}$ and $I^{42}_{21} \times I^{42}_{21} = I^{4}_{21}$ are also natural neutrosophic idempotents of Z^I_{42}.

There are 6 natural neutrosophic idempotents.

Now we study the natural neutrosophic finite complex modulo integer semigroup under × by some examples.

Example 2.18: Let $\{C^I(Z_4), \times\}$ = $\{0, 1, 2, 3, I^C_2, I^C_{2F}, I^C_{2+2i_F}, 1 + i_F, 2 + 2i_F, 1 + 3i_F, 2 + 3i_F, I^C_{1+i_F}, 3 + 2i_F, 3 + 3i_F, i_F, 2i_F, 3i_F, \times\}$ be a natural neutrosophic finite complex number of semigroup.

Clearly $I^C_2 \times I^C_2 = I^C_0$ is a natural neutrosophic nilpotent element of order two. Similarly I^C_{2F} and $I^C_{2+2i_F}$ are natural neutrosophic nilpotents.

$I^C_{1+i_F} \times I^C_{1+i_F} \times I^C_{1+i_F} \times I^C_{1+i_F} = I^C_0$ is a natural neutrosophic nilpotent element of order 4.

Example 2.19: Let $\{C^I(Z_2), \times\}$ = $\{0, 1, i_F, 1 + i_F, I^C_0, I^C_{1+i_F}, \times\}$ be a natural neutrosophic finite complex modulo integer semigroup of order 6.

Clearly $I^C_{1+i_F} \times I^C_{1+i_F} = I^C_0$ is a natural neutrosophic complex modulo integer nilpotent element of order two.

Example 2.20: Let
$\{C^I(Z_3), \times\} = \{0, 1, 2, i_F, 2i_F, 1 + i_F, 2 + i_F, 1 + 2i_F, 2i_F, I^C_0, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

This has no nontrivial natural neutrosophic zero divisors or nilpotents or idempotents.

Example 2.21: Let $\{C^I(Z_6), \times\} = \{0, 1, 2, \ldots, 5, i_F, 2i_F, \ldots, 5i_F, 1 + i_F, 2 + i_F, \ldots, 5 + i_F, 1 + 2i_F, 2 + 2i_F, 2 + 3i_F, 1 + 3i_F, 2 + 5i_F, 3 + 2i_F, 4 + 2i_F, 5 + 2i_F, 2 + 3i_F, 3 + 3i_F + 3 + 4i_F, 1 + 4i_F, 2 + 4i_F, 3 + 4i_F, 4 + 4i_F, 4 + 5i_F, 1 + 5i_F, 3 + 5i_F, 4 + 5i_F, 5 + 5i_F, I^C_0, I^C_2, I^C_3, I^C_4, I^C_{2i_F}, I^C_{3i_F}, I^C_{4i_F}, I^C_{2+2i_F}, I^C_{3+3i_F}, I^C_{4+4i_F}\ I^C_{2+4i_F}, I^C_{4+2i_F}, I^C_{1+3i_F}, I^C_{3+i_F}, I^C_{1+i_F}$ and so on, $\times\}$ be the natural neutrosophic semigroup.

$I^C_{2+2i_F} \times I^C_{3+3i_F} = I^C_0$ is a natural neutrosophic zero divisor.

$I^C_{3+3i_F} \times I^C_{3+3i_F} = I^C_0$ is a natural neutrosophic nilpotent of $C^I(Z_6)$ of order two

$I^C_3 \times I^C_3 = I^C_3$ is a natural neutrosophic idempotent of $C^I(Z_6)$.

Thus $C^I(Z_6)$ has natural neutrosophic idempotents zero divisors and nilpotents.

However it is important to note that $\{Z^I_6, \times\}$ has no neutrosophic nilpotents but $C^I(Z_6)$ has natural neutrosophic nilpotents of order two.

Clearly $Z^I_6 \subseteq C^I(Z_6)$.

Z^I_6 is a subsemigroup $C^I(Z_6)$.

Example 2.22: Let $\{C^I(Z_8), \times\} = \{0, 1, 2, \ldots, 7,\ I_0^C, I_2^C,\ I_4^C, I_6^C, i_F,$ $2i_F, \ldots, 5i_F, 6i_F, 7i_F, 1 + i_F, 2 + i_F, \ldots, 7 + i_F, 2 + 2i_F, 3 + 2i_F, 1 +$ $2i_F, 4 + 2i_F, \ldots, 7 + 2i_F, 1 + 3i_F, 2 + 3i_F, 3 + 3i_F, \ldots, 7 + 3i_F, 1 +$ $4i_F, 2 + 4i_F, \ldots, 7 + 4i_F, 1 + 5i_F, 2 + 5i_F, \ldots, 7 + 5i_F, 1 + 6i_F, 2 +$ $6i_F, \ldots, 7 + 6i_F, 1 + 7i_F, 2 + 7i_F + 3 + 7i_F, \ldots, 7 + 7i_F,$ $I_{2i_F}^C, I_{4i_F}^C, I_{6i_F}^C,\ I_{2+2i_F}^C, I_{4+4i_F}^C, I_{4+6i_F}^C,\ I_{6+6i_F}^C, I_{2+4i_F}^C, I_{4+2i_F}^C, \ldots,$ and so on, $\times\}$ be the natural neutrosophic complex modulo integer semigroup of finite order.

Finding $o(C^I(Z_n))$ or that of the collection of all natural neutrosophic elements of $C^I(Z_n)$ happens to be a difficult problem.

Thus $C^I(Z_8)$ has nilpotents and zero divisors but finding idempotents happens to be a difficult problem, when n in $C^I(Z_n)$ is of the form p^t, p a prime; $t > 1$.

Example 2.23: Let $\{C^I(Z_9), \times\} = \{0, 1, 2, \ldots, 8, i_F, 2i_F, 3i_F, \ldots,$ $8i_F, 1 + i_F, 2 + i_F, 3 + i_F, \ldots, 8 + i_F, 2 + 2i_F, 1 + 2i_F, \ldots, 8 + 2i_F, 1$ $+ 3i_F, \ldots, 8 + 3i_F, 1 + 4i_F, 2 + 4i_F, \ldots, 8 + 4i_F, 1 + 5i_F, 2 + 5i_F,$ $\ldots, 8 + 5i_F, 1 + 6i_F, 2 + 6i_F, 3 + 6i_F, \ldots, 8 + 6i_F, \ldots, 8 + 8i_F,$ $I_0^C, I_3^C, I_{3i_F}^C,\ I_{6i_F}^C, I_6^C, I_{3+3i_F}^C,\ I_{3+6i_F}^C, I_{6+3i_F}^C$, and so on, $\times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

Clearly $I_{3+3i_F}^C \times I_{3+3i_F}^C = I_0^C$ is a neutrosophic nilpotent element of order two.

However finding natural neutrosophic idempotents happens to be a challenging problem.

$I_{3+3i_F}^C, I_{6i_F+6}^C, I_0^C$ is a natural neutrosophic zero divisor.

So we leave open the following problem.

Problem 2.2: Let $\{C^I\ (Z_n), \times\}$, $n = p^t$ ($t > 1$, p a prime) be the natural neutrosophic finite complex modulo integer semigroup.

Can $C^I(Z_n)$ contain natural neutrosophic idempotents?

Example 2.24: Let $\{C^I(Z_{24}), \times\} = \{0, 1, 2, \ldots, 23, i_F, 2i_F, \ldots, 23i_F, I_0^C, I_2^C, \ldots, I_{22_F}^C, I_{2i_F}^C, I_{4i_F}^C, \ldots, I_{22i_F}^C, I_{3i_F}^C, I_3^C, I_9^C, I_{15}^C, I_{9i_F}^C, I_{15i_F}^C, I_{21}^C, I_{21i_F}^C$ and so on$\}$ be the natural neutrosophic complex finite modulo integer semigroup.

Clearly $C^I(Z_{24})$ has natural neutrosophic idempotents, nilpotents and zero divisors.

For $I_{12i_F+6i_F}^C \times I_4^C = I_0^C$ is a natural neutrosophic zero divisor.

$I_{9+9i_F}^C \times I_{8+8i_F}^C = I_0^C$ is again natural neutrosophic zero divisor.

It is important to note that in general the study of natural neutrosophic finite complex modulo integer semigroup $C^I(Z_n)$ under $\times$ happens to be innovative and interesting but at the same time difficult.

However it is kept on record that $\{Z_n^I, \times\} \subseteq \{C^I(Z_n), \times\}$ as a subsemigroup of finite order; $o(C^I(Z_n) < \infty$ is a finite semigroup.

For a given n finding the natural neutrosophic zero divisors of $C^I(Z_n)$, finding natural neutrosophic idempotents of $C^I(Z_n)$ and that of finding natural neutrosophic nilpotent elements happens to be a challenging open problem.

Problem 2.3: Let $\{C^I(Z_n), \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

i) Characterize those $C^I(Z_n)$ which has only natural neutrosophic zero divisors and no natural neutrosophic nilpotents and idempotents.

ii) Characterize those $C^I(Z_n)$ which has only natural neutrosophic zero divisors and natural neutrosophic nilpotents.

iii) Characterize those $C^I(Z_n)$ which has only natural neutrosophic idempotents and natural neutrosophic zero divisors but no natural neutrosophic nilpotents.

iv) Characterize those $C^I(Z_n)$ which has natural neutrosophic zero divisors, nilpotents and idempotents.

This task of studying some more properties about natural neutrosophic finite complex modulo integers semigroups is considered as a matter of routine and is left as an exercise to the reader.

Next we proceed onto give examples of natural neutrosophic-neutrosophic semigroups under product.

Example 2.25: Let $\{\langle Z_5 \cup I\rangle_I, \times\}$ = {0, 1, 2, 3, 4, 1 + I, I, 2I, 3I, 4I, 1 + 2I, 1 + 3I, 1 + 4I, 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 4 + I, 4 + 2I, 4 + 3I, 4 + 4I, I_0^I, I_I^I, I_{2I}^I, I_{3I}^I, I_{4I}^I and so on, ×} be the natural neutrosophic neutrosophic semigroup.

Finding natural neutrosophic zero divisors happens to be a very difficult problem in this case.

Example 2.26: Let $\{\langle Z_2 \cup I\rangle_I, \times\}$ = {0, 1, I + 1, I, I_0^I, I_I^I, I_{1+I}^I, ×} be the natural neutrosophic neutrosophic semigroup.

Clearly $o(\langle Z_2 \cup\rangle_I) = 7$ we see $I_I^I \times I_I^I = I_I^I$ and $I_{1+I}^I \times I_{1+I}^I = I_{1+I}^I$ are natural neutrosophic neutrosophic idempotents of $\langle Z_2 \cup I\rangle_I$.

We see $I_I^I \times I_{1+I}^I = I_0^I$ is a natural neutrosophic neutrosophic zero divisors.

Infact $\langle Z_2 \cup I\rangle_I$ has no natural neutrosophic neutrosophic nilpotent element associated with it.

Example 2.27: Let $\{\langle Z_3 \cup I\rangle_I, \times\} = \{0, 1, 2, I, 2I, 1 + I, I + 2I, 2 + I, 2 + 2I, I_0^I, I_I^I, I_{2I}^I, I_{1+2I}^I, I_{2+I}^C, \times\}$ be the natural neutrosophic neutrosophic semigroup.

$I_{1+2i_F}^C \times I_{1+2i_F}^C = I_{7+2i_F}^C$ and $I_I^C \times I_I^C = I_I^C$ are the natural neutrosophic-neutrosophic idempotents of $\langle Z_3 \cup I\rangle_I$.

However order of $\langle Z_3 \cup I\rangle_I$ is 14.

$I_{1+2i_F}^C \times I_I^C = I_0^C$ and $I_{2+I}^C \times I_I^C = I_0^C$ are natural neutrosophic neutrosophic zero divisors of $\langle Z_3 \cup I\rangle_I$.

Further $\langle Z_3 \cup I\rangle_I$ has not natural neutrosophic - neutrosophic nilpotents.

Example 2.28: Let $\{\langle Z_4 \cup I\rangle_I, \times\} = \{0, 1, 2, 3, I, 2I, 3I, 1 + I, I + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I, I_0^I, I_2^I, I_{2I}^I, I_{2+2I}^I, I_{1+I}^C, I_{1+3I}^C, I_{3+I}^I, I_{2+3I}, I_{2+I}^I, I_{3+3I}^I, I_{3I}, I_I\}$ be the natural neutrosophic-neutrosophic semigroup.

Clearly $o(\langle Z_4 \cup I\rangle_I) = 28$.

Now $I_{3+I}^I \times I_I^I = I_0^I$, $I_{2+2i_F}^I \times I_I^I = I_0^I$ there are several natural neutrosophic-neutrosophic zero divisor.

$I_{2+2i_F}^I \times I_{2+2I}^I = I_0^I$, $I_2^I \times I_2^I = I_0^I$ and $I_{2I}^I \times I_{2I}^I = I_0^I$ are some of the natural neutrosophic-neutrosophic nilpotents of order two.

$I_I^I \times I_I^I = I_I^I$ is a natural neutrosophic-neutrosophic idempotent of $\langle Z_4 \cup I\rangle_I$.

$I_{1+3I}^I \times I_{1+3I}^I = I_{1+3I}^I$ is also a natural neutrosophic neutrosophic idempotent of $\langle Z_4 \cup I\rangle_I$.

Thus $\langle Z_4 \cup I\rangle_I$ is a natural neutrosophic neutrosophic semigroup which has natural neutrosophic neutrosophic idempotent, natural neutrosophic neutrosophic zero divisors and natural neutrosophic neutrosophic nilpotents of order two.

Example 2.29: Let $\{\langle Z_7 \cup I\rangle_I, \times\}$ be the natural neutrosophic neutrosophic semigroup. $\langle Z_7 \cup I\rangle_I$ has natural neutrosophic neutrosophic idempotents to be $I_I^I \times I_I^I = I_I^I$ and has several natural neutrosophic-neutrosophic zero divisors given by $I_{3+4I}^I \times I_I^I = I_0^I$, $I_{6+I}^I \times I_I^I = I_0^I$, $I_{3I+4}^I \times I_I^I = I_0^I$ and so on.

However finding natural so neutrosophic-neutrosophic idempotents other than I_I^I is a difficult task.

We are not able to know whether $\langle Z_7 \cup I\rangle_I$, has natural neutrosophic-neutrosophic nilpotents.

In view of this we propose the following problem.

Problem 2.4: Let $\{\langle Z_p \cup I\rangle_I$, p a prime, $\times\}$ be the natural neutrosophic-neutrosophic semigroup.

i) Find $o(\langle Z_p \cup I\rangle_I)$.
ii) Find the number of natural neutrosophic neutrosophic elements of $\langle Z_p \cup I\rangle_I$
iii) Can $\langle Z_p \cup I\rangle_I$ have natural neutrosophic neutrosophic nilpotents?
iv) Prove $\langle Z_p \cup I\rangle_I$ has natural neutrosophic neutrosophic zero divisors.
Find the number of natural neutrosophic neutrosophic zero divisors.
v) Can $\langle Z_p \cup I\rangle_I$ have natural neutrosophic neutrosophic idempotents other than I_I^I?

Example 2.30: Let $\{\langle Z_6 \cup I\rangle_I, \times\}$ be the natural neutrosophic neutrosophic semigroup.

$\{\langle Z_6 \cup I\rangle_I, \times\}$ = {0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 5I, 1 + I, 2 + I, 3 + I, 4 + I, 5 + I, 1 + 2I, 2 + 2I, 3 + 2I, 4 + 2I, 5 + 2I,1 + 3I, 2 + 3I, 4 + 3I, 5 + 3I, 3 + 3I, 1 + 4I, 2 + 4I, 3 + 4I, 4 + 4I, 5 + 4I, 1 + 5I, 2 + 5I, 3 + 5I, 4 + 5I, 5 + 5I, I_0^I, I_I^I, $I_{5+I}^I, I_{5I+1}^I, I_{3+3I}^I$, I_{4+2I}^I, I_{2+4I}^I and so on}.

Clearly this has natural neutrosophic idempotents, natural neutrosophic neutrosophic nilpotents.

Thus we can prove the following theorem.

THEOREM 2.8: *Let $\{\langle Z_n \cup I\rangle_I, \times\}$ (n a composite number not of the form $n = p^t$, $t > 0$) be the natural neutrosophic finite complex modulo integer semigroup. $\langle Z_n \cup I\rangle_I$ has natural neutrosophic zero divisors, natural neutrosophic nilpotents and natural neutrosophic idempotents.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 2.9: *Let $\{\langle Z_n \cup I\rangle_I, \times\}$ be the natural neutrosophic semigroup. $\langle Z_n \cup I\rangle_I$ has zero divisors.*

Proof is direct and left as an exercise to the reader.

Next we proceed onto analyse the notion of natural neutrosophic dual number semigroups by examples.

Example 2.31: Let $\{\langle Z_{10} \cup g\rangle_I, \times\}$ be the dual number natural neutrosophic semigroup.

$\{\langle Z_{10} \cup g\rangle_I, \times\}$ = {0, 1, 2, …, 9g, 1 + g, 2 + g, …, 9 + g, 2g + 1, 2g + 2, …, 2g + 9, …, 9g + 1, 9g + 2, …, 9g + 9, I_0^g, I_g^g, I_{2g}^g, …, I_{9g}^g, I_{1+g}^g, I_{2g+1}^g, I_{1+3g}^g, . . ., $I_{1+9g}^g, I_{5+5g}^g, I_{2+2g}^g$, $I_{4+4g}^g, I_{6+6g}^g, I_{8+8g}^g$, $I_{2+4g}^g, I_{2g+4}^g, I_{2g+6}^g$, $I_{6g+2}^g, I_{8g+2}^g, I_{8+2g}^g$ and so on} be the natural neutrosophic dual number semigroup.

$I_g^g \times I_g^g = I_0^g$ is the natural neutrosophic nilpotent element of order two.

$I_{5g}^g \times I_{2g}^g = I_0^g$, $I_{5g}^g \times I_{4g}^g = I_0^g$, $I_{5g}^g \times I_{6g}^g = I_0^g$ and so on thus there are several natural neutrosophic zero divisors.

$I_{5g}^g \times I_{5g}^g = I_0^g$, $I_{3g}^g \times I_{3g}^g = I_0^g$, $I_{9g}^g \times I_{9g}^g = I_0^g$ and so on are the natural neutrosophic nilpotents of $\langle Z_{10} \cup g\rangle_I$.

$I_5^g \times I_5^g = I_5^g$ is a natural neutrosophic idempotent of $\langle Z_{10} \cup g\rangle_I$.

$\langle Z_{10} \cup g\rangle_I$; $I_6^g \times I_6^g = I_6^g$ is a natural neutrosophic idempotent of $\langle Z_{10} \cup g\rangle_I$.

Example 2.32: Let $\{\langle Z_7 \cup g\rangle_I, \times\}$ be the natural neutrosophic dual number semigroup.

$\langle Z_7 \cup g\rangle_I = \{0, 1, 2, \ldots, 6, g, 2g, \ldots, 6g, 1 + g, 2 + g, 6 + g, 2g + 1, 2g + 2, \ldots, 6 + 2g, 1 + 3g, \ldots, 6 + 3g, \ldots, 1 + 6g, \ldots, 6 + 6g, I_0^g, I_g^g, I_{2g}^g, I_{3g}^g, I_{4g}^g, I_{5g}^g, I_{6g}^g, I_{1+6g}^g, I_{6+g}^g, \ldots, I_{3+4g}^g, I_{4+3g}^g, I_{5+2g}^g, I_{2+5g}^g$ and so on$\}$ is a natural neutrosophic dual number semigroup which has natural neutrosophic dual number zero divisors and natural neutrosophic dual number nilpotents.

It is pertinent to keep on record that in case of $\langle Z_p \cup g\rangle_I$ we can have natural neutrosophic dual number zero divisors and natural neutrosophic dual number nilpotents even when p is a prime.

In view of this the following theorem is proved.

THEOREM 2.10: *Let $\{\langle Z_p \cup g\rangle_I;\ g^2 = 0$, p a prime, $\times\}$ be the natural neutrosophic dual number semigroup.*

i) *$\langle Z_p \cup g\rangle_I$ has atleast $p - 1$ natural neutrosophic nilpotents.*
ii) *$\langle Z_p \cup g\rangle_I$ has natural neutrosophic zero divisor.*

Proof is direct and hence left as an exercise to the reader.

We suggest the following problem.

Problem 2.5: Let $\{\langle Z_p \cup g\rangle_I, \times, p$ a prime, $g^2 = 0\}$ be the natural neutrosophic dual number semigroup.

Can $\langle Z_p \cup g\rangle_I$ have natural neutrosophic dual number idempotents?

Next we give one or two examples of natural neutrosophic dual number semigroups.

Example 2.33: Let $\{\langle Z_{24} \cup g\rangle_I, \times\}$ be the natural neutrosophic semigroups of dual numbers. $\langle Z_{24} \cup g\rangle_I = \{0, 1, 2, \ldots, 23, g, 2g, \ldots, 23g, I_0^g, I_g^g, I_{2g}^g, \ldots, I_{23g}^g, I_{2+22g}^g, I_{4+2g}^g, \ldots, I_{12g+22}^g$ and so on$\}$.

This has natural neutrosophic dual number idempotents.

$I_9^g \times I_9^g = I_9^g$.

This has atleast 23 natural neutrosophic dual number nilpotents.

$I_{2g}^g \times I_{2g}^g = I_0^g$, $I_{8g}^g \times I_{8g}^g = I_0^g$, $I_{23g}^g \times I_{23g}^g = I_0^g$ and so on.

This has several natural neutrosophic dual number zero divisors gives by

$I_{2g}^g \times I_{23g}^g = I_0^g$, $I_{4+4g}^g \times I_{6g}^g = I_0^g$, $I_{(8+8g)}^g \times I_{(3+3g)}^g = I_0^g$ and so on.

Thus we see $\langle Z_n \cup g\rangle_I$, n is a composite number of the $p_1p_2p_3 \ldots p_t$, p_i's distinct primes, we are sure to arrive at natural neutrosophic dual number nilpotents, natural neutrosophic dual number idempotents and natural neutrosophic dual number zero divisors.

Example 2.34: Let $\langle Z_{32} \cup g\rangle_I$, ×} be the natural neutrosophic dual number semigroup.

$\langle Z_{32} \cup g\rangle_I$ = {0, 1, 2, …, 31, g, 2g, …, 31g, 1 + g, 2 + g, …, 31+g, 2 + 2g, 1 + 2g, …, 31 + 2g, …, 31 + 3g, I_0^g, I_g^g, …, $I_{31g}^g, I_{8+8g}^g, I_{2+2g}^g$, $I_{4+4g}^g, I_{2g}^g, I_4^g$, I_6^g, I_{8g}^g, …, I_{30}^g and so on}.

This has natural neutrosophic zero divisors of dual numbers and natural neutrosophic dual number nilpotents of order two.

$I_{31g}^g \times I_{31g}^g = I_0^g$, $I_g^g \times I_g^g = I_0^g$, $I_{8g}^g \times I_{4g}^g = I_0^g$,

$I_{4g}^g \times I_{8g+8}^g = I_0^g$, $I_8^g \times I_8^g = I_0^g$, $I_{4+4g}^g \times I_{8g}^g = I_0^g$ are some of the natural neutrosophic dual number zero divisors and nilpotent.

The only challenging problem in this direction is finding the order of $\langle Z_n \cup g\rangle_I$ for a fixed n.

All other study is considered as a matter of routine and is left as an exercise to the reader.

Next we proceed onto study natural neutrosophic special dual like number semigroups by some examples.

Example 2.35: Let {$\langle Z_4 \cup h\rangle_I$, ×} = {0, 1, 2, 3, h, 2h, 3h, 1 + h, 2 + h, 3h, 1 + 2h, 2 + 2h, 3 + 2h, 1 + 3h, 2 + 3h, 3 + 3h, I_0^h, I_h^h, I_{2h}^h, $I_2^h, I_{2+2h}^h, I_{3h}^h$, $I_{2+h}^h, I_{3+hh}^h, I_{3h+1}^h$, I_{2+3h}^h, …, ×} be the natural neutrosophic special dual like number semigroup.

$I_{2+2h}^h \times I_{2+2h}^h = I_0^h$, $I_{3+h}^h \times I_h^h = I_0^h$, $I_{2h}^h \times I_{2h}^h = I_0^h$ and

$I^h_{3h+1} \times I^h_h = I^h_0$ are some of the natural neutrosophic special dual like number nilpotents and zero divisors of $\langle Z_4 \cup h\rangle_I$.

It is left for the reader to find the natural neutrosophic special dual like number idempotents of $\langle Z_4 \cup h\rangle_I$ other than I^h_h.

Example 2.36: Let $\{\langle Z_3 \cup h\rangle_I, \times\} = \{0, 1, 2, h, 2h, 1 + h, 1 + 2h, 2 + h, 2 + 2h, I^h_0, I^h_h, I^h_{2h}, I^h_{1+2h}, I^h_{2+h}, \times\}$ be the natural neutrosophic special dual like number semigroup.

$I^h_h \times I^h_h \times I^h_h$, $I^h_{1+2h} \times I^h_{1+2h} = I^h_{1+2h}$ are natural neutrosophic special dual like number idemopotents of $\langle Z^h_3 \cup h\rangle_I$.

$I^h_{1+2h} \times I^h_h = I^h_0$, $I^h_{2+h} \times I^h_h = I^h_0$ are some of the natural neutrosophic special dual like number zero divisors of $\langle Z_3 \cup h\rangle_I$.

Finding nilpotents of $\langle Z_p \cup h\rangle_I$, p a prime happens to be a difficult task.

This is left as a open problem.

Problem 2.6: Let $\langle Z_p \cup h\rangle_I, \times\}$ be the natural neutrosophic special dual like number semigroup p, a prime.

Can $\langle Z_p \cup h\rangle_I$ contain natural neutrosophic special dual like nilpotent elements?

Example 2.37: Let $\{\langle Z_6 \cup h\rangle_I, \times\}$ be the natural neutrosophic special dual like number semigroup.

$\langle Z_6 \cup h\rangle_I = \{0, 1, 2, \ldots, 5, h, 2h, \ldots, 5h, 1 + h, 2 + h, \ldots, 5 + h, 1 + 2h, 2 + 2h, \ldots, 5 + 2h, 3 + h, \ldots, 3h + 5, 1 + 4h, \ldots, 5 + 4h, 1 + 5h, 2 + 5h, \ldots, 5 + 5h, I^h_0, I^h_1, I^h_{2h}, I^h_{3h}, I^h_{4h}, \ldots, \times\}$ is the natural neutrosophic special dual like number semigroup.

$I^h_{3h} \times I^h_{2h} = I^h_0$, $I^h_{(1+5h)} \times I^h_h = I^h_0$, $I^h_{3+3h} \times I^h_h = I^h_0$ and $I^h_{3+h} \times I^h_4 = I^h_0$ are some of the natural neutrosophic special dual like number zero divisors.

$I^h_h \times I^h_h = I^h_h$, , $I^h_{3h} \times I^h_{3h} = I^h_{3h}$, $I^h_{4h} \times I^h_3 = I^h_3$ are some of the natural neutrosophic special dual like number of idempotents $\{\langle Z_6 \cup h\rangle_I, \times\}$.

Example 2.38: Let $\{\langle Z_{16} \cup h\rangle_I, \times\}$ be the natural neutrosophic special dual like number semigroup.

$\{\langle Z_{16} \cup h\rangle_I, \times\} = \{0, 1, 2, \ldots, 15, h, 2h, \ldots, 15, 1 + h, \ldots, 1 + 15h, 2 + h, \ldots, 2 + 15h, .., 15 + 15h, I^h_0, I^h_h, I^h_{2h}, I^h_{3h}, \ldots, I^h_{15h}$. I^h_{2+14h}, I^h_{1+15h} I^h_{h+15} and so on$\}$.

Clearly $I^h_{4h} \times I^h_{4h} = I^h_0$, $I^h_{6h} \times I^h_8 = I^h_0$ and so on.

There are natural neutrosophic special dual like number nilpotents and zero divisors.

Finding natural neutrosophic special dual like number idempotents happen to be a challenging problem barring the natural neutrosophic special dual like number idempotent I^h_h.

Hence finding the natural neutrosophic special dual like number in case of $\{\langle Z_{p^t} \cup h\rangle_I$, p a prime, t > 1 happens to be a challenging problem.

However the study of the natural neutrosophic special dual like number of semigroup is considered as a matter of routine.

In the following we proceed on to give examples of natural neutrosophic special quasi dual number semigroup.

Example 2.39: Let $\{\langle Z_4 \cup k\rangle_I, \times\} = \{0, 1, 2, 3, k, 2k, 3k, 1+k, 1+2k, 1+3k, 2 + k, 2 + 2k, 2 + 3k, 3 + k, 3 + 2k, 3 + 3k, I^k_0, I^k_k,$

I^k_{2k}, $I^k_{3k}, I^k_2, I^k_{2+2k}$, I^k_{1+k}, I^k_{1+3k}, I^k_{2+k}, I^k_{2+3k}, I^k_{3+k}, I^k_{3+3k}, $\times\}$ be the natural neutrosophic special quasi dual number semigroup.

This has natural neutrosophic special quasi dual number zero divisors.

$I^k_2 \times I^k_2 = I^k_0$, $I^k_2 \times I^k_{2k} = I^k_0$, $I^k_{2k} \times I^k_{2k} = I^k_0$, $I^k_{2+2k} \times I^k_2 = I^k_0$ are the natural neutrosophic zero divisors.

$I^k_{1+k} \times I^k_{1+k} = I^k_{k+1}$ is a natural neutrosophic special quasi dual number idempotent.

Example 2.40: Let $\{\langle Z_3 \cup k\rangle_I, \times\}$ be the natural neutrosophic special quasi dual number semigroup of finite order.

$\{\langle Z_3 \cup k\rangle_I, \times\}$ = {0, 1, 2, k, 2k, 1 + k, 1 + 2k, 2 + k, 2 + 2k, I^k_0, I^k_k, I^k_{2k}, $I^k_{1+k}, I^k_{1+2k}, I^k_{2+2k}$, $\times$} be the natural neutrosophic special quasi dual number semigroup.

$I^k_{1+k} \times I^k_{1+k} = I^k_{1+k}$ is a natural neutrosophic special quasi dual number idempotent.

$I^k_{2+2k} \times I^k_k = I^k_0$ is a natural neutrosophic special quasi dual number zero divisor.

Example 2.41: Let $\{\langle Z_{12} \cup k\rangle_I, \times\}$ = {0, 1,2, …, 11, k, 2k, …, 1 + k, 1 + 2k, …, 1 + 11k, 2 + k, 2 + 2k, …, 2 + 11k, 3 + k, 3 + 2k, 3 + 3k, …, 3 + 11k, …, 11 + k, 11 + 2k, …, 11 + 11k, I^k_0, I^k_k, …, I^k_{11k}, I^k_{6+6k}, I^k_{2+2k}, I^k_{3+3k}, I^k_{4+4k} and so on} be the natural neutrosophic special quasi dual number semigroup.

$I^k_4 \times I^k_4 = I^k_4$ are $I^k_9 \times I^k_9 = I^k_9$ natural neutrosophic special quasi dual number idempotent.

$I^k_4 \times I^k_{3+3k} = I^k_0$, $I^k_{6+6k} \times I^k_8 = I^k_0$, $I^k_{8+8k} \times I^k_3 = I^k_0$ are some of the natural neutrosophic special quasi dual number zero divisor.

$I^k_{6+6k} \times I^k_{6+6k} = I^k_0$, $I^k_6 \times I^k_6 = I^k_0$, $I^k_{6k} \times I^k_{6k} = I^k_0$ are some of the natural neutrosophic special quasi dual number nilpotents of order two.

The study is a matter of routine and the reader is expected to do this task.

Next we proceed onto study matrices with entries from Z^I_n or $C^I(Z_n)$ and so on.

The product is the natural product $\times_n$.

This is illustrated by examples.

Example 2.42: Let
A = {(x_1, x_2, x_3) / $x_i \in$ { Z^I_4, ×} = {0, 1, 2, 3, I^4_0, I^4_2 } $1 \leq i \leq 3$, ×} be the real natural neutrosophic matrix.

$(I^4_0, I^4_2, I^4_2) \times (I^4_0, I^4_2, I^4_0) = (I^4_0, I^4_0, I^4_0)$ is a real natural neutrosophic row matrix zero divisor.

This has several zero divisors idempotents and nilpotents.

However this has subsemigroups, both only real as well as real natural neutrosophic and so on.

B = {(x_1, x_2, x_3) / $x_i \in Z_4$; $1 \leq i \leq 3$} ⊆ A is a subsemigroup which is not ideal only a subsemigroup.

Example 2.43: Let

$$x = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle/ a_i \in \{ Z^I_7, \times\}; \{0, 1, 2, \ldots, 6, I^7_0\}, \times_n \right\}$$

be the real natural neutrosophic semigroup.

Clearly x is of finite order.

However x has several subsemigroups as well as ideals.

Example 2.44: Let

$$Y = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle/ a_i \in \{ Z^I_{12}, \times\}; \{0, 1, 2, \ldots, 1,\ I_0^{12},\ I_2^{12}, I_3^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_9^{12},\ I_{10}^{12}\}, \times_n \right\}$$

be the real natural neutrosophic matrix semigroup.

$$A = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle/ a_i \in \{ I_0^{12},\ I_2^{12}, I_3^{12}, I_4^{12},\ I_6^{12}, I_8^{12}, I_9^{12},\ I_{10}^{12}\}, 1 \leq i \leq 10; \times_n \right\} \subseteq Y$$

is a real natural neutrosophic subsemigroup of Y.

The task of finding subsemigroups and ideals of real natural neutrosophic matrix happens to be routine so left as an exercise to the reader.

Example 2.45: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle/ a_i \in \{Z_{19}^I, \times\} = \{0, 1, 2, \ldots, 18, I_0^{19}\}, \right.$$

$$\left. 1 \leq i \leq 12, \times_n \right\}$$

be the natural neutrosophic matrix semigroup under natural product.

This has subsemigroups, zero divisors, idempotents and ideals.

Example 2.46: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle/ a_i \in \{Z_8^I, \times\} = \{0, 1, 2, \ldots, 7, I_0^8, I_2^8, \right.$$

$$\left. I_4^8, I_6^8\}, \times_n, 1 \leq i \leq 16 \right\}$$

be the real natural neutrosophic semigroups under $\times_n$.

Clearly the usual product is not defined on S.

For if the usual product × is defined then elements like x + $I_t^8, I_t^8 + I_x^8$ and so on will occur and they do not in general belong to $\{Z_8^I, \times\}$.

$$\text{Let } x = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \in S.$$

$$x \times_n y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so is a zero divisor of S.}$$

Example 2.47: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \Bigg/ a_i \in \{ Z_{15}^I, \times\} = \{0, 1, 2, 3, \ldots, 14, \right.$$

$$\left. I_0^{15}, I_3^{15}, I_6^{15}, I_9^{15}, I_{12}^{15}, I_5^{15}, I_{10}^{15}, \times\}; 1 \leq i \leq 15, \times_n \right\}$$

be the real natural neutrosophic matrix semigroup under natural product.

$$x = \begin{bmatrix} 10 & I_0^{15} & 6 \\ I_6^{15} & I_{10}^{5} & I_0^{15} \\ 0 & 0 & 10 \\ 6 & 0 & I_6^{15} \\ 0 & I_6^{15} & 0 \end{bmatrix} \in B \text{ is such that}$$

$$x \times_n x = \begin{bmatrix} 10 & I_0^{15} & 6 \\ I_6^{15} & I_{10}^{15} & I_0^{15} \\ 0 & 0 & 10 \\ 6 & 0 & I_6^{15} \\ 0 & I_6^{15} & 0 \end{bmatrix}.$$

Thus this is the natural neutrosophic idempotent matrix of B.

$$\text{Let } y = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 10 \end{bmatrix} \text{ and}$$

$$x = \begin{bmatrix} 7 & 0 & 8 & 5 \\ 6 & 3 & 8 & 4 \\ 2 & 2 & 5 & 0 \\ 5 & 4 & 3 & 6 \\ 9 & 3 & 12 & 3 \end{bmatrix} \in B.$$

Clearly

$$x \times_n y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a real natural neutrosophic matrix zero divisor of B.

Example 2.48: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle/ a_i \in \{ Z_8^I, \times\} = \{0, 1, 2, \ldots, 7, I_0^8, I_2^8, I_4^8, I_6^8, \times\}, \times_n; 1 \leq i \leq 12 \right\}$$

be the real natural neutrosophic matrix semigroup under the natural product $\times_n$.

$$x = \begin{bmatrix} 4 & 0 \\ 0 & 4 \\ I_4^8 & I_0^8 \\ I_0^8 & 4 \\ 0 & 0 \\ 4 & I_4^8 \end{bmatrix} \in B \text{ is such that}$$

$$x \times_n x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_0^8 & I_0^8 \\ I_0^8 & 0 \\ 0 & 0 \\ 0 & I_0^8 \end{bmatrix}$$

is a natural neutrosophic nilpotent matrix of B.

However B has several natural neutrosophic matrix zero divisors.

We will one or two example of them.

$$x = \begin{bmatrix} 2 & 4 \\ 6 & I_4^8 \\ I_6^8 & I_2^8 \\ I_4^8 & I_0^8 \\ 2 & 6 \\ I_2^8 & I_6^8 \end{bmatrix} \text{ and } y = \begin{bmatrix} 4 & 2 \\ 4 & I_2^8 \\ I_4^8 & I_4^8 \\ I_4^8 & I_2^8 \\ 4 & 4 \\ I_4^8 & I_4^8 \end{bmatrix} \in B.$$

Clearly

$$x \times_n y = \begin{bmatrix} 0 & 0 \\ 0 & I_0^8 \\ I_0^8 & I_0^8 \\ I_0^8 & I_0^8 \\ 0 & 0 \\ I_0^8 & I_0^8 \end{bmatrix}$$

is the real natural neutrosophic matrix of B.

Example 2.49: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} / a_i \in \{ Z_{12}^I, \times\}; = \{0, 1, 2, \ldots, 11, \right.$$

$$\left. I_0^{12}, I_2^{12}, I_4^{12}, I_6^{12}, I_8^{12}, I_{10}^{12}, I_3^{12}, I_9^{12} \}; 1 \leq i \leq 12, \times_n \right\}$$

be the real natural neutrosophic matrix semigroup under the natural product $\times_n$.

P has natural neutrosophic nilpotent matrices, natural neutrosophic idempotent matrices and natural neutrosophic zero divisor matrices.

$$\text{Let } x = \begin{bmatrix} 6 & I_0^{12} & I_6^{12} & 0 \\ 0 & 0 & I_6^{12} & 6 \\ 0 & 6 & 0 & I_0^{12} \end{bmatrix} \in P.$$

Clearly

$$x \times_n x = \begin{bmatrix} 0 & I_0^{12} & I_0^{12} & 0 \\ 0 & 0 & I_0^{12} & 0 \\ 0 & 0 & 0 & I_0^{12} \end{bmatrix}$$

is the natural neutrosophic mixed zero matrix.

$(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ will be known as the real natural neutrosophic zero matrix.

$I_0^{12} = \begin{bmatrix} I_0^{12} & I_0^{12} & I_0^{12} & I_0^{12} \\ I_0^{12} & I_0^{12} & I_0^{12} & I_0^{12} \\ I_0^{12} & I_0^{12} & I_0^{12} & I_0^{12} \end{bmatrix}$ will be known as the natural pure neutrosophic zero matrix.

These two matrices are unique; however we have several natural neutrosophic mixed zero matrices.

A few of them are given in the following.

$$x_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I_0^{12} & 0 & 0 & I_0^{12} \\ 0 & 0 & I_0^{12} & 0 \\ 0 & I_0^{12} & 0 & 0 \end{bmatrix}, \; x_2 = \begin{bmatrix} I_0^{12} & I_0^{12} & I_0^{12} & 0 \\ 0 & 0 & 0 & I_0^{12} \\ I_0^{12} & 0 & I_0^{12} & 0 \\ 0 & I_0^{12} & 0 & I_0^{12} \end{bmatrix}$$

$$x_3 = \begin{bmatrix} I_0^{12} & 0 & 0 & I^{12} \\ 0 & 0 & I_0^{12} & 0 \\ I_0^{12} & 0 & 0 & I_0^{12} \\ 0 & 0 & I_0^{12} & 0 \end{bmatrix} \text{ and so on.}$$

This work defining all the three types of zero is left as a simple exercise to the reader.

However in case of Z_n^I there is one real zero that is 0 and only one natural neutrosophic zero that in I_0^n, but in case of real natural neutrosophic matrices there is one real matrix zero;

$$(0) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and one natural neutrosophic zero matrix

$$(I_0^n) = \begin{bmatrix} I_0^n & I_0^n & \dots & I_0^n \\ I_0^n & I_0^n & \dots & I_0^n \\ \vdots & \vdots & & \vdots \\ I_0^n & I_0^n & \dots & I_0^n \end{bmatrix}.$$

There are several mixed natural neutrosophic zeros.

They are not unique as is seen from the earlier example.

Example 2.50: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \middle/ a_i \in \{ Z^I_{10}, \times\} = \{0, 1, 2, \ldots, 9, I_2^{10}, I_4^{0}, I_6^{10}, I_8^{10}, I_5^{10}, \times\}, \times_n\ 1 \leq i \leq 8 \right\}$$

be the real natural neutrosophic matrix semigroup under natural product $\times_n$.

$$(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \left(I_0^{10}\right) = \begin{bmatrix} I_0^{10} \\ I_0^{10} \\ I_0^{10} \\ I_0^{10} \\ I_0^{10} \\ I_0^{10} \\ I_0^{10} \\ I_0^{10} \end{bmatrix}$$

are the real and natural neutrosophic zero matrices of M respectively.

$$p_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_0^{10} \end{bmatrix}, p_2 = \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} 0 \\ I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \end{bmatrix},$$

$$p_6 = \begin{bmatrix} 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \end{bmatrix} \text{ and so on.}$$

$$p_9 = \begin{bmatrix} I_0^{10} \\ I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_{10} = \begin{bmatrix} I_0^{10} \\ 0 \\ I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_{11} = \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \\ 0 \end{bmatrix}, p_{12} = \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \end{bmatrix} \text{ and so on.}$$

There are ${}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + \ldots + {}_{10}C_9 + {}_{10}C_{10}$ number of mixed natural neutrosophic zero matrices for this M.

$$\text{Let } x = \begin{bmatrix} I_5^{10} \\ 0 \\ 0 \\ 0 \\ I_4^{10} \\ 0 \\ 0 \\ I_6^{10} \end{bmatrix} \text{ and } y = \begin{bmatrix} I_6^{10} \\ 8 \\ 0 \\ 6 \\ I_5^{10} \\ 7 \\ 4 \\ I_5^{10} \end{bmatrix} \in M$$

$$x \times_n y = \begin{bmatrix} I_0^{10} \\ 0 \\ 0 \\ 0 \\ I_0^{10} \\ 0 \\ 0 \\ I_0^{10} \end{bmatrix}$$

is a natural neutrosophic mixed zero of M.

$$\text{Consider } x = \begin{bmatrix} 5 \\ 6 \\ I_6^{10} \\ I_5^{10} \\ 0 \\ 6 \\ 5 \\ I_0^{10} \end{bmatrix} \in M.$$

Clearly

$$x \times_n x = \begin{bmatrix} 5 \\ 6 \\ I_6^{10} \\ I_5^{10} \\ 0 \\ 6 \\ 5 \\ I_0^{10} \end{bmatrix} = x$$

is an natural neutrosophic idempotent of M.

$$x_1 = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 0 \\ 6 \\ 0 \\ 5 \\ 6 \end{bmatrix} \in M \text{ is such that } x_1 \times_n x_1 = \begin{bmatrix} 0 \\ 5 \\ 6 \\ 0 \\ 6 \\ 0 \\ 5 \\ 6 \end{bmatrix} = x_1$$

is an idempotent of M.

We call idempotents matrices of M whose elements are from Z_{10} as real idempotent matrix.

$$y_1 = \begin{bmatrix} I_0^{10} \\ I_6^{10} \\ I_5^{10} \\ I_0^{10} \\ I_0^{10} \\ I_6^{10} \\ I_5^{10} \\ I_6^{10} \end{bmatrix} \in M$$

as natural neutrosophic matrix idempotents of M.

However x given in this problem as an idempotent is a mixed natural neutrosophic matrix idempotent of M.

In fact all the three types of idempotents are several in number.

Thus only incase of natural neutrosophic matrix semigroups we can four types of idempotents.

$$(0) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & 0 \end{bmatrix}, (1) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

mixed real matrix idempotent with 0, 1 and any other idempotent of Z_n.

Apart from this natural neutrosophic idempotent matrices and mixed natural neutrosophic idempotent matrices.

These situations are described in the example 2.49 and by this example also.

Example 2.51: Let $\{ Z_{20}^{I}, \times\} = \{0, 1, 2, .., 19, I_0^{20}, I_2^{20}, I_4^{20}, I_6^{20}, I_8^{20}, \ldots, I_{18}^{20}, I_5^{20}, I_{15}^{20}, \times\}$ be the natural neutrosophic semigroup.

$P = \{(a_1, a_2, \ldots, a_7) / a_i \in (Z_{20}^{I}, \times); 1 \leq i \leq 7, \times\}$ be the natural neutrosophic row matrix semigroup

(0) = (0, 0, …, 0) is the zero element of P.

(1) = (1, 1, …, 1) is the identity matrix of P.

Both are trivial idempotents of P.

$(I_0^{20}, I_0^{20}, \ldots, I_0^{20}) = (I_0^{20})$ is also the natural neutrosophic idempotent of P.

(5) = (5, 5, …, 5) is again an idempotent of P.

$x_1 = (5, 0, 1, 0, 5, 5, 1) \in P$ is also an idempotent of P.

$y_1 = (I_5^{20}, I_5^{20}, I_5^{20}, I_5^{20}, I_5^{20}, I_5^{20}, I_5^{20}, I_5^{20}) \in P$ is again a natural neutrosophic idempotent row matrix of P.

$x_2 = (16, 16, 16, 16, 16, 16, 16) \in P$ is an real idempotent of P.

$x_3 = (16, 5, 0, 16, 16, 5, 0) \in P$ is again a real idempotent of P.

$x_4 = (I_{16}^{20}, I_{16}^{20}, I_{16}^{20}, I_{16}^{20}, I_{16}^{20}, I_{16}^{20}, I_{16}^{20}) \in P$ is a natural neutrosophic idempotent of P.

$x_5 = (I_5^{20}, I_{16}^{20}, I_{16}^{20}, I_0^{20}, I_5^{20}, I_0^{20}, I_{16}^{20}) \in P$ is a natural neutrosophic idempotent of P.

$x_6 = (I_5^{20}, I_{16}^{20}, 1, 16, 5, 0, 1)$ in P is a natural neutrosophic mixed idempotent of P.

Infact it is an interesting work to find out the number of all types of idempotents of P.

Now $I_{10}^{20} \times I_6^{20} = I_0^{20}$, $I_4^{20} \times I_5^{20} = I_0^{20}$, $I_4^{20} \times I_{10}^{20} = I_0^{20}$, $I_8^{20} \times I_5^{20} = I_0^{20}$ and so on are the natural neutrosophic idempotents.

$I_5^{20} \times I_5^{20} = I_5^{20}$ and $I_{16}^{20} \times I_{16}^{20} = I_{16}^{20}$ are the natural neutrosophic idempotents of $\{Z_{20}^I, \times\}$.

$I_{10}^{20} \times I_{10}^{20} = I_0^{20}$ is the natural neutrosophic nilpotent of $\{Z_{20}^I, \times\}$.

a = $(I_{10}^{20}, I_{10}^{20}, I_0^{20}, I_0^{20}, I_{10}^{20}, I_{10}^{20}, I_0^{20})$ is the natural neutrosophic nilpotent elements of P.

b = (10, 0, 0, 0, 10, 10, 0) $\in$ P is again a real nilpotent matrix of P.

c = $(I_{10}^{20}, I_0^{20}, 10, 10, 0, I_{10}^{20}, I_0^{20})$ is again a mixed natural neutrosophic nilpotent of order two in P.

d = $(I_{10}^{20}, I_{10}^{20}, I_{10}^{20}, I_{10}^{20}, I_{10}^{20}, I_{10}^{20}, I_{10}^{20})$ $\in$ P is the natural neutrosophic nilpotent of order two in P.

Let x = (0, 5, 0, 6, 2, 8, 4) and y = (8, 4, 0, 5, 10, 5, 5) $\in$ P.

It is easily verified.

$x \times y$ = (0, 0, 0, 0, 0, 0, 0) is a zero divisor.

Let v = $(I_{10}^{20}, I_4^{20}, I_6^{20}, I_5^{20}, I_0^{20}, I_8^{20}, I_{16}^{20})$ and

u = $I_2^{20}, I_{10}^{20}, I_{10}^{20}, I_4^{20}, I_{16}^{20}, I_{10}^{20}, I_5^{20}) \in$ P.

Clearly v × u = (0, 0, 0, 0, 0, 0, 0) known as the natural neutrosophic zero divisor of P.

Thus P has natural neutrosophic zero divisors.

Consider $p = (I_{10}^{20}, 6, 2, I_5^{20}, 4, I_{16}^{20}, I_8^{20})$ and
$q = (I_4^{20}, 10, 10, I_4^{20}, 5, I_5^{20}, I_5^{20}) \in P$.

$p \times q = (I_{10}^{20}, 0, 0, I_0^{20}, 0, I_0^{20}, I_0^{20}) \in P$ is the mixed natural neutrosophic zero divisor matrix.

We give one more example before we proceed onto work with $C^I(Z_n)$, $\langle Z_n \cup I\rangle_I$ and so on.

Example 2.52: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle/ a_i \in \{Z_{40}^I, \times\};\ 1 \leq i \leq 24, \times_n \right\}$$

be the real natural neutrosophic matrix semigroup.

This has natural neutrosophic zero divisors, idempotents and nilpotents.

Also S has real matrix idempotents, zero divisors and nilpotents.

The reader is left with the task of finding the order of S and all the special properties associated with S.

Now are proceed on to study natural neutrosophic complex modulo integer matrices by examples.

Example 2.53: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in \{C^I(Z_2) = \{0, 1, i_F, 1 + i_F, I_0^C, I_{1+i_F}^C, \right.$$

$$\left. \times\}; 1 \leq i \leq 15, \times_n \right\}$$

be the natural neutrosophic finite complex modulo integer semigroup under natural product $\times_n$.

Clearly W has mixed natural neutrosophic zero divisors.

However only $I_{1+i_F}^C \times I_{1+i_F}^C = I_0^C$ and $(1 + i_F) \times (1 + i_F) = 0$ are the only two nilpotents of W.

Finding zero divisor matrix, idempotent matrix etc. are considered as a matter of routine and hence left as an exercise to the reader.

Example 2.54: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle/ a_i \in \{C^I(Z_6); \times\}; 1 \leq i \leq 7, \times_n \right\}$$

be the natural neutrosophic finite complex modulo integer matrix semigroup under the natural product $\times_n$.

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2^C \\ I_3^C \\ I_4^C \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 5 \\ 4 \\ I_3^C \\ I_2^C \\ I_3^C \\ 5 \end{bmatrix} \in S \text{ is such that}$$

$$x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_0^C \\ I_0^C \\ I_0^C \\ 0 \end{bmatrix}$$

is a natural neutrosophic mixed zero matrix of S.

$$\text{Let } x = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 3 \\ 3 \\ 1 \\ 0 \end{bmatrix} \in S;$$

clearly $x \times_n x = x$ so x is a real idempotent matrix of S.

$$y = \begin{bmatrix} I_3^C \\ I_0^C \\ I_3^C \\ I_0^C \\ I_3^C \\ I_3^C \\ 0 \end{bmatrix} \in S \text{ is such that } y \times_n y = \begin{bmatrix} I_3^C \\ I_0^C \\ I_3^C \\ I_0^C \\ I_3^C \\ I_3^C \\ 0 \end{bmatrix} = y$$

is a natural neutrosophic mixed idempotent of S.

$$\text{Let } a = \begin{bmatrix} I_0^C \\ I_2^C \\ I_4^C \\ I_4^C \\ I_3^C \\ I_0^C \\ I_{3+3i_F}^C \end{bmatrix} \text{ and } b = \begin{bmatrix} I_{2+4i_F}^C \\ I_{3i_F}^C \\ I_{3+3i_F}^C \\ I_3^C \\ I_{4i_F+2}^C \\ I_{2i_F+4}^C \\ I_{4+2i_F}^C \end{bmatrix} \in S.\ a \times_n b = \begin{bmatrix} I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \end{bmatrix}$$

is a natural neutrosophic zero divisor of S.

Let

$$A = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \\ I_0^C \end{bmatrix} \Bigg/ a \;\; \{C^I(Z_6), \times\}, \times_n \right\} \subseteq S$$

is a subsemigroup which is also the natural neutrosophic subsemigroup and is not an ideal of S.

Constructing ideals happens to be difficult job.

Thus the natural neutrosophic complex modulo integer semigroup.

Example 2.55: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} / a_i \in \{C^I(Z_4), \times\};\ 1 \leq i \leq 4, \times_n\right\}$$

be the matrix of natural neutrosophic complex modulo integer semigroup under natural product $\times_n$.

Clearly M is not closed under the usual product $\times$.

$C^I(Z_4) = \{0, 1, 2, 3, i_F, 2i_F, 3i_F, 1 + i_F, 2 + i_F, 3 + i_F, 1 + 2i_F, 2 + 2i_F, 3 + 2i_F, 3 + 3i_F, 2 + 3i_F, 1 + 3i_F, I_0^C, I_{2iF}^C, I_{2+2i_F}^C, I_2^C, I_{1+i_F}^C, I_{3+i_F}^C, I_{3+3i_F}^C, I_{1+3i_F}^{20}, \times\}$ is a semigroup of natural neutrosophic finite complex modulo integers.

It is left as an exercise for the reader to find the order of M, real zero divisors matrices of M, complex zero divisor matrices of M, natural neutrosophic matrix zero divisors of M.

Similarly, find the idempotents and nilpotent matrices of any in M.

Next we proceed onto study the natural neutrosophic neutrosophic matrix semigroups built using $\langle Z_n \cup I\rangle_I$ by some examples.

Example 2.56: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix}\right. / a_i \in \{\langle Z_6 \cup I\rangle_I, \times\} = \{0, 1, 2, \ldots, 5, I, 2I,$$

3I, 4I, 5I, 1 + I, 2 + I, …, 5 + I, 2 + 2I, 1 + 2I, 3 + 2I, 4 + 2I, 5 + 2I, 1 + 3I, 2 + 3I, 3 + 3I, …, 5 + 3I, 1 + 4I, …, 5 + 4I, 1 + 5I, 2 + 5I, ..., 5 + 5I, I_0^I, I_I^I, I_{2I}^I, $I_{3I}^I, I_{4I}^I, I_{5I}^I$, $I_{1+5I}^I, I_{5+I}^I, I_{2+4I}^I$, $I_{4+2I}^I, I_{3+3I}^I, I_{4I+4}^I$, I_{2+2I}^I and so on, $\times\}$, $\times_n\}$ be the natural neutrosophic neutrosophic matrix semigroup under the natural product $\times_n$.

This M has natural neutrosophic-neutrosophic zero divisors, zeros and idempotents.

Finding these special elements is a matter of routine so left as an exercise to the reader.

Further the readers is left with the task of finding subsemigroups and ideals of M.

Example 2.57: Let

$$S = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right. / a_i \in \{\langle Z_4 \cup I\rangle_I, \times\} = \{0, 1, 2, 3, I, 2I, 3I, 1 + I,$$

2 + I, 3 + I, 1 + 2I, 2 + 2I, 3 + 2I, 1 + 3I, 2 + 3I, 3 + 3I, I_0^I, I_I^I, I_{2I}^I, $I_{3I}^I, I_{2+2I}^I, I_2^I$, $I_{3+I}^I, I_{3I+1}^I, I_{1+I}^I$, $I_{2+I}^I, I_{3+I}^I, I_{2+3I}^I$, I_{3+3I}, ×}, $1 \leq i \leq 5$, $\times_n$}
be the natural neutrosophic neutrosophic matrix semigroup.

S has real zero divisors and real nilpotents.

$$\text{For } x = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} \in S \text{ is such that } x \times_n x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a nilpotent element of order two.

$$y = \begin{bmatrix} 2I \\ 2I \\ 2I \\ 2+2I \\ 2I \end{bmatrix} \in S$$

is a neutrosophic nilpotent elements of order two as

$$y \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$x = \begin{bmatrix} I_2^I \\ I_2^I \\ I_2^I + 2I \\ I_2^I \\ I_0^I \end{bmatrix} \in S \text{ is such that } x \times_n x = \begin{bmatrix} I_0^I \\ I_0^I \\ I_0^I \\ I_0^I \\ I_0^I \end{bmatrix}$$

so is a natural neutrosophic neutrosophic nilpotent element of order two.

$$\text{Let } x_1 = \begin{bmatrix} I_{3+I}^I \\ I_{2I}^I \\ I_{3I+1}^I \\ I_2^I \\ I_{2+2I}^I \end{bmatrix} \text{ and } y_1 = \begin{bmatrix} I_I^I \\ I_2^I \\ I_I^I \\ I_{2I}^I \\ I_{2I}^I \end{bmatrix} \in S.$$

Clearly

$$x_1 \times_n y = \begin{bmatrix} I_0^I \\ I_0^I \\ I_0^I \\ I_0^I \\ I_0^I \end{bmatrix}$$

is a natural neutrosophic-neutrosophic matrix zero divisor of S.

The natural neutrosophic neutrosophic idempotent matrix of S is

$$x = \begin{bmatrix} I_I^I \\ I_0^I \\ I_I^I \\ I_I^I \\ I_0^I \end{bmatrix} \in S.$$

For $x \times_n x = x$.

Thus the task of finding the order of S and other special elements of S is left as an exercise to the reader.

The reader is also left with the task of finding subsemigroups and ideals of S.

Example 2.58: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \right. a_i \in \{\langle Z_5 \cup I\rangle_I, \times\} = \{0, 1, 2, 3, 4, I,$$

2I, 3I, 4I, 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 4 + I, 4 + 2I, 4 + 3I, 4 + 4I, I_0^I, I_I^I, I_{2I}^I, $I_{2I}^I, I_{3I}^I, I_{4I}^I$, $I_{3+2I}^I, I_{2+3I}^I, I_{1+4I}^I$, I_{4+I}^I and so on}, $\times_n$; $1 \leq i \leq 24\}$ be the natural neutrosophic neutrosophic matrix semigroup.

This has zero divisor real and natural neutrosophic neutrosophic.

Thus the reader is expected to find them.

Next we proceed onto study by examples.

The natural neutrosophic dual numbers matrix semigroup.

Example 2.59: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \right. / a_i \in \{\langle Z_6 \cup g\rangle_I, g^2 = 0, \times\} = \{0, 1, 2, \ldots, 5, g, 2g,$$

$\ldots, 5g, 1 + g, 2 + g, \ldots, 5 + g, 2g + 1, 2g + 2, \ldots, 5 + 2g, \ldots, 1 + 5g, \ldots, 5 + 5g, I_0^g, I_g^g, I_{5g}^g, I_{2g+4}^g, I_{4g+2}^g, I_{3+3g}^g, I_3^g, I_2^g, I_4^g$ and so on, $\times_n\}$ be the natural neutrosophic dual number matrix semigroup.

$$x = \begin{bmatrix} 2g \\ 3g+3 \\ 4g \\ 5g \\ 2+4g \end{bmatrix} \text{ and } y \begin{bmatrix} 1 \\ 2 \\ 3 \\ h \\ 3 \end{bmatrix} \text{ is such that } x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus B has dual number zero divisor matrices.

$$\text{Let } a = \begin{bmatrix} I_g^g \\ I_{3+3g}^g \\ I_{4g}^g \\ I_{2g}^g \\ I_{5g}^g \end{bmatrix} \text{ and } b = \begin{bmatrix} I_{5g}^g \\ I_2^g \\ I_3^g \\ I_g^g \\ I_{2g}^g \end{bmatrix}.$$

Clearly

$$a \times_n b = \begin{bmatrix} I_0^g \\ I_0^g \\ I_0^g \\ I_0^g \\ I_0^g \end{bmatrix}$$

is a natural neutrosophic dual number matrix zero divisor.

$$\text{We see } x = \begin{bmatrix} g \\ 3g \\ 2g \\ 4g \\ 5g \end{bmatrix} \in B \text{ is such that}$$

$$x \times_n x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a natural dual number nilpotent matrix of order two

$$y = \begin{bmatrix} I_g^g \\ I_{3g}^g \\ I_{2g}^g \\ I_{4g}^g \\ I_{5g}^g \end{bmatrix} \in B \text{ is such that } y \times_n y = \begin{bmatrix} I_0^g \\ I_0^g \\ I_0^g \\ I_0^g \\ I_0^g \end{bmatrix}$$

is a natural neutrosophic dual number nilpotent element of order two.

Study in this direction is interesting but however it is only a matter of routine

$$a_1 = \begin{bmatrix} I_3^g \\ I_0^g \\ I_0^g \\ I_3^g \\ I_3^g \end{bmatrix} \in B \text{ is such that } a_1 \times_n a_1 = \begin{bmatrix} I_3^g \\ I_0^g \\ I_0^g \\ I_3^g \\ I_3^g \end{bmatrix} = a_1.$$

Thus a_1 is a natural neutrosophic dual number idempotent matrix of B.

$$b_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} \in B$$

is such that $b_1 \times_n b_1 = b_1$ is real dual number idempotent matrix of B.

Example 2.60: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle/ a_i \in \{\langle Z_{10} \cup g\rangle_I; \times\}; 1 \leq i \leq 9, \times_n \right\}$$

be the natural neutrosophic dual number matrix semigroup.

Clearly on M the usual product operation cannot be defined.

$$A = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 6 \\ 6 & 0 & 5 \end{bmatrix} \in M \text{ is such that } a \times_n a = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 5 & 6 \\ 6 & 0 & 5 \end{bmatrix} = a$$

is a real idempotent matrix of M.

$$\text{Let } b = \begin{bmatrix} 2g & 3g & 9g \\ 8g & 6g & 4g \\ 0 & 2g & 5g \end{bmatrix} \in M; \text{ clearly } b \times_n b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in M$$

is a natural dual number nilpotent matrix of order two.

Let

$$c = \begin{bmatrix} 2g & 4g & 5 \\ 8+4g & 6g+4 & 2+6g \\ 4g+4 & 5g & 0 \end{bmatrix} \text{ and}$$

$$d = \begin{bmatrix} 5+5g & 5 & 2g \\ 5 & 5g & 5+5g \\ 5g & 8 & 6+5g \end{bmatrix} \in M.$$

Clearly

$$c \times_n d = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero divisor matrix of dual numbers.

$$a = \begin{bmatrix} I_{2g}^{g} & I_{0}^{g} & I_{4g}^{g} \\ I_{5g}^{g} & I_{6g}^{g} & I_{9g}^{g} \\ I_{8g}^{g} & I_{7g}^{g} & I_{3g}^{g} \end{bmatrix} \in M.$$

Clearly $a \times_n a = \begin{bmatrix} I_{0}^{g} & I_{0}^{g} & I_{0}^{g} \\ I_{0}^{g} & I_{0}^{g} & I_{0}^{g} \\ I_{0}^{g} & I_{0}^{g} & I_{0}^{g} \end{bmatrix}$ is the natural neutrosophic dual number nilpotent matrix of M.

Let $a_1 = \begin{bmatrix} 0 & I_{2g}^{g} & I_{0}^{g} \\ 0 & 0 & I_{4g}^{g} \\ I_{6g}^{g} & 0 & 0 \end{bmatrix}$ and $b_1 \begin{bmatrix} 2+2g & I_{3g}^{g} & I_{5g}^{g} \\ 6+4g & 8+9g & I_{5}^{g} \\ I_{5g}^{g} & 7g+7 & 9 \end{bmatrix} \in M$

$a_1 \times_n b_1 = \begin{bmatrix} 0 & I_{0}^{g} & I_{0}^{g} \\ 0 & 0 & I_{0}^{g} \\ I_{0}^{g} & 0 & 0 \end{bmatrix}$ is the mixed neutrosophic natural dual number matrix zero divisor.

Example 2.61: Let

$$N = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle/ a_i \in \{\langle Z_{18} \cup g\rangle_I; \times\}; 1 \leq i \leq 14, \times_n \right\}$$

be the natural neutrosophic dual number matrix semigroup under the natural product $\times_n$.

This N has natural neutrosophic dual number matrix zero divisors, idempotents and nilpotents.

All the properties associated with N can be studied as a matter of routine and hence left as an exercise to the reader.

Next we proceed onto study through examples the natural neutrosophic special dual like number matrix semigroup.

Example 2.62: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in \{\langle Z_8 \cup h\rangle_I, h^2 = h, \times\}; \; 1 \leq i \leq 15, \times_n \right\}$$

be the natural neutrosophic special dual like number matrix semigroup under the natural product $\times_n$.

One can as a matter of routine find the natural neutrosophic special dual like number matrix semigroup.

$$x = \begin{bmatrix} 4h & 0 & 4h \\ 4 & 4 & 0 \\ 0 & 0 & 4h \\ 4h & 0 & 4 \\ 4 & 4 & 4 \end{bmatrix} \in B \text{ is such that } x \times_n x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a nilpotent matrix of B.

Let a = $\begin{bmatrix} I_2^h & I_{4h}^h & I_0^h \\ I_{6h}^h & I_{2h}^h & I_{6h}^h \\ I_0^h & I_6^h & I_2^h \\ I_6^h & I_4^h & I_{2h}^h \\ I_0^h & I_{2h}^h & I_0^h \end{bmatrix}$ and b = $\begin{bmatrix} I_{4h}^h & I_4^h & I_{5h}^h \\ I_4^h & I_4^h & I_{4h}^h \\ I_{7h}^h & I_4^h & I_{4h}^h \\ I_{4h}^h & I_{4h}^h & I_4^h \\ I_{4+4h}^h & I_4^h & I_{2+4h}^h \end{bmatrix}$

is such that a $\times_n$ b = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a natural neutrosophic special dual like number matrix which is a zero divisor.

Study in this direction is realized is a matter of routine and left as an exercise to the reader.

Example 2.63: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \middle/ a_i \in \{\langle Z_{23} \cup h\rangle_I, h^2 = h, \times\};\ 1 \leq i \leq 8, \times_n \right\}$$

be the natural neutrosophic special dual like number matrix semigroup.

The reader is expected to find the order of M.

This M has natural neutrosophic zero divisors but finding idempotent and nilpotents matrix of M.

Example 2.64: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \middle/ a_i \in \{\langle Z_{14} \cup I \rangle_I, \right.$$

$$\left. h^2 = h;\ 1 \leq i \leq 21, \times_n \right\}$$

be the natural neutrosophic special dual like matrix semigroup.

Find all natural neutrosophic special dual like number matrix zero divisors, idempotents and nilpotents of P.

The study is left for the reader.

Next we proceed onto study the notion of natural neutrosophic special quasi dual number matrix semigroup which will be illustrated by examples.

Example 2.65: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in \{\langle Z_{12} \cup k \rangle_I, k^2 = 11k, \times\}; \right.$$

$$\left. 1 \leq i \leq 15, \times_n \right\}$$

be the natural neutrosophic special quasi dual number matrix semigroup.

$$\text{Let } x = \begin{bmatrix} 3 & 4 & 6 \\ 2 & 5+6k & 3k \\ 6k & 0 & 6 \\ 2k & 3+6k & 0 \\ 0 & 0 & 2+4k \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 4k & 3 & 2k+2 \\ 6k & 4k & 4 \\ 2+2k & 6+7k & 4+4k \\ 7+11k & 11+9k & 6+6k \end{bmatrix} \in S.$$

$$x \times_n y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a natural neutrosophic special quasi dual number matrix zero divisor.

$$b = \begin{bmatrix} 9 & 4 & 0 \\ 1 & 9 & 0 \\ 0 & 0 & 4 \\ 4 & 1 & 0 \\ 9 & 0 & 1 \end{bmatrix} \in S$$

is such that $b \times_n b = b$ so b is a natural special quasi dual number matrix idempotent.

Finding natural neutrosophic special quasi dual number matrix idempotents, zero divisors and nilpotents happen to be matter of routine so left as an exercise to the reader.

Example 2.66: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \;/\; a_i \in \{\langle Z_{24} \cup k\rangle_I, k^2 = 23k, \times\}, 1 \leq i \leq 4, \times_n \right\}$$

be the natural neutrosophic special quasi dual number matrix semigroup.

$$\text{Let } x = \begin{bmatrix} 12k \\ 3k \\ 4+4k \\ 2+4k \end{bmatrix} \text{ and } y = \begin{bmatrix} 2k+4 \\ 8k+16 \\ 6+12k \\ 12k+12 \end{bmatrix} \in M;\; x \times_n y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a natural special quasi dual number matrix zero divisors.

$$\text{Let } x = \begin{bmatrix} I_{12}^k \\ I_{6+6k}^k \\ I_{8+4k}^k \\ I_{16+16k}^k \end{bmatrix} \text{ and } y = \begin{bmatrix} I_{4k}^k \\ I_{8+8k}^k \\ I_{6+6k}^k \\ I_{3+3k}^k \end{bmatrix} \; x \times_n y = \begin{bmatrix} I_0^k \\ I_0^k \\ I_0^k \\ I_0^k \end{bmatrix}$$

is a natural neutrosophic special quasi dual number matrix zero divisor.

Study in this direction is left as an exercise to the reader as it is a matter of routine.

Now we proceed onto describe the notion of finite polynomial natural neutrosophic coefficient under × by examples.

Example 2.67: Let

$$B[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \,/\, a_i \in \{\langle Z_8^I, \times\rangle\} = B,\ 0 \leq i \leq 6,\ x^7 = 1,\right.$$
$$\left. x^8 = x \text{ and so on } x^{14} = 1, \times\right\}$$

be the real natural neutrosophic coefficient polynomial semigroup under $\times$.

$p(x) = 4x^6 + 6x^2 + 6$ and $q(x) = 4x^3 + 4 \in B[x]_6$.

Clearly $p(x) \times q(x) = 0$ is a zero polynomial.

Let $p(x) = x^3 \in B[x]_6$ there is no $q(x) \in B[x]_6$ such that $p(x) \times q(x) = 0$.

Let $p(x) = 2 + 2x^2 + 4x^3 + 6x^4 \in B[x]_6$, $q(x) = 4$ the constant polynomial of $B[x]_6$ is such that $p(x) \times q(x) = 0$.

However we are not able to find natural neutrosophic idempotents.

$$I_4^8 x^3 + I_2^8 x^4 + I_0^8 \in p(x)$$

$$q(x) = I_4^8 x^4 + I_0^8 \in B[x]_8.$$

Clearly $p(x) \times q(x) = I_0^8$ is the natural neutrosophic polynomial zero divisor of B.

Example 2.68: Let

$$D[x]_4 = \left\{\sum_{i=0}^{4} a_i x^i \,/\, a_i \in D = \langle Z_{12}^I, \times\rangle;\ 0 \leq i \leq 4,\ x^5 = 1,\ x^6 = x,\right.$$
$$\left. x^7 = x^2,\ x^8 = x^3,\ x^9 = x^4,\ x^{10} = 1 \text{ and so on}, \times\right\}$$

be the real natural neutrosophic coefficient polynomial semigroup under $\times$.

Let p(x) = $6x^3 + 3x^2 + 9$ and q(x) = $8x^4 + 4x^2 + 8 \in D[x]_4$;

p(x) × q(x) = 0 is a natural neutrosophic polynomial zero divisor.

Finding idempotents happens to be a challenging problem.

Example 2.69: Let

$$M[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \,/\, a_i \in M = \langle Z_6^I, \times\rangle; 0 \leq i \leq 10, x^{11} = 1, \times\right\}$$

be the real natural neutrosophic coefficient polynomial semigroup under ×.

Clearly $o(M[x]_{10}) < \infty$.

$M[x]_{10}$ has natural neutrosophic zero divisors and idempotents of the form 3, I_0^6 and I_3^6.

Example 2.70: Let

$$B[x]_5 == \left\{\sum_{i=0}^{4} a_i x^i \,/\, a_i \in B = \langle Z_{48}^I, \times\rangle; 0 \leq i \leq 5, \times\right\}$$

be the real natural neutrosophic coefficient polynomial semigroup of finite order.

I_4^{48}, I_{12}^{48}, I_{24}^{48}, I_2^{48}, …, I_{46}^{48}, I_{24}^{48} are idempotents of $B[x]_5$.

There are several zero divisors.

$I_{24}^{48} x^5 \times I_{24}^{48}\ x^5 = x_{24}^{48} x^4$ is not an idempotent.

The study of properties associated with $B[x]_5$ happens to be a matter of routine so left as an exercise to the reader.

Example 2.71: Let

$$D[x]_{24} = \left\{ \sum_{i=0}^{24} a_i x^i \,/\, a_i \in D = \langle Z_{43}^{I}, \times\rangle;\ x^{25} = 1,\ 0 \le i \le 24, \times \right\}$$

be the real natural neutrosophic coefficient polynomial semigroup.

$$o(D[x0_{24}) < \infty.$$

The task of finding ideals, subsemigroups idempotents, zero divisors and nilpotents are left as an exercise to the reader.

Next we study the natural neutrosophic finite complex modulo integer coefficient polynomial semigroups of finite order.

Example 2.72: Let

$$C[x]_3 = \left\{ \sum_{i=0}^{4} a_i x^i \,/\, a_i \in \langle C^I, (Z_{60}); \times\rangle \right\} = C,\ 0 \le i \le 3, \times\}$$

be the natural neutrosophic finite complex modulo integer coefficient polynomial semigroup $o([x]_3) < \infty$.

This has natural neutrosophic zero divisors and zero divisor polynomials.

$$\text{Let } p(x) = I_{15}^{C} x^3 + I_{30}^{C} x^2 + I_{45}^{C},$$

$$q(x) = I_{8}^{C} x^2 + I_{0}^{C} \in C[x]_3$$

$$p(x) \times q(x) = I_{0}^{C}.$$

The reader is given with the task of finding zero divisors, subsemigroups and ideals of $C[x]_3$.

Example 2.73: Let

$$D[x]_{16} = \left\{ \sum_{i=0}^{4} a_i x^i \,/\, a_i \in C^I(Z_{43}), \times\rangle\} = D;\ 0 \le i \le 16,\ x^{17} = 1;\ \times \right\}$$

be the natural neutrosophic finite complex modulo integer coefficient polynomial semigroup.

The reader is expected to study all special elements associated with $D[x]_{16}$.

Next we proceed onto study natural neutrosophic neutrosophic coefficient polynomial semigroups of finite order by examples.

Example 2.74: Let

$$S[x]_9 = \left\{ \sum_{i=0}^{9} a_i x^i \,/\, a_i \in S = \{\langle Z_9 \cup I\rangle_I, \times\rangle\};\ x^{10} = 1;\ 0 \le i \le 9, \times \right\}$$

be the natural neutrosophic-neutrosophic coefficient polynomial semigroup under ×.

Let $p(x) = I_3^I x^2 + I_{3I}^I$, $q(x) = I_{3+3I}^I x^8 + I_{6I}^8 x^6 + I_{6+6I}^I \in S[x]_9$.

Clearly $p(x) \times q(x) = I_0^I$.

That is they are the natural neutrosophic polynomial zero divisors.

Example 2.75: Let

$$B[x]_6 = \left\{ \sum_{i=0}^{6} a_i x^i \,/\, a_i \in \langle Z_{2^3 17^4 5^2 29^{10} 43^2} \cup I\rangle_I; = B\ ,\ x^7 = 1;\right.$$

$$\left. 0 \le i \le 6, \times \right\}$$

be the natural neutrosophic-neutrosophic coefficient polynomial semigroup under $\times$.

The reader is left with the task of finding ideals of $B[x]_6$ and that of the zero divisors and idempotents in any of $B[x]_6$.

Example 2.76: Let

$$M[x]_3 = \left\{ \sum_{i=0}^{3} a_i x^i \,/\, a_i \in M = \{\langle Z_{424} \cup I\rangle_I, \times\};\ x^4 = 1,\right.$$

$$\left. 0 \leq i \leq 3, \times \right\}$$

be the natural neutrosophic-neutrosophic polynomial coefficient semigroup.

This has zero divisor polynomial $Z_{424}[x]_3 \subseteq M[x]_3$ is a subsemigroup of $M[x]_3$ and not an ideal of $M[x]_3$.

Example 2.77: Let

$$P[x]_7 = \left\{ \sum_{i=0}^{4} a_i x^i \,/\, a_i \in P = \{\langle Z_{48} \cup g\rangle_I;\ x^8 = 1;\ g^2 = 9, \times\};\right.$$

$$\left. x^8 = 1;\ 0 \leq i \leq 7, \times \right\}$$

be the natural neutrosophic dual number polynomial coefficient semigroup.

This has subsemigroups which are zero square subsemigroups.

Example 2.78: Let

$$B[x]_5 = \left\{ \sum_{i=0}^{5} a_i x^i \,/\, a_i \in B = \{\langle Z_{403} \cup g\rangle_I;\ g^2 = 0, \times\};\ x^6 = 1;\right.$$

$$\left. 0 \leq i \leq 5, \times \right\}$$

be the natural neutrosophic dual number polynomial coefficient semigroup.

What is $o(B[x]_5)$?

This work is left as exercise to the reader. This has several zero divisors.

$Z_{403}[x]_5 \subseteq B[x]_5$ is only a subsemigroup and not an ideal of $B[x]_5$.

$(\langle Z_{403} \cup g\rangle)[x]_5 \subseteq B[x]_5$ is also a subsemigroup of $B[x]_5$ and not an ideal of $B[x]_5$ $(Z_{403}\, g)[x]_5 \subseteq B[x]_5$ is a subsemigroup which is also an ideal of $B[x]_5$.

Example 2.79: Let

$$D[x]_2 = \left\{\sum_{i=0}^{2} a_i x^i \,/\, a_i \in D = \langle Z_{2^2 3^5 19^7 29^2 37^3} \cup g\rangle_I,\ x^3 = 1, \times\};\right.$$

$$0 \le i \le 2, \times\}$$

be the natural neutrosophic dual number coefficient polynomial semigroup.

This has ideals and subsemigroup.

Example 2.80: Let

$$W[x]_5 = \left\{\sum_{i=0}^{5} a_i x^i \,/\, a_i \in \langle Z_{43} \cup h\rangle_I = W;\ x^6 = 1,\ 0 \le i \le 5, \times\}\right.$$

be the natural neutrosophic special dual like number coefficient polynomial semigroup under × $(Z_{43}h)[x]_5 \subseteq W[x]_5$ is an ideal of $W[x]_5$.

Example 2.81: Let

$$M[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \,/\, a_i \in M = \langle Z_{2^3 3^8 5^{12} 31^4} \cup h\rangle_I;\ x^{11} = 1,\right.$$
$$\left. 0 \leq i \leq 10, \times\right\}$$

be the natural neutrosophic special dual like number polynomial coefficient semigroup under $\times$.

The reader is left with the task of finding order of $M[x]_{10}$ and ideals of $M[x]_{10}$.

Next we proceed onto find natural neutrosophic special quasi dual number polynomial coefficient semigroup under $\times$.

Example 2.82: Let

$$W[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \,/\, a_i \in \langle Z_{423} \cup k\rangle_I = W,\ 0 \leq i \leq 6,\ x^7 = 1, \times\right\}$$

be the natural neutrosophic special quasi dual number polynomial coefficient semigroup.

$Z_{423}\ [x]_6 \subseteq W[x]_6$ is a subsemigroup and not an ideal of $W[x]_6$.

$(Z_{423})\ [x]_6\ k)\ [x] \subseteq W[x]_6$ is an ideal and is of finite order.

Example 2.83: Let

$$M[x]_5 = \left\{\sum_{i=0}^{5} a_i x^i \,/\, a_i \in M = \langle Z_{48} \cup k\rangle_I;\ 0 \leq i \leq 5,\ x^6 = 1; \times\right\}$$

be the natural neutrosophic special quasi dual number coefficient polynomial semigroup.

This has zero divisors and subsemigroups and ideals.

Next we proceed onto study special natural neutrosophic semigroup under addition are given the product operation.

We will first illustrate this by example.

Example 2.84: Let S = $\{\langle Z_4^I, +\rangle_I = \{0, 1, 2, 3, I, 2I, 3I, I_0^4, I_2^4, 1 + I, 2 + I, 3 + 3I, 2 + 2I, 1 + 2I, 3 + 2I, 3I + 1, 3I + 2, 3I + 3, I_{2I}^4, I_{2+2I}^4, I_{3+I}^4, I_{3I+1}^4 \ldots, \times\}$ be the special natural neutrosophic neutrosophic semigroup under ×.

$o(S) < \infty$.

This has natural neutrosophic-neutrosophic zero divisor and idempotents.

Example 2.85: Let M = $\{\langle Z_6^I, +\rangle_I = \{0, 1, 2, 3, \ldots, 5, I, 2I, 3I, \ldots, 5I, 1 + I, 2 + I, \ldots, 5 + I, 2 + 2I, 1 + 2I, \ldots, 5 + 2I, \ldots, I_0^6, I_2^6, I_4^6, I_3^6, I_I^6, I_{2I}^6, \ldots, I_0^6 + I_I^6 + I_3^6, a + I_0^6 \ (a \in Z_6) \ldots \}, \times\}$ be the natural neutrosophic neutrosophic special semigroup under ×.

Example 2.86: Let
$S = \{\langle Z_7^I, +\rangle = \{0, 1, 2, \ldots, 7, I_0^7, 1 + I_0^7, 2+ I_0^7, \ldots, 7+I_0^7\}, \times\}$ be the real natural neutrosophic special semigroup.

This is different from the real natural neutrosophic semigroup $\{Z_7^I, \times\}$.

Infact $\{Z_7^I, \times\} \subseteq S$ as a subsemigroup.

The task of finding ideals and subsemigroup of this semigroup S is left as an exercise to the reader.

Example 2.87: Let M $=\{\langle Z^I_{2^3 5^2 7^2}\rangle, \times\}$ be the real natural neutrosophic special semigroup under ×.

This has zero divisors, idempotents and subsemigroups as well as ideals.

Study in this direction is a matter of routine and hence left as an exercise to the reader.

Next we study special natural neutrosophic finite complex modulo integer semigroup by some examples.

Example 2.88: Let B = $\{\langle C^I(Z_2), +\rangle = \{0, 1, i_F, 1 + i_F, I_0^C, I_{1+i_F}^C, 1 + I_0^C, i_F + I_0^C, 1 + i_F + I_0^C, 1 + I_{1+i_F}^C, i_F + I_{1+i_F}^C, 1 + i_F + I_{1+i_F}^C, I_0^C + I_{1+i_F}^C, 1 + I_0^C + I_{1+i_F}^C, i_F + I_0^C + I_{1+i_F}^C, 1 + i_F + I_0^C + I_{1+i_F}^C\}, \times\}$ be the special natural neutrosophic finite complex modulo integer semigroup under ×.

Let $x = (1 + i_F) + I_{i_F+1}^C$ and $y = i_F + I_{1+i_F}^C + I_0^C \in B$.

$$x \times y = ((1 + i_F) + I_{i_F+1}^C)(i_F + I_0^C + I_{1+i_F}^C)$$

$$= i_F(1 + i_F) + I_{i_F+1}^C + I_0^C = 1 + i_F + I_0^C + I_{i_F+1}^C.$$

This is the way product operation is performed on B.

This study is also a matter of routine so left as an exercise to the reader.

Example 2.89: Let S = $\{\langle C^I(Z_8), +\rangle = \{0, 1, 2, \ldots, 7, i_F, 2i_F, \ldots, 7i_F, 1 + i_F, 2 + i_F, \ldots, 7 + i_F, 2 + 2i_F, 1 + 2i_F, \ldots, 7 + 2i_F, 1 + 3i_F, 2 + 3i_F, 3 + 3i_F, \ldots, 7 + 3i_F, \ldots, 1 + 7_F, 2 + 7i_F, \ldots, + 7 + 7i_F, I_0^C, I_2^C, I_4^C, I_{4i_F}^C, I_{6i_F}^C, I_6^C, I_{2+2i_F}^C, \ldots, I_{2i_F}^C, I_0^C + I_2^C, I_0^C + I_4^C, I_0^C + I_{2+2i_F}^C, I_0^C + I_{2i_F}^C, I_0^C + I_{6+6i_F}^C, \ldots, 5 + I_0^C + I_2^C + I_{2i_F}^C + I_4^C + I_{4i_F}^C + I_6^C + I_{6i_F}^C + I_{2+2i_F}^C + I_{2+4i_F}^C + \ldots + I_{7+i_F}^C, \ldots\}, \times\}$ be the special natural neutrosophic complex modulo integer semigroup under ×.

The study in this direction in considered as a matter of routine so left as an exercise to the reader.

Next we proceed onto study special natural neutrosophic dual number semigroup under product by some examples.

Example 2.90: Let M = $\{\langle C^I_{127}, +\rangle, \times\}$ be the special natural neutrosophic finite complex modulo integer semigroup under ×.

Example 2.91: Let W = $\{\langle C^I_{37^2 43^3 7^2}, +\rangle, \times\}$ be the special natural neutrosophic finite complex modulo integer semi group under ×.

Finding zero divisors idempotents nilpotents and substructures are considered as a matter of routine so is left as an exercise to the reader.

Next we give few examples of special natural neutrosophic dual number semigroup under ×.

Example 2.92: Let B = $\{\langle\langle Z_{10} \cup g\rangle_I, \times\rangle\}$ = {0, 1, 2, 3…, 9, g, 2g, …, 9g, 1 + g, 1 + 2g, …, 1 + 9g, 9 + g, 9 + 2g, …., 9 + 9g, I_0^g, I_2^g, I_4^g, I_5^g, I_6^g, I_8^g, I_g^g, I_{2g}^g, …, I_{9g}^g, …, $I_0^g + I_2^g$, $I_0^g + I_{4+4g}^g + I_{8+8g}^g$, …, I_0^g + … + $I_{9g}^g + I_{2+2g}^g$ + … + I_{8+8g}^g, +}, ×} be the special natural neutrosophic dual number semigroup under ×.

We see if a = 5 + 3g + I_0^g, I_{8+4g}^g and

$$b = 5 + 8g + I_4^g + I_{4+5g}^g \in B.$$

$$a \times b = (5 + 3g + I_0^g + I_{8+4g}^g) \times (5 + 8g + I_4^g + I_{4+5g}^g)$$

$$= 5 + 5g + I_0^g + I_{2+6g}^g + I_{2+6g}^g \in B.$$

This is the way product operation is performed on B.

As the working with this happens to be a matter of routine the reader is left with the task of finding zero divisors, idempotents and substructures.

Example 2.93: Let D = $\{\langle Z_{29} \cup g\rangle_I, +\rangle\}$ = {0, 1, 2, …, 28, g, 2g, …, 28g, I_0^g, I_g^g, I_{2g}^g, …, I_{28g}^g and so on $\times$} be the special natural neutrosophic dual number semigroup under product $\times$.

This has lots of zero divisors, idempotents, nilpotents and substructures.

This study is a matter of routine and is left as an exercise to the reader.

Next we proceed onto study by examples the notion of special natural neutrosophic special dual like number semigroup under $\times$.

Example 2.94: Let W = $\{\{\langle Z_{12} \cup h\rangle_I, +\rangle\}$ = {0, 1, 2, …, 11, h, 2h, …, 11h, 1 + h, 2 + h, …, 11 + h, 1 + 2h, …, 11 + 11h, I_0^h, I_h^g, I_{2h}^g, I_{3h}^g, I_{4h}^g, I_{6h}^g, …., I_{10}^g, I_{2+10h}^g, I_{h+11h}^g, I_{3h+9}^g, I_{10+2h}^g, …, I_{6h+6}^g, …, +}, $\times$} be the special natural neutrosophic special dual number like semigroup under $\times$.

Consider

$$a = (I_{10+2h}^h + I_{5+7h}^h, I_{6+6h}^h + 3 + 2h) \text{ and}$$

$$b = (I_h^h + I_{4h} + I_{5h}) \in W.$$

$$a \times b = (I_{10+2h}^h + I_{5+7h}^h, I_{6+6h}^h + 3 + 2h)\,(I_h^h + I_{4h}^h + I_{5h}^h)$$

$$= I_0^h + I_h^h + I_{4h}^h + I_{5h}^h \in W.$$

This is the way product operation is performed on W all these work is a matter of routine so the reader is left with this task.

Consider $a = (I^h_{10+2h} + I^h_{4+8h} + I^h_{5h+7} + I^h_{6h+6} + I^h_{3h+9h})$ and

$$b = I^h_0 + I^h_{3h} + I^h_{2h} \times W,$$

$$\text{Clearly } a \times b = I^h_0 .$$

Thus there are several natural neutrosophic zero division in W.

Example 2.95: Let M = $\{\{\langle Z_6 \cup h\rangle_I, +\} = \{0, 1, 2, \ldots, 5, h, 2h, \ldots, 5h, 1 + h, 2 + h, \ldots, 5 + h, \ldots, 5 + 5h, I^h_0, I^h_h, I^h_{2h}, \ldots, I^h_{5+h}, I^h_{6h+1}, \ldots, I^h_0 + I^h_h, I^h_h + I^h_{3+3h} + I^h_{2+4h} + I^h_{4+2h} + I^h_{5h+1}$ and so on $+\}, \times\}$ be the special natural neutrosophic special dual like number semigroup under ×.

Finding zero divisors, ideals, idempotents etc. of M is a matter of routine so left as an exercise to the reader.

Example 2.96: Let W = $\{\{\langle Z_{23^2 19^3 7^5 3^2} \cup h\rangle_I, h^2 = h, +\}; \times\}$ be the special natural neutrosophic special dual like number semigroup under product ×.

W has zero divisors.

The semigroup P = $\{\{\langle Z_{23^2 19^3 7^5 3^2} \cup h\rangle, \times\} \subseteq W$ as a subsemigroup.

Thus the term special makes are know the semigroup under + is given the product operation.

So as such the set of which the semigroup is defined is much larger than the one under product without first generated by the sum as a semigroup.

Next we proceed onto give examples of special natural neutrosophic special quasi dual number semigroup.

Example 2.97: Let M = $\{\{\langle Z_{43} \cup k\rangle_I, \times\};\ k^2 = 42k\}$ be the special quasi dual number natural neutrosophic semigroup under $\times$.

Finding zero divisors, idempotents of M is a difficult job.

However $\{\langle Z_{43} \cup k\rangle_I, \times\} \subseteq$ M as a subsemigroup.

In view of all these we prove the following theorem.

THEOREM 2.11: *Let S = $\{\langle Z_n^I, +\rangle, \times\}$ be the special real natural neutrosophic semigroup.*

i. *M = $\{\langle Z_n^I, \times\rangle\} \subseteq$ S is a subsemigroup.*
ii. *o(S) > o(M).*

Proof is direct and hence left as an exercise to the reader.

The above theorem is true.

if $\langle Z_n^I, +\rangle$ is replaced.

by $\langle C^I(Z_n), +\rangle$ or $\langle\langle Z_n \cup I\rangle_I, +\rangle$ or $\langle\langle Z_n \cup g\rangle_I, +\rangle$ or $\langle\langle Z_n \cup h\rangle_I, +\rangle$ or $\langle\langle Z_n \cup k\rangle_I, +\rangle$ which is just illustrated.

Example 2.98: Let M = $\{\langle\langle Z_{24} \cup k\rangle_I, +\rangle, \times\}$ be the special natural neutrosophic special quasi dual number semigroup under $\times$.

Clearly $(\langle Z_{24} \cup k\rangle_I, \times) \subseteq$ M as a subsemigroup.

If x = 2k + 4 and y = 12 k then x × y = 0.

$$I^k_{12k} \times I^k_{2k+4} = I^k_0 .$$

Thus we have usual and natural neutrosophic zero divisors.

$$I^k_{12k} \times I^k_{12k} = I^k_0 .$$

$$I^k_{12} \times I^k_{12} = I^k_0 .$$

12 × 12 = 0 and 12k × 12k = 0 are some natural neutrosophic nilpotents as well as usual nilpotents of order two.

$T_1 = \{Z_{34}, \times\} \subseteq M$ is a subsemigroup of M.

$T_2 = \{\langle Z_{24} \cup h\rangle; \times\} \subseteq M$ is a subsemigroup of M.

Clearly they are not ideals of M.

$T_3 = \{ I^k_0, I^k_{12}, I^k_{12k}, I^k_{12+12k} \} \subseteq M$ is a subsemigroup of M.

All these study is considered as a matter of routine so left as exercise to the reader.

Now we study the special natural neutrosophic coefficient polynomial semigroups.

They are polynomial semigroups built using the semigroups $\{\langle Z^I_n, +\rangle, \times\}$ or $\langle C^I(Z_n), +\rangle$ and so on.

Such study will be illustrate by one or two examples.

Example 2.99: Let

$$P[x]_9 = \left\{\sum_{i=0}^{9} a_i x^i \,/\, a_i \in P = \{\langle Z^I_{24}, + \rangle, \times\}; 0 \leq i \leq 9, \times\right\}$$

be the special real natural neutrosophic semigroup under $\times$.

Let

$$p(x) = (3 + I_0^{24} + I_{12}^{24} + I_2^{24})x^{+3} + (5 + I_6^{24} + I_8^{24})\, x^2 + (I_8^{24} + I_6^{24}) \text{ and}$$

$$q(x) = (4 + I_3^{24} + I_8^{24})\, x^6 + (I_9^{24} + 23) \in P[x]_9;$$

$p(x) \times q(x)$ is defined as a matter of routine.

The reader is expected to work with zero divisors, idempotents and nilpotents.

Example 2.100: Let

$$S[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \,/\, a_i \in S = \{\langle Z_9 \cup I\rangle_I, +\rangle, \times\}\ 0 \leq i \leq 10, \times\right\}$$

be the special natural neutrosophic neutrosophic coefficient polynomial semigroup under $\times$.

Let $p(x) = (3 + I_3^I + I_{3I}^I)x^3 + 4 + I_{6I}^I$ and

$q(x) = (2I + 7 + I_{6I+6}^I)\, x^2 + 2 \in S[x]_{10}$.

$p(x) \times q(x) = [(3 + I_3^I + I_{3I}^I)x^3 + 4 + I_{6I}^I]\, [(2I + 7 + I_{6I+6}^I)x^2 + 2]$

$= (6 + I_3^I + I_{3I}^I)x^3 + 8 + I_{6I}^I + (8I + 1 + I_{6I+6}^I + I_{6I}^I + I_0^I)x^2$, and so on are the routine ways the product operation is performed on $S[x]_{10}$.

Example 2.101: Let

$$S[x]_3 = \left\{\sum_{i=0}^{8} a_i x^i \,/\, a_i \in S = \{\langle Z_{10} \cup g \rangle_I, +, \times\} \times; 0 \leq i \leq 8\right\}$$

be the natural neutrosophic dual number coefficient polynomial semigroup.

S[x] has zero divisors, idempotents, subsemigroups and ideals.

The working is realized as a matter of routine so left as an exercise to the reader.

Next we see an example of special natural neutrosophic complex number coefficient polynomial semigroup under × and special natural neutrosophic special quasi dual number coefficient polynomial semigroup under ×.

Example 2.102: Let

$$W[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \,/\, a_i \in W = \langle\langle Z_{16} \cup k \rangle_I, +\rangle, \times\rangle; 0 \leq i \leq 8, \times\right\}$$

be the special natural neutrosophic special quasi dual number coefficient polynomial semigroup under ×.

Example 2.103: Let

$$M[x]_{19} = \left\{\sum_{i=0}^{19} a_i x^i \,/\, a_i \in M = \langle\langle C^I(Z_{14})\ +\rangle, \times\rangle; 0 \leq i \leq 19, \times\right\}$$

be the special natural neutrosophic finite complex modulo integer coefficient polynomial semigroup under ×.

The study of all these is a matter of routine and left as a simple exercise to the reader.

However we briefly describe the special natural neutrosophic matrix semigroups various types of examples.

Example 2.104: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle/ a_i \in \langle\langle Z_{10}^{I}, +\rangle, \times\rangle, \times_n;\ 1 \leq i \leq 5 \right\}$$

be the special natural neutrosophic matrix semigroup under natural product $\times_n$.

$$\text{Let } x = \begin{bmatrix} 3 + I_0^{10} + I_2^{10} \\ 0 \\ 5 + I_5^{10} \\ I_0^{10} \\ 6 \end{bmatrix} \text{ and } y = \begin{bmatrix} 4 + I_6^{10} \\ 5 + I_2^{10} \\ 3 + I_2^{10} \\ I_6^{10} \\ I_2^{10} + 8 \end{bmatrix} \in M$$

$$x \times_n y = \begin{bmatrix} 2 + I_0^{10} + I_2^{10} + I_6^{10} \\ I_2^{10} \\ 5 + I_5^{10} + I_2^{10} + I_0^{10} \\ I_0^{10} \\ 8 + 10_2^{10} \end{bmatrix} \in M.$$

This is the way $\times_n$ is performed on M.

Example 2.105: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \text{ where } a_i \in \langle\langle C^I (Z_4), + \rangle, \times\rangle\ 1 \leq i \leq 6, \times_n \right\}$$

be the special natural neutrosophic finite complex modulo integer matrix semigroup under natural product $\times_n$.

$$x = \begin{bmatrix} 0 & 2 + I_0^4 \\ 3i_F & I_{2i_F}^4 \\ 2i_F + 2 & I_{2+2i_f}^4 \end{bmatrix} \in S; \text{ we } x \times_n x = \begin{bmatrix} 0 & I_0^4 \\ 3 & I_0^4 \\ 0 & I_0^4 \end{bmatrix} \in S.$$

This is the way $\times_n$ operation is performed.

The task of finding subsemigroups, ideals etc. are left as an exercise to the reader.

Example 2.106: Let

$W = \{(a_1, a_2, a_3, a_4) / a_i \in \langle\langle\langle Z_{15} \cup I\rangle_I +\rangle, \times\rangle; 1 \leq i \leq 4, \times\}$ be the special natural neutrosophic-neutrosophic matrix semigroup.

Find o(W).

$x = (0, 4I, 8I + I_{3I+5}^I + 2 + 3I + I_{6I}^I, 0)$ and
$y = (7I + 9 + I_{3I}^I + I_0^I, 0, 0, 9 + 11I + I_{10}^I + I_{5I+10}^I + I_0^I) \in W.$

Clearly $x \times y = (0, 0, 0, 0)$ is a zero divisor.

The task of finding idempotents etc. are left as an exercise to the reader.

Example 2.107: Let

$$B = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \middle/ a_i \in \langle\langle\langle Z_7 \cup g\rangle_I , +\rangle \times\rangle, 1 \leq i \leq 28, \times_n\right\}$$

be the special natural neutrosophic dual number matrix semigroup.

The task of finding ideals, subsemigroups, zero divisors, idempotents etc. are left as an exercise to the reader.

Example 2.108: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle/ a_i \in \langle\langle\langle Z_{12} \cup h\rangle_I , +\rangle \times_n\rangle, 1 \leq i \leq 4, \times_n\right\}$$

be the special natural neutrosophic special dual like number matrix semigroup under $\times_n$.

This has idempotents, nilpotents and zero divisors.

Example 2.109: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle/ a_i \in \langle\langle\langle Z_6 \cup k\rangle_I , +\rangle \times\rangle, 1 \leq i \leq 5, \times_n \right\}$$

be the special natural neutrosophic special quasi dual number semigroup under $\times_n$.

Study all properties associated with this P.

We now proceed onto suggest problems some are difficult and some just exercise to the reader.

Problems

1. Obtain all special features associated with $\{ Z_n^I , \times\}$ the natural neutrosophic semigroup.

2. Let P = $\{ Z_{28}^I , \times\}$ be the natural neutrosophic semigroup.

   i) Find o($\{ Z_{28}^I , \times\}$.
   ii) Find all natural neutrosophic elements of Z_{28}^I.
   iii) Can Z_{28}^I have natural neutrosophic nilpotents?
   iv) Find all subsemigroups which are not ideals of P.
   v) Find all ideals of P.
   vi) Is P a S-semigroup.
   vii) Can P have natural neutrosophic idempotents which are S-idempotents?
   viii) Can P have natural neutrosophic zero divisors which are S-zero divisors?
   ix) Can P have S-ideals?
   x) Find all ideals of S which are not S-ideals.

xi) Find all natural neutrosophic zero divisors of P which are not S zero divisors.

3. Let M = { Z_{53}^{I}, ×} be the natural neutrosophic semigroup.

   Study questions (i) to (xi) of problem (2) for this M.

4. Let W = { $Z_{7^{21}}^{I}$, ×} be the natural neutrosophic semigroup.

   i) Study questions (i) to (xi) of problem two for this W.
   ii) Obtain all other special and distinct features enjoyed by W.

5. Let X = { $Z_{23^{10}}^{I}$, ×} be the natural neutrosophic semigroup.

   i) Study questions (i) to (xi) of problem two for this X.
   ii) Show X has more number of natural neutrosophic elements then W in problem (4).

6. Let Y = { $Z_{2^3 3^4 5^2 7^3 13^2 19}^{I}$, ×} be the natural neutrosophic semigroup under ×.

   Study questions (i) to (xi) of problem (2) for this Y.

7. Let S = {$C^I(Z_{248})$, ×} be the natural neutrosophic complex modulo integer semigroup under ×.

   i) Study questions (i) to (xi) or problem (2) for this S.
   ii) Compare S with P_1 = { Z_{248}^{I} , ×}.

8. Let M = {C^I($Z_{2^{24}}$, ×} be the natural neutrosophic complex modulo integer semigroup under ×.

   i) Study questions (i) to (xi) of problem (2) for this M.
   ii) Enumerate all special features associated with this M.

9. Let B = $\{C^I(Z_{13^6}), \times\}$ be the natural neutrosophic complex modulo integer semigroup under ×.

   Study questions (i) to (xi) of problem (2) for this B.

10. Let W = $\{C^I(Z_{2.35.7.11.13}), \times\}$ be the natural neutrosophic complex modulo integer semigroup under ×.

    i) Study questions (i) to (xi) of problem (2) for this W.
    ii) Compare this W with B of problem.

11. Let T = $\{C^I(Z_{7^{10}13^{16}19^2 23^5}), \times\}$ be the natural neutrosophic complex modulo integer semigroup.

    Study questions (i) to (xi) of problem (2) for this T.

    Compare this T with B in problem 9.

12. Let S = $\{\langle Z_{148} \cup I\rangle_I, \times\}$ be the natural neutrosophic-neutrosophic semigroup.

    i) Study questions (i) to (xi) of problem (2) for this S.
    ii) Compare this S with W = $\{C^I(Z_{148}), \times\}$.

13. Let B = $\{\langle Z_{5^{121}} \cup I\rangle_I, \times\}$ be the natural neutrosophic-neutrosophic semigroup.

    Study questions (i) to (xi) of problem (2) for this B.

14. Let E = $\{\{\langle Z_{5.2.3.13.17.23} \cup I\rangle_I, \times\}$ be the natural neutrosophic semigroup.

    Study questions (i) to (xi) of problem (2) for this B.

15. Let F = $\{\langle Z_{3^4 2^7 13^5 11^7 29^2} \cup I\rangle_I, \times\}$ be the natural neutrosophic-neutrosophic semigroup.

Study questions (i) to (xi) of problem (2) for this F.

16. Obtain all special features enjoyed natural neutrosophic quasi dual number semigroups.

17. Let L = $\{\langle Z_{124} \cup g\rangle_I, \times\}$ natural neutrosophic dual number semigroup.

    i) Study questions (i) to (xi) of problem (2) of this L.
    ii) Prove L has zero square subsemigroups.
    iii) Can L have zero square ideals?
    iv) Obtain any other special feature associated with L.

18. Let S = $\{\langle Z_{43} \cup g\rangle_I, \times\}$ be the natural neutrosophic dual number semigroup.

    Study questions (i) to (xi) of problem (2) for this S.

19. Let W = $\{\langle Z_{2.35.13.7.11.19} \cup g\rangle_I, \times\}$ be the natural neutrosophic dual number semigroup.

    Study questions (i) (xi) of problem (2) for this W.

20. Let M = $\{\langle Z_{19^3 11^2 7^5 5^{10} 3^7 2^{11} 23^2} \cup g\rangle_I, \times\}$ be the natural neutrosophic dual number semigroup.

    Study questions (i) to (xi) of problem (2) for this M.

21. Let W = $\{\langle Z_{14^3} \cup h\rangle_I, \times\}$ be the natural neutrosophic special dual like number semigroup.

    i) Study questions (i) to (xi) of problem (2) for this W.
    ii) Compare W with the semigroups $S_1 = \{\langle Z^I_{14^3}, \times\}$, $S_2 = \{\langle Z_{14^3} \cup g\rangle_I, \times\}$ and $S_3 = \{C_I(Z_{14^3}), \times\}$.

22. Let M = $\{\langle Z_{2.3.5.7.11.19} \cup h\rangle_I, h^2 = h, \times\}$ be the natural neutrosophic special dual like number semigroup.

Study questions (i) to (xi) of problem (2) for this M.

23. Let V = $\{\langle Z_{19^3 23^4 29^2 11^4 17^5} \cup h\rangle_I, \times\}$ be the natural neutrosophic special dual like number semigroup.

    Study question (i) to (xi) of problem (2) for this M.

24. Let M = $\{\langle Z_{124} \cup k\rangle_I,\ k^2 = 123k,\ \times\}$ be the natural neutrosophic special quasi dual number semigroup.

    i) Study questions (i) to (xi) of problem (2) for this M.
    ii) Obtain any other special feature associated with M.
    iii) Compare the semigroup M with the semigroup $B_1 = \{C^I(Z_{124}), \times,\ i_F^2 = 123\}$, $B_2 = \{\langle Z_{124} \cup I\rangle_I, I^2 = I, \times\}$ and B_3 $\{\langle Z_{124} \cup g\rangle_I, g^2 = 0, \times\}$.

25. Let W = $\{\langle Z_{127} \cup k\rangle_I,\ \times,\ k^2 = 126k\}$ be the natural neutrosophic special quasi dual number semigroup.

    i) Study questions (i) to (xi) of problem (2) for this W.
    ii) Compare W with B = $\{\langle Z_{128} \cup k\rangle_I, k^2 = 127k, \times\}$.

26. Let S = $\{\langle Z_{2^6 3^7 13^4 23^2 29^6 5^4} \cup k\rangle_I, \times\}$ be the natural neutrosophic special quasi dual number semigroup.

    Study questions (i) to (ix) of problem (2) for this S.

27. Let P = $\{\langle Z_{2.3.17.23.43.59} \cup k\rangle_I, \times\}$ be the natural neutrosophic special quasi dual number semigroup.

    i) Compare P with M, S and W of problems 24, 25 and 26 respectively.
    ii) Study questions (i) to (ix) of problem (2) for this P.

28. Find all special features associated with the natural real neutrosophic matrix semigroup.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle/ a_i \in \langle Z^I_{480}, \times\rangle, 1 \leq i \leq 12, \times_n \right\}.$$

i) What is o(P)?
ii) How many matrix idempotents are in P?
iii) Find the total number of natural neutrosophic of nilpotents in P.
iv) Find all natural neutrosophic zero divisors of P.
v) Find all ideals of P.
vi) Find all subsemigroups which are not ideals in P.
vii) Is P a S-semigroup?
viii) Can P have S-zero divisors?
ix) Find S-ideals of P.
x) Can there be a S-subsemigroup which is not an S-ideal of P?

29. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix} \middle/ a_i \in \langle Z^I_{239}, \times\rangle, \right.$

$1 \leq i \leq 18, \times_n\}$

be the natural neutrosophic real matrix semigroup.

i) Study questions (i) to (x) of problem (28) for this P.

ii) Compare this B with P of problem (28).

30. Let L = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle/ a_i \in \langle Z^I_{3^{21}7^{10}}, \times\rangle, 1 \leq i \leq 6, \times_n\right\}$ be the natural neutrosophic real matrix semigroup.

Study questions (i) to (x) of problem (28) for this L.

Compare L with B of problem (29).

31. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle/ a_i \in \{C^I(Z_{29}), i_F^2 = 28, \times_n\}, 1 \leq i \leq 9\right\}$

be the natural neutrosophic complex modulo integer semigroup under natural product.

i) Study questions (i) to (x) of problem (28) for this T.
ii) Can T be compatible under the usual product ×.
iii) Compare T with L of problem (30).

32. Let G = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle/ a_i \in \{C^I(Z_{3^2 5^3 13^4 19^7 7^2}), \times\}, \times_n; 1 \leq i \leq 24\right\}$

be the natural neutrosophic finite complex modulo integer semigroup under the natural product $\times_n$.

Study questions (i) to (x) of problem (28) for this G.

33. Let H = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \middle/ a_i \in \right.$

$(C^I(Z_{13^{21}}), \times\}, 1 \leq i \leq 21, \times_n\}$

be the natural neutrosophic finite complex modulo integer semigroup under the natural product x_n.

i) Study questions (i) to (x) of problem (28) for this H.
ii) Compare H with G and T of problems (32) and (31) respectively.

34. Let
S = $\{(a_1, a_2, a_3, a_4, a_5, a_6, a_7) / a_i \in \{C_I(Z_{24}), \times\}, 1 \leq i \leq 7, \times\}$ be the natural neutrosophic finite complex modulo integer semigroup.

Study questions (i) to (x) of problem (28) for this S.

35. Let W = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle/ a_i \in \{\langle Z_{295} \cup I\rangle_I, \times\}; 1 \leq i \leq 14, \times_n\right\}$

be the natural neutrosophic neutrosophic matrix semigroup under the natural product $\times_n$.

i) Study questions (i) to (x) of problem (28) for this W.

ii) Obtain any of the special features associated with this W.

36. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{29} & a_{30} & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{36} & a_{37} & a_{38} & a_{39} & a_{40} & a_{41} & a_{42} \end{bmatrix} \middle/ a_i \in \right.$

$\langle Z_{8273} \cup I\rangle_I, \times\}; 1 \leq i \leq 42, \times_n\}$

be the natural neutrosophic matrix semigroup under the natural product $\times_n$.

a) Study questions (i) to (x) of problem (28) for this V.
b) Find the total number of $\{ I_t^t$ got using division on $\langle Z_{8273} \cup I\rangle$.

37. Let K = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \middle/ a_i \in \{\langle Z_{5^3 7^5 19^2} \cup I\rangle_I, \times\}; 1 \leq i \leq 8, \times_n\}\right.$ be

the natural neutrosophic neutrosophic matrix semigroup under natural product.

i) Study questions (i) to (x) of problem (28) for this K.
ii) Find all natural neutrosophic elements in K.

38. Let R = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix} \middle/ a_i \in \langle Z_{43} \cup g\rangle_I, \times\}; 1 \leq i \leq 36, \times_n\right\}$

be the natural neutrosophic dual number matrix semigroup under the natural product $\times_n$.

i) Prove R is not even closed under the usual product $\times$.
ii) Study questions (i) to (x) of problem (28) for this R.

39. Let S = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle/ a_i \in \{\langle Z_{3^8 5^7 29^2 37^5} \cup g\rangle_I, \times\}; 1 \leq i \leq 7, \times_n\right\}$ be

the natural neutrosophic dual number matrix semigroup under the natural product $\times_n$.

i) Study questions (i) to (x) of problem (28) for this S.
ii) Obtain any other special feature associated with S.

40. Let W = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle/ a_i \in \langle Z_{27} \cup h\rangle_I; \times\}; 1 \leq i \leq 24, \times_n\right\}$

be the natural neutrosophic special dual like number matrix semigroup.

Study questions (i) to (x) of problem (28) for this W.

41. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle/ a_i \in \{\langle Z_{2^3 11^5 19^2 29^4 3^2} \cup h\rangle_I, \times\},\right.$

$1 \leq i \leq 16, \times_n\}$

be the natural neutrosophic special dual like number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (x) of problem (28) for this B.

42. Let K = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix} \middle/ a_i \in\right.$

$\langle Z_{47} \cup k\rangle, k^2 = 46k, \times\}, 1 \leq i \leq 14, \times_n\}$

be the natural neutrosophic special quasi dual number semigroup under the natural product $\times_n$.

i) Study questions (i) to (x) of problem (28) for this k.
ii) Compare this k with
 S_1 = {2 × 7 matrix semigroup with entries from $\langle Z_{47} \cup I\rangle_I, \times_n\}$,

 S_2 = {2 × 7 matrix semigroup with entries from $C^I(Z_{47}), \times_n\}$,

 S_3 = {2 × 7 matrix semigroup with entries from $\langle Z_{47} \cup g\rangle_I, \times_n\}$ and

S_4 = {2 × 7 matrix semigroup with entries from $\{ Z_{47}^I \}$, $\times_n$ }.

43. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right.$ / $a_i \in \langle Z_{3^6 2^7 11^5 29^2} \cup k \rangle_I$, $k^2 = k$,

$\times_n \}$; $1 \le i \le 18$, $\times_n \}$

be the natural neutrosophic special quasi dual number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (x) of 28 of problem (28) for this W.

44. Let $B[x]_{21} = \left\{ \sum_{i=0}^{21} a_i x^i \;/\; a_i \in \{ Z_{43}^I, \times \} = B, 0 \le i \le 21, \right.$

$x^{22} = 1, \times \}$

be the real natural neutrosophic coefficient polynomial ring.

i) What is the $o(B[x]_{21})$.
ii) Find all ideals of $B[x]_{21}$.
iii) Can $B[x]_{21}$ has zero divisors?
iv) Can $B[x]_{21}$ have subsemigroups which not ideals?
v) How many polynomials in $B[x]_{21}$ have natural neutrosophic coefficient?

45. Let $B[x]_{15} = \left\{ \sum_{i=0}^{15} a_i x^i \;/\; a_i \in \{ C^I(Z_{240}), \times \} = B; 0 \le i \le 15, \right.$

$x^{16} = 1, \times \}$

be the natural neutrosophic complex modulo integer coefficient polynomial semigroup of finite order.

Study questions (i) to (v) of problem (44) for this $B[x]_{15}$.

46. Let $D[x]_6 = \left\{\sum_{i=0}^{6} a_i x^i \,/\, a_i \in \{C^I(Z_{2^3 7^4 19^2 29^5 43^2}), \times\}, 0 \leq i \leq 6, x^7 = 1, \times\right\}$

be the natural neutrosophic complex modulo integer coefficient polynomial semigroup under ×.

Study questions (i) to (v) of problem (44) for this $D[x]_6$.

47. Let $P[x]_3 = \left\{\sum_{i=0}^{3} a_i x^i \,/\, a_i \in P = \{\langle Z_{12} \cup I\rangle_I, \times\} = \times; x^4 = 1, 0 \leq i \leq 3\right\}$

be the natural neutrosophic neutrosophic coefficient polynomial semigroup and ×.

Study questions (i) to (v) of problem (44) for this $P[x]_3$.

48. Let $S[x]_{28} = \left\{\sum_{i=0}^{28} a_i x^i \,/\, a_i \in S\ \{\langle Z_{47^3 53^2 61^4} \cup I\rangle_I \times\}; x^{29} = 1, 0 \leq i \leq 28, \times\right\}$

be the natural neutrosophic neutrosophic coefficient polynomial semigroup under ×.

Study questions (i) to (v) of problem (44) for this $S[x]_{28}$.

49. Let $T[x]_{10} = \left\{\sum_{i=0}^{10} a_i x^i \,/\, a_i \in T = \{\langle Z_{43} \cup g\rangle_I, \times\}, x^{11} = 1, \times\right\}$

be the natural neutrosophic dual number coefficient polynomial semigroup under ×.

Study questions (i) to (v) of problem (44) for this $T[x]_{10}$.

50. Let $W[x]_7 = \left\{\sum_{i=0}^{7} a_i x^i \,/\, a_i \in W = \{\langle Z_{11^3 17^2 29^4 53^5 61^2} \cup g\rangle_I, \times\}\right.$

$\left. x^8 = 1, 0 \leq i \leq 7, x\right\}$

be the natural neutrosophic dual number coefficient polynomial semigroup under ×.

Study questions (i) to (v) of problem (44) for this $w[x]_7$.

51. Let $B[x]_{24} = \left\{\sum_{i=0}^{24} a_i x^i \,/\, a_i \in B = \{\langle Z_{424} \cup h\rangle_I, \times\}; x^{25} = 1,\right.$

$\left. 0 \leq i \leq 24, \times\right\}$

be the natural neutrosophic special dual like number coefficient polynomial under ×.

Study questions (i) to (v) of problem (44) for this $B[x]_{24}$.

52. Let $W[x]_{20} = \left\{\sum_{i=0}^{20} a_i x^i \,/\, a_i \in W = \{\langle Z_{23^4 19^6 53^5 67^2} \cup h\rangle_I, \times\}\right.$

$\left. x^2 = 1, 0 \leq i \leq 20, \times\right\}$

be the natural neutrosophic special dual like number coefficient polynomial semigroup under ×.

Study questions (i) to (v) of problem (44) for this $W[x]_{20}$.

53. Let $M[x]_{12} = \left\{\sum_{i=0}^{12} a_ix^i / a_i \in M = \{\langle Z_{16} \cup k\rangle_I;\ k^2 = 15k, \times\},\right.$

$0 \leq i \leq 12,\ x^{13} = 1, \times\}$

be the natural neutrosophic special quasi dual number coefficient polynomial semigroup.

Study questions (i) to (v) of problem (44) for this $M[x]_{12}$.

54. Let $N[x]_7 = \left\{\sum_{i=0}^{7} a_ix^i / a_i \in N\ \{\langle Z_{2^7 5^{10} 11^3 19^2 47^3 53^2} \cup k\rangle_I \times\};\right.$

$0 \leq i \leq 7,\ x^8 = 1, \times\}$

be the natural neutrosophic special quasi dual number polynomial coefficient semigroup under ×.

Study questions (i) to (v) of problem (44) for this $N[x]_7$.

55. Let $V = \{\langle Z_{10}^I, +\rangle, \times\}$ be the special real natural neutrosophic semigroup under ×

i) Find o(V).

ii) Find all natural neutrosophic zero divisor.

iii) $(I_5^{10} + I_0^{10}) = p\ (I_2^{10} + I_4^{10} + I_6^{10} + I_8^{10})\ q \in V$ is $p \times q = I_0^{10}$?

iv) Find all natural neutrosophic idempotents of V.

v) Find all subsemigroups of V.

vi) Find all ideals of V.

vii) Is V a S-semigroup?

viii) Can V have nilpotent elements?

ix) Can V have S-zero divisors?

x) Can V have S-idempotents?

56. Let W = $\{ Z_{19\times 23^2\times 29^3}$, $+\rangle$, $\times\}$ be the special natural neutrosophic semigroup.

    Study questions (i), (ii), (iv) to (x) of problem (44) for this W.

57. Let S = $\{\{\langle Z_{43\times 47} \cup I\rangle_I, \times\}$ be the special natural neutrosophic neutrosophic semigroup under $\times$.

    Study questions (i) to (x) of problem (44) for this S.

58. Let x = $\{\langle\langle Z_{424} \cup I\rangle_I, +\rangle, \times\}$ be the special natural neutrosophic neutrosophic semigroup under +.

    Study questions (i) to (x) of problem (44) for this $\times$.

59. Let Z = $\{\langle\langle Z_{24} \cup g\rangle_I, +\rangle, \times\}$ be the special natural neutrosophic dual number semigroup under $\times$.

    Study questions (i), (ii) and (iv) to (x) of problem (44) for this Z.

60. Let D = $\{\langle\langle Z_{43^2 29^3 53^9} \cup g\rangle_I, +\rangle, \times\}$ be the special natural neutrosophic dual number semigroup under ×.

    Study questions (i), (ii) and (iv) to (x) of problem (44) for this D.

61. Let M = $\{\langle Z_{24} \cup h\rangle_I, +\rangle, \times\}$ be the special natural neutrosophic special dual like number semigroup under ×.

    Study questions (i), (ii) and (iv) to (x) of problem (44) for this M.

62. Let P = $\{\langle\langle Z_{2^3 5^2 13^3 29^2 47^4} \cup k\rangle_I +\rangle, \times\}$ be the special natural neutrosophic special quasi dual number semigroup under ×.

    Study questions (i), (ii) and (iv) to (x) of problem (44) for this P.

63. Study the special features enjoyed by the special natural neutrosophic matrix M semigroup under product where

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \middle/ a_i \in \right.$$

$$\left.\{\langle Z^I_{13^2 \times 17^3 \times 23^4 \times 27^2 \times 47^2 \times 53^9}, +\rangle\} \quad 1 \le i \le 28, \times_n\right\}.$$

64. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right.$ / $a_i \in \langle\langle C^I(Z_{43}), +\rangle, \times\rangle$;

$1 \leq i \leq 16, \times_n\}$

be the special natural neutrosophic complex modulo integer matrix semigroup under the natural product $\times_n$.

i) Find o(W).

ii) How many elements are pure natural neutrosophic?

iii) Find zero divisor of W.

iv) Can W have idempotents?

v) Find ideals of W.

vi) Is W a S-semigroup?

vii) Find subsemigroups which are not ideals of W.

65. Let $M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right.$ / $a_i \in \{\langle\langle Z_7 \cup g\rangle_I, \times\}; 1 \leq i \leq 6, \times_n\}$ be the

special natural neutrosophic dual number semigroup matrix under natural product $\times_n$.

Study questions (i) to (vii) of problem 64 for this M.

66. Let
$S = \{(a_1, a_2, a_3, a_4) / a_i \in \langle\langle Z_{3^2 7^4 13^6 29^2 53^4} \cup g\rangle, +\rangle; 1 \leq i \leq 4, \times\}$
be the special natural neutrosophic dual number matrix under the product ×.

Study questions (i) to(vii) of problem (64) for this S.

67. Let $T = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{30} \end{bmatrix} / a_i \in \right.$

$\langle\langle Z_{43\times 59 \times 61 \times 101} \cup h\rangle + \rangle, \times_n, 1 \leq i \leq 30\}$

be the special natural neutrosophic special dual like number matrix semigroup under natural product $\times_n$.

Study questions (i) to (vii) of problem (64) for this T.

68. Let $B = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} / a_i \ \langle\langle Z_{2.3.7^3.11^2} \cup k \rangle + \rangle, 1 \leq i \leq 14, \times_n\right\}$

be the special natural neutrosophic special quasi dual number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (vii) of problem (64) for this B.

69. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} \end{bmatrix} \right.$ / $a_i \in$

$\langle\langle Z_{425 \times 13^2 \times 19^3} \cup k\rangle_I, +\rangle, \times_n; 1 \leq i \leq 21\}$

be the special natural neutrosophic special quasi dual number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (vii) of problem (64) for this S.

70. Let $B[x]_8 = \left\{\sum_{i=0}^{8} a_i x^i \right.$ / $a_i \in$ B $\langle\langle Z^I_{484}, +\rangle, \times\rangle$, $0 \leq i \leq 8$, $\times$;

$x^9 = 1\}$

be the special natural neutrosophic coefficient polynomial semigroup under $\times$.

i) Find $o(B[x]_8)$.
ii) How many zero divisors are in $B[x]_8$?
iii) Find all special natural neutrosophic elements of $B[x]_8$.
iv) Is $B[x]_8$ a S-semigroup?
v) Find all ideals of $B[x]_8$.
vi) How many subsemigroups of $B[x]_8$ are not ideals?
vii) Can $B[x]_8$ have nilpotents?
viii) Can $B[x]_8$ have idempotents?

71. Let $S[x]_{20} = \left\{\sum_{i=0}^{20} a_i x^i \right.$ / $a_i \in$ $\{\langle\langle Z_{48} \cup I\rangle_I + \rangle, \times\} = S$;

$0 \leq i \leq 20$, $x^{21} = 1, \times\}$

be the special natural neutrosophic neutrosophic coefficient polynomial semigroup under $\times$.

Study questions (i) to (viii) of problem 70 for this $S[x]_{20}$.

72. Let $P[x]_5 = \left\{ \sum_{i=0}^{5} a_i x^i \,/\, a_i \in P = \{\langle\langle Z_{5^2 7^3 19^2 29^4 43^2 53} \cup I \rangle_I, + \rangle, +\}; 0 \leq i \leq 5, x^6 = 1, \times \right\}$

be the special natural neutrosophic neutrosophic coefficient polynomial semigroup under ×.

Study questions (i) to (viii) of problem (70) for this $P[x]_5$.

73. Let $M[x]_{12} = \left\{ \sum_{i=0}^{12} a_i x^i \,/\, a_i \in M = \{\langle\langle Z_{28} \cup g \rangle_I, + \rangle, +\}; 0 \leq i \leq 12, x^{13} = 1, \times \right\}$

be the special natural neutrosophic dual number coefficient polynomial semigroup under product.

Study questions (i) to (viii) of problem (70) for this $M[x]_{12}$.

Obtain any other special feature associated with this $M[x]_{12}$.

74. Let $W[x]_3 = \left\{ \sum_{i=0}^{3} a_i x^i \,/\, a_i \in W = \{\langle Z_{3.5.7.13^2.17^2.29^2.53^4} \cup g\rangle_I, +\}, \times\}; 0 \leq i \leq 3, x^4 = 1, \times \right\}$

be the special natural neutrosophic dual number coefficient polynomial semigroup under product.

Study questions (i) to (viii) of problem (70) for this $W[x]_3$

75. Let $V[x]_7 = \left\{ \sum_{i=0}^{7} a_i x^i \,/\, a_i \in V = \{\langle\langle Z_{2.3.5^3.49} \cup h \rangle, + \rangle, \times\}; 0 \leq i \leq 7, x^8 = 1, \times \right\}$

be the special natural neutrosophic special dual like number coefficient polynomial semigroup under product.

Study questions (i) to (viii) of problem (70) for this $V[x]_7$.

76. Let $P[x]_{12} = \left\{ \sum_{i=0}^{12} a_i x^i \,/\, a_i \in P = \{\langle\langle C^I(Z_{4224}), +\rangle, \times\}; \right.$

$$0 \leq i \leq 12,\ x^{13} = 1, \times\}$$

be the special natural neutrosophic complex modulo integer coefficient polynomial semigroup under $\times$.

Study questions (i) to (viii) of problem (70) for this $P[x]_{12}$.

77. Let $T[x]_{17} = \left\{ \sum_{i=0}^{6} a_i x^i \,/\, a_i \in T = \{\langle\langle Z_{2.5.7.23.47^2} \cup k\rangle_I, +\rangle, \times\}; \right.$

$$0 \leq i \leq 6,\ x^7 = 1\}$$

be the special natural neutrosophic special quasi dual number coefficient polynomial semigroup under $\times$.

Study questions (i) to (viii) of problem (70) for this $T[x]_{17}$.

Chapter Three

MOD-SEMIGROUPS USING NATURAL NEUTROSOPHIC ELEMENTS OF [0, n)

In this chapter we proceed on to study semigroups under + on the natural neutrosophic elements in [0, n). Similarly we define product operation on natural neutrosophic elements in [0, n).

We will illustrate this by examples.

Example 3.1: Let $\{[^{I}[0,6) = \{0, 1, 2, \ldots, 5.999, \ldots, I_2^{[0,6)}, I_4^{[0,6)}, I_3^{[0,6)}, I_5^{[0,6)}, I_{1.2}^{[0,6)}, \ldots\}$. It is recalled 5 is a unit element in [0,6) as $5 \times 5 = 1 \pmod 6$, however in [0,6) $1.2 \times 5 = 6 \pmod 6$ so is a zero divisor. This sort of occurrences is not true in case of Z_6.

A unit in R can never contribute to zero divisors, it is an impossibility so we choose to call such zero divisors as MOD pseudo zero divisors [24].

However it is a open conjecture to find pseudo zero divisors of any interval [0, n); n any integer.

So study in this direction is very new.

Further we are not able to find a routine method to obtain all pseudo zero divisor and nilpotent elements if any of [0, n).

We just put forth result that there can be more zero divisors when n in [0, n) is a prime.

For when we lake [0, 7) the interval 4 is a unit, 2 is a unit. 3.5 is a pseudo zero divisor. 1.75 is again a pseudo zero divisor.

We are not in a position to find more pseudo zero divisors. Consider the MOD interval [0, 11), we see 10 is a unit in [0, 11).

But $1.1 \in [0, 11)$ is a pseudo zero divisor as

$1.1 \times 10 = 11 \equiv 0 \pmod{11}$.

So $I_{10}^{[0,11)}$ and $I_{1.1}^{[0,11)}$ are MOD natural neutrosophic pseudo zero divisors of [0, 11).

Likewise $I_{2.2}^{[0,11)}$ and $I_{5}^{[0,11)}$ are MOD natural neutrosophic pseudo zero divisors as $2.2 \times 5 = 11 = 0 \pmod{11}$ is a pseudo zero divisor of [0, 11).

Similarly $5.5, 2 \in [0, 11)$ is such that
$5.5 \times 2 = 11 \equiv 0 \pmod{11}$. But 2 is a unit of [0, 11).

So they are pseudo zero divisors contributing to natural neutrosophic pseudo zero divisors $I_{2}^{[0,11)}$ and $I_{5.5}^{[0,11)}$.

Consider $4 \in [0, 11)$ 4 is a unit but $4 \times 2.75 \equiv 0 \pmod{11}$ so 4 is a pseudo zero divisor and this contributes to the natural neutrosophic pseudo zero divisor of [0, 11), that is $I_{4}^{[0,11)}$ and

$I_{2.75}^{[0,11)}$ are such that $I_4^{[0,11)} \times I_{2.75}^{[0,11)} = I_0^{[0,11)}$ is a pseudo natural neutrosophic zero divisor.

Consider the unit 8 in the MOD interval [0, 11), 8 in [0, 11) is such that $8 \times 7 \equiv 1 \pmod{11}$.

$1.375 \in [0, 11)$ is such that $1.375 \times 8 \equiv 0 \pmod{11}$ so are a pair of pseudo zero divisors leading to the natural neutrosophic zero divisor.

Let us consider the interval [0, 11) there are several pseudo zero divisors.

In fact we do not have zero divisors from Z_{11} as it is a field.

Consider $I_{1.1}^{[0,11)}$; $I_{1.1}^{[0,11)} \times I_{1.1}^{[0,11)} = I_{1.21}^{[0,11)}$,

$$I_{1.1}^{[0,11)} \times I_{1.21}^{[0,11)} = I_{1.331}^{[0,11)}$$

$$I_{1.331}^{[0,11)} \times I_{1.21}^{[0,11)} = I_{1.61051}^{[0,11)}$$

$$I_{1.61051}^{[0,11)} \times I_{1.331}^{[0,11)} = I_{2.1435881}^{[0,11)}$$

$$I_{2.1435881}^{[0,11)} \times I_{1.61051}^{[0,11)} = I_{3.452271214}^{[0,11)}$$

$$I_{3.452271214}^{[0,11)} \times I_{3.452271214}^{[0,11)} = I_{11.9181}^{[0,11)}$$

$$I_{3.452271214}^{[0,11)} \times I_{2.1435881}^{[0,11)} = I_{7.400247492}^{[0,11)}.$$

A natural question would be should all these elements produced with $I_{1.1}^{[0,11)}$ find their place in the natural neutrosophic element of the MOD interval [0,11).

For even $I_{10}^{[0,11)} \times I_{10}^{[0,11)} = I_1^{[0,11)}$ makes it difficult to define product. For $I_1^{[0,11)}$ is not a natural neutrosophic element.

So we over come this problem in two ways.

i) Except the pseudo zero divisor and the unit which makes it a pseudo zero divisor. Arrange them only as pair and not include them in any form of product ($I_{10}^{[0,11)}, I_{1.1}^{[0,11)}$) is a pseudo zero divisor unit pair.

No sensible operation can be performed on them.

They only show mathematics in case of MOD-interval is different from the usual reals.

Such study is not only innovative but fascinating.

In fact it is conjectured that the more pseudo zero divisors in the MOD interval [0, n) then it implies the n in [0, n) is such n is a prime.

Consider [0,12) the MOD interval. The pseudo zero divisors of [0,12) are $I_5^{[0,12)}$ and $I_{2.4}^{[0,12)}$ are such that $I_{2.4}^{[0,11)} \times I_5^{[0,12)} = I_0^{[0,12)}$. Thus ($I_5^{[0,12)}, I_{2.4}^{[0,12)}$) is a pseudo zero divisor unit pair.

For 5 × 5 = 1 (mod 12). Consider $I_{1.2}^{[0,12)} \times I_{10}^{[0,12)} = I_0^{[0,12)}$ is a zero divisor.

Clearly 10 is a zero divisor in [0,12). $I_{1.5}^{[0,12)} \times I_8^{[0,12)} = I_0^{[0,12)}$ and so are only zero divisors and not pseudo zero divisors.

This gives one a surprise.

Can we make a conclusion if in [0, n), n is a composite number then we cannot have many pseudo zero divisors or a pseudo zero divisor unit pair?

Consider [0,8) the MOD interval. Can [0,8) have pseudo zero divisor-unit pair?

Let x = 1.6 and y = 5 ∈ [0.8).

$I_{1.6}^{[0,8)} \times I_{5}^{[0,8)} = I_{0}^{[0,8)}$ is a pseudo zero divisor unit pair.

This has one zero divisor unit pair given by ($I_{1.6}^{[0,11)}, I_{5}^{[0,8)}$). Consider the MOD interval [0,9), 4.5, 2 $\in$ [0,9) is such that it is a pseudo zero divisor for $4.5 \times 2 \equiv 0 \pmod 9$, 2 is a unit as $2 \times 5 = 1 \pmod 9$.

Let $I_{4.5}^{[0,9)}$ and $I_{2}^{[0,9)}$ be the pseudo natural neutrosophic zero divisor as $I_{4.5}^{[0,9)} \times I_{2}^{[0,9)} = I_{0}^{[0,9)}$.

Thus $\{I_{4.5}^{[0,9)}, I_{2}^{[0,9)}\}$ is the MOD natural neutrosophic zero divisor-unit pair.

Let 4 and 2.25 $\in$ [0,9). Clearly $4 \times 2.25 = 0 \pmod 9$. Four is a unit as $4 \times 7 = 1 \pmod 9$.

So $I_{4}^{[0,9)}$ and $I_{2.25}^{[0,9)}$ are such that $I_{4}^{[0,9)} \times I_{2.25}^{[0,9)} = I_{0}^{[0,9)}$.

Hence $\{I_{2.25}^{[0,9)}, I_{4}^{[0,9)}\}$ is a MOD natural neutrosophic zero divisor unit pair of $^{I}[0,9)$.

Now x = 1.125 and y = 8 $\in$ [0,9) are such that
$x \times y = 1.125 \times 8 = 0 \pmod 9$.

8 is a unit of [0,9) as $8 \times 8 = 1 \pmod 9$. We see $I_{8}^{[0,9)}$ and $I_{1.125}^{[0,9)} \in {}^{I}[0,9)$ are such that $I_{8}^{[0,9)} \times I_{1.125}^{[0,9)} = I_{0}^{[0,9)}$ is a natural neutrosophic zero divisor and $\{I_{1.125}^{[0,9)}, I_{8}^{[0,9)}\}$ is a natural neutrosophic pseudo zero divisor-unit pair of $^{I}[0,9)$.

It is observed that if in the interval [0, n), n is odd we get more pseudo zero divisor-unit pair than when [0, n) when n is a even composite number. Consider the MOD interval [0,10).

$I_{2}^{10}, I_{4}^{10}, I_{8}^{10}$ and I_{5}^{10} are the natural neutrosophic elements of [0, 10).

Take x = 2.5 and y = 4 ∈ [0, 10) then x × y = 0 (mod 10).

Consider x_1 = 1.25 and y_1 = 8 ∈ [0, 10) the
$x_1 \times y_1 = 0$ (mod 10).

They are again MOD natural neutrosophic zero divisors of [0, 10).

Can [0, 10) have pseudo zero divisor that is pseudo zero divisor-unit pair?

Consider the interval [0,14). This has MOD pseudo zero divisor-unit pair.

For take 1.4 and 10 ∈ [0,14), clearly
1.4 × 10 = 14 ≡ 0 (mod 14) is not a MOD pseudo zero divisor as 10 is not a unit in [0,14).

Consider 2.8 and 5 ∈ [0, 14). Clearly 2.8 × 5 ≡ 0 (mod 14) and $I_5^{[0,14)} \times I_{2.8}^{[0,14)} = I_0^{[0,14)}$ is a MOD pseudo divisor of [0,14) as 5 × 3 ≡ 1 (mod 14) is a unit in [0,14).

Consider [0,16) the MOD interval.

Clearly 1.6 × 10 ≡ 16 ≡ 0 (mod 16), but 10 is a zero divisor of [0, 16).

Consider 3.2, 5 ∈ [0,16), 3.25 × 5 = 16 ≡ (0) mod 16. But 5 × 13 ≡ 1 (mod 16).

Thus [0,16) has MOD pseudo zero divisor unit pair given by $\{ I_{3.2}^{[0,16)}, I_5^{[0,16)} \}$.

Can this interval [0,16) have more number of MOD pseudo zero divisor-unit pair?

Consider the MOD interval [0,17).

Consider 8.5, 2 ∈ [0, 17), 2 is a unit of [0, 17) get
8.5 × 2 = 17 ≡ 0 (mod 17) is a pseudo zero divisor.

So ($I_{8.5}^{[0,17)} \times I_{2}^{[0,17)}$) is a MOD natural neutrosophic pseudo zero divisor-unit pair.

Let x = 4 and y = 4.25 ∈ [0, 17); 4 is a unit of [0,17) yet x × y = 4 × 4.25 = 0 (mod 17).

So { $I_{4}^{[0,17)}, I_{4.25}^{[0,17)}$ } is MOD natural neutrosophic unit-pseudo zero divisor pair.

Consider x = 2.125 and y = 8 ∈ [0,17);
x × y = 2.125 × 8 = 17.000 ≡ 0 (mod 17) is a pseudo zero divisor.

So { $I_{2.125}^{[0,17)}, I_{8}^{[0,17)}$ } is a MOD natural neutrosophic pseudo zero divisor unit pair.

Also x = 1.7 and y = 10 ∈ [0, 17) is a pseudo zero divisor and x × y = 1.7 × 10 ≡ 0 mod 17 is a MOD natural neutrosophic pseudo zero divisor-unit pair given by { $I_{1.7}^{[0,17)} \times I_{10}^{[0,17)}$ }.

Further x = 3.4 and y = 5 ∈ [0,17) is such that
x × y = 3.4 × 5 = 0 (mod 17) is a MOD-pseudo zero divisor and { $I_{3.4}^{[0,17)}$, $I_{5}^{[0,17)}$ } is a MOD natural neutrosophic pseudo zero divisor unit pair.

Considering [0, p), p a prime we see these are several MOD natural neutrosophic pseudo zero divisor unit pair.

However if p is even we get only very few MOD natural neutrosophic pseudo zero divisor unit pair.

For p an odd number these are some MOD natural neutrosophic pseudo zero divisor unit pairs.

Consider [0,33) the MOD interval.

To find the pseudo zero divisors and MOD natural neutrosophic pseudo zero divisors.

Clearly 2 is a unit of [0,33) 16.5 ∈ [0, 33) is such that 2 × 16.5 = 0 (mod 33) so is a pseudo zero divisor of [0, 33).

Consider 4 ∈ [0, 33) is a unit of [0, 33) 8.15 ∈ [0, 33) is such that 8.15 × 4 = 0 (mod 33) is a pseudo zero divisor.

Also for 8 ∈ [0,33) is a unit but 4.125 × 8 ≡ 0 (mod 33) so is a pseudo zero divisor.

Likewise 16 and 2.0625 is a pseudo zero divisor.

All these pairs contribute to MOD natural neutrosophic pseudo zero divisor-unit pair.

$$\{ I_{16.5}^{[0,33)} \times I_{2}^{[0,33)} \}, \{ I_{8.25}^{[0,33)} \times I_{4}^{[0,33)} \}, \{ I_{4.125}^{[0,33)} \times I_{8}^{[0,33)} \} \text{ and}$$

$$\{ I_{2.0625}^{[0,33)} \times I_{16}^{[0,33)} \}.$$

Further 5 is a unit of [0,33) but 5 × 6.6 is a pseudo zero divisor.

Thus $\{ I_{6.6}^{[0,33)} \times I_{5}^{[0,33)} \}$ is a MOD natural neutrosophic pseudo zero divisor-unit pair.

Also 10 is a unit and a pseudo zero divisor as

10 × 3.3 ≡ 0 (mod 3.3) and

$I_{10}^{[0,8)} \times I_{3.3}^{[0,33)} = I_{0}^{[0,33)}$ so $\{ I_{3.3}^{[0,33)} \times I_{10}^{[0,33)} \}$ is a MOD natural neutrosophic pseudo zero divisor unit pair of ${}^{I}[0,33)$.

Consider 20 and 1.65 ∈ [0, 33).

Clearly $20 \times 1.65 = 0 \pmod{33}$.

Further $I_{1.65}^{[0,33)} \times I_{20}^{[0,33)} = I_0^{[0,33)}$ is the MOD natural neutrosophic pseudo zero divisor-unit pair of ${}^{I}[0, 33)$.

Now 13.2 and 25 ∈ [0.33) and $13.2 \times 25 = 0 \pmod{33}$.

Further $I_{13.2}^{[0,33)}$ and $I_{25}^{[0,33)} \in {}^{I}[0,33)$ is such that $I_{13.2}^{[0,33)} \times I_{25}^{[0,33)} = I_0^{[0,33)}$ is a MOD natural neutrosophic pseudo zero divisor unit pair of ${}^{I}[0,33)$.

It is observed from these illustration that ${}^{I}[0,33)$ has more number of MOD natural neutrosophic pseudo zero divisor-unit pair than ${}^{I}[0,32)$.

For ${}^{I}[0,32)$ has only two MOD natural neutrosophic pseudo zero divisor-unit pair given by { $I_{6.4}^{[0,32)}, I_5^{[0,32)}$ } and { $I_{1.28}^{[0,32)}, I_{25}^{[0,32)}$ }.

The reader is left with the task of finding if any MOD natural neutrosophic pseudo zero divisor-unit pair in ${}^{I}[0,32)$.

Thus it is conjectured that p is a prime [0, p) has more number MOD natural neutrosophic pseudo zero divisor unit pairs than [0, p – 1) and [0, p + 1).

Now we try to characterize the type of probable MOD natural neutrosophic pseudo zero divisor pairs in [0, n) by the following theorem.

THEOREM 3.1: *Let [0, n); n odd be the MOD interval. The elements which are units in [0, n) and which contribute for MOD natural neutrosophic pseudo zero divisor pairs are*

i) $2; 2^2, 2^3, \ldots, 2^t > n$
ii) If $5 \nmid n$ *and 5 is a unit then 5, 10, 20, 25, 40 and so on. t5 (t even or t = 5).*

Proof: The proof exploits only simple number theoretic techniques so left as an exercise to the reader.

THEOREM 3.2: *Let [0, n); n even be the MOD interval. The elements which contribute to the MOD natural neutrosophic pseudo zero divisor-unit pair are 5, 5^2, 5^3, …, provided (5, n) = 1.*

Proof: Follows by adopting simple number theoretic techniques.

It is yet another open problem to find the existence of primes p which will not be recurrent decimals when division by them is done on n. n / p is not a recurrent decimal; (n, p) = 1.

Next we proceed onto study the algebraic structure enjoyed by this collection of MOD neutrosophic natural pseudo zero divisor-unit pair by some examples.

Example 3.2: Let ${}^{I}[0, 4) = \{0, 1, 2, 3, [0, 4), I_0^{[0,4)}, I_2^{[0,4)}$ and so on$\}$ be the MOD natural neutrosophic interval.

To the best of our knowledge we are not able to find pseudo all zero divisor-unit pair of natural neutrosophic numbers.

Example 3.3: Let ${}^{I}[0,3) = \{0, 1, 2\ [0,3);\ I_0^{[0,3)}, I_{1.5}^{[0,3)}, I_2^{[0,3)}, \ldots,\}$ be the MOD natural neutrosophic interval. Clearly $\{ I_{1.5}^3 \times I_2^3 \}$ is the MOD natural neutrosophic pseudo zero divisor-unit pair.

We feel there exist one and only one pair of MOD natural neutrosophic pseudo zero-divisor unit.

Example 3.4: Consider ${}^{I}[0,6) = \{[0, 6),\ I_0^{[0,6)}, I_2^{[0,6)}, I_3^{[0,6)}, I_4^{[0,6)}, I_{1.5}^{[0,6)}, I_{2.25}^{[0,6)}, I_{3.375}^{[0,6)}, I_{0.75}^{[0,6)}, I_{0.375}^{[0,6)}, I_{0.1875}^{[0,6)}, I_{0.046875}^{[0,6)}$ and so on$\}$.

Thus we may get infinite number of MOD natural neutrosophic elements in ${}^{I}[0,6)$.

Example 3.5: Let ${}^{I}[0,5)$ = {[0, 5), $I_0^{[0,5)}, I_{2.5}^{[0,5)}$, $I_{1.25}^{[0,5)}, I_{0.625}^{[0,5)}$, $I_{0.3125}^{[0,8)}, I_{0.15625}^{[0.5)}$ and so on}.

There are some MOD natural neutrosophic pseudo zero divisor unit pair.

We guess there may be infinite number of such MOD natural neutrosophic elements.

Example 3.6: Let us consider ${}^{I}[0,7)$ = {[0,7), $I_0^{[0,7)}, I_{3.5}^{(0,7)}$, $I_{1.75}^{[0,7)}, I_{0.875}^{[0,7)}, I_{0.4375}^{[07)}, \ldots\}$.

Infact we feel there may be infinitely many such MOD natural neutrosophic elements.

Example 3.7: Let ${}^{I}[0,8)$ = {[0,8), $I_0^{[0.8)}$, $I_2^{[0,8)}$, $I_4^{[0,8)}, I_6^{[0,8)}$, $I_{1.6}^{[0,8)}, I_{3.2}^{[0,8)}$, $I_{6.4}^{[0,8)}, I_{4.8}^{[0,8)}, I_{0.8}^{[0,8)}$ and so on} be the MOD natural neutrosophic elements of ${}^{I}[0,8)$.

***Example 3.8**:* Let
${}^{I}[0,9)$ = {[0,9), $I_0^{[0,9)}, I_{4.5}^{[0,9)}, I_3^{[0,9)}$, $I_6^{[0,4)} \times I_{2.25}^{[0,9)}$, $I_{1.125}^{[0,9)}$ and so on}.

Thus at this stage we leave it as a open conjecture to find the number of MOD natural neutrosophic elements of ${}^{I}[0,9)$.

Thus we see in general we can have for any ${}^{I}[0,n)$ n odd or even or prime, we have MOD natural neutrosophic elements to be infinite in number.

However we see the number of MOD natural neutrosophic pseudo zero divisor unit pair are finite in number.

So study in this direction is also interesting and innovative.

For now we can define the notion of pseudo idempotent unit.

We call x, y ∈ [0, n) to be a pseudo zero divisor if y is a zero divisor. x ∈ [0, n) to be a pseudo zero divisor if y is a zero. x ∈ [0, n) \ Z_n gives x × y to be an idempotent of Z_n.

This definition has meaning provided there is an idempotent in Z_n.

We will illustrate this situation by some examples.

Example 3.9: Let $^I[0,6)$ = [[0, 6), $I_0^{[0,6)}$, …, $I_3^{[0,6)}, I_{1.5}^{[0,6)}, I_2^{[0,6)}$, …} be the MOD natural neutrosophic elements set.

x = 1.5 and y = 2 in [0,6) is such that x × y = 3 is an idempotent.

Thus $I_{1.5}^{[0,6)} \times I_2^{[0,6)} = I_3^{[0,6)}$ which is the MOD natural neutrosophic idempotent.

This pair $\{I_{1.5}^{[0,6)}, I_2^{[0,6)}\}$ is defined as the MOD natural neutrosophic pseudo zero divisor-idempotent pair.

Example 3.10: Let $^I[0,15)$ = {[0,15), $I_{0,}^{[0,15)}, I_3^{[0,15)}$, $I_6^{[0,15)}, I_9^{[0,15)}$, $I_5^{[0,15)}, I_{7.5}^{[0,15)}$, $I_{10}^{[0,15)}, I_{12}^{[0,15)}$, $I_{3.6}$, $I_{4.5}^{[0,15)}$, … and so on}.

$$I_{3.6}^{[0,15)} \times I_{10}^{[0,15)} = I_6^{[0,15)}.$$

The reader is left with the task of finding such elements.

Example 3.11: Let $^I[0,10)$ = {[0,10), $I_0^{[0,10)}, I_2^{[0,10)}, I_4^{[0,10)}$, $I_6^{[0,10)}, I_8^{[0,10)}$, $I_5^{[0,10)}$, $I^{[0,10)}$, $I_{2.5}^{[0,10)}, I_{1.25}^{[0,10)}, I_5^{[0,10)}$ and so on} be the MOD natural neutrosophic element set.

$$I_{2.5}^{[0,10)} \times I_2^{[0,10)} = I_5^{[0,10)} \qquad \text{I}$$

2 is a zero divisor of Z_{10} and 5 is an idempotent of Z_{10}.

$$I_{1.25}^{[0,10)} \times I_4^{[0,10)} = I_5^{[0,10)} \qquad \text{II}$$

$$I_{0.625}^{[0,10)} \times I_8^{[0,10)} = I_5^{[0,10)} \qquad \text{III}$$

From I, II and III it is evident that the same idempotent can create pseudo zero divisor-idempotent pair.

Study in this direction is open for a researcher as most of the results are number theoretic in nature.

Example 3.12: Let ${}^{I}[0,20) = \{[0,20),\ I_0^{[0,20)}, I_2^{[0,20)}, I_4^{[0,20)},\ I_6^{[0,20)},\ I_8^{[0,20)}, I_{10}^{[0,20)}, \ldots,\ I_{18}^{[0,20)},\ I_5^{[0,20)}, I_{10}^{[0,20)},\ I_{15}^{[0,20)}, I_{2.5}^{[0,20)}$ and so on$\}$.

$5^2 = 5 \pmod{20}$ is an idempotent $16^2 = 16 \pmod{20}$.

$$I_{2.5}^{[0,20)} \times I_{10}^{[0,20)} = I_5^{[0,20)} \qquad \text{I}$$

$$I_{10.75}^{[0,20)} \times I_8^{[0,20)} = I_{16}^{[0,20)} \qquad \text{II}$$

Thus II and I are the desired pairs.

Can we say in case of even numbers 2p such that p is a prime always has idempotents and the MOD natural neutrosophic pseudo zero divisor-idempotent pair.

We do not know whether [0, n) where $n = 2^t$ has any such pairs.

This is extend to any [0, n) where $n = p^m$; $m > 1$ can contain such pairs.

This study is open to researchers.

Next we proceed onto study other intervals. $[0, n)i_F$ is not taken in this study as $i_F^2 = n - 1 \notin [0, n)i_F$.

Barring this complex number interval we study all other intervals.

Consider the neutrosophic MOD interval [0, n)I.

Clearly $^{I}[0, n)I = \{[0, n)I, I_t^{[0,n)}; t \in [0, n)I\}$.

$$I_I^{[0,n)I} \times I_I^{[0,n)I} = I_I^{[0,n)I}.$$

Let $^{I}[0,12)I = \{[0, 12)I, I_a^{[0,12)I}; a \in [0, 12)I\}$.

$$I_{4I}^{[0,12)I} \times I_{4I}^{[0,12)I} = I_{4I}^{[0,12)}$$

$$I_{9I}^{[0,12)I} \times I_{9I}^{[0,12)I} = I_{9I}^{[0,12)I}$$

$$I_{2.4}^{[0,12)I} \times I_{10I}^{[0,12)I} = I_{0}^{[0,12)I}$$

is a MOD natural neutrosophic zero divisor.

$$I_{4.8I}^{[0,12)} \times I_{5I}^{[0,12)I} = I_{0}^{[0,12)I}$$

$$I_{1.2I}^{[0,12)I} \times I_{10I}^{[0,12)I} = I_{0}^{[0,12)I}$$

$$I_{1.5I}^{[0,12)I} \times I_{8I}^{[0,12)I} = I_{0}^{[0,12)I}$$

$$I_{1.5}^{[0,12)I} \times I_{6I}^{[0,12)I} = I_{9I}^{[0,12)I}$$

is a MOD natural neutrosophic pseudo zero divisor-idempotent pair.

Next we study the MOD neutrosophic interval [0,10)I.

$^{I}[0,10)I = \{[0, 10)I, I_t^{[0,10)I}; t \in [0,10)I\}$

$$I_{5I}^{[0,10)I} \times I_{5I}^{[0,10)I} = I_{5I}^{[0,1)I}$$

$$I_{4I}^{[0,10)I} \times I_{2.5I}^{[0,10)I} = I_{0}^{[0,10)I}$$

$$I_{2}^{[0,10)I} \times I_{2.5I}^{[0,10)I} = I_{5I}^{[0,10)I}$$

is a MOD natural neutrosophic pseudo zero divisor-idempotent pair.

This is the way operations are performed.

In contrast with ${}^{I}[0,n)$ this [0,n)I makes every element to be contributing to the MOD natural neutrosophic element of ${}^{I}[0, n)I$.

Study on this direction is has lots of scope for every element in [0,n)I contribute to a MOD natural neutrosophic number.

Next we study the notion of MOD natural neutrosophic elements of [0,n)g; $g^2 = 0$.

As in case of [0,n)I every element in this is a MOD natural neutrosophic element.

Further the main difference is [0,n)I can have idempotents pseudo zero divisor-unit pair and pseudo zero divisor-idempotent pair but in case of [0, n)g every element only contributes to MOD natural neutrosophic nilpotent element of order two.

So there are not many interesting properties that can be associated with [0,n)g.

Next we study [0,n)h.

${}^{I}[0,n)h$ has MOD natural neutrosophic pseudo zero divisor-unit pairs and MOD natural interval neutrosophic pseudo zero divisor-idempotent pair depending on n.

We will give some examples of this situation.

Example 3.13: Let ${}^{I}[0, 15)h = \{[0, 15)h,\ I_t^{[0,15)h}\ ;\ t \in [0,15)\ h\}$ be the MOD natural neutrosophic special dual like interval number elements.

$I_{10h}^{[0,15)h} \times I_{10h}^{[0,15)h} = I_{10h}^{[0,15)h}$ is the MOD natural neutrosophic idempotent.

$I_{2h}^{[0,15)h} \times I_{7.5}^{[0,15)h} = I_{0}^{[0,15)h}$ and $I_{8h}^{[0,15)h} \times I_{1.875}^{[0,15)h} = I_{0}^{[0,15)h}$ are some MOD pseudo zero divisor-non zero divisor pairs.

For $2h \times x \neq 0$ for any $x \in Z_{15}h$, $4h \times x \neq 0$ for any $x \in Z_{15}h$ and so on.

Now $I_{1.5h}^{[0,15)h} \times I_{10h}^{[0,15)I} = I_{0}^{[0,15)h}$ and so on.

$I_{2.5}^{[0,15)h} \times I_{4h}^{[0,15)h} = I_{10}^{[0,15)h}$ is the MOD pseudo idempotent non zero divisor pair.

This is the way operations are performed on ${}^{I}[0,n)h$.

Example 3.14: Let ${}^{I}[0,12)h = \{[0,12)h,\ I_t^{[0,12)h}\ ;\ t \in [0,12)h\}$ be the MOD natural neutrosophic zero divisor, pseudo divisors pairs.

Working with them is a matter of routine so left as an exercise to the reader, however some examples are given.

$I_{4.5h}^{[0,12)I} \times I_{2h}^{[0,12)h} = I_{9h}^{[0,12)h}$ is the pseudo zero divisor-idempotent pair $I_{5h}^{[0,12)h} \times I_{1.2h}^{[0,12)h} = I_{6h}^{[0,12)h}$ is a pseudo non zero divisor-nilpotent of order two pair.

$I_{5h}^{[0,12)h} \times I_{2.4h}^{[0,12)h} = I_{0}^{[0,12)h}$ is the pseudo zero divisor non zero divisor pair.

Since 5h is a nonzero divisor of Z_{12h}.

Thus we can get all such types of special MOD natural neutrosophic elements in ${}^{I}[0,12)h$.

Such in this direction is also a matter of routine so left as an exercise to the reader.

Next we proceed onto study MOD natural neutrosophic special quasi dual number elements of ${}^{I}[0,n)k$; $k^2 = (n-1)k$ by the following examples.

Example 3.15: Let ${}^{I}[0,10)k = \{[0,10)k,\ I_t^{[0,10)k}\ ;\ t \in [0,10)\ k\}$ be the collection of all MOD natural neutrosophic special quasi dual number elements.

Let $I_{2.5k}^{[0,10)1} \times I_{4k}^{[0,10)k} = I_0^{[0,10)k}$ and

$I_{1.25k}^{[0,10)k} \times I_{4k}^{[0,10)k} = I_{5k}^{[0,10)k}$ is a MOD natural neutrosophic pseudo zero divisor-idempotent pair.

$$I_{4k}^{[0,10)k} \times I_{4k}^{[0,10)k} = I_{4k}^{[0,10)k}.$$

4k is also an idempotent of $Z_{10}k$; further $4k \in Z_{10}k$ is also a zero divisor as $4k \times 5k = 0 \pmod{10}$.

Study in this type is interesting and innovative and left as an exercise to the reader.

Next we proceed onto study MOD natural neutrosophic numbers in MOD planes.

Consider $R_n(m)$; $R_n([0,m))$; $2 \leq m < \infty$ to be the MOD real number plane.

We have MOD natural neutrosophic elements associated with it.

Consider $Z_{10} \times Z_{10} = \{(a, b) / a, b \in Z_{10}\}$.

All natural neutrosophic numbers associated with $Z_{10} \times Z_{10}$ by convention we consider it as a matrix and work.

We cannot get any planes out of it. This study is done in Chapter I and II of this book.

However $R_n(m)$ is a MOD plane. So we define all, $I_{(a,0)}^{R_n(m)}$ and $I_{(0,a)}^{R_n(m)}$ where $a \in [0, m)$ also MOD natural neutrosophic elements.

We will illustrate this situation by some examples.

Example 3.16: Let $R_n(4)$ be the MOD neutrosophic real plane.

$I(R_n(4)) = \{R_n(4),\ I_{(a,0)}^{[0,4)},\ I_{(0,a)}^{[0,4)}, I_{(2.2)}^{[0,4)}, I_{(0,0)}^{[0,4)}\ a \in [0,4)\}$ is the set of all MOD natural neutrosophic elements.

Thus the collection is an infinite collection.

Example 3.17: Let $R_n(5)$ be the MOD neutrosophic real plane built using [0,5). The set of all MOD natural neutrosophic elements are

$$I(R_n(5)) = \{R_n(5),\ I_{(0,a)}^{R_n(5)}, I_{(a,0)}^{R_n(5)}, I_{(2.5,2.5)}^{R_n(5)}, I_{(2.2)}^{R_n(g)}, I_{(2.5,2)}^{R_n(5)},$$

$$I_{(2,2.5)}^{R_n(5)}, I_{(1.25,4)}^{R_n(5)}, I_{(4,1.25)}^{R_n(5)}, I_{(4,4)}^{R_n(5)}, I_{(1.25,1.25)}^{R_n(5)} \text{ and so on } a \in [0, 4)\}.$$

We see some are MOD natural neutrosophic zero divisors. Some contribute to the MOD natural neutrosophic pseudo zero divisor-unit pair also.

Example 3.18: Let

$$I(R_n(6)) = \{R_n(6),\ I_{(a,0)}^{R_n(6)}, I_{(0,a)}^{R_n(6)},\ I_{(2,3)}^{R_n(6)}, I_{(3,3)}^{R_n(6)}, I_{(3,2)}^{R_n(6)},$$

$$I_{(2,2)}^{R_n(6)}, I_{(4,2)}^{R_n(6)}, I_{(2,3)}^{R_n(6)},\ I_{(4,4)}^{R_n(6)}, I_{(2,4)}^{R_n(6)}, I_{(1.5,1.5)}^{R_n(6)},\ I_{(3,1.5)}^{R_n(6)}, I_{(1.5,3)}^{R_n(6)}\}$$

are the MOD natural neutrosophic zero divisors, some of which are pseudo zero divisor-idempotents.

Study in this direction is interesting and left as exercise for the reader.

Example 3.19: Let $I(R_n(15)) = \{R_n(15),\ I_{(a,0)}^{R_n(15)}, I_{(0,a)}^{R_n(15)}, I_{(7.5,7.5)}^{R_n(15)}, I_{(7.5,2)}^{R_n(15)}, I_{(2,7.5)}^{R_n(15)},\ I_{(2,2)}^{R_n(15)}, I_{(7.5,4)}^{R_n(15)}, I_{(4,7.5)}^{R_n(15)}, I_{(5,5)}^{R_n(15)},\ I_{(7.5,8)}^{R_n(15)}, I_{(8,7.5)}^{R_n(15)}, I_{(4,4)}^{R_n(15)},\ I_{(10,10)}^{R_n(15)}, I_{(8,8)}^{R_n(15)}, I_{(12,12)}^{R_n(15)}, I_{(3,9)}^{R_n(15)}, I_{(6,6)}^{R_n(15)}, I_{(6,9)}^{R_n(15)}, I_{(9,6)}^{R_n(15)}, \ldots$ and so on$\}$ be the MOD natural neutrosophic elements.

This has MOD natural neutrosophic pseudo zero divisors-unit pairs and MOD-natural neutrosophic pseudo zero divisor-idempotent pairs apart from MOD natural neutrosophic zero divisors and idempotents.

Example 3.20: Let $I(R_n(7)) = \{R_n(7),\ I_{(a,0)}^{R_n(7)}, I_{(0,a)}^{R_n(7)}, I_{(2,3.5)}^{R_n(7)}, I_{(3.5,2)}^{R_n(7)}, I_{(2,2)}^{R_n(7)}, I_{(3.5,3.5)}^{R_n(7)},\ I_{(1.75,4)}^{R_n(7)}, I_{(4,1.75)}^{R_n(7)}, I_{(4,4)}^{R_n(7)}, I_{(1.75,1.75)}^{R_n(7)}$ and so on $a \in R_n(7)\}$ are the MOD natural neutrosophic zero divisors. MOD neutrosophic natural pseudo zero divisor unit pair.

We see $I(R_n(7))$ has no MOD natural neutrosophic idempotents or nilpotents.

Study of $I(R_n(m))$; m a prime happens to give only pseudo zero divisors unit pairs.

There are no idempotents or nilpotents so study in this direction can lead to more research.

Next we proceed onto study the MOD natural neutrosophic elements of MOD-finite complex modulo integer planes by examples.

Example 3.21: Let $I(C^I(9)) = \{C^I(9),\ I^{C^I(9)}_{4.5+4.5i_F}, I^{C^I(9)}_{2+2i_F}$ and so on$\}$ be the MOD natural neutrosophic finite complex modulo integer elements.

Study in this direction is a matter of routine.

For study in this direction is a matter of routine and hence left as an exercise to the reader.

Example 3.22: Let $I(C^I(16)) = \{C^I(16),\ I^{C^I(16)}_{5+5i_F}, I^{C^I(16)}_{3.2i_F}, I^{C^I(16)}_{3.2+3.2i_F}, I^{C^I(16)}_{1.6+1.6i_F}\ I^{C^I(16)}_{10+10i_F}$ and so on$\}$ are the MOD natural neutrosophic finite complex modulo integer elements.

This has pseudo zero divisors, zero divisors and nilpotents as MOD natural neutrosophic finite complex modulo integers.

The rest of the study is left as an exercise to the reader.

Example 3.23: Let $I(C^I(24))$ be the collection of all MOD natural neutrosophic finite complex modulo integer elements. $I(C^I(24))$ has MOD natural neutrosophic zero divisors, idempotents, nilpotents apart from MOD natural neutrosophic pseudo zero divisor unit pair and MOD natural neutrosophic pseudo zero divisor idempotent pair.

Study in this direction is considered as a matter of routine so left as an exercise to the reader.

Next we proceed on to describe by examples the MOD natural neutrosophic elements of dual number planes.

Example 3.24: $I(\langle[0,3 \cup g\rangle_I\rangle = \{\langle[0,3) \cup g\rangle_I,\ I^{(\langle[0,3)\cup g\rangle_I)}_{(1.5,2)},\ I^{(\langle[0,3)\cup g\rangle_I)}_{(2,2)},\ I^{(\langle[0,3)\cup g\rangle_I)}_{(2,1.5)},\ I^{(\langle[0,3)\cup g\rangle_I)}_{(1.5,1.5)}, I^{\langle[0,3)\cup g\rangle_I}_{(0,ag)},\ I^{(\langle[0,3)\cup g\rangle_I)}_{(a,0)}$ and so on$\}$ is the MOD natural neutrosophic dual number zero divisors, and pseudo zero divisors, unit pairs.

Since 3 is a prime there does not exist any MOD natural neutrosophic dual number idempotents so the question of MOD natural neutrosophic dual number zero divisor idempotents does not arise.

Example 3.25: Let $I(\langle[0,13) \cup g\rangle_I) = \{\langle[0,13) \cup g\rangle_I$, $I_{(6.5+6.5g)}^{\langle[0,13)\cup g\rangle}, I_{(a,0)}^{\langle[0,13)\cup g\rangle}$, $I_{(0,ag)}^{\langle[0,13)\cup g\rangle}, I_{(2+2g)}^{\langle[0,13)\cup g\rangle}$ and so on$\}$ be the collection of MOD natural neutrosophic dual number zero divisors, pseudo zero divisors unit pair.

We are not in a position to confirm whether there can be MOD natural neutrosophic dual number idempotents.

Apart from this the study is a matter of routine and is left as an exercise to the reader.

Next we proceed onto study the MOD natural neutrosophic special dual like number elements of the MOD natural neutrosophic special dual like number plane by some examples.

Example 3.26: Let $I(\langle[0,10) \cup h\rangle_I) = \{\langle[0,10) \cup h\rangle_I$, $I_{5+5h}^{\langle[0,10)\cup h\rangle}$, , $I_{2+2h}^{\langle[0,10)\cup h\rangle}, I_{(4+4h)}^{\langle[0,10)\cup h\rangle}$, $I_{(8+8h)}^{\langle[0,10)\cup h\rangle}$ and so on$\}$ are the MOD natural neutrosophic special dual like number zero divisors, idempotent etc.

Study in this direction is a matter of routine so left as an exercise to the reader.

Example 3.27: Let $I(\langle[0,23) \cup h\rangle_I) = \{\langle[0,23) \cup h\rangle_I$, $I_{(0,ah)}^{\langle[0,23)\cup h\rangle}, I_{(11.5,11.5h)}^{\langle[0,23)\cup h\rangle}, I_{(a,0)}^{\langle[0,23)\cup h\rangle}$, $I_{2+2h}^{\langle[0,23)\cup h\rangle}, I_{(5.75+5.75h)}^{\langle[0,23)\cup h\rangle}$, $I_{(4+4h)}^{\langle[0,23)\cup h\rangle}$, $I_{(10+10h)}^{\langle[0,23)\cup h\rangle}$, $I_{(2.3+2.3h)}^{\langle[0,23)\cup h\rangle}, I_{8+8h}^{\langle[0,23,\cup h\rangle}$, $I_{(2.875+2.875h)}^{\langle[0,23)\cup h\rangle}$ and so on$\}$ be the MOD natural neutrosophic special dual like number elements of $I(\langle[0,23) \cup h\rangle_I)$.

Next we proceed onto study the MOD natural neutrosophic special quasi dual number elements of $I(\langle[0,15)\cup k\rangle_I$ by some examples.

Example 3.28: Let $I(\langle[0,15) \cup k\rangle_I) = \{\langle[0,15) \cup k\rangle_I$, $I_{7.5+7.5k}^{\langle[0,15)\cup k\rangle}, I_{2+2k}^{\langle[0,15)\cup k\rangle}, I_{4+4k}^{\langle[0,5)\cup k\rangle}, I_{8+8k}^{\langle[0,15)\cup k\rangle}, I_{(3.75+3.75)}^{\langle[0,15)\cup k\rangle}, I_{(12+12k)}^{\langle[0,15)\cup k\rangle}$, and so on$\}$ be the collection of all MOD natural neutrosophic special quasi dual numbers.

Study in this direction is innovative.

$I_{3.75+3.75k}^{\langle[0,15)\cup h\rangle} \times I_{8}^{\langle[0,15)\cup k\rangle} = I_{0}^{\langle[0,15)\cup k\rangle}$ is the MOD natural neutrosophic special quasi dual number pseudo zero divisor unit pair.

Study in this direction is interesting and important.

Since all these study is a matter of routine we leave it as an exercise to the reader.

Now we proceed onto study the notion of MOD natural neutrosophic matrices built using $R_n(n)$, $C_n(m)$, $R^I(m)$, $R_n^g(m)$, $R_n^k(m)$ and $R_n^h(m)$; $2 \leq m < \infty$.

All these will be illustrated by some examples.

Example 3.29: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} / a_i \in I(R(9)); 0 \leq i \leq 12\right\}$$

be the MOD natural neutrosophic matrix with entries from I(R(9)).

i) This M is only semigroup under +.
ii) M is a semigroup under $\times_n$, the natural product.

iii) M has idempotents with respect to + and $\times_n$.
vi) M is of infinite order.

The verification is a matter of routine.

Example 3.30: Consider

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \Bigg/ a_i \in I(R_n(19))\ 1 \leq i \leq 6, \times_n \right\}$$

be the MOD natural neutrosophic real matrix semigroup. This has many MOD natural neutrosophic pseudo zero divisor-unit pairs.

Under the operation '+', B is only a semigroup as B has matrices x such that x + x = x.

Example 3.31: Let W = {(a_1, a_2, a_3) / $a_i \in$ I{C(81)); $1 \leq i \leq 3$} be the MOD complex finite modulo integer neutrosophic matrix semigroup both under + and ×.

Clearly x = $(I_9^{C(81)}, I_{40.5}^{C(81)}, I_{20.25}^{C(81)}) \in$ W is such that x + x = x so W is a semigroup under +.

Consider y = $(I_9^{C(81)}, I_2^{C(81)}, I_4^{C(81)}) \in$ W; we see x × y = $(I_0^{C(81)}, I_0^{C(81)}, I_0^{C(81)})$ is a pseudo zero divisor.

Let x = $(I_{20.25}^{C(81)}, I_{8.1}^{C(81)}, I_{16.2}^{C(81)})$ and y = $\{I_{4+4i_F}^{C(81)}, I_{10i_F}^{C(81)}, I_{5i_F}^{C(81)}) \in$ W is such that x × y = $(I_0^{C(81)}, I_0^{C(81)}, I_0^{C(81)})$ is a MOD natural neutrosophic complex finite modulo integer interval matrix pseudo divisor non zero divisor pair.

Study in this direction is also considered as a matter of routine so left as an exercise to the reader.

Example 3.32: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} / a_i \in I(C(43)), 1 \leq i \leq 8\right\}$$

be the MOD natural neutrosophic finite complex modulo integer matrix semigroup.

We have x ∈ S such that x + x = x.

Clearly S has matrices a, b such that a $\times_n$ b is a MOD natural neutrosophic zero divisor so the pair (a, b) is a MOD natural neutrosophic finite complex modulo integer matrix pseudo zero divisor unit pair. These are infact several such pairs.

The rest of the work is left as an exercise to the reader.

Let

$$a = \begin{bmatrix} I^{C(43)}_{21.5+21.5i_F} & I^{C(43)}_{10.75} \\ I^{C(43)}_{4.3+4.3i_F} & I^{C(43)}_{8.6+8.6i_F} \\ I^{C(43)}_{5.375} & I^{C(43)}_{21.5+10.75} \\ I^{C(43)}_{8.6+8.6i_F} & I^{C(43)}_{8.6+4.3i_F} \end{bmatrix} \text{ and } b = \begin{bmatrix} I^{C(43)}_{2} & I^{C(43)}_{4} \\ I^{C(43)}_{10} & I^{C(43)}_{5} \\ I^{C(430}_{8} & I^{C(43)}_{4} \\ I^{C(43)}_{5} & I^{C(43)}_{10} \end{bmatrix} \in S$$

is such that

$$a \times_n b = \begin{bmatrix} I^{C(43)}_{0} & I^{C(43)}_{0} \\ I^{C(43)}_{0} & I^{C(43)}_{0} \\ I^{C(43)}_{0} & I^{C(43)}_{0} \\ I^{C(43)}_{0} & I^{C(43)}_{0} \end{bmatrix}$$

so (a,b) is a MOD natural neutrosophic finite complex modulo integer matrix pseudo zero divisor-unit pair.

Example 3.33: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle/ a_i \in I(\langle[0,10 \cup\rangle_I; 1 \leq i \leq 10\} \right.$$

be the MOD natural neutrosophic neutrosophic matrix.

Clearly W has MOD natural neutrosophic-neutrosophic matrix.

Further W has MOD natural neutrosophic-neutrosophic idempotent matrices under +.

Infact W has MOD natural neutrosophic-neutrosophic pseudo zero divisor-unit pair and MOD natural neutrosophic pseudo zero divisor-idempotent pair.

Reader is left with the task of finding such elements from W.

Example 3.34: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle/ a_i \in I(\langle[0,19) \cup g\rangle_I); 1 \leq i \leq 9\} \right.$$

be the MOD natural neutrosophic dual number plane matrices.

Clearly {P, +} is only a semigroup under + for P has every matrices A such that A + A = A.

Further {P, $\times_n$} is a semigroup and this has pseudo zero divisor-unit pairs.

Example 3.35: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle/ a_i \in I(\langle[0,24) \cup g\rangle_I);\ 1 \leq i \leq 7 \right\}$$

be the MOD natural neutrosophic dual number matrices M has MOD natural neutrosophic dual number idempotents with respect to +.

Further $\{M, \times_n\}$ is an infinite MOD natural neutrosophic dual number matrix semigroup under natural product $\times_n$.

We see M has several MOD natural neutrosophic pseudo zero divisor-unit pairs as well as MOD natural neutrosophic pseudo zero divisors-idempotent pairs of matrices under the natural product $\times_n$.

Next we give examples of MOD natural neutrosophic matrices with entries from $I(\langle[0,n) \cup h\rangle_I)$ the MOD natural neutrosophic special dual like number.

Example 3.36: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle/ a_i \in (\langle[0,20) \cup h\rangle_I);\ 1 \leq i \leq 21 \right\}$$

be the MOD natural neutrosophic special dual like number matrices.

P is only a semigroup under addition as P contains elements x such that x + x = x.

P is again a MOD natural neutrosophic semigroup of matrices under the natural product $\times_n$.

Also P has MOD natural neutrosophic special dual like number pseudo zero divisor-idempotent pairs and many or infact an infinite collection of MOD natural neutrosophic pseudo zero divisors.

Example 3.37: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in I(\langle [0,29) \cup h \rangle_I); \; 1 \leq i \leq 21 \right\}$$

be the MOD natural neutrosophic special dual like number matrices.

(S, +) is only a semigroup and in fact, S has several x such that x + x = x.

(S, ×) is also only a commutative semigroup which has many natural neutrosophic special dual like number pseudo zero divisors unit pairs.

Such elements can be found by the interested reader.

Example 3.38: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \\ a_{15} & a_{16} \end{bmatrix} \in I(\langle[0,43) \cup h\rangle_I),\ 1 \leq i \leq 16\right\}$$

be the MOD natural neutrosophic special dual like number matrices M has several MOD natural neutrosophic idempotents under +.

M has also several MOD natural neutrosophic pseudo zero divisor-units pair and pseudo zero divisor-idempotent pairs.

Finding these elements are left as an exercise to the reader.

Next we proceed onto study MOD natural neutrosophic special quasi dual number matrix semigroups under + and under $\times_n$ by some examples.

Example 3.39: Let

$$B = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \Bigg/\ a_i \in I(\langle[0,19) \cup k\rangle_I;\ 1 \leq i \leq 18\right\}$$

be the MOD natural neutrosophic special quasi dual number matrices.

$\{B, \times_n\}$ is an infinite semigroup and has MOD natural neutrosophic pseudo zero divisor unit pair under natural product $\times_n$.

$\{B, +\}$ is an infinite semigroup under + and has MOD-natural neutrosophic idempotents under +.

Finding them is a matter of routine and this task is left as an exercise to the reader.

Example 3.40: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} / a_i \in I(\langle[0,22) \cup k\rangle_I);\ 1 \leq i \leq 8\right\}$$

be the MOD natural neutrosophic special quasi dual number matrices.

$\{S, +\}$ is a semigroup of infinite order and $a + a = a$ for many $a \in S$.

$\{S, \times_n\}$ is a commutative semi group of infinite order which has MOD natural neutrosophic special quasi dual number pseudo zero divisor-unit pair and pseudo zero divisor- idempotent pair.

Finding such elements are left as an exercise to the reader.

Next we proceed onto just describe how MOD natural neutrosophic coefficient polynomials are defined by some examples.

Example 3.41: Let

$$L = \left\{\sum_{i=0}^{\infty} a_i x^i / a_i \in I(R_n(10))\right\}$$

be the MOD natural real neutrosophic coefficient polynomials. (L, +) is only a semigroup of infinite order.

These semigroups have several idempotents of natural real neutrosophic number under +.

Let p(x) = $I_2^{I(R_n(10))} + I_5^{I(R_n(10))} x^3 \in L$; p(x) + p(x) = p(x) so p(x) is a natural neutrosophic idempotent polynomial.

The reader is left with the task of finding them (L, ×) is a semigroup having zero divisors and neutrosophic zero divisors.

Example 3.42: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(C_n(24))\right\}$$

be the MOD natural real neutrosophic finite complex modulo integer coefficient polynomials.

The semigroup {S, +) has infinite number of natural neutrosophic idempotents.

(S, ×) the semigroup has several MOD natural neutrosophic pseudo zero divisor-unit pair.

This is left as an exercise to the reader.

Example 3.43: Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0, 7) \cup I\rangle_I)\right\}$ be the collection of all MOD natural neutrosophic neutrosophic coefficient polynomials.

{S, +} is a semigroup with infinite number of MOD natural neutrosophic neutrosophic idempotent polynomials.

(S, ×) is the MOD natural neutrosophic-neutrosophic infinite polynomial semigroup which has pseudo zero MOD natural neutrosophic-neutrosophic zero divisors.

Next we describe by examples other MOD natural neutrosophic polynomials with coefficients form. $I(\langle[0, n) \cup g\rangle_I$, $I(\langle[0, n) \cup k\rangle_I$ and $I(\langle[0,n) \cup h\rangle_I$.

Example 3.44: Let

$$M = \left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,5) \cup g\rangle_I)\right\}$$

be the MOD natural real neutrosophic dual number coefficient polynomials.

This has infinite number of MOD natural neutrosophic idempotents under + and has infinite number of natural neutrosophic zero divisors under ×.

Further study on these semigroups of polynomials is left as an exercise to the reader.

Next we briefly give examples of MOD natural neutrosophic special dual like number coefficient polynomial semigroups under + and × and MOD natural neutrosophic special quasi dual number coefficients polynomial semigroups under + and ×.

Example 3.45: Let

$$P = \left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,9) \cup h\rangle_I)\right\}$$

be the MOD natural neutrosophic special dual like number coefficient polynomials.

(P, +) is an infinite MOD natural neutrosophic special dual like number coefficient polynomial semigroup which has infinite number of idempotents under +.

{P, ×} is an infinite commutative MOD natural neutrosophic special dual like number coefficient polynomial semigroup which has infinite number of MOD natural neutrosophic zero divisor and pseudo zero divisor polynomials.

Example 3.46: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;/\; a_i \in I(\langle[0,13) \cup k\rangle_I \right\}$$

be the collection of all MOD natural neutrosophic special quasi dual number of coefficient polynomials

{S, +} is an infinite MOD natural neutrosophic special quasi dual number polynomial semigroups which has infinite number of idempotents under +.

{S, ×} is the MOD natural neutrosophic special quasi dual number coefficient polynomial semigroup which has infinite number of zero divisors and pseudo zero divisors.

Finding all these are left as exercise for the reader.

However we suggest many problems in this direction to the reader.

Problems.

1. Given the MOD interval [0,n) find all pseudo divisors and MOD natural neutrosophic zero divisors.

2. Study question (1) in case of [0,17).

3. Study question (1) in case of [0,16).

4. Study question (1) in case of [0, 18).

5. Compare the solutions in problems (2), (3) and (4).

6. Can we say [0, n) if n is a prime has more number of MOD natural neutrosophic pseudo zero divisor-unit pairs?

7. Compare the MOD natural neutrosophic elements in [0, n –1), [0, n) and [0, n + 1) where n is a prime.

   i) Which MOD interval has maximum number of MOD natural neutrosophic elements?
   ii) Which MOD interval has the minimum number of MOD natural neutrosophic elements?

8. Let [0, 20) be the MOD real interval I[0, 20)) be the collection of all MOD natural neutrosophic elements of [0, 20).

   i) Prove {I([0, 20)), +} is a semigroup.
   ii) Is the number of MOD natural neutrosophic idempotents in I([0, 20)) infinite in number?
   iii) Can {I([0, 20)), +} have subsemigroups of finite order?
   iv) Find all special features enjoyed by I([0, 20)).
   v) Prove P = {([0, 20)), ×} is an infinite order semigroup.
   vi) Can the collection of all MOD natural neutrosophic elements of P form a subsemigroup?
   vii) Can ideals of P be finite?
   viii) Is P a S-semigroup?
   ix) What are the special features enjoyed by P?
   x) Find the number of MOD natural neutrosophic pseudo zero divisor-unit pairs of P.
   xi) Find all the MOD natural neutrosophic pseudo zero divisor-idempotent pairs of P?
   xii) Find all MOD natural neutrosophic pseudo zero divisors.
   xiii) Find all MOD natural neutrosophic pseudo idempotents.

9. Let ${}^{I}[0, 24)$ be the collection of all MOD real natural neutrosophic elements associated with ${}^{I}[0,24)$.

   Study questions (i) to (xiii) of problem 8 for this ${}^{I}[0,24)$.

10. Let ${}^{I}[0, 47)$ be the MOD real natural neutrosophic elements.

    Study questions (i) to (xiii) of problem 8 for this I([0, 47)).

11. Compare the properties enjoyed by I([0, 20)), I([0, 24)), I([0, 4)) in problems 8, 9 and 10 respectively.

12. Let ${}^{I}[0,14)I$ be the MOD natural neutrosophic- neutrosophic elements.

    Study questions (i) to (xiii) of problem (8) for this I([0, 14)I).

13. Let ${}^{I}[0, 13)I$ be the MOD natural neutrosophic neutrosophic elements.

    Study questions (i) to (xiii) of problem (8) for this ${}^{I}[0, 13)I$.

14. Let ${}^{I}[0, 12)I$ be the MOD natural neutrosophic neutrosophic elements.

    Study questions (i) to (xiii) of problem (8) for this ${}^{I}[0, 12)I$.

15. Compare ${}^{I}[0, 14)I$, ${}^{I}[0, 13)I$ and ${}^{I}[0, 12)I$ of problems 12, 13 and 14 respectively with each other.

16. Prove ${}^{I}[0, n)i_F$ ($i_F^2 = n - 1$) $2 \leq n < \infty$ the MOD natural neutrosophic finite complex modulo integers is not closed under $\times$.

17. Find the special features enjoyed by ${}^{I}[0, 17)i_F$ under +.

18. Let $^I[0, 9)g$ be the collection of all MOD natural neutrosophic dual number elements.

    Study questions (i) to (xiii) of problem (8) for this $^I[0, 29)g$.

19. Compare $^I[0, 29)g$ with $I([0, 30)g)$ and study questions (i) to (xiii) of problem (8) for this $^I[0, 30)g$.

20. Let $^I[0, 48)h$ be the collection of all MOD natural neutrosophic special dual like numbers.

    Study questions (i) to (xiii) of problem (8) for this $^I[0, 48)h$.

21. Let $^I[0, 49)h$ be the collection of all MOD natural neutrosophic special dual like numbers.

    Study questions (i) to (xiii) of problem (8) for this $^I[0, 49)h$.

22. Let $^I[0, 47)h$ be the collection of all natural neutrosophic special dual like numbers.

    Study questions (i) to (xiii) of problem (8) for this $^I[0, 47)h$.

23. Study problems (20), (21) and (22) compare them.

24. Let $^I[0, 52)k$ be the collection of all MOD natural neutrosophic special quasi dual number elements.

    Study questions (i) to (xiii) of problem (8) for this $^I[0, 52)k$.

25. Let $^I[0, 53)k$ be the collection of all MOD natural neutrosophic special quasi dual number elements.

Study questions (i) to (xiii) of problem (8) for this ${}^{I}[0, 53)k$.

26. Let ${}^{I}[0, 54)k$ be the collection of all MOD natural neutrosophic special quasi dual number elements.

    Study questions (i) to (xiii) of problem (8) for this ${}^{I}[0, 54)$ k.

27. Compare results in problems (24), (25) and (26) of ${}^{I}[0, 52)k$, ${}^{I}[0, 53)k$ and ${}^{I}[0, 54)k$ respectively.

28. Let $M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle/ a_i \in I([0, 47); 1 \leq i \leq 5, \times_n\right\}$ be the MOD

    natural neutrosophic matrix semigroup under $\times_n$.

    i) Prove M is of infinite order.
    ii) Find subsemigroup of finite order.
    iii) Find ideals of M of M.
    iv) Prove all ideals of M are of infinite order.
    v) Show there are matrices which result in pseudo zero divisor-unit pair.
    vi) Find all MOD natural neutrosophic idempotents of any of M.
    vii) Enumerate all special features enjoyed by M.
    viii) Can M have matrices nilpotent of order two?

29. Let $B = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle/ a_i \in I([0, 273));\right.$

    $1 \leq i \leq 12, \times_n\}$

be the MOD natural neutrosophic matrix semigroup.

Study questions (i) to (viii) of problem (28) for this B.

30. Let W = $\{(a_1, a_2, a_3, a_4, a_5, a_6) / a_i \in I([0, 48)); 1 \leq i \leq 6, \times\}$ be the MOD natural neutrosophic semigroup.

Study questions (i) to (viii) of problem (28) for this W.

31. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix} / a_i \in I(C_n(10); 1 \leq i \leq 20, \times_n\right\}$.

Study questions (i) to (viii) of problem (28) for this S.

32. Prove $^I[0, n)$ under + is only a semigroup.

33. Can $^I[0, n)$ have infinite number of idempotents under +?

34. Prove $^I[0, n)$ under × is a semigroup of infinite order.

35. Can $^I[0, 17)$ have MOD natural neutrosophic pseudo zero divisor-idempotent pair?

36. Prove $^I[0, 17)$ has MOD natural neutrosophic pseudo zero divisor-unit pairs.

37. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix}\right.$ / $a_i \in I(C_n(28))$ $1 \leq i \leq 18, \times_n\}$ be

    the MOD natural neutrosophic semigroup under the natural product $\times_n$.

    Study questions (i) to (viii) of problem (28) for this V.

38. Prove $I(\langle 0, 47) \cup g\rangle_I)$ has MOD natural neutrosophic dual number pseudo zero divisor unit pairs.

39. Let S = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \\ a_{15} & a_{16} \end{bmatrix}\right.$ / $a_i \in I([0, 18) g)$ $g^2 = 0$, $1 \leq i \leq 16, \times_n\}$

    be the MOD natural neutrosophic dual number matrix semigroup.

    Study questions (i) to (viii) of problem (28) for this S.

40. Let B = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}\right.$ / $a_i \in$

    $I([0, 27)g)$; $g^2 = 0$, $1 \leq i \leq 14, \times_n\}$

be the MOD natural neutrosophic dual number matrix semigroup under natural product $\times_n$.

Study questions (i) to (viii) of problem (28) for this B.

41. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} \\ a_{29} & a_{30} & a_{31} & a_{32} \end{bmatrix} \right.$ / $a_i \in I([0, 43)\, h)$, $h^2 = h$; $1 \leq i \leq 32, \times_n\}$

be the MOD natural neutrosophic special dual like number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (viii) of problem (28) for this W.

42. Let $M = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{pmatrix} \right.$ / $a_i \in I([0, 56)\, h)$; $h^2 = h$, $1 \leq i \leq 15, \times_n\}$

be the MOD natural neutrosophic special dual like number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (viii) of problem (28) for this M.

43. Let y = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right.$ / a_i I([0, 24) k) $k^2 = 23k$;

$1 \leq i \leq 16, \times_n\}$

be the MOD natural neutrosophic special quasi dual number matrix semigroup under the natural product $\in_n$.

i) Study questions (i) to (viii) of problem (28) for this Y.
ii) Prove Y is not defined under the usual product $\times$.

44. Let I([0, 23)k), $k^2 = 22k$ be the MOD natural neutrosophic special quasi dual number interval.

Find all MOD natural neutrosophic pseudo zero divisor-unit elements.

45. Let S = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right.$ / $a_i \in$ I(R_n(29)) $1 \leq i \leq 6, \times_n\}$ be the MOD natural neutrosophic real matrix semigroup.

Study questions (i) to (viii) problem (28) for this S.

46. Let

B = $\left\{\begin{pmatrix} a_1 & a_2 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{24} \end{pmatrix}\right.$ / $a_i \in$ I(C_n^I (93));

$1 \leq i \leq 24, \times_n\}$

be the MOD natural neutrosophic complex modulo integer matrix semigroup under the natural product $\times_n$.

Study questions (i) to (viii) of problem (28) for this B.

47. Let

$$D = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \middle/ a_i \in I\left(\langle [0, 26) \cup I\rangle_I\right); 1 \leq i \leq 24, \times_n \right\}$$

be the MOD natural neutrosophic neutrosophic matrix semigroup under the natural product $\times_n$.

Study questions (i) to (viii) of problem (28) for this D.

48. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle/ a_i \in \right.$

$\left. I(\langle [0, 43) \cup g\rangle_I]; 1 \leq i \leq 16, \times_n \right\}$

be the MOD natural neutrosophic dual number matrix semigroup under $\times_n$, the natural product.

Study questions (i) to (viii) of problem (28) for this B.

49. Let $Z = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix} \middle/ a_i \in (\langle[0, 47) \cup h\rangle_I); 1 \leq i \leq 18, \times_n\right\}$

    be the MOD natural neutrosophic special dual like number matrix semigroup under the natural product $\times_n$.

    Study questions (i) to (viii) of problem (28) for this Z.

50. Let $S = \left\{\begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{12} \\ a_{13} & a_{14} & a_{15} & \ldots & a_{24} \end{pmatrix} \middle/ a_i \in I(\langle[0, 23) \cup k\rangle_I); k^2 = 22k; 1 \leq i \leq 24, \times_n\right\}$

    be the MOD natural neutrosophic special quasi dual number matrix semigroup.

    Study questions (i) to (viii) of problem (28) for this S.

51. Let $P = \left\{\sum_{i=0}^{\infty} a_i x^i \;/\; a_i \in I([0,27))\right\}$ be the MOD natural neutrosophic coefficient polynomial.

    i) Study (P, +) for MOD natural neutrosophic idempotents ideals and subsemigroups
    ii) Study (P, ×) for MOD natural neutrosophic pseudo zero divisor unit pairs and pseudo zero divisor-idempotent pairs.
    iii) Study (P, ×} for any other special property.

52. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(R_n(33)\right\}$ be the MOD natural neutrosophic real pair coefficient polynomial.

    Study questions (i) to (iii) of problem (51) for this S.

53. Let B = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,293)\ I\rangle)\right\}$MOD natural neutrosophic-neutrosophic coefficient polynomial.

    Study questions (i) to (iii) of problem (51) for this B.

54. Let M = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(R^I(9))\right\}$ MOD natural neutrosophic-neutrosophic coefficient polynomial.

    Study questions (i) to (iii) for this M.

55. Let D = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,129)\ h\rangle_I)\right\}$ be the MOD natural neutrosophic special dual like number coefficient polynomial semigroup.

    Study questions (i) to (iii) of problem (51) for this D.

56. Let V = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,43)\ k\rangle_I)\right\}$ be the MOD natural neutrosophic special quasi dual number coefficient polynomial semigroup.

    Study questions (i) to (iii) problem (51) for this V.

57. Let B = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,28) \cup I\rangle_I)\right\}$ be the MOD natural neutrosophic-neutrosophic coefficient polynomial.

    Study questions (i) to (iii) of problem (51) for this B.

58. Let A = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,123) \cup g\rangle_I)\right\}$ be the MOD natural neutrosophic dual number coefficient polynomial.

    Study questions (i) to (iii) of problem (51) for this A.

59. Let E = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,197) \cup h\rangle_I)\right\}$ be the MOD natural neutrosophic special dual like number coefficient polynomial.

    Study questions (i) to (iii) of problem (51) for this E.

60. Let F = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,293) \cup k\rangle_I)\right\}$ be the MOD natural neutrosophic quasi dual number coefficient polynomial.

    Study questions (i) to (iii) of problem (51) for this F.

61. Study all special and distinct features associated with S = $\left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,215) \cup h\rangle_I)\right\}$ be the MOD natural neutrosophic special dual like number coefficient polynomial.

i) Is (S, ×) a S-semigroup?
ii) Can (S, ×) have ideals of finite order?
iii) Can S have MOD natural neutrosophic pseudo zero divisor-unit pair?
iv) Can S have MOD natural neutrosophic pseudo zero divisor idempotent pair?
v) Can S have S-ideals?
vi) Can S have S-MOD natural neutrosophic idempotents?

62. Study questions (i) to (vi) of problem (61) for the

$$B = \left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(C^I(251)), \times\right\}$$

63. Let $M = \left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,433) \cup I\rangle_I), \times\right\}$ be the MOD natural neutrosophic-neutrosophic coefficient polynomial semigroup.

Study questions (i) to (vi) of problem (61) for this M.

64. Let $G = \left\{\sum_{i=0}^{\infty} a_i x^i \,/\, a_i \in I(\langle[0,273) \cup g\rangle_I), \times\right\}$ be the MOD natural neutrosophic dual number coefficient polynomial semigroup.

i) Study questions (i) to (vi) of problem (61) for this G.
ii) Obtain any other special feature associated with G.

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book the notion of semigroups under + is constructed using the MOD natural neutrosophic integers or MOD natural neutrosophic-neutrosophic numbers or MOD natural neutrosophic finite complex modulo integer or MOD natural neutrosophic dual number integers or MOD natural neutrosophic special dual like number or MOD natural neutrosophic special quasi dual numbers are analysed in a systematic way.

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