Smarandache BE-Algebras

$$P_{n-1} \subset P_{n-2} \subset \ldots \subset P_2 \subset P_1 \subset S$$

$$\{w_{n-1}\} > \{w_{n-2}\} > \ldots > \{w_2\} > \{w_1\} > \{w_0\}$$

$$P_{n-1} \subset P_{n-2} \subset \ldots \subset P_2 \subset P_1 \subset S$$

$$\{w_{n-1}\} < \{w_{n-2}\} < \ldots < \{w_2\} < \{w_1\} < \{w_0\}$$

Arsham Borumand Saeid

Arsham Borumand Saeid

Smarandache BE-Algebras

Education Publisher

2013

Education Publisher
1313 Chesapeake Avenue
Columbus, Ohio 43212
USA
Tel. (614) 485-0721

Peer Reviewers:
Dr. Linfan Mao, Institute of Systems Science, Academy of Mathematics and Systems, Chinese Academy of Sciences, Beijing 100080, P. R. China.
Prof. Xingsen Li, School of Management, Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, P. R. China.
Dr. Rajesh Singh, Department of Statistics, Faculty of Science, Banaras Hindu University, Varanasi (U.P.), India, PIN - 221005.

EAN: 9781599732411
ISBN: 978-1-59973-241-1

Dedicated to the memory of my father

Dedicated to my mother

Dedicated to my wife

Arsham Borumand Saeid

CONTENTS

Preface

In this book, we study the notions of Smarandache n-structure CI-algebras and Smarandache weak BE-algebras.

Smarandache algebraic structures have been studied in a series of eleven books by W. B. Vasantha Kandasamy, one book by Jaiyeola Temitope Gbolahan, and tens of more books by W. B. Vasantha Kandasamy and Florentin Smarandache, including the neutrosophic algebraic structures as well. The study of Smarandache algebraic structures has caused a shift of paradigm in the study of algebraic structures. Smarandache algebraic structures simultaneously analyses two or more distinct algebraic structures.

There are three types of such structures: a) Smarandache Strong Structures, b) Smarandache Weak Structures), and c) Smarandache Strong-Weak Structures.

A Smarandache Strong Structure on a set S means a structure on S that has a proper subset P with a stronger structure.

By proper subset of a set S, we mean a subset P of S, different from the empty set, from the original set S, and from the idempotent elements if any.

In any field, a Smarandache strong n-structure on a set S means a structure $\{w_0\}$ on S such that there exists a chain of proper subsets

$$P_{n-1} < P_{n-2} < \quad < P_2 < P_1 < S$$

where $' <'$ means 'included in', whose corresponding structures verify the inverse chain

$$\{w_{n-1}\} > \{w_{n-2}\} > \quad > \{w_2\} > \{w_1\} > \{w_0\}$$

where $' >'$ signifies 'strictly stronger' (i.e., structure satisfying more axioms).

And by structure on S we mean the strongest possible structure $\{w\}$ on S under the given operation(s).

As a particular case, a Smarandache strong 2-algebraic structure (two levels only of structures in algebra) on a set S, is a structure $\{w_0\}$ on S such that there exists a proper subset P of S, which is embedded with a stronger structure $\{w_1\}$.

For example, a Smarandache strong semigroup is a semigroup that has a proper subset which is a group. Also, a Smarandache strong ring is a ring that has a proper subset which is a field.

Properties of Smarandache strong semigroups, groupoids, loops, bigroupoids, biloops, rings, birings, vector spaces, semirings, semivector spaces, non-associative semirings, bisemirings, near-rings, non-associative near-ring, binear-rings, fuzzy algebra and linear algebra are presented in the below books together with examples, solved and unsolved problems, and theorems.

Also, applications of Smarandache strong groupoids, near-rings, and semirings in automaton theory, in error correcting codes, in the construction of S-sub-biautomaton, in social and economic research can be found in the literatures.

A Smarandache Weak Structure on a set S means a structure on S that has a proper subset P with a weaker structure.

By proper subset of a set S, we mean a subset P of S, different from the empty set, from the original set S, and from the idempotent elements if any.

In any field, a Smarandache weak n-structure on a set S means a structure $\{w_0\}$ on S such that there exists a chain of proper subsets

$$P_{n-1} < P_{n-2} < ... < P_2 < P_1 < S$$

, where $' >'$ means 'included in', whose corresponding structures verify the chain

$$\{w_{n-1}\} < \{w_{n-2}\} < \ < \{w_2\} < \{w_1\} < \{w_0\}$$

where $' <'$ signifies 'strictly weaker' (i.e., structure satisfying less axioms).

And by structure on S we mean a structure $\{w\}$ on S under the given operation(s). As a particular case, a Smarandache weak 2-algebraic structure (two levels only of structures in algebra) on a set S, is a structure $\{w_0\}$ on S such that there exists a proper subset P of S, which is embedded with a weaker structure $\{w_1\}$.

For example, a Smarandache weak monoid is a monoid that has a proper subset which is a semigroup. Also, a Smarandache weak ring is a ring that has a proper subset which is a near-ring.

A Smarandache Strong-Weak Structure on a set S means a structure on S that has two proper subsets: P with a stronger structure and Q with a weaker structure. By proper subset of a set S, we mean a subset P of S, different from the empty set, from the original set S, and from the idempotent elements if any.

In any field, a Smarandache strong-weak n-structure on a set S means a structure $\{w_0\}$ on S such that there exist two chains of proper subsets

$$P_{n-1} < P_{n-2} < ... < P_2 < P_1 < S$$

and

$$Q_{n-1} < Q_{n-2} < ... < Q_2 < Q_1 < S$$

where $' <'$ means 'included in', whose corresponding stronger structures verify the chain

$$\{w_{n-1}\} > \{w_{n-2}\} > ... > \{w_2\} > \{w_1\} > \{w_0\}$$

and respectively the weaker structures verify the chain

$$\{v_{n-1}\} < \{v_{n-2}\} < ... < \{v_2\} < \{v_1\} < \{v_0\}$$

where $' >'$ signifies 'strictly stronger' (i.e. structure satisfying more axioms) and $' <'$ signifies 'strictly weaker' (i.e. structure satisfying less axioms). And by structure on S we mean a structure $\{w\}$ on S under the given operation(s).

As a particular case, a Smarandache strong-weak 2-structure (two levels only of structures in algebra) on a set S, is a structure $\{w_0\}$ on S such that there exist two proper subsets P and Q of S, where P is embedded with a stronger structure than $\{w_0\}$, while Q is embedded with a weaker structure than $\{w_0\}$.

For example, a Smarandache strong-weak monoid is a monoid that has a proper subset which is a group, and another proper set which is a semigroup. Also, a Smarandache strong-weak ring is a ring that has a proper subset which is a field, and another proper subset which is a near-ring.

See

http://fs.gallup.unm.edu/SmarandacheStrongStructures.htm

http://fs.gallup.unm.edu/SmarandacheWeakStructures.htm

http://fs.gallup.unm.edu/SmarandacheStrong-WeakStructures.htm

This book gives a lot of scope for the reader to developing the subject. This book has three chapters.

In Chapter one, an elaborate recollection of BE/CI-algebras are given. It also gives notions about these algebraic structures and their properties in details.

Smarandache structure on CI-algebras, Smarandache filter and Smarandache ideals are defined and discussed in Chapter two.

Chapter three defines the notion of Smarandache weak BE-algebra, Q-Smarandache filters and Q-Smarandache ideals.

Arsham Borumand Saeid

2013- Kerman

1 Introduction and Preliminaries

The processing of certain information, especially inferences, is based on certain information, and it is done on classical two-valued logic. Logic appears in a 'scared' (respectively 'profane') form which is dominant in proof theory (especially in model theory). The role of logic in mathematics and computer science is two folded - as a tool for applications in both areas, and a technique for laying the foundations.

Non-classical logic including many-valued logic, fuzzy logic, neutrosophic logic, etc. takes advantage in front of the classical logic to handle information with various facets of uncertainty, such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy and especially neutrosophic information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life.

In recent years, motivated by both theory and application, the study of t-norm and N-norm-based logic systems and the corresponding pseudo-logic systems has been become a greater focus in the field of logic. Here, t-norm and N-norm-based algebraic investigations were first done according to the corresponding algebraic investigations, and in the case of pseudo-logic systems, the algebraic development was first done according to the corresponding logical development.

As it is well known, BCK/BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [11, 12]. BCI-algebras are generalizations of BCK-algebras. Most of the algebras related to the t-norm based logic, such as MTL-algebras, BL-algebras [9], hoop, MV-algebras and Boolean algebras et al., are extensions of BCK-algebras.

The Smarandache algebraic structures theory was introduced in 1973 by Florentin Smarandache. In [14], Kandasamy studied the Smarandache groupoids, sub-groupoids, ideal of groupoids, seminormal sub groupoids, Smarandache Bol groupoids, and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and they were studied by F.

Smarandache since 1973. R. Padilla studied the first Smarandache algebraic structures in 1998.

In [13] Y. B. Jun discussed the Smarandache structure in BCI–algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI–algebras, Smarandache subalgebras and Smarandache ideals and investigated some related properties. In [5], Borumand Saeid and Namdar studied the concept of Smarandache BCH–algebras and obtained many interesting results about Smarandache (fresh, clean and fantastic) ideal in a BCH–algebras. Smarandache BL–algebras have been invented by Borumand Saeid et. al [6], and they deal with Smarandache ideal structures in Smarandache BL–algebras. A. Ahadpanah and A. Borumand Saeid introduced the notion of Smarandache hyper BCC-algebras [1].

Recently, H. S. Kim and Y. H. Kim defined a BE–algebra [16]. S. S. Ahn and K. S. So defined the notion of ideals in BE–algebras, and then stated and proved several characterizations of such ideals [3]. In [18], B.

L. Meng introduced the notion of an CI–algebra as a generation of a BE–algebra.

The notion of BCK-algebra was formulated first in 1966 by K. Iseki, a Japanese mathematician. This notion is originated from two different ways. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations introduced by L. Kantorovic and E. Livenson to make a new set from the old sets. These fundamental operations are union, intersection and the set difference. Then, as a generalization of those three operations and properties, we have the notion of Boolean algebra. If we take both the union and the intersection as a general algebra, the notion of distributive lattice is obtained. Moreover, if we consider the notion of union or intersection, we have the notion of an upper semilattice or a lower semilattice. But the notion of set difference was not considered systematically before K. Iseki.

Another motivation is taken from classical and non-classical propositional calculus. There are some systems which contain the only implication functor among the logical functors. These examples are the systems of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C. AS. Meredith.

We know the following simple relations in set theory:

$$(A - B) - (A - C) \subset C - B$$

$$A - (A - B) \subset B$$

In propositional calcului, these relations are denoted by

$$(p \longrightarrow q) \longrightarrow ((q \longrightarrow r) \longrightarrow (p \longrightarrow r))$$

$$p \longrightarrow ((p \longrightarrow q) \longrightarrow q)$$

From these relationships, K. Iseki introduced a new notion called a BCK-algebra.

Definition 1.1 [10] *Let X be a set with a binary operation "$*$" and a constant "0". Then $(X, *, 0)$ is called a BCK-algebra if it satisfies the following conditions:*

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $0 * x = 0$,

(V) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

for all $x, y, z \in X$.

For brevity we also call X a BCK-algebra. If in X we define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$, then $(X, *, 0)$ is a BCK-algebra if and only if it satisfies the following: For all $x, y \in X$;

(i) $(x * y) * (x * z) \leq z * y$,

(ii) $x * (x * y) \leq y$,

(iii) $x \leq x$,

(iv) $0 \leq x$,

(v) $x \leq y$ and $y \leq x$ imply $x = y$.

Theorem 1.2 [10] *In a BCK-algebra $(X, *, 0)$, we have the following properties:*

For all $x, y, z \in X$;

(i) $x \leq y$ *implies* $z * y \leq z * x$,

(ii) $x \leq y$ *and* $y \leq z$ *implies* $x \leq z$,

(iii) $(x * y) * z = (x * z) * y$,

(iv) $x * y \leq z$ *implies* $x * z \leq y$,

(v) $(x * z) * (y * z) \leq x * y$,

(vi) $x \leq y$ *implies* $x * z \leq y * z$,

(vii) $x * y \leq x$,

(vii) $x * 0 = x$.

Definition 1.3 [10] *If there is an element* $1 \in X$ *of a* BCK*-algebra* X *satisfying* $x \leq 1$ *for all* $x \in X$*, then the element 1 is called the unit of* X*. A* BCK*-algebra with a unit is called to be bounded.*

Theorem 1.4 [10] *Let* $(X, *, 0)$ *be a* BCK*-algebra and* $1 \notin X$*. We define the operation* " $*'$ " *on* $\overline{X} = X \cup \{1\}$ *as follows*

$$x *' y = \begin{cases} x * y & \text{if } x, y \in X \\ \{0\} & \text{if } x \in X \text{ and } y = 1 \\ 1 & \text{if } x = 1 \text{ and } y \in X \\ 0 & \text{if } x = y = 1, \end{cases}$$

Then $(\overline{X}, *', 0)$ *is a bounded* BCK*-algebra with unit 1.*

Definition 1.5 [10] *Let* $(X_1, *_1, 0)$ *and* $(X_2, *_2, 0)$ *be two* BCK*-algebras and* $X_1 \cap X_2 = \{0\}$*. Suppose* $X = X_1 \cup X_2$*, define* $*$ *on* X *as follows:*

$$x * y = \begin{cases} x *_1 y & \text{if } x \text{ and } y \text{ belong to } X_1 \\ x *_2 y & \text{if } x \text{ and } y \text{ belong to } X_2 \\ x & \text{if } x \text{ and } y \text{ do not belong to the same algebra} \end{cases}$$

Next we will verify that $(X, *, 0)$ *is a* BCK*-algebra, this algebra is called to be the union of* $(X_1, *_1, 0)$ *and* $(X_2, *_2, 0)$*, denoted by* $X_1 \bigoplus X_2$.

Theorem 1.6 [10] *Let* $(X_i, *_i, 0_i)$*,* $(i \in I)$ *be an indexed family of* BCK*-algebras and let* $\prod_{i \in I} X_i$ *be the set of all mappings* $f : I \longrightarrow \cup_{i \in I} X_i$*, where*

$f(i) \in X_i$ for all $i \in I$. For $f, g \in \prod_{i \in I} X_i$, we define $f * g$ by $(f * g)(i) = f(i) * g(i)$ for every $i \in I$, and by 0 we mean $0(i) = 0_i, \forall i \in I$. Then $(\prod_{i \in I} X_i, *, 0)$ is a BCK-algebra, which is called the direct product of $X_i (i \in I)$.

Definition 1.7 [10] *Let $(X, *, 0)$ be a BCK-algebra and S be a non-empty subset of X. Then S is called to be a subalgebra of X if, for any $x, y \in S, x * y \in S$, i.e., S is closed under the binary operation $*$ of X.*

Definition 1.8 [10] *A non-empty subset I of a BCK-algebra X is called an ideal of X if for all $x, y \in X$:*

(i) $0 \in I$

(ii) $x * y \in I$ *and* $y \in I$ *imply that* $x \in I$.

Definition 1.9 [10] *Let X be a non-empty set with a binary operation "$*$" and a constant "0". Then $(X, *, 0)$ is called a BCI-algebra if it satisfies the following conditions:*

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,

for all $x, y, z \in X$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a BCK-algebra.

A nonempty subset S of X is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the book [10] for further information regarding BCK/BCI-algebras.

Definition 1.10 *[16] An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called a BE–algebra if*

(BE1) $x * x = 1$ *for all* $x \in X$ *;*

(BE2) $x * 1 = 1$ *for all* $x \in X$ *;*

(BE3) $1 * x = x$ *for all* $x \in X$ *;*

(BE4) $x * (y * z) = y * (x * z)$ *for all* $x, y, z \in X$ *(exchange).*

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

Proposition 1.11 *[16] If* $(X; *, 1)$ *is a* BE*–algebra, then* $x*(y*x) = 1$, *for any* $x, y \in X$.

Definition 1.12 *[17] An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called a* CI*–algebra if*

(CI1) $x * x = 1$ *for all* $x \in X$*;*

(CI2) $1 * x = x$ *for all* $x \in X$*;*

(CI3) $x * (y * z) = y * (x * z)$ *for all* $x, y, z \in X$.

Definition 1.13 *[15] An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called a dual* BCK*–algebra if*

(BE1) $x * x = 1$ *for all* $x \in X$*;*

(BE2) $x * 1 = 1$ *for all* $x \in X$*;*

(dBCK1) $x * y = y * x = 1 \Longrightarrow x = y$;

(dBCK2) $(x * y) * ((y * z) * (x * z)) = 1$;

(dBCK3) $x * ((x * y) * y) = 1$.

Lemma 1.14 *[15] Let* $(X; *, 1)$ *be a dual* BCK*–algebra and* $x, y, z \in X$. *Then:*

(a) $x * (y * z) = y * (x * z)$,

(b) $1 * x = x$.

Proposition 1.15 *[24] Any dual* BCK*–algebra is a* BE*–algebra.*

Example 1.16 *[24] Let* $X = \{1, 2, \ldots\}$ *and the operation* $*$ *be defined as follows:*

$$x * y = \begin{cases} y & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

then $(X; *, 1)$ *is a* BE*–algebra, but it is not a dual* BCK*–algebra.*

Definition 1.17 *[8] An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called an implication algebra if for all* $x, y, z \in X$ *the following identities hold:*

(I1) $(x * y) * x = x$;

(I2) $(x * y) * y = (y * x) * x$;

(I3) $x * (y * z) = y * (x * z)$.

Proposition 1.18 *[8] If $(X; *, 1)$ is an implication algebra, then $(X; *, 1)$ is a dual BCK–algebra.*

Proposition 1.19 *[24] Any implication algebra is a BE–algebra.*

Definition 1.20 *[16] Let $(X, *, 1)$ be a BE–algebra and F be a non-empty subset of X. Then F is said to be a filter of X if*

$(F1)$ $1 \in F$;

$(F2)$ $x * y \in F$ *and* $x \in F$ *imply* $y \in F$.

Definition 1.21 *[2] A non-empty subset I of X is called an ideal of X if it satisfies:*

(I1) $\forall x \in X$ *and* $\forall a \in I$ *imply* $x * a \in I$ *, i.e,* $X * I \subseteq I$;

(I2) $\forall x \in X$, $\forall a, b \in I$ *imply* $(a * (b * x)) * x \in I$.

Lemma 1.22 *[3] A nonempty subset I of X is an ideal of X if and only if it satisfies*

1. $1 \in I$;
2. $(\forall x, y \in X)$ $(\forall y \in I)$ $(x * (y * z) \in I \Rightarrow x * z \in I)$.

Proposition 1.23 *[2] Let I be an ideal of X. If $a \in I$ and $a \leq x$, then $x \in I$.*

Definition 1.24 *[17] Let X be a CI–algebra and $x, y \in X$. Define $A(x, y)$ by*

$$A(x, y) := \{z \in X : x * (y * z) = 1\}$$

We call $A(x, y)$ an upper set of x and y.

Definition 1.25 *[24] A CI–algebra X is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$*

Definition 1.26 *[16] A BE–algebra X is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$*

Proposition 1.27 *[23] Let X be a self distributive BE–algebra. Then for all $x, y, z \in X$ the following statements hold:*

(1) if $x \leq y$, then $z * x \leq z * y$;

(2) $y * z \leq (x * y) * (x * z)$;

Definition 1.28 *[24] A BE–algebra X is called commutative if $(x*y)*y = (y*x)*x$ for any $x, y \in X$.*

Theorem 1.29 *[24] If $(X, *, 1)$ is a commutative BE–algebra, then $(X, *, 1)$ is a dual BCK–algebra.*

Definition 1.30 *[16] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE–algebra if*

(BE1) $x * x = 1$ for all $x \in X$;

(BE2) $x * 1 = 1$ for all $x \in X$;

(BE3) $1 * x = x$ for all $x \in X$;

(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (exchange).

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

Proposition 1.31 *[16] If $(X; *, 1)$ is a BE–algebra, then $x * (y * x) = 1$, for any $x, y \in X$.*

Definition 1.32 *[17] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a CI–algebra if*

(CI1) $x * x = 1$ for all $x \in X$;

(CI2) $1 * x = x$ for all $x \in X$;

(CI3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

Definition 1.33 *[15] An algebra $(X; *, 1)$ of type $(2, 0)$ is called a dual BCK–algebra if*

(BE1) $x * x = 1$ for all $x \in X$;

(BE2) $x * 1 = 1$ for all $x \in X$;

(dBCK1) $x * y = y * x = 1 \Longrightarrow x = y$;

(dBCK2) $(x * y) * ((y * z) * (x * z)) = 1$;

(dBCK3) $x * ((x * y) * y) = 1$.

Lemma 1.34 *[15] Let* $(X; *, 1)$ *be a dual* BCK*–algebra and* $x, y, z \in X$*. Then:*

(a) $x * (y * z) = y * (x * z)$,

(b) $1 * x = x$.

Proposition 1.35 *[24] Any dual* BCK*–algebra is a* BE*–algebra.*

Example 1.36 *[24] Let* $X = \{1, 2, \ldots\}$ *and the operation* $*$ *be defined as follows:*

$$x * y = \begin{cases} y & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

then $(X; *, 1)$ *is a* BE*–algebra, but it is not a dual* BCK*–algebra.*

Definition 1.37 *[8] An algebra* $(X; *, 1)$ *of type* $(2, 0)$ *is called an implication algebra if for all* $x, y, z \in X$ *the following identities hold:*

(I1) $(x * y) * x = x$;

(I2) $(x * y) * y = (y * x) * x$;

(I3) $x * (y * z) = y * (x * z)$.

Proposition 1.38 *[8] If* $(X; *, 1)$ *is an implication algebra, then* $(X; *, 1)$ *is a dual* BCK*–algebra.*

Proposition 1.39 *[24] Any implication algebra is a* BE*–algebra.*

Definition 1.40 *[16] Let* $(X, *, 1)$ *be a* BE*–algebra and* F *be a non-empty subset of* X*. Then* F *is said to be a filter of* X *if*

$(F1)$ $1 \in F$;

$(F2)$ $x * y \in F$ *and* $x \in F$ *imply* $y \in F$.

Definition 1.41 *[2] A non-empty subset* I *of* X *is called an ideal of* X *if it satisfies:*

(I1) $\forall x \in X$ *and* $\forall a \in I$ *imply* $x * a \in I$ *, i.e,* $X * I \subseteq I$ *;*

(I2) $\forall x \in X$ *,* $\forall a, b \in I$ *imply* $(a * (b * x)) * x \in I$.

Lemma 1.42 *[3] A nonempty subset* I *of* X *is an ideal of* X *if and only if it satisfies*

1. $1 \in I$;

2. $(\forall x, y \in X)$ $(\forall y \in I)$ $(x * (y * z) \in I \Rightarrow x * z \in I)$.

Proposition 1.43 *[2] Let I be an ideal of X. If $a \in I$ and $a \leq x$, then $x \in I$.*

Definition 1.44 *[17] Let X be a CI–algebra and $x, y \in X$. Define $A(x, y)$ by*

$$A(x, y) := \{z \in X : x * (y * z) = 1\}$$

We call $A(x, y)$ an upper set of x and y.

Definition 1.45 *[24] A CI–algebra X is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$*

Definition 1.46 *[16] A BE–algebra X is said to be self distributive if $x * (y * z) = (x * y) * (x * z)$, for all $x, y, z \in X$*

Proposition 1.47 *[23] Let X be a self distributive BE–algebra. Then for all $x, y, z \in X$ the following statements hold:*

(1) if $x \leq y$*, then* $z * x \leq z * y$ *;*

(2) $y * z \leq (x * y) * (x * z)$ *;*

Definition 1.48 *[24] A* BE*–algebra* X *is called commutative if* $(x * y) * y = (y * x) * x$ *for any* $x, y \in X$*.*

Theorem 1.49 *[24] If* $(X, *, 1)$ *is a commutative* BE*–algebra, then* $(X, *, 1)$ *is a dual* BCK*–algebra.*

2 Smarandache Weak BE-algebras

Note that every BE–algebra is a CI–algebra, but the converse is not true. A CI–algebra which is not a BE–algebra is called a proper CI–algebra.

Definition 2.1 *A Smarandache weak BE–algebra X is said to be a BE–algebra X in which there exists a proper subset Q of X such that*

(S1) $1 \in Q$ and $|Q| \geq 2$;

(S2) Q is a CI–algebra under the operation of X.

Example 2.2 *Let $X := \{1, a, b, c\}$ be a set with the following table.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	1
c	1	1	1	1

Then $(X, *, 1)$ is a BE–algebra. If Q is one of the sets $\{1, a\}$, $\{1, b\}$, $\{1, c\}$, $\{1, a, b\}$ or $\{1, a, c\}$, then Q is a CI–algebra. So X is a Smaran-

dache weak BE–algebra.

Example 2.3 *Let $X := \{1, a, b, c\}$ be a set with the following table.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	1	1

Then it is a BE–algebra, but $Q = \{1, b, c\}$ is not a CI-algebras and X is not a Smarandache weak BE-algebras.

Definition 2.4 *A nonempty subset F of a BE–algebra X is called a Smarandache filter of X related to Q (or briefly, Q–Smarandache filter of X) if it satisfies:*

(SF1) $1 \in F$;

(SF2) $(\forall y \in Q)(\forall x \in F)(x * y \in F \Rightarrow y \in F)$.

Example 2.5 *In Example 3.2, if $Q = \{1, a\}$, then $F = \{1, b\}$ is a Q-Smarandache filter of X, but $F = \{1, a\}$ is not a Smarandache filter of*

X.

Note. If F is a Smarandache filter of a BE–algebra X related to every CI–algebra contained in X, we simply say that F is a Smarandache filter of X.

Proposition 2.6 *If $\{F_\lambda : \lambda \in \Delta\}$ is an indexed set of Q–Smarandache filters of a BE–algebra X, where $\Delta \neq \emptyset$, then $F = \cap\{F_\lambda : \lambda \in \Lambda\}$ is a Q–Smarandache filter of X.*

Proposition 2.7 *Any filter F of a BE–algebra X is a Q–Smarandache filter of X.*

Note. By the following example we show that the converse of above proposition is not correct in general.

Example 2.8 *Let $X := \{1, a, b, c\}$ be a set with the following table.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

Then X is a BE–algebra and $Q = \{1, b\}$ is a CI–algebra which is properly contained in BE–algebra X and $F = \{1, b\}$ is a Q–Smarandache filter of X but it is not a filter of X because $b * a \in F$ and $b \in F$, but $a \notin F$.

Proposition 2.9 *If F is a Q–Smarandache filter F of a self distributive BE–algebra X, then $(\forall x, y, z \in Q)(z*(y*x) \in F, z*y \in F \Rightarrow z*x \in F)$.*

Proof. Since $z * (y * x) = (z * y) * (z * x) \in F$ and $z * y \in F$, by $(SF2)$ we have $z * x \in F$.

Proposition 2.10 *If F is a Q–Smarandache filter of a self distributive BE–algebra X, then $(\forall x, y \in Q)(y * (y * x) \in F \Rightarrow y * x \in F)$.*

Proof. Assume that $y * (y * x) \in F$ for all $x, y \in Q$ since $y * y = 1 \in F$, by $(SF1)$ and Proposition 3.9 we have $y * x \in F$.

Proposition 2.11 *Let F be a Q–Smarandache filter of a BE-algebra X. Then:*

(1) $F \neq \emptyset$.

(2) If $x \in F$, $x \leq y$, $y \in Q$, then $y \in F$.

(3) If X is a self distributive CI–algebra and $x, y \in F$, then $x * y \in F$.

Proof. (1) Since F is a Q–Smarandache filter of X, therefore by $(SF1)$ we have $1 \in F$. Hence $F \neq \emptyset$.

(2) Let $x \in F$, $x \leq y$ and $y \in Q$. Then $x*y = 1 \in F$, therefore by $(SF2)$ we get that $y \in F$.

(3) We have $y * (x * (x * y)) = x * (y * (x * y)) = x * (x * 1) = x * 1 = 1$, and hence $y * (x * (x * y)) \in F$. Since $y \in F$, by $(SF2)$ $x * (x * y) \in F$. It follows from Proposition 3.10 $x * y \in F$.

Proposition 2.12 *If F is a Q–Smarandache filter of a BE–algebra X*

and if Q *satisfies* $X*Q \subseteq Q$*, then* $(\forall x, y \in F)(\forall z \in Q)(x*(y*z)=1 \Rightarrow z \in F)$

Proof. Assume that $X*Q \subseteq Q$. Let F be a Q–Smarandache filter of X. Suppose that $x*(y*z)=1$ where $x, y \in F$ and $z \in Q$. Then $y*z \in Q$ by the hypothesis and $x*(y*z) \in F$ we have $y*z \in F$. By $(SF2)$, $z \in F$. This completes the proof.

Theorem 2.13 *Let* Q_1 *and* Q_2 *are* CI*–algebras which are properly contained in a* BE*–algebra* X *and* $Q_1 \subseteq Q_2$*. Then every* Q_2*–Smarandache filter is a* Q_1*–Smarandache filter of* X*.*

Note. By the following example we show that the converse of above theorem is not correct in general.

Example 2.14 *In Example 3.8,* $Q_1 = \{1, b\}$*,* $Q_2 = \{1, b, c\}$ *are* CI*–algebras which are properly contained in* X*. It is easily checked that* $F = \{1, b\}$ *is a* Q_1*–Smarandache filter of* X *and is not a* Q_2*–Smarandache filter of* X*, since* $c*a = c \in F$*, but* $a \notin F$*.*

Proposition 2.15 *Let X be a self distributive BE–algebra and F be a Q–Smarandache filter of X. Then $F_a = \{x : a * x \in F\}$ is a Q–Smarandache filter, for any $a \in X$.*

Proof. Since $a * a = 1 \in F$, $a \in F_a$ and so $\emptyset \neq F_a$. Assume $x * y \in F_a$ and $x \in F_a$. Then $a * (x * y) \in F$ and $a * x \in F$. By self distributivity we have $(a * x) * (a * y) \in F$ and $a * x \in F$. Thus $a * y \in F$ and so $y \in F_a$. Therefore F_a is a Q–Smarandache filter of X.

Definition 2.16 *Let X be a BE–algebra and $x, y \in X$, $Q \subset X$ be a CI–algebra. Define*

$$A(x, y) := \{z \in Q : x * (y * z) = 1\}$$

We call $A(x, y)$ a Q–Smarandache upper set of x and y.

Note. It is easy to see that $1, x, y \in A(x, y)$. The set $A(x, y)$, where $x, y \in Q$, need not be a Q–Smarandache filter of X in general. In Example 3.2, it is easy to check that $A(1, a) = \{a\}$, which means that $A(1, a)$ is not a Q–Smarandache filter of X.

Proposition 2.17 *Let X be a self distributive BE–algebra. Then the Q–Smarandache upper set $A(x, y)$ is a Q–Smarandache filter of X, where $x, y \in Q$.*

Proof. Since $x * 1 = 1$, for all $x \in Q$, $1 \in A(x, y)$. Let $a * b \in A(x, y)$ and $a \in A(x, y)$, where $b \in Q$. Then $1 = x * (y * (a * b))$ and $1 = x * (y * a)$. It follows from the definition of the self distributivity

$$1 = x * (y * (a * b)) = x * ((y * a) * (y * b)) = (x * (y * a)) * (x * (y * b)) =$$

$$1 * (x * (y * b)) = x * (y * b)$$

Therefore $b \in A(x, y)$. This proves that $A(x, y)$ is a Q–Smarandache filter of X.

Theorem 2.18 *If F is a non-empty subset of a BE–algebra X, then F is a Q–Smarandache filter of X if and only if $A(x, y) \subseteq F$, for all $x, y \in F$.*

Proof. Assume that F is a Q–Smarandache filter of X and $x, y \in F$. If $z \in A(x, y)$, then $x * (y * z) = 1 \in F$. Then by Proposition 3.12, $z \in F$.

Hence $A(x, y) \subseteq F$.

Conversely, suppose that $A(x, y) \subseteq F$ for all $x, y \in F$. Since $x * (y * 1) = x * 1 = 1$, $1 \in A(x, y) \subseteq F$. Assume $a * b$, $a \in F$. Since $(a * b) * (a * b) = 1$, we have $b \in A(a * b, a) \subseteq F$. Hence F is a Q–Smarandache filter of X.

Theorem 2.19 *If F is a Q–Smarandache filter of a BE–algebra X, then*

$$F = \cup_{x,y \in F} A(x, y).$$

Proof. Let F be a Q–Smarandache filter of X and $z \in F$. Since $z * (1 * z) = z * z = 1$, we have $z \in A(z, 1)$. Hence

$$F \subseteq \cup_{z \in F} A(z, 1) \subseteq \cup_{x,y \in F} A(x, y)$$

If $z \in \cup_{x,y \in F} A(x, y)$, then there exist $a, b \in F$ such that $z \in A(a, b)$. It follows from Theorem 3.18 that $z \in F$. This means that $\cup_{x,y \in F} A(x, y) \subseteq F$.

Theorem 2.20 *If F is a Q–Smarandache filter of BE–algebra X, then*

$$F = \cup_{x \in F} A(x, 1).$$

Proof. Let F be a Q–Smarandache filter of X and $z \in F$. Since $z * (1 * z) = z * z = 1$, we have $z \in A(z, 1)$. Hence

$$F \subseteq \cup_{z \in F} A(z, 1)$$

If $z \in \cup_{x \in F} A(x, 1)$, then there exists $a \in F$ such that $z \in A(a, 1)$, which means that $a * z = a * (1 * z) = 1 \in F$. Since F is a Q–Smarandache filter of X and $a \in F$, we have $z \in F$. This means that $\cup_{x \in F} A(x, 1) \subseteq F$.

Definition 2.21 *A nonempty subset I of a BE–algebra X is called a Smarandache ideal of X related to Q (or briefly, a Q–Smarandache ideal of X) if it satisfies:*

(SI1) $x \in Q$ and $a \in I$ imply $x * a \in I$, i.e., $Q * I \subseteq I$

(SI2) $(\forall x \in Q)(\forall a, b \in I)$ imply $(a * (b * x)) * x \in I$.

Example 2.22 *In Example 3.2, $Q = \{1, a\}$ is a CI–algebra of X and $I = \{1, a, b\}$ is a Smarandache ideal of Q, but $J = \{1, c\}$ is not a Q–Smarandache ideal of X because $c, 1 \in J$ and $a \in Q$, but $(c * (1 * a)) * a = (c * a) * a = 1 * a = a \notin J$.*

Lemma 2.23 *Let X be a CI–algebra. Then*

(1) Every Q–Smarandache ideal I of X contains 1*;*

(2) If I is a Q–Smarandache ideal of X, then $(a * x) * x \in I$ for all $a \in I$ and $x \in Q$.

Proof. (1) Let $\emptyset \neq I$ be a Q–Smarandache ideal of X. For $x \in I$, $1 = x * x \in I * I \subseteq Q * I \subseteq I$. Thus $1 \in I$.

(2) Let $b := 1$ in $(SI2)$. Then $(a * (1 * x)) * x \in I$. Hence $(a * x) * x \in I$.

Lemma 2.24 *A nonempty subset I of a BE–algebra X is a Q–Smarandache ideal of X if and only if it satisfies*

1. $1 \in I$*;*

2. *$(\forall x, z \in Q)$ $(\forall y \in I)$ $(x * (y * z) \in I \Rightarrow x * z \in I)$.*

Theorem 2.25 *Let Q_1 and Q_2 are CI–algebras which are properly contained in a BE-algebra X and $Q_1 \subseteq Q_2$. Then every Q_2–Smarandache ideal of X is a Q_1–Smarandache ideal.*

3 Smarandache Strong CI-Algebras

Note that every BE–algebra is a CI–algebra, but the converse is not true. A CI–algebra which is not a BE–algebra is called a proper CI–algebra.

Definition 3.1 *A Smarandache* Strong *CI–algebra X is defined to be a CI–algebra X in which there exists a proper subset Q of X such that*

(S1) $1 \in Q$ *and* $|Q| \geq 2$*;*

(S2) Q *is a BE–algebra under the operation of X.*

Example 3.2 *Let* $X := \{1, a, b, c, d\}$ *be a set with the following table.*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

then $(X;*,1)$ *is a* CI*–algebra but is not a* BE*–algebra. If* $Q := \{1, a, b, c\}$,

then Q *is a* BE*–algebra. So* X *is a* Q*-Smarandache* CI*–algebra.*

Example 3.3 *Let* $X := \{1, a, b\}$ *be a set with the following table.*

$*$	1	a	b
1	1	a	b
a	a	1	b
b	b	b	1

then $(X;*,1)$ *is a* CI*–algebra but is not a Smarandache* CI*–algebra.*

Definition 3.4 *A nonempty subset* F *of* CI*–algebra* X *is called a Smarandache filter of* X *related to* Q *(or briefly,* Q*–Smarandache filter of* X*) if it satisfies:*

(SF1) $1 \in F$;

(SF2) $(\forall y \in Q)(\forall x \in F)(x * y \in F \Rightarrow y \in F)$.

Example 3.5 *In Example 3.2,* $\{1, a\}$ *and* $\{1, b\}$ *are Smarandache filter of* X.

Note. If F is a Smarandache filter of CI–algebra X related to every BE–algebra contained in X, we simply say that F is a Smarandache filter of X.

Proposition 3.6 *If $\{F_\lambda : \lambda \in \Delta\}$ is an indexed set of Q–Smarandache filters of X, where $\Delta \neq \emptyset$, then $F = \cap\{F_\lambda : \lambda \in \Lambda\}$ is a Q–Smarandache filter of X.*

Proposition 3.7 *Any Filter F of CI–algebra X is a Q–Smarandache filter of X.*

Note. By the following example we show that the converse of above proposition is not correct in general.

Example 3.8 *Let* $X := \{1, a, b, c\}$ *be a set with the following table.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

Then X *is a* CI*–algebra and* $Q = \{1, a\}$ *is* BE*–algebra which is properly contained in* CI*–algebra* X *and* $F = \{1, b, c\}$ *is a* Q*–Smarandache filter of* X *but it is not a filter of* X *because* $c * a = c \in F$ *and* $c \in F$*, but* $a \notin F$*.*

Proposition 3.9 *If* F *is a* Q*–Smarandache filter* F *of self distributive* CI*–algebra* X*, then* $(\forall x, y, z \in Q)(z * (y * x) \in F, z * y \in F \Rightarrow z * x \in F)$*.*

Proof. Since $z * (y * x) = (z * y) * (z * x) \in F$ and $z * y \in F$, then by $(SF2)$ we have $z * x \in F$.

Proposition 3.10 *If* F *is a* Q*–Smarandache filter* F *of self distributive* CI*–algebra* X*, then* $(\forall x, y \in Q)(y * (y * x) \in F \Rightarrow y * x \in F)$*.*

Proof. Assume that $y * (y * x) \in F$ for all $x, y \in Q$ since $y * y = 1 \in F$, by $(SF1)$ and Proposition 3.9 we have $y * x \in F$.

Proposition 3.11 *Let F be a Q–Smarandache filter of X. Then:*

(1) $F \neq \emptyset$.

(2) If $x \in F$, $x \leq y$, $y \in Q$, then $y \in F$.

(3) If X is self distributive BE–algebra and $x, y \in F$, then $x * y \in F$.

Proof. (1) Since F is a Q–Smarandache filter of X, therefore by $(SF1)$ we have $1 \in F$, then $F \neq \emptyset$.

(2) Let $x \in F$, $x \leq y$ and $y \in Q$. Then $x * y = 1 \in F$, therefore by $(SF2)$ we get that $y \in F$.

(3) We have $y * (x * (x * y)) = x * (y * (x * y)) = x * (x * 1) = x * 1 = 1$, thus $y * (x * (x * y)) \in F$, also $y \in F$, then by $(SF2)$ $x * (x * y) \in F$, therefore by Proposition 3.10, $x * y \in F$.

Proposition 3.12 *If F is a Q–Smarandache filter of CI–algebra X, and Q satisfies $X * Q \subseteq Q$, then* $(\forall x, y \in F)(\forall z \in Q)(x * (y * z) = 1 \Rightarrow z \in F)$

Proof. Assume that $X * Q \subseteq Q$ and F be a Q–Smarandache filter of X. Suppose that $x * (y * z) = 1$, for all $x, y \in F$ and $z \in Q$, then $y * z \in Q$ by hypothesis and $x * (y * z) \in F$ we have $y * z \in F$. By $(SF2)$ since $y \in F$, it follows that $z \in F$. This completes the proof.

Theorem 3.13 *Let Q_1 and Q_2 are BE –algebras which are properly contained in CI–algebra X and $Q_1 \subseteq Q_2$. Then every Q_2–Smarandache filter is a Q_1–Smarandache filter.*

Note. By the following example we show that the converse of above theorem is not correct in general.

Example 3.14 *In Example 3.8, $Q_1 = \{1, a\}$, $Q_2 = \{1, a.b\}$ are BE–algebra which are properly contained in X. It is easily checked that $F = \{1, b, c\}$ is a Q_1–Smarandache filter of X and is not a Q_2–Smarandache filter of X, since $c * a = c \in F$, but $a \notin F$.*

Proposition 3.15 *Let X be a self distributive Smarandache CI–algebra and F be a Q–Smarandache filter of X. Then $F_a = \{x : a * x \in F\}$ is a Q–Smarandache filter, for any $a \in X$.*

Proof. Since $a * a = 1 \in F$, then $a \in F_a$ and so $\emptyset \neq F_a$. Assume $x * y \in F_a$ and $x \in F_a$, then $a * (x * y) \in F$ and $a * x \in F$. By self distributivity we have $(a * x) * (a * y) \in F$ and $a * x \in F$. Thus $a * y \in F$ and so $y \in F_a$. Therefore F_a is a Q–Smarandache filter of X.

Definition 3.16 *Let X be a CI–algebra and $x, y \in X$, $Q \subset X$ be a BE–algebra. Define*

$$A(x, y) := \{z \in Q : x * (y * z) = 1\}$$

We call $A(x, y)$ a Q–Smarandache upper set of x and y.

Note. It is easy to see that $1, x, y \in A(x, y)$. The set $A(x, y)$, where $x, y \in Q$, need not be a Q–Smarandache filter of X in general. In Example 3.2, it is easy to check that $A(1, d) = \{d\}$, which means that $A(1, a)$ is not a Q–Smarandache filter of X.

Proposition 3.17 *Let X be a self distributive Smarandache CI–algebra. Then Q–Smarandache upper set $A(x, y)$ is a Q–Smarandache filter of X, where $x, y \in Q$.*

Proof. Since $x*1=1$ for all $x \in Q$, then $1 \in A(x,y)$. Let $a*b \in A(x,y)$ and $a \in A(x,y)$, where $b \in Q$. Then $1 = x*(y*(a*b))$ and $1 = x*(y*a)$. It follows from the definition of self distributivity have

$$1 = x*(y*(a*b)) = x*((y*a)*(y*b)) = (x*(y*a))*(x*(y*b)) =$$
$$1*(x*(y*b)) = x*(y*b)$$

Therefore $b \in A(x,y)$. This proves that $A(x,y)$ is a Q–Smarandache filter of X.

Theorem 3.18 *If F is a non-empty subset of a CI–algebra X, then F is a Q–Smarandache filter of X if and only if $A(x,y) \subseteq F$, which $A(x,y)$ is a Q–Smarandache upper set.*

proof. Assume that F is a Q–Smarandache filter of X and $x, y \in F$. If $z \in A(x,y)$, then $x*(y*z) = 1 \in F$. Then by Proposition 3.12, $z \in F$. Hence $A(x,y) \subseteq F$.

Conversely, suppose that $A(x,y) \subseteq F$ for all $x, y \in F$. Since $x*(y*1) = x*1 = 1$, $1 \in A(x,y) \subseteq F$. Assume $a*b$, $a \in F$. Since $(a*b)*(a*b) = 1$, we have $b \in A(a*b, a) \subseteq F$. Hence F is a Q–Smarandache filter of X.

Theorem 3.19 *If F is a Q–Smarandache filter of a CI–algebra X, then*

$$F = \cup_{x,y \in F} A(x, y).$$

Proof. Let F be a Q–Smarandache filter of X and $z \in F$. Since $z * (1 * z) = z * z = 1$, we have $z \in A(z, 1)$. Hence

$$F \subseteq \cup_{z \in F} A(z, 1) \subseteq \cup_{x,y \in F} A(x, y)$$

If $z \in \cup_{x,y \in F} A(x, y)$, then there exist $a, b \in F$ such that $z \in A(a, b)$. It follows from Theorem 3.18, that $z \in F$. this means that $\cup_{x,y \in F} A(x, y) \subseteq F$.

Theorem 3.20 *If F is a Q–Smarandache filter of CI–algebra X, then*

$$F = \cup_{x \in F} A(x, 1).$$

Proof. Let F be a Q–Smarandache filter of X and $z \in F$. Since $z * (1 * z) = z * z = 1$, we have $z \in A(z, 1)$. Hence

$$F \subseteq \cup_{z \in F} A(z, 1)$$

If $z \in \cup_{x \in F} A(x,1)$, then there exists $a \in F$ such that $z \in A(a,1)$, which means that $a * z = a * (1 * z) = 1 \in F$. Since F is Q–Smarandache filter of X and $a \in F$, we have $z \in F$. This means that $\cup_{x \in F} A(x,1) \subseteq F$.

Definition 3.21 *A nonempty subset I of Smarandache CI–algebra X is called a Smarandache ideal of X related to Q (or briefly, Q–Smarandache ideal of X) if it satisfies:*

(SI1) $\forall x \in Q$ and $\forall a \in I$ imply $x * a \in I$, i.e., $Q * I \subseteq I$

(SI2) $(\forall x \in Q)(\forall a, b \in I)$ imply $(a * (b * x)) * x \in I$.

Example 3.22 *In Example 3.2, $Q = \{1, a, b\}$ is a BE–algebra of X and $I = \{1, a\}$ is a Smarandache ideal of Q, but $J = \{1, c\}$ is not a Q–Smarandache ideal of X because $c, 1 \in J$ and $a \in Q$, but $(c*(1*a))*a = (c * a) * a = 1 * a = a \notin J$.*

Lemma 3.23 *Let X be a CI–algebra. Then*

(1) Every Q–Smarandache ideal I of X contains 1;

(2) If I is a Q–Smarandache ideal of X, then $(a*x)*x \in I$ for all $a \in I$ and $x \in Q$.

Proof. (1) Let $\emptyset \neq I$ is a Q–Smarandache ideal of X. For $x \in I$, $1 = x*x \in I*I \subseteq Q*I \subseteq I$. Thus $1 \in I$.

(2) Let $b := 1$ in $(SI2)$. Then $(a*(1*x))*x \in I$. Hence $(a*x)*x \in I$.

Lemma 3.24 *A nonempty subset I of Q–Smarandache of X is a Q–Smarandache ideal of X if and only if it satisfies*

1. $1 \in I$;
2. $(\forall x, y \in Q)$ $(\forall y \in I)$ $(x*(y*z) \in I \Rightarrow x*z \in I)$.

Theorem 3.25 *Let Q_1 and Q_2 are BE–algebras which are properly contained in X and $Q_1 \subseteq Q_2$. Then every Q_2–Smarandache ideal of X is a Q_1–Smarandache ideal.*

Definition 3.26 *If X is a Q–Smarandache CI–algebra, X is said to be a Q–Smarandache commutative if Q is a commutative BE–algebra, i.e, for all $x, y \in Q$, $(x*y)*y = (y*x)*x$.*

Example 3.27 *Let* $X := \{1, a, b, c, d\}$ *be a set with the following table.*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	a	d
b	1	1	1	a	d
c	1	1	a	1	d
d	d	d	d	d	1

then $(X; *, 1)$ *is* CI*–algebra, but* $Q := \{1, a, b, c\} \subseteq X$ *is a commutative* BE*–algebra, so* X *is a* Q*–Smarandache commutative* BE*–algebra.*

Proposition 3.28 *If* X *is a* Q*–Smarandache commutative* CI*–algebra, then for all* $x, y \in Q$*,* $x * y = 1$ *and* $y * x = 1$ *imply* $x = y$*.*

Proof. Let $x, y \in Q$ and suppose that $x * y = y * x = 1$. Then

$$x = 1 * x = (y * x) * x = (x * y) * y = 1 * y = y.$$

Theorem 3.29 *An algebra* X *is a* Q*–Smarandache commutative* CI*–algebra if and only if the following identities hold: for any* $x, y, z \in Q$

(1) $(y * 1) * x = x$;

(2) $(y * x) * (z * x) = (x * y) * (z * y)$;

(3) $x * (y * z) = y * (x * z)$.

Proof. Necessity. It suffices to prove (2). By $(BE4)$ and commutativity we have $(z*x)*(y*x) = y*((z*x)*x) = y*((x*z)*z) = (x*z)*(y*z)$.

Sufficiency. By (1) we have $1 * x = ((1 * 1) * 1) * x = x$. $(BE3)$

From (1) and $(BE3)$ we conclude $1 = 1 * 1 = ((1 * x) * 1) * (1 * 1) = (1 * (1 * x) * (1 * (1 * x)) = (1 * x) * (1 * x) = x * x$. $(BE1)$

By $(BE1)$ we have $1 = (x * 1) * (x * 1) = x * 1$, hence $(BE2)$ hold. It suffices to prove commutativity. From (1), (2), (3), we have

$(y * x) * x = (y * x) * ((y * 1) * x) = (x * y) * ((y * 1) * y) = (x * y) * y$.

Then Q is a commutative BE–algebra.

Definition 3.30 *A Smarandache BE–algebra X is defined to be a BE–algebra X in which there exists a proper subset Q of X such that*

(S1) $1 \in Q$ *and* $|Q| \geq 2$;

(S2) Q is a dual BCK–algebra under the operation of X.

Example 3.31 *Let $X := \{1, a, b, c\}$ be a set with the following table.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	1
c	1	1	1	1

then $(X, *, 1)$ is a BE–algebra, but it is not a dual dual BCK–algebra because $b * c = 1$ and $c * b = 1$ but $c \neq b$. On the other hand $Q := \{1, a, b, c\} \subset X$ is a dual BCK–algebra. Then X is a Q–Smarandache BE–algebra.

Definition 3.32 *A Smarandache dual BCK–algebra X is defined to be a dual BCK–algebra X in which there exists a proper subset Q of X such that*

(S1) $1 \in Q$ and $|Q| \geq 2$;

(S2) Q is an implication algebra under the operation of X.

Example 3.33 *Let* $X := \{1, a, b\}$ *be a set with the following table.*

$*$	1	a	b
1	1	a	b
a	1	1	a
b	1	1	1

then $(X, *, 1)$ *is a dual* BCK*–algebra but it is not an implication algebra because* $(a * b) * a = a * a = 1 \neq a$*. On the other hand* $Q := \{1, a\} \subset X$ *is an implication algebra. Then* X *is a* Q*–Smarandache dual* BCK*–algebra.*

Note. A Smarandache strong n-structure on a set S means a structure $\{W_0\}$ on a set S such that there exists a chain of proper subsets $P_{n-1} < P_{n-2} < ... < P_2 < P_1 < S$ where $<$ means "included in" whose corresponding structures verify the inverse chain $W_{n-1} > W_{n-2} > ... > W_2 > W_1 > W_0$ where $>$ signifies strictly stronger (i.e structure satisfying more axioms)

Definition 3.34 *A Smarandache strong* 3*–structure of* CI*–algebra* X *is* *a chain $X_1 > X_2 > X_3 > X_4$ where X_1 is a CI–algebra, X_2 is a BE–algebra, X_3 is a dual BCK–algebra, X_4 is an implication algebra.*

Example 3.35 *Let $X := \{1, a, b, c, d\}$ be a set with the following table.*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	a	d
b	1	1	1	a	d
c	1	1	1	1	d
d	d	d	d	d	1

hen $X_1 = \{1, a, b, c, d\}$ is a CI–algebra, $X_2 = \{1, a, b, c\}$ is a BE–algebra, $X_3 = \{1, a, b\}$ is a dual BCK–algebra, $X_4 = \{1, a\}$ is an implication algebra. So, X is a Smarandache strong 3–structure.

References

[1] A. Ahadpanah and A. Borumand Saeid, *Smarandache Hyper BCC-algebra*, Comp. Math with Appl., 61 (2011) 2490-2497.

[2] Ahn S. S. and So K. S., *On generalized upper sets in BE–Algebras*, Bull, Korean math. Soc. 46(2009), no. 2, 281-287.

[3] Ahn S. S. and So K. S., *On ideals and upper sets in BE–algebras*, Sci. Math. Jpn. 68(2008), no. 2, 279-285.

[4] Borumand Saeid A., *Smarandache weak BE–algebras*, Commun. Korean Math. Soc. 27 (2012), No. 3, 489-496.

[5] Borumand Saeid A. and Namdar A., *Smarandache BCH–algebra*, World Applied Sciences Journal 7(2009), 77-83.

[6] Borumand Saeid A., Ahadpanah A. and Torkzadeh L., *Smarandache BL–algebra*, Journal of Applied Logic 8(2010), 235-261.

[7] Borumand Saeid A. and Rezaei A., *Smarandache n-structure on CI-algebras*, Results in Math., 2013, Vol. 63, Issue 1-2, 209-219.

[8] Borzooei R. A. and Khosravi Shoar S., *Implication algebras are eguivalent to the dual implicative BCK–algebras*, Sci. Math. Jpn. 63(2006), 429-431.

[9] P. H'ajek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[10] Huang Y., *BCI-algebra,* Science Press, China (2006).

[11] Imai Y. and K. Iseki, *On axiom systems of propositional calculi*, XIV Proc. Japan Academy, 42 (1966), 19-22.

[12] Iseki K. and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japanica, 23 (1978), 1-26.

[13] Jun Y. B., *Smarandache BCI–algebras*, Sci. Math. Jpn. 62(1)(2005), 137-142.

[14] KANDASAMY W. B. V., *Smarandache gropoids*, http://www.gallup.unm.edu/Smarandache/Groupoids.pdf.

[15] KIM H. K., YON Y. H., *Dual BCK–algebra and MV–algebra*, Sci. Math. Jpn. 66(2007), 247-253.

[16] KIM H. S. AND KIM Y. H., *On BE–Algebras*, Sci. Math. Jpn. Online e-2006, 1299-1302.

[17] KYUNG H. K., *A Not on CI–algebras*, Int. Math. Forum, Vol. 6, 2011, no. 1, 1-5.

[18] MENG B. L., *CI–algebra*, Sci. Math. Jpn. Online, e-2009,695-701.

[19] PADILLA R., *Smarandache algebraic structures*, Bull. Pure Appl. Sci. Sect. E Math. Stat. 17E (1998), no. 1, 119-121.

[20] REZAEI A. AND BORUMAND SAEID A., *On fuzzy subalgebras of BE–algebras*, Afr. Mat. Vol. 22, 2 (2011), 115-127.

[21] Rezaei A. and Borumand Saeid A., *Some results in BE- algebras*, Analele Universitatii Oradea Fasc. Matematica, Tom XIX (2012), 33-44.

[22] Smarandache Florentin, *Special Algebraic Structures*, Arhivele Statului Filiala Vlcea, Rm. Vlcea, Romania, 1973.

[23] Smarandache Florentin, *Special Algebraic Structures*, 78-81, in Collected Papers, Vol. III, Abaddaba, Oradea, 2000.

[24] Walendziak A., *On commutative BE–algebra*, Sci. Math. Jpn. online e-(2008), 585-588.

There are three types of Smarandache Algebraic Structures:

1. **A <u>Smarandache Strong Structure</u> on a set S means a structure on S that has a proper subset P with a stronger structure.**
2. **A <u>Smarandache Weak Structure</u> on a set S means a structure on S that has a proper subset P with a weaker structure.**
3. **A <u>Smarandache Strong-Weak Structure</u> on a set S means a structure on S that has two proper subsets: P with a stronger structure, and Q with a weaker structure.**

By *proper subset* of a set S, one understands a subset P of S, different from the empty set, from the original set S, and from the idempotent elements if any.

Having two structures {u} and {v} defined by the same operations, one says that *structure {u} is stronger than structure {v}*, i.e. {u} > {v}, if the operations of {u} satisfy more axioms than the operations of {v}.

Each one of the first two structure types is then generalized from a 2-level (the sets $P \subset S$ and their corresponding strong structure $\{w_1\}>\{w_0\}$, respectively their weak structure $\{w_1\}<\{w_0\}$) to an n-level (the sets $P_{n-1} \subset P_{n-2} \subset \ldots \subset P_2 \subset P_1 \subset S$ and their corresponding strong structure $\{w_{n-1}\} > \{w_{n-2}\} > \ldots > \{w_2\} > \{w_1\} > \{w_0\}$, or respectively their weak structure $\{w_{n-1}\} < \{w_{n-2}\} < \ldots < \{w_2\} < \{w_1\} < \{w_0\}$). Similarly for the third structure type, whose generalization is a combination of the previous two structures at the n-level.

A *Smarandache Weak BE-Algebra* X is a BE-algebra in which there exists a proper subset Q such that $1 \in Q$, $|Q| \geq 2$, and Q is a CI-algebra.

And a *Smarandache Strong CI-Algebra* X is a CI-algebra X in which there exists a proper subset Q such that $1 \in Q$, $|Q| \geq 2$, and Q is a BE-algebra.

The book elaborates a recollection of the BE/CI-algebras, then introduces these last two particular structures and studies their properties.

ISBN 9781599732411

9 781599 732411 90000 >