W.B. VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

SPECIAL DUAL LIKE NUMBERS AND LATTICES

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W. B. Vasantha Kandasamy
Florentin Smarandache

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DEDICATION

A.P.Kotelnikov

We dedicate this book to A.P. Kotelnikov. The algebra of dual numbers has been originally conceived by W.K. Clifford, but its first applications to mechanics are due to A.P. Kotelnikov. The original paper of A.P. Kotelnikov, published in the Annals of Imperial University of Kazan (1895), is reputed to have been destroyed during the Russian revolution.

PREFACE

In this book the authors introduce a new type of dual numbers called special dual like numbers.

These numbers are constructed using idempotents in the place of nilpotents of order two as new element. That is $x = a + bg$ is a special dual like number where a and b are reals and g is a new element such that $g^2 = g$. The collection of special dual like numbers forms a ring. Further lattices are the rich structures which contributes to special dual like numbers. These special dual like numbers $x = a + bg$; when a and b are positive reals greater than or equal to one we see powers of x diverge on; and every power of x is also a special dual like number, with very large a and b. On the other hand if a and b are positive reals lying in the open interval (0, 1) then we see the higher powers of x may converge to 0.

Another rich source of idempotents is the Neutrosophic number I, as $I^2 = I$. We build several types of finite or infinite rings using these Neutrosophic numbers. We also define the notion of mixed dual numbers using both dual numbers and special dual like numbers. Neither lattices nor the Neutrosophic number I can contribute to mixed dual numbers. The two sources are the linear operators on vector spaces or linear algebras and the modulo integers Z_n; n a suitable composite number, are the ones which contribute to mixed dual numbers.

This book contains seven chapters. Chapter one is introductory in nature. Special dual like numbers are introduced in chapter two. Chapter three introduces higher dimensional special dual like numbers. Special dual like neutrosophic numbers are introduced in chapter four of this book. Mixed dual numbers are defined and described in chapter five and the possible applications are mentioned in chapter six. The last chapter has suggested over 145 problems.

We thank Dr. K.Kandasamy for proof reading and being extremely supportive.

W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this book the authors for the first time introduce the new notion of special dual like numbers. Dual numbers were introduced in 1873 by W.K. Clifford.

We call a number a + bg to be a special dual like number if a, b $\in$ R (or Q or Z_n or C) and g is a new element such that $g^2 = g$.

We give examples of them.

The natural class of special dual like numbers can also be got from $\langle Z \cup I \rangle = \{a + bI \mid a, b \in Z, I^2 = I$, I the indeterminate$\}$ ($\langle Q \cup I \rangle$ or $\langle Z_n \cup I \rangle$ or $\langle R \cup I \rangle$ or $\langle C \cup I \rangle$).

Thus introduction of special dual like numbers makes one identify these neutrosophic rings as special dual like numbers.

Apart from this in this book we use distributive lattices to build the special dual like numbers.

For S = $\{a + bg \mid a, b \in R$ and $g \in L$, L a lattice$\}$ paves way to a special dual like number as $g \cap g = g$ and $g \cup g = g$ that is every element in L is an idempotent under both the operations on L. However if we are using only two dimensional special dual like numbers we do not need the notion of distributivity in lattices. Only for higher dimensional special dual like numbers we need the concept of distributivity.

Further the modulo numbers Z_n are rich in idempotents leading one to construct special dual like numbers.

We in this book introduce another concept called the mixed dual numbers. We call $x = a_1 + a_2g_1 + a_3g_2$, $a_1, a_2, a_3 \in Q$ (or Z or C or Z_n or R) and g_1 and g_2 are new elements such that $g_1^2 = 0$ and $g_2^2 = g_2$ with $g_1g_2 = g_2g_1 = 0$ (or g_1 or g_2 'or' used in the mutually exclusive sense) as a mixed dual number.

We generate mixed dual numbers only from Z_n. However we can use linear operators of vector spaces / linear algebras to get mixed dual numbers.

Study in this direction is also carried out. We construct mixed dual numbers of any dimension. However the dimension of mixed dual numbers are always greater than or equal to three. Only Z_n's happen to be a rich source of these mixed dual numbers. We have constructed other algebraic structures using these two new numbers.

For more about vector spaces, semivector spaces and rings refer [19-20].

Chapter Two

SPECIAL DUAL LIKE NUMBERS

In this chapter we introduce a new notion called a special dual like number.

The special dual like numbers extend the real numbers by adjoining one new element g with the property $g^2 = g$ (g is an idempotent). The collection of special dual like numbers forms a particular two dimensional general ring.

A special dual like number has the form $x = a + bg$, a, b are reals, with $g^2 = g$; g a new element.

Example 2.1: Let $g = 4 \in Z_{12}$, $a, b \in R$ any real $x = a + bg$ is a special dual like number

$$x^2 = (a + bg)(a + bg) = a^2 + (2ab + b^2)g$$

$$= A + Bg \text{ (using } g^2 = g) \text{ only if } 2a = -b \text{ (as } b \neq 0).$$

If $b = -2a$ then we see $x = a - 2ag$ and $x^2 = a^2 + (4a^2 - 4a^2)g = a^2$ only the real part of it.

However if $x = a + bg$ and $y = c + dg$, $xy \neq bg$ for any real a, b, c, d in R or Q or Z as $a \neq 0$ $b \neq 0$, $c \neq 0$ and $d \neq 0$.

We just describe the operations on special dual like numbers.

Suppose $x = a_1 + b_1g$ and $y = c_1 + d_1g$ then $x \pm y = (a_1 \pm c_1) + (b_1 \pm d_1)g$, the sum can be a special dual like number or a pure number. If $a_1 = \pm c_1$ then $x \pm y$ is a pure part of the special dual like and is of the form $(b_1 \pm d_1)g$.

If $b_1 = \pm d_1$ then $x \pm y$ is a pure number $a_1 \pm c_1$.

We see unlike dual numbers in case of pure part of dual like number the product is again a pure dual number as $g^2 = 0$; where as in case of dual number the product will be zero as $g^2 = 0$.

We will show by some simple examples.

Let $g = 5 \in Z_{10}$ we see $g^2 = g$. Consider $x = 7 + 6g$ and $y = –7 + 3g$ any two special dual like numbers.

$x + y = 9g$ and $x – y = 14 + 3g$ so $x + y$ is a pure dual number where as $x – y$ is again special dual like number. Now take $x = 7 + 6g$ and $y = –7 + 3g$ we find the product of two special dual like numbers.

$$\begin{aligned} x \times y &= (7 + 6g) \times (–7 + 3g) \\ &= –49 – 42g + 21g + 18g^2 \qquad (\because g = g^2) \\ &= –49 – 3g \text{ is again a special dual like number.} \end{aligned}$$

This if $x = a + bg$ and $y = c + dg$ be any two special dual like numbers then $x \times y = (a + bg)(c+dg) = ac + bcg + dag + bdg^2$

$$= ac + (bc + da \;\; bd)g.$$

Now the product of two special dual like numbers can never be a pure dual number for $ac \neq 0$ as a and c are reals.

The product xy is a real number only if bc + da + bd = 0, that is

$$c + d = \frac{-da}{b} \text{ or}$$

$$a + b = \frac{-bc}{d}$$

For (3 + 2g) (5 – 2g) = 15 so that it is a pure real number.

THEOREM 2.1: *Let x = a + bg be a given special dual like number where $g^2 = g$; a, b ∈ R. We have infinitely many y = c+dg such that xy = real and is not a special dual like number.*

The proof is direct.

However for the reader to follow we give an example.

Example 2.2: Let x = 3 + 5g where g = 3 ∈ Z_6 be a special dual like number.

Let y = a + bg (a, b ∈ R), such that xy = A + 0g

$$\begin{aligned} \text{Consider } x \times y &= (3 + 5g)(a + bg) \\ &= 3a + 5ag + 3bg + 5bg \\ &= 3a + g(5a + 8b) \end{aligned}$$

Given 5a + 8b = 0 so that we get 5a = –8b we have infinite number of non zero solutions.

Thus for a given special dual like number we can have infinite number of special dual like numbers such that the product is real that is only real part exist.

Further it is pertinent to mention the convention followed in this book.

If $x = a + bg$ ($g = g^2$) $a, b \in R$ we call a the pure part of the special dual like number and b as the pure dual part of the special dual like number.

THEOREM 2.2: *Let $x = a + bg$ be a special dual like number ($a, b \in R \setminus \{0\}$) then for no special dual like number $y = c + dg$; $c, d \in R \setminus \{0\}$; we have the pure part of the product to be zero. That is the pure product of xy is never zero.*

Now we see this is not the case with '+' or '–'.

For if $x = -7 + 8g$ and $y = 7 - 5g$ be two special dual like numbers then $x + y = 3g$, this special dual like numbers sum has only pure dual part and pure part of $x + y$ is 0.

However for a given $x = a+bg$ we have a infinitely many $y = c+dg$ such that $x + y = 0 + (b+d)g$. This y's are defined as the additive inverse of the pure parts of x and vice versa.

Similarly if $x = 3 - 5g$ and $y = 8 + 5g$ be any two special dual like numbers we see $x + y = 11 - (0) g$ that is $x + y$ is only the pure part of the special dual like number.

Thus we have the following to be true. For every $x = a + bg$ there exists infinitely many y; $y = c+dg$ such that $x + y = (a + c) + (0)g$ these y's will be called as additive inverse of the x.

Now for a given special dual like number $x = a + bg$ we have a unique $y = -a - bg$ such that $x + y = (0) + (0)g$. This y is unique and is defined as the additive inverse of x.

Inview of all these we have the following theorem the proof of which is left as an exercise to the reader.

THEOREM 2.3: *Let $x = a + bg$ be a special dual like number $g^2 = g$ ($a, b \in R$ or Q or Z).*

(i) we have infinitely many $y = d + (-b)g$; $d \in R \setminus \{0, -a\}$ such that $x + y = a + d + (0)g$ pure part.

(ii) for $x = a + bg$ are have infinitely may $y = -a + dg$, $d \in R \setminus \{0, -b\}$ such that $x + y = 0+(b+d)g$, the pure dual part.

(iii) for a given special dual like number $x = a+ bg$ we have a unique $y = -a -bg$ such that $x + y = (0) + (0)g$. This y is defined as the additive inverse of x.

Now we proceed onto give some notations followed in this book.

$R(g) = \{a + bg \mid a, b \in R; g^2 = g\}$,
$Q(g) = \{a + bg \mid a, b \in Q, g^2 = g\}$,
$Z(g) = \{a + bg \mid a, b \in Z \text{ and } g^2 = g\}$ and
$Z_n(g) = \{a + bg \mid a, b \in Z_n, g^2 = g \text{ and p a prime}\}$.

Following these notation we see that
$R(g) = \{\text{collection of all special dual like numbers}\}$.

Clearly $R \underset{\neq}{\subset} R(g)$ ($Q(g)$ or $Z(g)$ or $Z_n(g)$, n a prime and $g^2 = g$).

THEOREM 2.4: *$R(g) = \{a + bg \mid a, b \in R$ where $g^2 = g\}$ be $Z_n(g)$ the collection of special dual like numbers, $R(g)$ is an abelian group under addition.*

The proof is direct and hence left as an exercise to the reader.

Now we just see how product $\times$ occurs on the class of special dual like numbers.

Let $x = a + bg$ and $y = c + dg$ be any two special dual like numbers. $xy = ac + (ad + bc + db)g$, we see if $a, b, c, d \in R \setminus \{0\}$, $xy \neq (0)$ for all $x, y \in R(g)$. If a or $c = 0$ then $xy = bdg$

$\neq (0)$. If b or d = 0 then xy = ac $\neq$ (0). Thus xy $\neq$ (0) whatever be a, b, c, d $\in R \setminus \{0\}$. However in the product xy the pure dual part can be zero if ad + bc + db = 0.

Thus if 3 + 2g = x is a special dual like number then the inverse of x is a unique y such that xy = 1 + 0 (g). That is y = 1/3 – 2/5g is the special dual like number such that xy = (3+2g) (1/3 – 2/15g)

$$= 3 \times \frac{1}{3} + \frac{29}{3} - \frac{3.2}{15} g - \frac{2.29}{15}$$

$$= 1 + \left(\frac{2}{3} - \frac{6}{15} - \frac{4}{15}\right) g$$

$$= 1 + 0.g$$

$$= 1.$$

But all elements in R(g) is not invertible. For take 5g $\in$ R(g) we do not have a y in R(g) such that y × 5g = 1. Hence only numbers of the for x = a + bg with a, b $\in R\setminus\{0\}$ has inverse. If b = 0 of course x $\in$ R has a unique inverse. If a + bg, a $\neq$ –b then only we have inverse.

Inview of all these observations we have the following theorems.

THEOREM 2.5: *Let R(g) (or Q(g)) be the collection of all special dual like numbers.*

(i) Every x $\in \{a + bg \mid a, b \in R \setminus \{0\}$ and $g^2 = g, a \neq -b\}$ has a unique inverse with respect to product ×.

(ii) R(g) has zero divisors with respect to ×.

(iii) x $\in \{bg \mid b \in R \setminus \{0\}, g^2 = g\}$ has no inverse in R(g).

The proof of this theorem is direct and need only simple number theoretic techniques. All element a – ag are zero divisors for (a – ag)g = ag – ag = 0.

THEOREM 2.6: *Let R(g) (Q(g) or Z(g)) be the collection of all special dual like numbers (R(g), ×) is a semigroup and has zero divisors.*

This proof is also direct and hence left as an exercise to the reader.

THEOREM 2.7: *Let*
$(R(g), \times, +) = \{a + bg \mid a, b \in R, g^2 = g, \times, +\}$. $\{R(g), \times, +)$ is a commutative ring with unit 1 = 1 + 0.g.

This proof is also direct.

Corollary 2.1: (R(g), +, ×) is not an integral domain.

We can have for g matrices which are idempotent linear operators or g can be the elements of the standard basis of a vector space.

We will illustrate these situations by some examples.

Example 2.3: Let

$$R(g) = \{a + bg \mid g = (1, 1, 0, 0, 1, 1, 0, 1);\ a, b \in R\}$$

be the general ring of special dual like numbers.

Example 2.4: Let

$$Q(g) = \{a + bg \mid g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix},$$

$$g \times_n g = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, a, b \in Q\}$$

be the general ring of special dual like numbers.

Example 2.5: Let

$$Z(g) = \{a + bg \mid g = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, g \times_n g = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, a, b \in Z\}$$

be the general ring of special dual like numbers.

Example 2.6: Let

$$Z_5(g) = \{a + bg \mid g \times g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}$$

is the general ring of special dual like numbers. $Z_5(g)$ has zero divisors, for $1 + 4g$, $g \in Z_5(g)$ and $g(1+4g) = g + 4g = 5g = 0$ (mod 5) as $g^2 = g$.

Example 2.7: Let

$$Z_{11}(g) = \{a + bg \mid a, b \in Z_{11}, g = (1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0)\}$$

be a general ring of special dual numbers.

$(1 + 10g)g = g + 10g \equiv 0 \pmod{11}$. Thus g is a zero divisor in $Z_{11}(g)$.

Inview of this we have the following theorem.

THEOREM 2.8: *Let $Z_p(g) = \{a + bg \mid g^2 = g$ and $a, b \in Z_p\}$ be a general ring of special dual numbers. $Z_p(g)$ is of finite order and has zero divisors.*

Proof: Clearly order of $Z_p(g)$ is p^2 and for $1 + (p-1)g$ and $g \in Z_p(g)$ we have $(1+(p-1)g)g = g + (p-1)g \equiv 0 \pmod p$ as $g^2 = 0$. Hence the claim.

Suppose t + rg and g are in $Z_p(g)$ with $t + r \equiv p \equiv 0 \pmod p$ we see $(t + rg)\, g = tg + rg \equiv 0 \pmod p$. It is pertinent to note that in R(g) all element of the form a – ag, $a \in R \setminus \{0\}$ are zero divisors for $(a–ag) \times g = ag – ag = 0$ as $g^2 = 0$.

Inview of all these we have the following result.

THEOREM 2.9: *Let R(g) (Z(g) or Q(g) or $Z_p(g)$) be general special dual like number ring. R(g) has zero divisors and infact g is a zero divisor.*

Proof: We know $a – ag \in R(g)$ where $a \in R \setminus \{0\}$.

We see $g \in R(g)$ (as $1–g = g.1 = g$) $(a – ag)\, g = ag – ag \equiv 0$ as $g^2 = 0$. Hence the claim.

Now we have the following observations about special dual like number general rings.

Example 2.8: Let $Z_7(g) = \{a + bg \mid g = (1, 1, 0, 1, 0), a, b \in Z_7\}$ be a ring of 7^2 elements. $Z_7(g)$ is the general special dual like number ring.

Consider S = {1 + 6g, 6+g, 2+5g, 5+2g, 3+4g, 4+3g, 0} is a subring of $Z_7(g)$. Clearly $1 + 6g \in S$ is an idempotent of S as

$(1+6g)^2 = 1+6g+6g+36g \pmod 7$.
$= 1 + 6g + 6g + g = 1 + 6g$.

Infact 1 + 6g generates the subring as

$1 + 6g + 1 + 6g = 2 + 5g \pmod 7$
$1 + 6g + 2 + 5g = 3 + 4g \pmod 7$
$3 + 4g + 1 + 6g = 4 + 3g \pmod 7$
$4 + 3g + 1 + 6g = 5 + 2g \pmod 7$
$5 + 2g + 1 + 6g = 6 + g \pmod 7$
$6 + g + 1 + 6g = 0 \pmod 7$.

Hence 1 + 6g generates S additively.

Infact 1 + 6g acts as the multiplicative identity.

For (1+6g) (s) = s for all $s \in S$.

Consider P = {0, g, 2g, 3g, 4g, 5g, 6g} $\subseteq Z_7(g)$. It is easily verified P is also a subring and g acts as the multiplicative identity.

For $2g \times 4g = g \pmod 7$
$3g \times 5g = g \pmod 7$
$6g \times 6g = g \pmod 7$.

So 2g is the inverse of 4g with g as its identity and so on.

Likewise in S we see for (2+5g); (4+3g) is its inverse as (2+5g) (4+3g) = 1+6g.

(6+g) (6+g) = 1 + 6g.
(5+2g) (3+4g) = 1 + 6g
So for 5 + 2g; 3 + 4g is its inverse.

We see the subrings S and P are such that

$S \times P = \{sp \mid \text{for all } s \in S \text{ and } p \in P\} = \{0\}$. We call such subrings as orthogonal subrings. Infact these two are fields of order 7 and infact their product is zero.

Let M = {1 + g, 2+2g, 3+3g, 4+4g, 5+5g, 6+6g, 0} $\subseteq Z_7(g)$. M is an abelian group under addition how ever it is not multiplicatively closed.

For $(1+g)^3 = 1$ and $1 \notin M$.

Also $(1+g)^2 = 1 + 3g \notin M$.
$(3+2g)^3 = 1$ and $(2+2g)^2 = 4 + 5g \notin M$.
$(3+3g)^2 = 2 + 6g \notin M$.
$(3+3g)^3 = 6 \notin M$.
$(4+4g)^2 = 2 + 6g \notin M$.
$(4+4g)^3 = 1$.
$(6+6g)^3 = 6 \notin M$.

Consider $4 + 5g \in Z_7(g)$
$(4+5g)^2 = 2 + 2g$
$(4 + 5g)^3 = 1$.

For $5 + 4g \in Z_7(g)$ we have $3 + g \in Z_7(g)$ is such that $(5 + 4g)(3+g) = 1 \pmod 7$.

Thus $Z_7(g)$ has units subrings, orthogonal subrings and zero divisors.

Example 2.9: Let $Z_5(g) = \{a + bg \mid 5 = g \in Z_{20}, a, b \in Z_5\}$ be the general ring of special dual like numbers.

Take $S = \{0, 1+4g, 2+3g, 3+2g, 4+g\} \subseteq Z_5(g)$, S is a subring of $Z_5(g)$.

$M = \{0, g, 2g, 3g, 4g\} \subseteq Z_5(g)$ is also a subring of $Z_5(g)$.

Take $P = \{0, 1, 2, 3, 4\} \subseteq Z_5(g)$ is a subring.
P is not an ideal of $Z_5(g)$. M is an ideal of $Z_5(g)$.

Consider the subring T generated by $1 + g$; T = {0, 1+g, 2+2g, 3+3g, 4 + 4g, 1+3g, 4+2g, 2+4g, 3+g, 1+2g, 1, 2, 3, 4, 2+g, 3+4g, 3g, g, 2g, 4g, 1+4g, 4+g, 2+3g, 3+2g, 4+3g}.

Now in view of these two examples we have the following result.

THEOREM 2.10: *Let*

$$Z_p(g) = \{a + bg \mid a, b \in Z_p, p \text{ a prime}, g^2 = g\}$$

be the general ring of special dual like numbers.

(i) $S = \{0, g, \ldots, (p-1)g\} \subseteq Z_p(g)$ *is a subring of* $Z_p(g)$ *which is also an ideal of* $Z_p(a)$.

(ii) $T = \{0, 1, 2, \ldots, p-1\}\ Z_p \subseteq Z_p(g)$, *is a subring of* $Z_p(g)$ *which is not an ideal.*

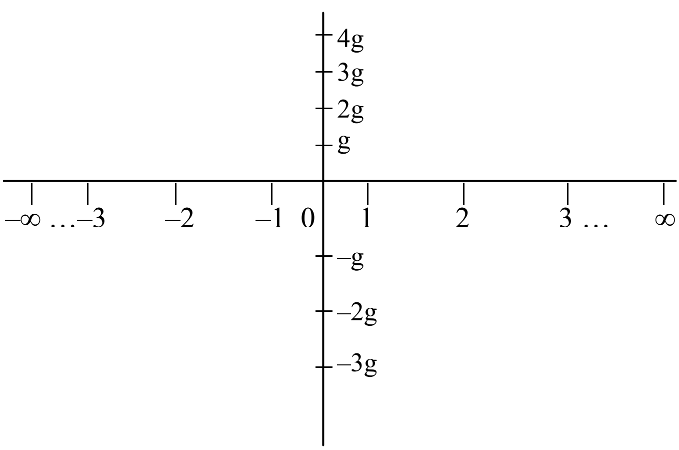
(iii) $P = \{a + bg \mid a + b \equiv 0 \ (\text{mod } p),\ a, b \in Z_p(g) \setminus \{0\}\} \subseteq Z_p(g)$ *is a subring as well as an ideal of* $Z_p(g)$.

(iv) As subrings (or ideals) P and S are orthogonal P.S. = (0). $P \cap S = \{0\}$ *but* $P + S \neq Z_p(g)$.

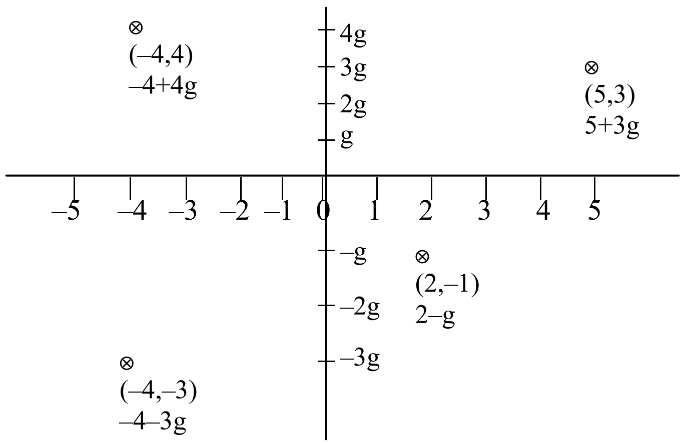
The proof is direct and hence is left as an exercise to the reader.

Consider $R(g) = \{a + bg \mid a, b \in R$; g the new element such that $g^2 = g\}$; the general ring of special dual like numbers.

R the set of reals. Taking the reals on the x-axis and g's on y axis we get the plane called the special plane of dual like numbers.

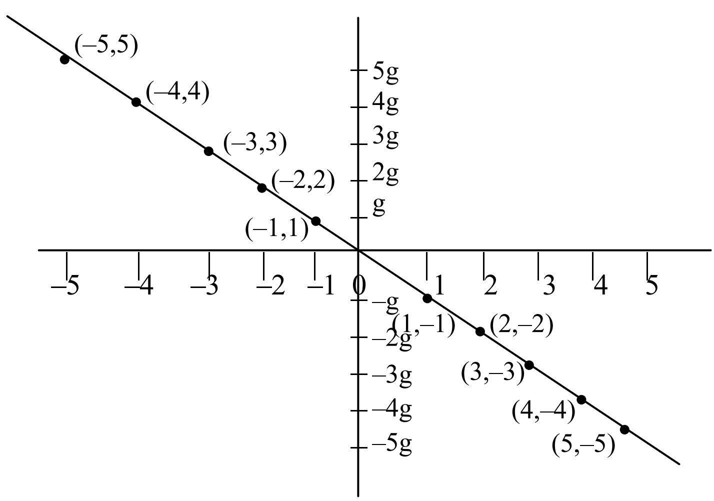


Suppose $-4 - 3g$, $2-g$, $-4+4g$ and $5+3g$ are special dual like numbers then we plot them in the special dual like plane as follows.



We call the y-axis as g-axis.

Now consider the line 1–g, 2–2g, 3–3g, 4–4g, …, 0, –1+g, –2+2g, –3+3g, –4+4g, …, then this can be plotted as follows:



We see in this set represented by the line –a+ag and a–ag for all $a \in R^+$ every element mg on the g-axis is such that $mg \times (-a + ag) = -mag + mag$ (as $g^2 = g$) = 0.

Likewise mg (a–ag) = 0.

Further the set $S = \{-a + ag \mid a \in R^+\}$ is a subring of R(g) known as the orthogonal like line of the line $\{\pm mg \mid m \in R\}$ = P, the g-axis. Further the g-axis is also a subring of R(g).

$$P.S = \{0\} \text{ and } P \cap S = \{0\}.$$

This is another feature of the special dual like numbers which is entirely different from dual numbers.

Now we proceed onto explore other properties related with special dual like numbers.

We can have special dual like number matrices where the matrices will take its entries from R(g) or Q(g) or Z(g) or $Z_p(g)$.

Now we can also form polynomials with special dual like number coefficients $R(g)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in R(g) \right\}$; R(g)[x] is a ring called the general ring of polynomial special dual like number coefficients.

Now we will illustrate how special dual like number matrices with examples.

Example 2.10: Let
$M = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i = x_i + y_i g \in R(g); g^2 = g, 1 \leq i \leq 5\}$ be the collection of row matrices with entries from R(g) M will be also known as the special dual like number row matrices.

We can write $M_1 = \{(x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)g \mid x_i, y_i \in R; 1 \leq i \leq 5 \text{ and } g^2 = g\}$. Clearly both are isomorphic as general ring of special dual like numbers.

If $A = (5 + 2g, 3–g, 4+2g, 0, 1+g)$ and
$B = (8 + g, 3g, 0, 4+g, 1+3g)$ are in M then

$A = (5, 3, 4, 0, 1) + (2, –1, 2, 0, 1)g \in M_1$.
$B = (8, 0, 0, 4, 1) + (1, 3, 0, 1, 3)g \in M_1$.

$$\begin{aligned} \text{Now } A + B &= (13+3g, 3+2g, 4+2g, 4+g, 2+4g) \\ &= (13, 3, 4, 4, 2) + (3g, 2g, 2g, g, 4g). \end{aligned}$$

$$\begin{aligned} \text{Also } A + B &= (5, 3, 4, 0, 1) + (8, 0, 0, 4, 1) + \\ &\quad [(2, –1, 2, 0, 1) + (1, 3, 0, 14)]g. \\ &= (13, 3, 4, 4, 2) + (3, 2, 2, 1, 4)g \end{aligned}$$

We see $A + B \in M (M_1)$.

$$\begin{aligned} \text{Now } A \times B &= (40, 0, 0, 0, 1) + (2, –3, 0, 0, 3)g + \\ &\quad (5, 9, 0, 0, 3)g + (16, 0, 0, 0, 1) \\ &= (40, 0, 0, 0, 1) + (23, 6, 0, 0, 7)g. \end{aligned}$$

$$\begin{aligned} \text{Now } A \times B &= ((5 + 2g) (8 +g), (3–g)3g, \\ &\quad (4+2g)0, 0 \times (4+g), (1+g) (1+3g)) \\ &= (40 + 23g, 6g, 0, 0, 1+7g). \end{aligned}$$

Thus both ways the product is the same. $(M (M_1), +, \times)$ is the general ring of special dual like numbers of row matrices.

Example 2.11: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i = x_i + y_i g, x_i, y_i \in Q; 1 \le i \le 6 \text{ and } g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

be the general ring of special dual like number column matrices.

$$\text{Let } A = \begin{bmatrix} 3+8g \\ -2+g \\ 1+4g \\ 0 \\ -8g \\ 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2g \\ -4 \\ 0 \\ 6+g \\ 1-g \\ -5+2g \end{bmatrix}$$

be any two elements in M.

$$A + B = \begin{bmatrix} 3+10g \\ -6+g \\ 1+4g \\ 6+g \\ 1-9g \\ 1+2g \end{bmatrix} \in M.$$

$$A \times B = \begin{bmatrix} (3+8g)2g \\ (-2+g)-4 \\ (1+4g)\times 0 \\ 0\times(6+g) \\ -8g(1-g) \\ 6\times(-5+2g) \end{bmatrix} = \begin{bmatrix} 22g \\ -8-4g \\ 0 \\ 0 \\ 0 \\ -30+12g \end{bmatrix}.$$

Now A can be represented as

$$A = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \\ 4 \\ 0 \\ -8 \\ 0 \end{bmatrix} g \text{ and B is represented as}$$

$$B = \begin{bmatrix} 0 \\ -4 \\ 0 \\ 6 \\ 1 \\ -5 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} g.$$

$$\text{Now AB} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \\ 0 \\ 6 \\ 1 \\ -5 \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \\ 4 \\ 0 \\ -8 \\ 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} g^2 +$$

$$\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} g + \begin{bmatrix} 8 \\ 1 \\ 4 \\ 0 \\ -8 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \\ 0 \\ 6 \\ 1 \\ -5 \end{bmatrix} g$$

$$= \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ 0 \\ -30 \end{bmatrix} + \begin{bmatrix} 16+6+0 \\ 0+0-4 \\ 0+0+0 \\ 0+0+0 \\ 8+0-8 \\ 0+12+0 \end{bmatrix} g$$

$$= \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ 0 \\ -30 \end{bmatrix} + \begin{bmatrix} 22 \\ -4 \\ 0 \\ 0 \\ 0 \\ 12 \end{bmatrix} g.$$

Thus we see we can write

$$A = \begin{bmatrix} 3+8g \\ -2+g \\ 1+4g \\ 0 \\ -8g \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \\ 4 \\ 0 \\ -8 \\ 0 \end{bmatrix} g.$$

Both the representations are identical or one and the same.

Now we give examples of a general ring of special dual like number square matrices.

Example 2.12: Let

$$S = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \middle| \text{ where } a_i = x_i + y_i g \in Q(g) \text{ with } x_i, y_i \in Q, \right.$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \; g^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \times_n \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\left. = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = g; \; 1 \le i \le 9 \right\}$$

be the general ring of special dual like number square matrices.

$$\text{Let } A = \begin{pmatrix} 8-g & 9g & 0 \\ 1+5g & 2 & -3+2g \\ 0 & -4-g & 1+3g \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} 0 & 2 & 7+9g \\ 3-g & g & 5 \\ -7+2g & g+1 & 0 \end{pmatrix} \in S.$$

$$\text{Now } A + B = \begin{pmatrix} 8-g & 2+9g & 7+9g \\ 4+4g & 2+g & 2+2g \\ -7+2g & -3 & 1+3g \end{pmatrix} \text{ is in S.}$$

Now we can define two types of products on S, natural product $\times_n$ and usual product $\times$. Under natural product $\times_n$, S is a commutative ring and where as under usual product $\times$, S is a non commutative ring.

We will illustrate both the situations.

$A \times_n B = B \times_n A$ for all A, B $\in$ S. Thus (S, +, $\times_n$) is a commutative ring.

Now we find A × B =

$$\begin{pmatrix} 8-g\times 0+9g\times 3-g+0\times -7+2g & 8-g\times 2+9g\times g+0\times g+1 \\ 1+5g\times 0+2\times 3-g+-3+2g\times -7+2g & 1+5g\times 2+2g+-3+2g\times g+1 \\ 0\times 0+-4-g\times 3-g+1+3g\times -7+2g & 0\times 2+-4-g\times g+1+3g\times g+1 \end{pmatrix.$$

$$\begin{matrix} 8-g\times 7+9g+9g\times 5+0\times 0 \\ 1+5g\times 7+9g+2\times 5+-3+2g\times 0 \\ 0\times 7+9g-4-g\times 5+1+3g\times 0 \end{matrix}\right)$$

$$= \begin{pmatrix} 18g & 16+7g & 56+101g \\ 27-18g & -1+13g & 17+89g \\ -19-11g & 1+2g & -20-5g \end{pmatrix} \text{ is in S.}$$

Clearly $A \times B \neq A \times_n B$, further it is easily verified $A \times B \neq B \times A$.

Now we can write A as

$$A = \begin{pmatrix} 8-g & 9g & 0 \\ 1+5g & 2 & -3+2g \\ 0 & -4-g & 1+3g \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 1 & 2 & -3 \\ 0 & -4 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 9 & 0 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} g$$

$$\text{and } B = \begin{pmatrix} 0 & 2 & 7+9g \\ 3-g & g & 5 \\ -7+2g & g+1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 7 \\ 3 & 0 & 5 \\ -7 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 9 \\ -1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} g .$$

$$\text{Now } A \times_n B = \begin{pmatrix} 8 & 0 & 0 \\ 1 & 2 & -3 \\ 0 & -4 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 7 \\ 3 & 0 & 5 \\ -7 & 1 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 8 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & -4 & 1 \end{pmatrix} \times_n \begin{pmatrix} 0 & 0 & 9 \\ -1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} g +$$

$$\begin{pmatrix} -1 & 9 & 0 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 0 & 2 & 7 \\ 3 & 0 & 5 \\ -7 & 1 & 0 \end{pmatrix} g +$$

$$\begin{pmatrix} -1 & 9 & 0 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \times_n \begin{pmatrix} 0 & 0 & 9 \\ -1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} g$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 2 & -15 \\ 0 & -4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -4 & 0 \end{pmatrix} g +$$

$$\begin{pmatrix} 0 & 18 & 0 \\ 15 & 0 & 10 \\ 0 & -1 & 0 \end{pmatrix} g + \begin{pmatrix} 0 & 0 & 0 \\ -5 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} g$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 3 & 2 & -15 \\ 0 & -4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 18 & 0 \\ 9 & 2 & 10 \\ 0 & -6 & 0 \end{pmatrix} g.$$

Now both way natural products are the same

$A \times B =$

$$\begin{pmatrix} 8 & 0 & 0 \\ 1 & 2 & -3 \\ 0 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 7 \\ 3 & 0 & 5 \\ -7 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 1 & 2 & -3 \\ 0 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 9 \\ -1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} g +$$

$$\begin{pmatrix} -1 & 9 & 0 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 7 \\ 3 & 0 & 5 \\ -7 & 1 & 0 \end{pmatrix} g + \begin{pmatrix} -1 & 9 & 0 \\ 5 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 9 \\ -1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} g$$

$$= \begin{pmatrix} 0 & 16 & 56 \\ 27 & -1 & 17 \\ -19 & 1 & -20 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 72 \\ -8 & -1 & 9 \\ 6 & -3 & 0 \end{pmatrix} g +$$

$$\begin{pmatrix} 27 & -2 & 38 \\ -14 & 12 & 35 \\ -24 & 3 & -5 \end{pmatrix} g + \begin{pmatrix} -9 & 9 & -9 \\ 4 & 2 & 45 \\ 7 & 2 & 0 \end{pmatrix} g$$

$$= \begin{pmatrix} 0 & 16 & 56 \\ 27 & -1 & 17 \\ -19 & 1 & -20 \end{pmatrix} + \begin{pmatrix} 18 & 7 & 101 \\ -18 & 13 & 89 \\ -11 & 2 & -5 \end{pmatrix} g$$

is the same as $A \times B$ taken the other way.

Example 2.13: Let

$$P = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix} \middle| \; a_i = x_i + y_i g \in Q(g), \right.$$

$$\left. g = 3 \in Z_6, x_i, y_i \in Q; 1 \leq i \leq 10 \right\}$$

be the general ring of special dual like number 2 × 5 matrix. $(P, +, \times_n)$ is a commutative ring.

$$\text{Let } A = \begin{pmatrix} 2+g & 3 & -4+2g & 0 & g \\ 0 & 5-g & 0 & 1+7g & 3-2g \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} 0 & 8g & 3-g & 0 & 1+5g \\ 1+g & 7 & 0 & 2+7g & -5 \end{pmatrix}$$

be two elements of P.

$$A + B = \begin{pmatrix} 2+g & 3+8g & -1+g & 0 & 1+6g \\ 1+g & 12-g & 0 & 3+14g & -2-2g \end{pmatrix} \in P.$$

$A \times_n B =$

$$\begin{pmatrix} 0 & 24g & -4+2g \times 3-g & 0 & g \times 1+5g \\ 0 & 5-g \times 7 & 0 \times 0 & 1+7g \times 2+7g & 3-2g \times -5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 24g & -12+8g & 0 & 6g \\ 0 & 35-7g & 0 & 2+70g & -15+10g \end{pmatrix}.$$

Now A can also be written as

$$A = \begin{pmatrix} 2 & 3 & -4 & 0 & 0 \\ 0 & 5 & 0 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 7 & -2 \end{pmatrix} g \text{ and}$$

$$B = \begin{pmatrix} 0 & 0 & 3 & 0 & 1 \\ 1 & 7 & 0 & 2 & -5 \end{pmatrix} + \begin{pmatrix} 0 & 8 & -1 & 0 & 5 \\ 1 & 0 & 0 & 7 & 0 \end{pmatrix} g.$$

$$\text{Now } A \times_n B = \begin{pmatrix} 2 & 3 & -4 & 0 & 0 \\ 0 & 5 & 0 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 3 & 0 & 1 \\ 1 & 7 & 0 & 2 & -5 \end{pmatrix} +$$

$$\begin{pmatrix} 2 & 3 & -4 & 0 & 0 \\ 0 & 5 & 0 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 0 & 8 & -1 & 0 & 5 \\ 1 & 0 & 0 & 7 & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 7 & -2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 3 & 0 & 1 \\ 1 & 7 & 0 & 2 & -5 \end{pmatrix} g +$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & 7 & -2 \end{pmatrix} \times \begin{pmatrix} 0 & 8 & -1 & 0 & 5 \\ 1 & 0 & 0 & 7 & 0 \end{pmatrix} g$$

$$= \begin{pmatrix} 0 & 0 & -12 & 0 & 0 \\ 0 & 35 & 0 & 2 & -15 \end{pmatrix} + \begin{pmatrix} 0 & 24 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \end{pmatrix} g +$$

$$\begin{pmatrix} 0 & 0 & 6 & 0 & 1 \\ 0 & -7 & 0 & 14 & 10 \end{pmatrix} g + \begin{pmatrix} 0 & 0 & -2 & 0 & 5 \\ 0 & 0 & 0 & 49 & 0 \end{pmatrix} g$$

$$= \begin{pmatrix} 0 & 0 & -12 & 0 & 0 \\ 0 & 35 & 0 & 2 & -15 \end{pmatrix} + \begin{pmatrix} 0 & 24 & 8 & 0 & 6 \\ 0 & -7 & 0 & 70 & 10 \end{pmatrix} g .$$

We use the second method for the simplification is easy. Thus we see both are the equivalent way of representation.

Now having seen examples of general ring of special dual like number matrices we now represent when the entries are from $Z_p(g)$.

Let

$Z_p(g) = \{a + bg \mid a, b \in Z_p$, g is a new element such that $g^2 = 0\}$ be the general modulo integer ring of special dual like numbers.

We now give examples of them.

Example 2.14: Let

$$V = \left\{a + b\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 12 \\ 12 & 12 & 12 \\ 12 & 0 & 12 \end{bmatrix} \middle| \; a, b \in Z_5,\ 12 \in Z_{132},\right.$$

$$\left. 12 \text{ is the new element as } 12^2 \equiv 12 \pmod{132}\right\}$$

be the general modulo integer ring of special dual like numbers. V is finite, that is V has only finite number of elements in it.

Example 2.15: Let

$$S = \{(a_1, a_2, a_3) + (b_1, b_2, b_3)g \mid a_i, b_j \in Z_{11},$$

$$1 \le i, j \le 3,\ g = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \text{ where } 3 \in Z_9\}$$

be the general modulo integer ring of dual numbers.

Suppose

$$x = (3, 7, 2) + (5, 10, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \text{ and}$$

$$y = (8, 2, 10) + (3, 4, 2)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \in S.$$

We see

$$x + y = (0, 9, 1) + (8, 3, 2) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \text{ and}$$

$$x \times y = (3, 7, 2)\,(8, 2, 10) + (3, 7, 2)\,(3, 4, 2) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} +$$

$$(5, 10, 0)\,(8, 2, 10) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} +$$

$$(5, 10, 0)\,(3, 4, 2) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= (2, 3, 9) + (9, 6, 4) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} +$$

$$(7, 9, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + (0, 0, 0)$$

$$= (2, 3, 9) + (5, 4, 4) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus S is a ring of finite order and of characteristic eleven. S has zero divisors, units, subrings and ideals.

Take $I = \{(a, 0, 0) + (a, 0, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \; a \in Z_{11}\} \subseteq S$, I is an ideal of S.

Consider $M = \{(a, 0, 0) + (0, b, 0) \times \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \mid a, b \in Z_{11}\} \subseteq S$ is only a group under ‘+’ of S.

For x = (a, 0, 0) + (0, b, 0) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and

y = (c, 0, 0) + (0, d, 0) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ we have x + y $\in$ M.

But x × y = (a, 0, 0) (c, 0, 0) + (a, 0, 0) (0, d, 0) × $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ +

$$(0, b, 0)\ (c, 0, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} +$$

$$(0, b, 0)\ (0, d, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \quad (ac, 0, 0) + (o, bd, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

M is only a subring as M is a semigroup under '+'.

Take z = $(x_1, x_2, x_3) + (y_1, y_2, y_3) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

now xz = $(ax_1, 0, 0) + (ay_1, 0, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + (0, x_2b, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$= (ax, 0, 0) + (ay_1, x_2b, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Clearly xz $\notin$ M. Thus M is a subring and not an ideal of S.

Let x = (0, a, 0) + (b, 0, 0) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\text{and } y = (0, 0, c) + (0, 0, d) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

be in S. Clearly $x \times y = (0, 0, 0) + (0, 0, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 0.$

Thus x, y are zero divisors in S for different a, b, c, d $\in Z_{11}$.

However we compare this with $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ where $3 \in Z_6$.

Clearly

$$T = \{(a_1, a_2, a_3) + (b_1, b_2, b_3) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \text{ where } a_i, b_j \in Z_{11},$$

$$1 \leq i, j \leq 3,\ 3 \in Z_6 \text{ so that}$$

$$\left.\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\}$$

is a general ring of special dual like numbers.

Now consider $P = \left\{(a, 0, 0) + (b, 0, 0) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \middle|\ a, b \in Z_{11}\right\} \subseteq T$.

Is P is an ideal of T?

Now (P, +) is an abelian group.

(P, ×) is a semigroup. So $(P, +, \times) \subseteq (T, +, \times)$ is a subring.

Consider $z = (x_1, x_2, x_3) + (y_1, y_2, y_3) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \in T$ and

$$\text{let } x = (a, 0, 0) + (b, 0, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \in P.$$

$$\text{Now } xz = (x_1a, 0, 0) + (y_1b, 0, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} +$$

$$(ay_1\ 0\ 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + (x_1b, 0, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= (x_1a, 0, 0) + (y_1b + ay_1 + x_1b \text{ (mod 11)}, 0, 0\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \in P$$

has P is an ideal of T.

Consider

$$N = \{(x, 0, 0) + (0, y, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \text{ where } x, y \in Z_{11}\} \subseteq T.$$

Is N an ideal of T?

We see (N, +) is an additive abelian group.

Further (N, ×) is a semigroup under ×.

However for $s \in T$ and $n \in N$ we see $sn \notin T$, that is if

$$s = (x_1, x_2, x_3) + (y_1, y_2, y_3)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\text{and } n = (x, 0, 0) + (0, y, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\text{Then sn} = (x_1x, 0, 0) + (xy_1, 0, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} +$$

$$[0, x_2\, y, 0]\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + (0, yy_2, 0)\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= (x_1x\ 0\ 0) + (xy_1, x_2y + yy_2, 0) \times \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \notin N.$$

Thus N is only a subring and not an ideal of T.

Thus we have compared how the general ring of special dual like numbers and general ring of dual number behave.

Example 2.16: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} (4,9,0,4,9) \,\middle|\, 4, 9 \in Z_{12} \text{ and} \right.$$

$$\left. b_i, a_j \in Z_{19}, 1 \leq i, j \leq 5 \right\}$$

be a general ring of special dual like numbers.

We just show how this has zero divisors under the natural product $\times_n$ of M.

M is finite and M has zero divisors and M is commutative.

Further if x = $\begin{bmatrix}3\\0\\1\\2\\4\end{bmatrix}+\begin{bmatrix}5\\4\\2\\1\\3\end{bmatrix}$ (4, 9, 0, 4, 9) ∈ M then

$$x^2 = \begin{bmatrix}9\\0\\1\\4\\16\end{bmatrix} + \left(\begin{bmatrix}15\\0\\2\\2\\12\end{bmatrix}+\begin{bmatrix}15\\0\\2\\2\\12\end{bmatrix}+\begin{bmatrix}6\\16\\4\\1\\9\end{bmatrix}\right) \times (4, 9, 0, 4, 9)$$

$$= \begin{bmatrix}9\\0\\1\\4\\16\end{bmatrix} + \begin{bmatrix}17\\16\\8\\5\\14\end{bmatrix} (4, 9, 0, 4, 9) \in M.$$

Suppose y = $\begin{bmatrix}1\\2\\0\\3\\0\end{bmatrix}+\begin{bmatrix}0\\2\\0\\0\\1\end{bmatrix}$ (4, 9, 0, 4, 9)

and z = $\begin{bmatrix}0\\0\\1\\0\\7\end{bmatrix}+\begin{bmatrix}3\\0\\1\\2\\0\end{bmatrix}$ (4, 9, 0, 4, 9) are in M, then

$$xz = \begin{bmatrix}1\\2\\0\\3\\0\end{bmatrix} \times_n \begin{bmatrix}0\\0\\1\\0\\7\end{bmatrix} + \begin{bmatrix}1\\2\\0\\3\\0\end{bmatrix} \times_n \begin{bmatrix}3\\0\\1\\2\\0\end{bmatrix} (4, 9, 0, 4, 9)$$

$$+ \begin{bmatrix}0\\2\\0\\0\\1\end{bmatrix} \times_n \begin{bmatrix}0\\0\\1\\0\\7\end{bmatrix} (4, 9, 0, 4, 9) + \begin{bmatrix}0\\2\\0\\0\\1\end{bmatrix} \times_n \begin{bmatrix}3\\0\\1\\2\\0\end{bmatrix} (4, 9, 0, 4, 9)$$

$$= \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix} + \left(\begin{bmatrix}3\\0\\0\\6\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\7\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right) (4, 9, 0, 4, 9)$$

$$= \begin{bmatrix}3\\0\\0\\6\\7\end{bmatrix} (4, 9, 0, 4, 9) \in M$$

has no pure part only pure special dual like number part.

Consider $x = \begin{bmatrix}0\\0\\4\\0\\0\end{bmatrix} + \begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix}(4, 9, 0, 4, 9)$

and $y = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\3\\0\end{bmatrix}(4, 9, 0, 4, 9) \in M$.

We see $xy = \begin{bmatrix}0\\0\\4\\0\\0\end{bmatrix} \times_n \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\4\\0\\0\end{bmatrix} \times_n \begin{bmatrix}0\\0\\0\\3\\0\end{bmatrix}(4, 9, 0, 4, 9) +$

$$\begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix} \times_n \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}(4, 9, 0, 4, 9) + \begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix} \times_n \begin{bmatrix}0\\0\\0\\3\\0\end{bmatrix}(4, 9, 0, 4, 9)$$

$$= \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}(4, 9, 0, 4, 9) \in M.$$

Thus M has zero divisors.

We can easily verify M has ideals and subrings which are not ideals.

Example 2.17: Let S =

$$\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \\ b_{10} & b_{11} & b_{12} \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 0 \\ 4 \\ 0 \end{bmatrix} \middle| \; a_i, b_j \in Z_2, 4, 9 \in Z_{12}, \; 1 \leq i, j \leq 12\right\}$$

be a commutative general ring of special dual like numbers.

Suppose x = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ and

y = $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ are in S.

We see x + y = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 0 \\ 4 \\ 0 \end{bmatrix}$.

$x \times_n y =$

$$\begin{bmatrix}1&0&0\\0&1&1\\1&0&1\\0&1&0\end{bmatrix} \times_n \begin{bmatrix}0&1&1\\1&1&1\\1&1&1\\0&0&1\end{bmatrix} + \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\\0&1&0\end{bmatrix} \times_n \begin{bmatrix}1&0&0\\1&1&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}4\\9\\0\\4\\0\end{bmatrix} +$$

$$\begin{bmatrix}1&1&1\\0&1&0\\0&1&1\\1&0&0\end{bmatrix} \times_n \begin{bmatrix}0&1&1\\1&1&1\\1&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}4\\9\\0\\4\\0\end{bmatrix} + \begin{bmatrix}1&1&1\\0&1&0\\0&1&1\\1&0&0\end{bmatrix} \times_n \begin{bmatrix}1&0&0\\1&1&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}4\\9\\0\\4\\0\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0\\0&1&1\\1&0&1\\0&0&0\end{bmatrix} + \left(\begin{bmatrix}1&0&0\\0&1&1\\1&0&0\\0&1&0\end{bmatrix} + \begin{bmatrix}0&1&1\\0&1&0\\0&1&1\\0&0&0\end{bmatrix} + \begin{bmatrix}1&0&0\\0&1&0\\0&1&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}4\\9\\0\\4\\0\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0\\0&1&1\\1&0&1\\0&0&0\end{bmatrix} + \begin{bmatrix}0&1&1\\0&1&1\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}4\\9\\0\\4\\0\end{bmatrix} \in M.$$

This general ring has zero divisors, subrings which are not ideals and ideals.

Example 2.18: Let

$$V = \left\{ \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \right) \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \right| a_i, b_j \in Z_7,$$

$$1 \leq i, j \leq 9, 4, 9 \in Z_{12}\}$$

be a non commutative general ring of special dual like numbers.

Here

$$g = \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \text{ with } 4, 9 \in Z_{12} \text{ and}$$

$$g^2 = g \times_n g = \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \times_n \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} = g$$

is the new element that makes special dual like numbers.

$$\text{Now let } x = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix}$$

$$\text{and } y = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \text{ be in V.}$$

$$\text{Now } x + y = \begin{bmatrix} 5 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 4 \\ 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \in V.$$

$$\text{Consider } x \times y = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 3 \end{bmatrix} +$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \right)^2$$

$$= \begin{bmatrix} 0 & 5 & 5 \\ 1 & 4 & 4 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 1 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 5 & 2 & 6 \\ 1 & 4 & 4 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 4 \\ 2 & 1 & 4 \\ 6 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 5 \\ 1 & 4 & 4 \\ 2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 \\ 5 & 6 & 5 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \in V.$$

Consider $y \times x = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} +$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \left(\begin{bmatrix} 5 & 2 & 1 \\ 0 & 1 & 4 \\ 5 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 & 4 \\ 2 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 4 \\ 1 & 4 & 5 \end{bmatrix} \right) \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 1 \\ 2 & 3 & 3 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 & 9 & 0 \\ 0 & 4 & 0 & 9 \\ 0 & 4 & 9 & 0 \end{bmatrix} \in V.$$

Cleary $xy \neq yx$, this leads to a non commutative general ring of special dual like numbers.

Example 2.19: Let M = $\{(a_{ij}) + (b_{ij})g \mid g$ is a new element such that $g^2 = g$ and (a_{ij}) and (b_{ij}) are 7×7 matrices with entries from $Z_3\}$ be a general non commutative ring of special dual like numbers.

Clearly M is of finite order of characteristic three and has subrings which are not ideals, one sided ideals, ideals and zero divisors.

If on M we define the natural product $\times_n$ then M becomes a commutative general ring of special dual like numbers.

Next we proceed onto define vector spaces using special dual like numbers.

Recall if
$X = \{a + bg \mid g$ is a new element such that $g^2 = g$ and $a, b \in Q\}$, X is an additive abelian group.

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} g \;\middle|\; \text{where } g^2 = g, a_i, b_j \in R, 1 \le i, j \le 4 \right\}$$

is again an additive abelian group.

Let $S = \{(a_1, a_2, \ldots, a_{10}) + (b_1, b_2, \ldots, b_{10})g \mid g^2 = g, a_i, b_j \in Q$ with $1 \le i, j \le 10\}$ is again an additive abelian group.

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \ldots & b_7 \\ b_8 & b_9 & \ldots & b_{14} \\ b_{15} & b_{16} & \ldots & b_{21} \end{bmatrix} g \;\middle|\; g^2 = g; \right.$$

$$\left. a_i, b_j \in Q, 1 \le i, j \le 21 \right\}$$

is again an additive abelian group.

Finally $P = \left\{ \left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} g \right| g^2 = g; \right.$

$a_i, b_j \in Q, 1 \leq i, j \leq 9\}$

is again an abelian group under addition.

Now using these additive groups if define vector spaces over the appropriate fields then we define these vector spaces as special dual like number vector spaces. If there is some product compatible on them we define them as special dual like number linear algebras.

We will illustrate this situation by some examples.

Example 2.20: Let V = {$(a_1, a_2) + (b_1, b_2)$ where $g = 10 \in Z_{30}$, $g^2 = (100) \bmod 30 = 10 = g$ and $a_i, b_j \in Q$, $1 \leq i \leq 2$} be a special dual like number vector space over the field Q.

V has $W = \{(a_1, 0) + (b_1, 0)\, g \mid a_1, b_1 \in Q;\ g^2 = g = 10 \in Z_{30}\} \subseteq V$ and $P = \{(0, a) + (0, b)g \mid a, b \in Q;\ g^2 = g = 10 \in Z_{30}\} \subseteq V$ as subspaces, that is special dual like number vector subspaces of V over the field Q.

Clearly $W \cap P = (0)$ and $W + P = V$, that is V the direct sum of subspaces of V.

Example 2.21: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_7 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_7 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q, 1 \leq i, j \leq 7 \text{ and} \right.$$

$g = (4, 9), 4, 9 \in Z_{12}, g^2 = (4, 9)^2 = (16, 81) \pmod{12} = (4, 9) = 9\}$

be a special dual number vector space over the field Q.

Consider

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q; 1 \le i, j \le 3, g = (4, 9) \right\} \subseteq P,$$

M_1 is a special dual number like vector subspace of P over Q.

Let

$$M_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q; 1 \le i, j \le 2, g = (4, 9) \right\} \subseteq P,$$

M_1 is a special dual number like vector subspace of P over Q.

Consider

$$M_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ b_1 \\ b_2 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q; 1 \leq i, j \leq 2, g = (4, 9) \right\} \subseteq P$$

is a special dual like number vector subspace of P.

Clearly $M_i \cap M_j = (0)$ if $i \neq j$, $1 \leq i, j \leq 3$.

Further $V = M_1 + M_2 + M_3$, that V is a direct sum of special dual like number vector subspaces of P over Q.

Let

$$N_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q; 1 \leq i, j \leq 2, g = (4, 9) \right\} \subseteq P,$$

$$N_2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} g \;\middle|\; a_i, b_j \in Q; 1 \leq i, j \leq 2, g = (4, 9) \right\} \subseteq P,$$

$$N_3 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ b_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} g \;\right|\; a_i, b_j \in Q; g = (4, 9)\ 1 \leq i, j \leq 2 \right\} \subseteq P,$$

$$N_4 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} g \;\right|\; a_i, b_j \in Q; g = (4, 9)\ 1 \leq i, j \leq 2 \right\} \subseteq P \text{ and}$$

$$N_5 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ a_2 \\ 0 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \\ b_2 \\ 0 \\ b_3 \end{bmatrix} g \;\right|\; a_i, b_j \in Q; g = (4, 9)\ 1 \leq i, j \leq 2 \right\} \subseteq P$$

be special dual like number vector subspaces of P.

Clearly $P_i \cap P_j \neq (0)$ if $i \neq j$, $1 \leq i, j \leq 5$.

Further $P \subseteq N_1 + N_2 + N_3 + N_4 + N_5$. Thus P is a pseudo direct sum of subspaces of P over Q.

Example 2.22: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} (3,3,0,3,0) \middle| a_i, b_j \in Q, \right.$$

$$\left. 1 \leq i, j \leq 9, 3 \in Z_6 \right\}$$

be a special dual like number vector space over the field Q. V has subspaces. If on V we define usual matrix product V becomes linear algebra of special dual like numbers which is non commutative.

If on V be define the natural product $\times_n$, V becomes a commutative linear algebra of special dual like numbers.

Example 2.23: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \\ b_{10} & b_{11} & b_{12} \end{bmatrix} g \middle| a_i, b_j \in Q, \right.$$

$$\left. 1 \leq i, j \leq 12, g = (9, 4), 9, 4 \in Z_{12} \right\}$$

be a vector space of special dual like numbers over the field Q. S is a commutative linear algebra if on S we define the natural product.

Now having seen examples of vector spaces and linear algebras of special dual like numbers we can find basis, linear operator, subspaces and linear functionals using them, which is treated as a matter of routine and hence left as an exercise to the reader.

Now we proceed onto define semiring of special dual like numbers and develop their related properties.

For properties of semirings, semifields and semivector spaces refer [19-20].

Let S = {a + bg | a, b ∈ $R^+ \cup \{0\}$, g is the new element, $g^2 = g$}. It is easily verified S is a semiring which is a strict semiring. Infact S is a semifield. The same result holds good if in S, $R^+ \cup \{0\}$ is replaced by $Z^+ \cup \{0\}$ and $Q^+ \cup \{0\}$.

We will illustrate this situation by some examples.

Example 2.24: Let P = {a + bg | a, b ∈ $Z^+ \cup \{0\}$ g = (4, 9) where 4, 9 ∈ Z_{12}, $g^2 = (4, 9)^2 = g$} be the semifield of special dual like numbers.

Example 2.25: Let

$$M = \left\{a + bg \mid a, b \in Q^+ \cup \{0\}, g = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, 3 \in Z_6\right\}$$

be the semifield of special dual like numbers.

Example 2.26: Let

$$M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in Z^+ \cup \{0\}, 1 \leq i \leq 4, \right.$$

$$\left. g \text{ is the new element } (4, 4) \text{ such that } 4 \in Z_{12}\right\}$$

be the semiring of special dual like numbers.

Clearly M is not a semifield for if

$$x = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ 0 \end{bmatrix} \text{ are in M then } x \times_n y = (0).$$

So M is only a commutative strict semiring.

Example 2.27: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & \ldots & b_6 \\ b_7 & b_8 & \ldots & b_{12} \end{bmatrix} g \,\middle|\, 6 = g \in Z_{30} \right.$$

so that $g^2 = 6 \times 6 \pmod{30} = 6 = g$. $a_i, b_j \in Z^+ \cup \{0\}$, $1 \leq i, j \leq 12\}$ be the semiring of special dual like numbers under natural product $\times_n$.

S is not a semifield as S has zero divisors.

Example 2.28: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \\ b_7 & b_8 \\ b_9 & b_{10} \end{bmatrix} g \,\middle|\, a_i, b_j \in Q^+ \cup \{0\}, \right.$$

$$1 \leq i, j \leq 10;\ g = 10 \in Z_{30}\}$$

be the semiring of special dual like numbers. Clearly P is a strict semiring but P is not a semifield as P has zero divisors.

Example 2.29: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i + y_i g \text{ with } g = (4, 4, 4, 4, 9, 9); \right.$$

$$\left. 9, 4 \in Z_{12}; 1 \leq i \leq 9; x_i, y_i \in Q^+ \cup \{0\} \right\}$$

be the matrix semiring of special dual like numbers. S has zero divisors and S is a strict non commutative semiring under usual matrix product and a commutative semiring of matrices under the natural product.

Example 2.30: Let M = $\{(a_1, a_2, \ldots, a_6)$ where $a_i = x_i + y_i g$ with $x_i, y_i \in Z^+$, $1 \leq i \leq 6$, $g = 4 \in Z_{12}\} \cup \{(0, 0, 0, 0, 0, 0)\}$ be a semiring of row matrices of special dual like numbers. M is also a semifield of dual like numbers.

Example 2.31: Now if we take

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \text{with } a_i = x_i + y_i g \ 1 \leq i \leq 5; \right.$$

$$\left. x_i, y_i \in Q^+ \cup \{0\}, g = 6 \in Z_{30} \right\}$$

be the semiring of column vectors under natural product $\times_n$ of special dual like numbers. Clearly P is only a strict semiring and is not a semifield.

Example 2.32: Let W = $\{(a_1, a_2, a_3) \mid a_i = x_i + y = g$ with $x_i, y_i \in R^+$, $1 \leq i \leq 3$, $g = 9 \in Z_{12}\} \cup \{(0, 0, 0)\}$ be a semifield of special dual like numbers.

Example 2.33: Let

$$S = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{array} \right] \right| a_i = x_i + y_i g \text{ with } x_i, y_i \in R^+,$$

$$1 \leq i \leq 15\} \cup \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \right\}$$

be a semifield of special dual like numbers.

Example 2.34: Let

$$S = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{array} \right] \right| a_i = x_i + y_i g \text{ with } x_i, y_i \in Z^+ \cup \{0\},$$

$$1 \leq i \leq 15, g = 6 \in Z_{30}\}$$

be a semiring of special dual like numbers. S is a strict semiring but is not a semifield S has non trivial zero divisors.

Example 2.35: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i+y_ig \text{ where } g = 3 \in Z_6, \right.$$

$$\left. x_i, y_i \in Z^+\cup\{0\}\ 1 \leq i, j \leq 9 \right\}$$

be the non commutative semiring of special dual like numbers. P is not a semifield as P contains zero divisors and P is non commutative.

Example 2.36: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i+y_ig,\ g = 4 \in Z_{12}, \right.$$

$$\left. x_i, y_i \in Z^+\cup\{0\},\ 1 \leq i, j \leq 9 \right\}$$

be the commutative semiring of special dual like numbers under the natural product $\times_n$. M is not a field for M contains zero divisors.

Example 2.37: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i+y_ig,\ g = 4 \in Z_{12}, \right.$$

$$\left. x_i, y_i \in Z^+\cup\{0\}\ 1 \leq i \leq 9 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

be the non commutative semiring which has no zero divisors. Clearly S is not a semifield as the usual product on S is non commutative.

Example 2.38: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i+y_ig,\ g = 4 \in Z_6,\ x_i, y_i \in Q^+, \right.$$

$$\left. 1 \leq i \leq 9 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

S under the natural product $\times_n$ is a semifield.

Now having seen examples of semifields and semirings we wish to bring a relation between S and P. Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i+y_ig,\ g = 9 \in Z_{12},\ x_i, y_i \in Q^+; \right.$$

$$\left. 1 \leq i \leq 9 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ and}$$

$$P = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix} g \middle| x_i, y_i \in Q^+; \right.$$

$$g = 9 \in Z_{12},\ 1 \leq i \leq 9\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g \right\}$$

be two semifields under natural product, $\times_n$.

We can map f : S → P such that for any A ∈ S in the following way.

$$f(A) = f\left(\begin{bmatrix} x_1 + y_1 g & x_2 + y_2 g & x_3 + y_3 g \\ x_4 + y_4 g & x_5 + y_5 g & x_6 + y_6 g \\ x_7 + y_7 g & x_8 + y_8 g & x_9 + y_6 g \end{bmatrix} \right)$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix} g,$$

f is a one to one map so the semifields are isomorphic, be it under natural product $\times_n$ or under usual product, ×.

Consider η : P → S such that

$$\eta\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix} g \right)$$

$$= \begin{bmatrix} x_1 + y_1 g & x_2 + y_2 g & x_3 + y_3 g \\ x_4 + y_4 g & x_5 + y_5 g & x_6 + y_6 g \\ x_7 + y_7 g & x_8 + y_8 g & x_9 + y_6 g \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}.$$

Clearly η is a one to one map of P onto S. P is isomorphic to S as semifield be it under the natural product $\times_n$ or be it under usual product.

Now we will show how addition and natural product / usual product are performed on square matrices with entries from special dual like numbers.

Let $A = \begin{bmatrix} 3+2g & 6+g \\ 5-7g & 1+3g \end{bmatrix}$

$$= \begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix} g \text{ and}$$

$$B = \begin{bmatrix} 1+g & 3-g \\ 4+3g & 5+2g \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g.$$

Now $A \times B = \begin{bmatrix} 3+2g & 6+g \\ 5-7g & 1+3g \end{bmatrix} \times \begin{bmatrix} 1+g & 3-g \\ 4+3g & 5+2g \end{bmatrix}$

$$= \begin{bmatrix} (3+2g)(1+g)+(6+g)(4+3g) \\ (5-7g)(1+g)+(1+3g)(4+3g) \end{bmatrix.$$

$$\begin{matrix} (3+2g)(3-g)+(6+g)(5+2g) \\ (5-7g)(3-g)+(1+3g)(5+2g) \end{matrix}\Bigg]$$

$$= \begin{bmatrix} 3+2g+3g+2g+24+4g+18g+3g & 9+6g-3g-2g+30+12g+5g+2g \\ 5+5g-7g-7g+4+9g+12g+3g & 15-21g-5g+7g+5+15g+6g+2g \end{bmatrix}$$

$$= \begin{bmatrix} 27+32g & 39+20g \\ 9+15g & 20+4g \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 39 \\ 9 & 20 \end{bmatrix} + \begin{bmatrix} 32 & 20 \\ 15 & 4 \end{bmatrix} g \quad \ldots \text{ I}$$

Consider

$$\left(\begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix} g\right)\left(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g\right)$$

$$= \begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} g +$$

$$\begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g$$

$$= \begin{bmatrix} 27 & 39 \\ 9 & 20 \end{bmatrix} + \left(\begin{bmatrix} 6 & 11 \\ 5 & -6 \end{bmatrix} + \begin{bmatrix} 21 & 9 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 2 & 13 \end{bmatrix}\right) g$$

$$= \begin{bmatrix} 27 & 39 \\ 9 & 20 \end{bmatrix} + \begin{bmatrix} 32 & 20 \\ 15 & 4 \end{bmatrix} g \qquad \ldots \text{ II}$$

Clearly I and II are the same.

Now we will find $A \times_n B$

$$= \begin{bmatrix} 3+2g & 6+g \\ 5-7g & 1+3g \end{bmatrix} \times_n \begin{bmatrix} 1+g & 3-g \\ 4+3g & 5+2g \end{bmatrix}$$

$$= \begin{bmatrix} (3+2g)(1+g) & (6+g)(3-g) \\ (5-7g)(4+3g) & (1+3g)(5+2g) \end{bmatrix}$$

$$= \begin{bmatrix} 3+2g+3g+2g & 18+3g-6g-g \\ 20-28g-21g+15g & 5+15g+2g+6g \end{bmatrix}$$

$$= \begin{bmatrix} 3+7g & 18-4g \\ 20-34g & 5+23g \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 18 \\ 20 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -34 & 23 \end{bmatrix} g \qquad \ldots \text{ I}$$

$$A \times_n B = \left(\begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix} g \right) \times_n \left(\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g \right)$$

$$= \begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix} \times_n \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix} \times_n \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} g$$

$$+ \begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix} \times_n \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g + \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix} \times_n \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} g$$

$$= \begin{bmatrix} 3 & 18 \\ 20 & 5 \end{bmatrix} + \left(\begin{bmatrix} 2 & 3 \\ -28 & 15 \end{bmatrix} + \begin{bmatrix} 3 & -6 \\ 15 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -21 & 6 \end{bmatrix} \right) g$$

$$= \begin{bmatrix} 3 & 18 \\ 20 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -34 & 23 \end{bmatrix} g \qquad \ldots \text{ II}$$

I and II are equal.

Now if we consider

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i = x_i + y_i g \text{ with } x_i, y_i \in Z^+, \right.$$

$$\left. g = 3 \in Z_6, 1 \leq i \leq 5 \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be the semifield of special dual like numbers.

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} g \middle| x_i, y_i \in Z^+, \right.$$

$$\left. 1 \leq i \leq 5, g = 3 \in Z_6 \right\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be a the semifield of special dual like numbers.

We see S and P are isomorphic as semifields.

Similarly if

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \end{bmatrix} \middle| \; a_i = x_i + y_i g;\; x_i, y_i \in Q^+, \right.$$

$$\left. 1 \leq i \leq 15,\; g = 3 \in Z_6 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} \right\}$$

be the semifield of special dual like numbers.

Let

$$P = \left\{ \begin{bmatrix} x_1 & x_2 & \ldots & x_5 \\ x_6 & x_7 & \ldots & x_{10} \\ x_{11} & x_{12} & \ldots & x_{15} \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & \ldots & y_5 \\ y_6 & y_7 & \ldots & y_{10} \\ y_{11} & y_{12} & \ldots & y_{15} \end{bmatrix} g \; \middle| \right.$$

$$\left. x_i, y_i \in Q^+,\; 1 \leq i \leq 15,\; g = 3 \in Z_6 \right\}$$

$$\cup \left\{ \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} g \right\}$$

be the semifield of special dual like numbers. As semifields S and P are isomorphic.

Now using this fact either we represent elements as in S or as in P both are equivalent.

Now we can proceed on to define the notion of semiring of polynomial of dual numbers.

Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = t_i + s_i g \text{ with } t_i, s_i \in Q^+, \right.$$

$$\left. g \text{ such that } g^2 = g \right\} \cup \{0\},$$

S is a semifield of polynomials with special dual like numbers as its coefficients.

We can also have the coefficients to be matrices.

For consider $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i = \begin{bmatrix} d_i^1 \\ d_i^2 \\ d_i^3 \\ d_i^4 \end{bmatrix} \right.$ with $d_i^t = m_i^t + n_i^t g$

where $g^2 = g$ and $m_i^t, n_i^t \in Z^+ \cup \{0\}$, $1 \leq t \leq 4\}$; P is only a semiring and is not a semifield as this special dual like number coefficient matrix polynomial ring has zero divisors.

Suppose $M = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_s = \begin{bmatrix} d_i^1 \\ d_i^2 \\ d_i^3 \\ d_i^4 \end{bmatrix} \right.$ with $d_i^t = m_i^t + n_i^t g$

where $g^2 = g$ and $m_i^t, n_i^t \in Z^+$, $1 \leq t \leq 4\} \cup \{0\}$;

M is a semifield with matrix polynomial special dual like number coefficients.

Thus we can have polynomials with matrix coefficients where the entries of the matrices are special dual like numbers.

We give examples of them.

Example 2.39: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} s_1^i & s_5^i \\ s_2^i & s_6^i \\ s_3^i & s_7^i \\ s_4^i & s_8^i \end{bmatrix} \; s_t^i = x_t^i + y_t^i g \text{ with } x_t^i, y_t^i \in Z^+, \right.$$

$$\left. g \text{ is the new element with } g^2 = g \text{ and } 1 \leq t \leq 8 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

be a semifield of special dual like number matrix coefficients.

Example 2.40: Let

$$V = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} p_1^i & p_5^i \\ p_2^i & p_6^i \\ p_3^i & p_7^i \\ p_4^i & p_8^i \end{bmatrix} \text{ where } p_i = x_t^i + y_t^i g \right.$$

$$\text{with } x_t^i, y_t^i \in Z^+ \cup \{0\}$$

and g is the new element such that $g^2 = g$; $1 \leq i \leq 8\}$ be the semiring of special dual like number polynomials with matrix coefficients. Clearly M is not a semifield.

Example 2.41: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_k = (m_{ij})_{6 \times 6},\, m_{ij} = t_{ij} + s_{ij}\, g \text{ with } t_{ij}, s_{ij} \in R^+, \right.$$

$$\left. 1 \leq i, j \leq 36,\, g = 3 \text{ is in } Z_6 \text{ with } g^2 = g = 3 \right\} \cup$$

$$\left\{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\}$$

be the semifield of special dual like number with square matrix coefficient polynomials under the natural product $\times_n$. If the usual product '$\times$' of matrices is taken P is only a semiring as the operation '$\times$' on P is non commutative.

Also if in P, t_{ij}, $s_{ij} \in R^+ \cup \{0\}$, $1 \leq i, j \leq 36$, $g = 3 \in Z_6$ then also P is only a semiring even under natural product $\times_n$ as P has zero divisors.

Thus we have seen examples of various types of semirings and semifields of special dual like numbers.

Now we describe how we get special dual like numbers. In the first place the modulo integers happen to be a very rich structure that can produce the new element 'g' with $g^2 = g$, which is used to construct special dual like numbers.

For take any Z_n, n not a prime and $n \geq 6$ then in most cases we get atleast one new element $g \in Z_n$ such that $g^2 = g \pmod n$.

We just give illustrations.

Consider Z_6, 3, $4 \in Z_6$ are such that $3^2 \equiv 3 \pmod 6$ and $4^2 = 4 \pmod 6$ 3 and 6 are new elements. Consider Z_7, Z_{11} or any Z_p they do not have new elements such that they are idempotents.

In view of this we see if $x \in Z_n$ is an idempotent then $x^2 = x$ so that $x^2 - x = 0$ that is $x^2 + (n-1)x = 0$.

Hence $x (x+n-1) = 0$ as $x \neq 0$ and $x + n-1 \not\equiv 0$.

We see $3^2 = 3 \pmod 6$ $3^2 - 3 \equiv 0 \pmod 6$ that is $3^2 + 5 \times 3 \equiv 0 \pmod 6$ that is $3\,[3 + 5] \equiv 0 \pmod 6$ $3 \times 2 \equiv 0 \pmod 6$. So Z_6 has zero divisors.

$4 \in Z_6$ is such that $4^2 \equiv 4 \pmod 6$ $4 \times (4 + 5) \equiv 0 \pmod 6$ so that $4 \times 3 \equiv 0 \pmod 6$ is a zero divisor. We have 3 and 4 in Z_6 are idempotents. These serve to build special dual like numbers.

Not only we get $a + bg$ and $c + dg_1$, $g = 3$ and $g_1 = 4$ are special dual like numbers but elements like

$$p = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 3 \end{bmatrix} \text{ and } q = \begin{bmatrix} 3 & 4 \\ 4 & 3 \\ 4 & 4 \\ 3 & 3 \\ 3 & 0 \\ 0 & 3 \\ 4 & 0 \end{bmatrix}$$

are also such that $p \times_n p = p \pmod 6$ and $q \times_n q = q \pmod 6$.

If $A = \begin{bmatrix} 3 & 4 & 0 & 3 \\ 4 & 4 & 0 & 4 \\ 3 & 4 & 0 & 4 \end{bmatrix}$ we see $A \times_n A \equiv A \pmod 6$ and so on.

Thus this method leads us to get from these two new elements 3 and 4 infinitely many new elements or to be more in mathematical terminology we see we can using these two idempotents with 0 construct infinitely many $m \times n$ matrices $m, n \in Z^+$ which are idempotents.

Thus using these collection of idempotents we can build special dual like numbers.

Clearly Z_8 has no idempotents, Z_9 has no idempotents, however Z_{10} has idempotents 5, 6 $\in Z_{10}$ are idempotents. Z_{11} has no idempotent. Consider Z_{12}, Z_{12} has 4 and 9 to be idempotents. Z_{14} has 7 and 8 to be idempotents. In Z_{15}, 6 and 10 are idempotents. Z_{18} has 9 and 10 to be their idempotents.

In view of this we have the following three theorems.

THEOREM 2.11: *Let Z_p be the finite prime field of characteristic p. Z_p has no idempotents.*

Proof: Clear from the fact a field cannot have idempotents.

THEOREM 2.12 : *Let Z_{p^2} be the finite modulo integers, p a prime Z_{p^2} has no idempotents.*

Simple number theoretic methods yields the result for if $n \in Z_{p^2}$ is such that $n^2 = n \pmod{p^2}$ then $n(n-1) \equiv 0 \pmod{p^2}$.

Using the fact p is a prime $n^2 \equiv n$ is impossible by simple number theoretic techniques.

However this is true for any Z_{p^n} p a prime, $n \geq 2$.

Example 2.42: Let Z_{27} be the ring of modulo integers. Z_{27} has no idempotents $Z_{27} = Z_{3^3}$.

Example 2.43: Let S = Z_{10} be the ring 5, 6 $\in Z_{10}$ are such that $5^2 = 25 = 5 \pmod{10}$, $6^2 = 36 = 6 \pmod{10}$.

So 5, 6 are idempotents of Z_{10}.

Example 2.44: Let S = Z_{14} be the ring of modulo integers 7, 8 $\in Z_{14}$ are such that $7^2 = 49 = 7 \pmod{14}$, $8^2 = 64 \equiv 8 \pmod{14}$, 8 and 7 are the only idempotents of Z_{14}.

Example 2.45: Let S = Z_{34} be the ring of modulo integers. 17, 18 $\in Z_{34}$ are such that $17^2 \equiv 17 \pmod{34}$ and $18^2 = 8 \pmod{34}$. Thus only 17 and 18 are the idempotents of Z_{34} which is used in the construction of special dual like numbers.

Inview all these examples we have the following theorem.

THEOREM 2.13: *Let S = Z_{2p} (where p is a prime) be the ring of modulo integers. Clearly p, p+1 are idempotents of S.*

Proof is direct using simple number theoretic techniques.

Example 2.46: Let Z_{15} be the ring of modulo integers 6 and 10 are idempotents of Z_{15}.

Example 2.47: Let Z_{21} be the ring of modulo integers. 7 and 15 are the idempotents of Z_{21}.

Example 2.48: Let Z_{33} be the ring of modulo integers. 12 and 22 are idempotents of Z_{33}.

Example 2.49: Let Z_{39} be the ring of modulo integers. 13 and 27 are idempotents of Z_{39}.

Example 2.50: Let Z_{35} be the ring of integers the idempotents in Z_{35} are 15 and 21.

Inview of all these we make the following theorem.

THEOREM 2.14: *Let Z_{pq} (p and q two distinct primes) be the ring of modulo integers Z_{pq} has two idempotent t and m such that t = ap and q = bm, a $\geq$ 1 and m $\geq$ 1.*

The proof is straight forward and uses only simple number theoretic methods.

Example 2.51: Let Z_{30} be the ring of integers. 6, 10, 15, 16, 21 and 25 are idempotents of Z_{30}.

Example 2.52: Let Z_{42} be the ring of integers. 7, 15, 21, 22, 28 and 36 are idempotents of Z_{42}.

Thus we have the following theorem.

THEOREM 2.15: *Let Z_n be the ring of integers. n = pqr where p, q and r are three distinct primes.*

Then Z_n has atleast 6 non trivial idemponents which are of the form ap, bq and cr ($a \geq 1$, $b \geq 1$ and $c \geq 1$).

The proof exploits simple number theoretic techniques.

Example 2.53: Let Z_{210} be the ring of modulo integers. 15, 21, 36, 60, 70, 105, 106, 196, 175, 120, 126, and 85 are some of the idempotents in Z_{210}.

Example 2.54: Let Z_{50} be the ring of modulo integers. 25 and 26 are the only idempotent of Z_{50}.

Now using these idempotents we can construct many special dual like numbers.

Next we proceed on to study the algebraic structures enjoyed by the collection of idempotents in Z_n.

Example 2.55: Let Z_{42} be the ring of modulo integers. We see S = {7, 0, 15, 21, 22, 28 and 36} are idempotents of Z_{42} we give the table under ×. However under '+' we see S is not even closed.

×	0	7	15	21	22	28	36
0	0	0	0	0	0	0	0
7	0	7	21	21	28	28	0
15	0	21	15	21	36	0	36
21	0	21	21	21	0	0	0
22	0	28	36	0	22	28	36
28	0	28	0	0	28	28	0
36	0	0	36	0	36	0	36

(S, ×) is a semigroup. Thus product of any two distinct idempotents in S is either an idempotent or a zero divisor.

That is for a, b ∈ S.
We have $a \times b = 0 \pmod{42}$
or $(a \times b) = c \pmod{42}$, $0 \neq c \in S$
or $a \times b = b \pmod{42}$
or $a \times b = a \pmod{42}$.

We call this semigroup as special dual like number associated component semigroup of S.

Example 2.56: Let Z_{30} be the ring of modulo integers.

S = {0, 6, 10, 15, 16, 21, 25} ⊆ Z_{30} be the collection of idempotents of Z_{30}. Clearly S is not closed under '+' modulo 30.

The table for S under × is as follows:

×	0	6	10	15	16	21	25
0	0	0	0	0	0	0	0
6	0	6	0	0	6	6	0
10	0	0	10	0	10	0	10
15	0	0	0	15	0	15	15
16	0	6	10	0	16	6	10
21	0	6	0	15	6	21	15
25	0	10	10	15	21	15	25

(S, ×) is a semigroup which is the special dual like number associated semigroup. If we want we can adjoin '1'. The unit element as $1^2 = 1 \pmod{n}$. Now we cannot give any other structure. Further S is not an idempotent semigroup also.

We can call it as an idempotent semigroup provided we accept '0' as the idempotent and xy = 0 (x ≠ 0 and y ≠ 0) then interpret 'xy = 0' as not zero divisor but again an idempotent.

THEOREM 2.16: *Let Z_m be the ring of modulo integers. m = 2p where p is a prime. S = {0, p, p+1} $\subseteq Z_m$ is a semigroup with p(p+1) = 0 (mod m).*

Proof : $p(p+1) = p^2 + p = p(p+1)$ as p+1 is even as p is a prime. So $p(p+1) \equiv 0 \pmod{m}$. Hence the claim. (S, ×) is a semigroup.

We see in case of Z_{33}, 22 and 12 are the idempotents of Z_{33}. We see $22 \times 12 \equiv 0 \pmod{33}$. Further S = {0, 12, 22} $\subseteq Z_{33}$ is a semigroup.

Thus we see as in case of Z_{2p} the ring Z_{3p}, p a prime also behaves. Infact for Z_{35}, 15 and 21 are idempotents and
$15 \times 21 \equiv 0 \pmod{35}$.

Hence S = {0, 15, 21} $\subseteq Z_{35}$ is a semigroup under product ×.

In view of all these we have the following theorem.

THEOREM 2.17: *Let Z_{pq} (p and q be two distinct primes) be the ring of modulo integers. Let x, y be idempotents of Z_{pq} we see $x \times y \equiv 0 \pmod{pq}$ and S = {0, x, y} $\subseteq Z_{pq}$ is a semigroup.*

The proof requires only simple number theoretic techniques hence left as an exercise to the reader.

Let S = Z_m where m = $p_1p_2 \ldots p_t$, p_i are distinct that m is the product of t distinct primes.

(i) How many idempotents does $Z_m \setminus \{0,1\}$ contain?

(ii) Is P = {$s_1, \ldots, s_n$, 0, 1}, a semigroup where $s_1, \ldots, s_n$ are idempotents of Z_m?

This is left as an open problem for the reader.

Now we proceed on to describe semivector spaces and semilinear algebras of special dual like numbers.

Let M = $\{(a_1, a_2, \ldots, a_9) \mid a_i = x_i + y_ig$ where $x_i, y_i \in Z^+ \cup \{0\}$, g such that $g^2 = g$; $1 \leq i \leq 9\}$ be a semivector space of special dual like numbers over the semifield.

M is also known as the special dual like number semivector space over the semifield $Z^+ \cup \{0\}$.

Clearly M is not a semivector space over the semifields $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$.

Example 2.57: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in \{x_i + y_ig \mid x_i, y_i \in Q^+ \cup \{0\}, \right.$$

$$\left. g = 3 \in Z_6, g^2 = g \; 1 \leq i \leq 5\}\right\}$$

be the semivector space of special dual like numbers over the semifield $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$. If on V we can define $\times_n$ the natural product, V becomes a semilinear algebra.

Example 2.58: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i = \{x_i + y_ig \text{ where } x_i, y_i \in Q^+ \cup \{0\}, \right.$$

$$\left. g = 7 \in Z_{14}, 1 \leq i \leq 4\}\right\}$$

be the semivector space over the semifield $Z^+ \cup \{0\}$.

If we define the usual matrix product × on S then S is a non commutative semilinear algebra.

If on S we define the natural product $\times_n$ then S is a commutative semilinear algebra special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Let $A = \begin{bmatrix} 3+2g & 0 \\ 4+5g & 2+g \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1+3g \\ 2+g & 4+2g \end{bmatrix}$ be in S.

$$A \times B = \begin{bmatrix} 3+2g & 0 \\ 4+5g & 2+g \end{bmatrix} \times \begin{bmatrix} 0 & 1+3g \\ 2+g & 4+2g \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (3+2g)(1+3g) \\ (2+g)^2 & (4+5g)(1+3g)+(2+g)(4+2g) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3+2g+9g+6g^2 \\ 4+4g+g^2 & 4+12g+5g+15g^2+8+4g+4g+2g^2 \end{bmatrix}$$

(using $g^2 = g$)

$$= \begin{bmatrix} 0 & 3+17g \\ 4+5g & 12+42g \end{bmatrix} \in S.$$

Suppose instead of the usual product $\times$ we define the natural product $\times_n$;

$$A \times_n B = \begin{bmatrix} 3+2g & 0 \\ 4+5g & 2+g \end{bmatrix} \times_n \begin{bmatrix} 0 & 1+3g \\ 2+g & 4+2g \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 8+10g+4g+5g^2 & 8+4g+4g+2g^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 8+19g & 8+10g \end{bmatrix} \in S.$$

However we see $A \times B \neq A \times_n B$.

Example 2.59: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i = x_i + y_i g \right.$$

$$\left. \text{where } g = 5 \in Z_{10},\ x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 10 \right\}$$

be a semivector space of special dual like number over the semifield $Z^+ \cup \{0\}$.

On P we can define the usual product, however under the natural product $\times_n$, P is a semilinear algebra.

Consider

$$M_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 2 \right\} \subseteq P,$$

$$M_2 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 2 \right\} \subseteq P,$$

$$M_3 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_1 & a_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right| a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10},$$

$$x_i, y_i \in R^+ \cup \{0\}, 1 \le i \le 2\} \subseteq P,$$

$$M_4 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_1 & a_2 \\ 0 & 0 \end{bmatrix} \right| a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10},$$

$$x_i, y_i \in R^+ \cup \{0\}, 1 \le i \le 2\} \subseteq P \text{ and}$$

$$M_5 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_1 & a_2 \end{bmatrix} \right| a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10},$$

$$x_i, y_i \in R^+ \cup \{0\}, 1 \le i \le 2\} \subseteq P$$

be semivector subspaces of the semivector space P. Infact M_1, M_2, M_3, M_4 and M_5 are semivector subspaces of special dual like numbers over the semifield $Z^+ \cup \{0\}$ of P.

$$\text{Clearly } M_i \cap M_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } i \neq j,\ 1 \leq i, j \leq 5 \text{ and}$$

$P = M_1 + M_2 + M_3 + M_4 + M_5$, that is P is the direct sum of special dual like number semivector subspaces of P over the semifield $R^+ \cup \{0\}$.

Suppose

$$T_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 3 \right\} \subseteq P,$$

$$T_2 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \\ a_3 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 3 \right\} \subseteq P,$$

$$T_3 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & 0 \\ 0 & a_2 \\ a_3 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 3 \right\} \subseteq P,$$

$$T_4 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & 0 \\ 0 & 0 \\ 0 & a_2 \\ a_3 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\}, 1 \leq i \leq 3 \right\} \subseteq P,$$

and

$$T_5 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & 0 \\ 0 & 0 \\ 0 & a_2 \\ a_3 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\}, 1 \leq i \leq 3 \right\} \subseteq P$$

be special dual like number semivector subspaces of P over the semifield $R^+ \cup \{0\}$.

We see $T_i \cap T_j = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \right.$ if $i \neq j$, $1 \leq i, j \leq 5$, $a = x + yg$;

$$\left. g \in Z_{10}, x, y \in R^+ \cup \{0\} \right\}.$$

Only in one case

$$T_4 \cap T_5 = \left\{ \begin{bmatrix} 0 & 0 \\ a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ a_2 & 0 \end{bmatrix} \middle| \; a_i = x_i + y_i g,\ g = 5 \in Z_{10}, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \le i \le 2 \right\} \subseteq P.$$

Thus P $\underset{\neq}{\subset}$ $T_1 + T_2 + T_3 + T_4 + T_5$, so P is the pseudo direct sum of special dual like number semivector subspaces of P over the semifield $R^+ \cup \{0\}$.

We have several semivector subspaces of P. P can be represented as a direct sum or as a pseudo direct sum depending on the subsemivector spaces taken under at that time.

Example 2.60: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in \{x_i + y_i g \mid x_i, y_i \in Q^+ \cup \{0\}\}, \right.$$

$$\left. 1 \le i \le 5,\ g = 10 \in Z_{30} \right\}$$

be a semivector space of special dual like numbers over the semifield $Q^+ \cup \{0\}$.

W = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i = \{x_i + y_i g \mid x_i, y_i \in Q^+ \cup \{0\}\}, 1 \le i \le 5, g = 6 \in Z_{30}\}$ be a semivector space of special dual like numbers over the semifield $Q^+ \cup \{0\}$.

Consider T : V → W

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4, a_5),$$

then T is defined as a semilinear transformation from V to W.

Likewise we can define the notion of semilinear operator and semilinear functional of a semivector space of special dual like numbers.

For if A = (3 + 2g, 4 + g, 15 + g, 2g, 0) ∈ V then if f is a semilinear functional from V to $Q^+ \cup \{0\}$, we see
$f(A) = 3 + 4 + 15 + 0 + 0 = 22 \in Q^+ \cup \{0\}$.

So we can define f as a semilinear functional of V.

Thus the study of semilinear functional, semilinear operator and semilinear transformation can be treated as a matter of routine. This task of defining / describing the related properties of these structures and finding $\text{Hom}_{Q^+\cup\{0\}}(V, W)$, $\text{Hom}_{Q^+\cup\{0\}}(V, V)$ and L (V, $Q^+ \cup \{0\}$) are left as exercise to the reader.

We can also define projection and semiprojection on vector spaces and semivector spaces of special dual numbers respectively.

Further both projections as well semiprojections themselves can be used to construct special dual like numbers.

One can do all the study by replacing the semivector space of special dual like numbers by the semilinear algebra of special dual like numbers over the semifield. This study is also simple and hence left for the reader as exercise.

Finally we can define the notion of basis, linearly dependent set and linearly independent set of a semivector space / semilinear algebra of special dual like numbers.

We can also define the notion of set vector space of special dual like numbers and semigroup vector space of special dual like numbers over the field F. We have two or more dual numbers and they are not related in any way we use the concept of set vector space of special dual like numbers.

All these concepts we only describe by examples.

Example 2.61: Let M = $\{a + bg_1, c + dg_2 \mid a, b, c, d \in R, g_1 = 5 \in Z_{10}$ and $g_2 = 3 \in Z_6\}$ be a set vector space of special dual like numbers over the set S = 3Z.

Example 2.62: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, (a_1, a_2, a_3, a_4) \mid a_i = \{x_i + y_i g \text{ with} \right.$$

$$x_i, y_i \in R\}, 1 \leq i \leq 4, g = (3, 4, 3, 4, 3, 4) \text{ where } 3, 4 \in Z_6\}$$

be a set vector space of special dual like numbers over the set S = $\{3Z \cup 5Z \cup 7Z\}$.

Example 2.63: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \end{bmatrix}, (a_1, a_2, a_3) \mid \right.$$

$a_i = \{x_i + y_i g$ with $x_i, y_i \in R\}$, $1 \leq i \leq 20$,

$g = (10, 10, 0, 10, 0)$ where $10 \in Z_{30}\}$

be a set vector space of special dual like numbers over the set $F = 5Z$.

Example 2.64: Let

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i, \sum_{i=0}^{\infty} b_i x^i \,\middle|\, a_i = \{x_i + y_i g_1 \text{ with} \right.$$

$x_i, y_i \in Q$, $g_1 = 5 \in Z_{10}\}$, and $b_j = x_j + y_j g_2$,

$g_2 = 10 \in Z_{30}$, $x_j, y_j \in 3Z\}$

be the set vector space of special dual like numbers over the set $S = 5Z \cup 3Z^+$.

It is pertinent to mention here that we can define subset vector subspaces of special dual like numbers and set vector subspaces of special dual like numbers.

Example 2.65: Let

$M = \{a + bg_1, d + cdg_2, e + fg_3 \mid a, b \in 3Z$, $c, d \in 5Z$

and $e, f \in 11Z^+ \cup \{0\}$ where $g_1 = 4 \in Z_{11}$,

$$g_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 4 \end{bmatrix}, 3, 4 \in Z_6 \text{ and } g_3 = (6, 10, 6, 10), 6, 10 \in Z_{30}\}$$

be the set vector space of special dual like numbers over the set $S = 5Z$.

Example 2.66: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, c + dg_2, \sum_{i=0}^{\infty} d_i x^i \,\middle|\, a_i = \{x_i + y_i g_1 \text{ with} \right.$$

$x_i, y_i \in 13Z$, $1 \leq i \leq 4$, $g_1 = 6 \in Z_{30}$, c, d $\in$ Q,

$$g_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \end{bmatrix}, 4, 3 \in Z_6, d_i = m_i + n_i g_3 \text{ where}$$

$g_3 = (5, 5, 5, 6, 0, 5, 6)$, $5, 6 \in Z_{10}$, $m_i, n_i \in 12Z\}\}$

be a set vector space of special dual like numbers over the set $5Z^+ \cup 3Z$.

Example 2.67: Let

$S = \{a + bg_1, d + cdg_2$ and $e + fg_3 \mid a, b \in Z^+, c, d \in Q^+$ and

$e, f \in 14Z^+$, where $g_1 = (0, 4, 9, 0, 4, 9)$, $4, 9 \in Z_{12}$,

$$g_2 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \\ 3 \end{bmatrix}, 3, 4 \in Z_6 \text{ and } g_3 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} \text{ where } 10, 6 \in Z_{30}\}$$

be the set vector space of special dual like numbers over the set $S = 5Z^+ \cup 8Z^+$.

All properties associated with set vector spaces can be developed in case of set vector spaces of special dual like number without any difficulty. This task is left as an exercise to the interested reader.

Now we proceed onto define a very special set vector spaces which we choose to call as strong special set like vector spaces of special dual like numbers.

DEFINITION 2.1: *Let S = {collection of algebraic structures using special dual like numbers} be a set. Let F be a field if for every x $\in$ S and a $\in$ F*

(i) ax = xa $\in$ S.
(ii) (a + b)x = ax + bx
(iii) a (x+y) = ax + ay
(iv) a.0 = 0
(v) 1.s = s for all x, y, s $\in$ S and a, b, 0 $\in$ F, then we define S to be a strong special set like vector space of special dual like numbers.

We will illustrate this situation by some examples.

Example 2.68: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \begin{bmatrix} x_1 & x_5 & x_9 \\ x_2 & x_6 & x_{10} \\ x_3 & x_7 & x_{11} \\ x_4 & x_8 & x_{12} \end{bmatrix}, (d_1, d_2, \ldots, d_{10}) \mid a_i = m_i + n_i g_1, \right.$$

$d_j = t_j + s_j\, g_3$ and $x_k = p_k + r_k\, g_2$ where $m_i, n_i \in Q$, $1 \leq i \leq 4$, $p_k, r_k \in R$, $1 \leq k \leq 12$ and $t_j, s_j \in Q$; $1 \leq j \leq 10$; with $g_1 = (4, 3, 4)$, $4, 3 \in Z_6$, $g_2 = (17, 18)$, $17, 18 \in Z_{34}$ and

$$\left. g_3 = \begin{pmatrix} 7 & 8 & 7 & 8 \\ 7 & 0 & 8 & 7 \end{pmatrix}, 7, 8 \in Z_{14} \right\}$$

be the strong special set like vector space of special dual like numbers over the field Q. Clearly no addition can be performed on M.

Example 2.69: Let

$$S = \{a + bg_1, c+dg_2, c+fg_3, m + ng_4, x + yg_5 \mid a, b, e, f \in R, c,$$

$$d, m, n, x, y \in Q, g_1 = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \end{bmatrix}, 4, 3 \in Z_6,$$

$$g_2 = \begin{pmatrix} 7 & 8 & 7 \\ 8 & 7 & 8 \\ 8 & 8 & 8 \end{pmatrix}, 8, 7 \in Z_{14}, g_3 = \begin{pmatrix} 10 & 6 & 10 & 6 \\ 6 & 10 & 6 & 10 \end{pmatrix}, 10, 6 \in Z_{30},$$

$$g_4 = \begin{bmatrix} 5 & 6 \\ 6 & 5 \\ 5 & 5 \\ 6 & 6 \end{bmatrix}, 5, 6 \in Z_{10} \text{ and}$$

$$g_5 = \begin{bmatrix} 4 & 9 & 4 & 9 & 4 & 9 \\ 9 & 4 & 9 & 4 & 9 & 4 \end{bmatrix} \text{ with } 9, 4 \in Z_{12}\}$$

be a strong special set like vector space of special dual like numbers over the field Q. We see g_1, g_2, g_3, g_4 and g_5 are idempotents which are unrelated for they take values from distinct Z_n's. No type of compatability can be achieved as it is not possible to define operations on them.

Example 2.70: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, (x_1, x_2, x_3), m + ng_3, \sum_{i=0}^{\infty} t_i x^i \;\middle|\; a_i = r_i + s_i g_1, \right.$$

$x_j = c_j + d_j g_2$, $t_k = q_k + p_k\, g_4$ such that $r_i, s_i, c_j, d_j, q_k, p_k$,

m and n $\in$ Q; $1 \le i \le 3$, $1 \le j \le 3$, $1 \le k \le \infty$; $g_1 = (6, 10, 6)$,

$$g_2 = \begin{bmatrix} 10 \\ 6 \\ 10 \end{bmatrix},\ g_3 = \begin{bmatrix} 6 & 10 & 6 & 10 \\ 6 & 10 & 6 & 10 \\ 6 & 6 & 10 & 10 \end{bmatrix} \text{ and } g_4 = (10, 6)$$

with $10, 6 \in Z_{30}\}$

be a strong special set vector space of special dual like numbers over the field Q.

Though the g_i's are elements basically from Z_{30} that using the idempotents 6 and 10 of Z_{30}, still we see we cannot define any sort of compatible operation on M.

Now on same lines we can define strong special set like semivector space of special dual like numbers over the semifield F.

We only give some examples for this concept.

Example 2.71: Let

$$P = \{a + bg_1,\ c + dg_2,\ m + nd_3 \text{ where } a, b \in Q^+ \cup \{0\},$$
$$c, d \in 3Z^+ \cup \{0\} \text{ and } m, n \in R^+ \cup \{0\};$$

$$g_1 = (3, 4),\ 3, 4 \in Z_6,\ g_2 = \begin{bmatrix} 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 & 4 & 3 & 4 & 3 \end{bmatrix}$$

$$4, 3 \in Z_6 \text{ and } g_3 = \begin{bmatrix} 4 & 3 & 4 \\ 3 & 4 & 4 \\ 4 & 4 & 3 \\ 3 & 4 & 3 \end{bmatrix},\ 4, 3 \in Z_6\}$$

be the strong special set like semivector space of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Clearly no compatible operation on P can be defined. Further P is not a semivector space over $Q^+ \cup \{0\}$ or $R^+ \cup \{0\}$.

Example 2.72: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}, (d_1, d_2, d_3, d_4, d_5), \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \right. \text{ where}$$

$a_i = x_i + y_i g$, $x_i, y_i \in 3Z^+ \cup \{0\}$, $1 \leq i \leq 6$, $d_j = m_j + n_j g$; $m_j, n_j \in 5Z^+ \cup \{0\}$, $1 \leq j \leq 5$, $x_t = r_t + s_t g$; $1 \leq t \leq 4$, $r_t, s_t \in 17Z^+ \cup \{0\}$ and $p_s = q_s + t_s q$, $q_s, t_s \in 43Z^+ \cup \{0\}$; $1 \leq s \leq 3$ with $g = 4 \in Z_{12}\}$ be a strong special set like semivector space of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.73: Let

$$W = \left\{ \sum_{i=0}^{\infty} a_i x^i, \sum_{i=0}^{\infty} b_i x^i, \sum_{i=0}^{\infty} m_i x^i \,\middle|\, a_i = t_i + s_i g_1 + n_j = m_j + n_j g_2 \right.$$

and $m_k = c_k + d_k g_3$ where $t_i, s_i \in 3Z^+ \cup \{0\}$, $m_j, n_j \in 47Z^+ \cup \{0\}$

$$\text{and } c_k, d_k \in 10Z^+ \cup \{0\} \text{ with } g_1 = \begin{bmatrix} 3 & 4 \\ 4 & 3 \\ 3 & 4 \\ 4 & 3 \\ 3 & 4 \end{bmatrix}; 4, 3 \in Z_6,$$

$g_2 = (10, 6, 10, 6, 10, 6)$, $10, 6 \in Z_{30}$ and

$$g_3 = \begin{bmatrix} 11 & 12 & 11 & 12 & 11 \\ 12 & 11 & 12 & 11 & 12 \\ 11 & 11 & 12 & 12 & 12 \end{bmatrix}; 11, 12 \in Z_{22}\}$$

be the strong special set like semivector space of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Example 2.74: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_5 \\ a_2 & a_6 \\ a_3 & a_7 \\ a_4 & a_8 \end{bmatrix}, \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, x + yg \,\middle|\, x, y \in Q^+ \cup \{0\}, \right.$$

$a_i = x_i + y_ig$, $x_i, y_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 8$, $b_j = t_j + s_jg$; $t_j, s_j \in Q^+ \cup \{0\}$, $d_m = a_m + b_mg$, $a_m, b_m \in Q^+ \cup \{0\}$; $1 \leq m \leq 9$, $1 \leq j \leq 4$ and $g = 10 \in Z_{30}\}$ be the strong special set like semivector space of special dual like numbers over the semifield $Q^+ \cup \{0\}$.

Example 2.75: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}, \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \,\middle|\, a_i = x_i + y_ig, b_j \right.$$

$= m_j + n_jg_2$, $c_k = s_k + r_kg_3$ and $d_m = a_m + b_mg_4$ where x_i, x_j, y_i, n_j, s_k, r_k, a_m and $b_m \in \in Q^+ \cup \{0\}$, $1 \leq i, j, k, m \leq 4$.

$$g_1 = (4\ 3\ 4\ 3);\ 4, 3 \in Z_6,\ g_2 = \begin{bmatrix} 10 & 6 \\ 10 & 6 \\ 6 & 10 \end{bmatrix}, 10, 6 \in Z_{30},$$

$$g_3 = \begin{bmatrix} 11 & 12 & 11 & 12 & 11 \\ 12 & 11 & 12 & 11 & 12 \end{bmatrix}; 11, 12 \in Z_{22} \text{ and}$$

$$g_4 = \begin{bmatrix} 6 & 10 & 6 & 10 & 6 & 10 \\ 10 & 6 & 10 & 6 & 10 & 6 \end{bmatrix}, 10, 6 \in Z_{30}\}$$

be the strong special set like semivector space of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

The study of substructures, writing them as direct sum of subspaces, expressing them as a direct sum of pseudo vector subspaces, linear transformation, linear operator and linear functionals happen to be a matter of routine, hence left as an exercise to the reader.

Example 2.76: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, (a_1, a_2, \ldots, a_{12}) \;\middle|\; a_i \in \{x_i + y_i g \right.$$

$$\text{where } x_i, y_i \in Q^+ \cup \{0\} \text{ and } g = \begin{bmatrix} 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{bmatrix} \text{ with}$$

$$3, 4 \in Z_{12}\}; 1 \leq i \leq 12\}$$

be the strong special set like semivector space of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

$$\text{Take } M_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \;\middle|\; a_i \in \{x_i + y_i g \right.$$

$$\text{where } x_i, y_i \in Q^+ \cup \{0\} \; 1 \leq i \leq 8\} \subseteq S,$$

$$M_2 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in \{x_i + y_i g \text{ where } x_i, y_i \in Q^+ \cup \{0\}, \right.$$

$$g = \begin{bmatrix} 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{bmatrix}, \; 1 \leq i \leq 9\} \subseteq S$$

and

$$M_3 = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in \{x_i + y_i g$$

$$\text{where } x_i, y_i \in Q^+ \cup \{0\}, g = \begin{bmatrix} 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{bmatrix}, \; 1 \leq i \leq 12\} \subseteq S$$

are strong special set like semivector subspaces of special dual like numbers of S over the semifield $Z^+ \cup \{0\}$.

Clearly $S = M_1 + M_2 + M_3$ and $M_i \cap M_j = \phi$ if $i \neq j$; $1 \leq i, j \leq 3$. Thus S is the direct sum of semivector subspaces.

Now consider

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}, (a_1, a_2, \ldots, a_{12}) \; \middle| \; a_i = x_i + y_i g \right.$$

$$\text{where } x_i, y_i \in Q^+ \cup \{0\}; 1 \leq i \leq 12,$$

$$g = \begin{bmatrix} 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{bmatrix}, 3, 4 \in Z_{12}\} \subseteq S,$$

$$P_2 = \{(a_1, a_2, \ldots, a_{12}), \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \Bigg| \; a_i = x_i + y_i g$$

with $x_i, y_i \in Q^+ \cup \{0\}; 1 \leq i \leq 12,$

$$g = \begin{bmatrix} 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{bmatrix}, 3, 4 \in Z_{12}\} \subseteq S \text{ and}$$

$$P_3 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \right| \; a_i = x_i + y_i g$$

with $x_i, y_i \in Q^+ \cup \{0\}, 1 \leq i \leq 9, g = \begin{bmatrix} 3 & 4 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{bmatrix},$

$3, 4 \in Z_{12}\} \subseteq S$

be strong special set like semivector subspaces of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Clearly $P_i \cap P_j \neq \phi$ if $i \neq j$; $1 \leq i, j \leq 3$.

Thus $S \supseteq P_1 + P_2 + P_3$ so S is only a pseudo direct sum of semivector subspaces of S over $Z^+ \cup \{0\}$.

We can define $T : S \to S$

$$\text{where } T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}\right) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix},$$

$$T(a_1, a_2, a_3, a_4, a_5, a_6, \ldots, a_{12}) = \begin{bmatrix} a_2 & a_4 & a_6 \\ a_8 & a_{10} & a_{12} \\ a_3 & a_6 & a_9 \end{bmatrix}$$

and

$$T(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}) = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}).$$

Thus T is a special set linear operator on S.

Similarly we can define

$f : S \to Z^+ \cup \{0\}$ as follows:

$$f(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}) = [x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8]$$

where $a_i = x_i + y_ig$; $x_i, y_i \in Q^+ \cup \{0\}$ that is if $\sum x_i = n$ if n is a fraction we near it to a integer.

For instance n = t/s t , s but t/s > ½ = 0.5 then n = 1 if t/s < ½ = 0.5 then n = 0 if t/s = m r/s with r / s < 0.5 then t/s = m if t/s = m+r/s r/s > 0.5 t/s = m+1.

f is a set linear functional on S.

Interested reader can study the properties of basis, linear independent element and linearly dependent elements and so on.

Now we just show we can write a matrix with entries $a_i = x_i + y_ig$ in the form of two matrices that is A + Bg where A and B are matrices with $g^2 = g$, we can define this as the special dual like matrix number.

We will illustrate this situation only by examples.

Example 2.77: Let
$M = \{(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)g \mid x_i, y_i \in Q^+ \cup \{0\}, g^2 = g\}$ be a special dual like row matrix number semiring.

We see $N = \{(a_1, a_2, a_3, a_4) \mid a_i = x_i + y_i g, x_i, y_j \in Q^+ \cup \{0\}, 1 \leq i, j \leq 4, g^2 = g\}$ is a special dual like row matrix number semiring such that M is isomorphic to N, by an isomorphism

$\eta : M \to N$ such that
$\eta((x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4)g)$
$= (x_1 + y_1g, x_2+g_2g, x_3+y_3g, x_4+y_4g) = (a_1, a_2, a_3, a_4).$

Example 2.78: Let

$$T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{10} \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{10} \end{bmatrix} g \;\middle|\; g = \begin{bmatrix} 3 & 4 & 3 & 4 & 3 \\ 4 & 3 & 4 & 3 & 4 \end{bmatrix} \right.$$

$$\left. \text{with } 3, 4 \in Z_6, x_i, y_i \in Z^+ \cup \{0\}, 1 \leq i \leq 10 \right\}$$

be the special dual like column matrix number semiring such that T is isomorphic with

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \;\middle|\; a_i = x_i, y_i g + \ 1 \leq i \leq 10 \text{ and} \right.$$

$$\left. g = \begin{bmatrix} 3 & 4 & 3 & 4 & 3 \\ 4 & 3 & 4 & 3 & 4 \end{bmatrix}, 3, 4 \in Z_6 \text{ with } x_i, y_i \in Z^+ \cup \{0\} \right\}.$$

Example 2.79: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ with} \right.$$

$$\left. g = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, 4, 3 \in Z_6, x_i, y_i \in Z^+ \cup \{0\}, 1 \leq i \leq 9 \right\}$$

be the special dual like square matrix number semiring such that S is isomorphic with

$$P = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix} g \; \middle| \; g = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}, \right.$$

$$\left. 4, 3 \in Z_6, x_i, y_i \in Z^+ \cup \{0\}, 1 \leq i, j \leq 9 \right\}$$

the special dual like square matrix number semiring.

Finally consider the following example.

Example 2.80: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \end{bmatrix} \middle| \; x_i, y_i\, g; \; g = \begin{bmatrix} 7 & 8 & 7 & 8 & 7 \\ 8 & 7 & 8 & 7 & 8 \end{bmatrix}, \right.$$

$$\left. 7, 8 \in Z_{14}; \; x_i, y_i \in Z^+ \cup \{0\}, 1 \leq i \leq 30 \right\}$$

be the special dual like rectangular matrix number semiring. P is isomorphic with

$$Q = \left\{ \left. \begin{bmatrix} x_1 & x_2 & \dots & x_{10} \\ x_{11} & x_{12} & \dots & x_{20} \\ x_{21} & x_{22} & \dots & x_{30} \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & \dots & y_{10} \\ y_{11} & y_{12} & \dots & y_{20} \\ y_{21} & y_{22} & \dots & y_{30} \end{bmatrix} g \right| \right.$$

$$g = \begin{bmatrix} 7 & 8 & 7 & 8 & 7 \\ 8 & 7 & 8 & 7 & 8 \end{bmatrix} \quad 8,7 \in Z_{14};\ x_i, y_i \in Z^+ \cup \{0\},$$

$1 \leq i, j \leq 30\}$ as a semiring.

Now we just show if

$$S[x] = \left\{ \left. \sum_{i=0}^{\infty} a_i x^i \right| a_i = t_i + s_i g \text{ with } g = 7 \in Z_{14};\ t_i, s_i \in Q^+ \cup \{0\} \right\}$$

then S[x] isomorphic with

$$P = \left\{ \sum_{i=0}^{\infty} (t_i) x^i + \sum_{i=0}^{\infty} s_i g x^i \text{ with } g = 7 \in Z_{14};\ t_i, s_i \in Q^+ \cup \{0\} \right\}.$$

For define $\eta : S[x] \rightarrow P$ by $\eta\,(p(x)) = \eta\left(\sum_{i=0}^{\infty} a_i x^i\right)$

$$= \eta \left(\sum_{i=0}^{\infty} (t_i + s_i g) x^i \right) = \sum_{i=0}^{\infty} t_i x^i + \left(\sum_{i=0}^{\infty} s_i x^i \right) g \in P.$$

η is 1-1 and is an isomorphism of semirings.

The results are true if coefficients of the polynomials are matrices with special dual like number entries.

Chapter Three

HIGHER DIMENSIONAL SPECIAL DUAL LIKE NUMBERS

In this chapter we for the first time introduce the new notion of higher dimensional special dual like numbers. We study the properties associated with them. We also indicate the method of construction of any higher dimensional special dual like number space.

Let $x = a + bg_1 + cg_2$ where g_1 and g_2 are idempotents such that $g_1g_2 = 0 = g_2g_1$ and a, b, c are reals. We call x as the three dimensional special dual like number.

We first illustrate this situation by some examples.

Example 3.1: Let $x = a + bg_1 + cg_2$ where $g_1 = 3$ and $g_2 = 4$; 3, $4 \in Z_6$. x is a three dimensional special dual like number.

We see if $y = c + dg_1 + eg_2$ another three dimensional dual like number then $x \times y = (a + bg_1 + cg_2)\ (c + dg_1 + eg_2)$

$= ac + bcg_1 + c^2g_2 + adg_1 + bd\,g_1^2 + cdg_2g_1 + aeg_2$
$+ beg_1g_2 + ce\,g_2^2$

$= ac + (bc + ad + bd)g_1 + (c^2 + ae + ce)g_2.$

We see once again xy is a three dimensional special dual like number.

Thus if g_1 and g_2 are two idempotents such that $g_1^2 = g_1$ and $g_2^2 = g_2$ with $g_1g_2 = g_2g_1 = 0$ then
$R(g_1, g_2) = \{a + bg_1 + cg_2 \mid a, b, c \in R\}$ denotes the collection of all three dimensional special dual like numbers.

Clearly $R(g_1) = \{a + bg_1 \mid a, b \in R\} \subseteq R(g_1, g_2)$ and $R(g_2) = \{a + bg_2 \mid a, b \in R\} \subseteq R(g_1, g_2)$ we see $(R(g_1, g_2), +)$ is an abelian group under addition.

For if $x = a + bg_1 + cg_2$ and $y = c + dg_1 + eg_2$ are in $R(g_1, g_2)$ then $x + y = a + c\ (b+d)g_1 + (c+d)g_2$ is in $R(g_1, g_2)$.

Likewise $x - y = (a-c) + (b-d)g_1 + (c-e)g_2$ is in $R(g_1, g_2)$. Further $x + y = y + x$ for all $x, y \in R(g_1, g_2)$.

$0 = 0 + 0g_1 + 0g_2 \in R(g_1, g_2)$ is the additive identity in $R(g_1, g_2)$. Clearly for every $x = a + bg_1 + cg_2$ in $R(g_1,g_2)$ we have $-x = -a - bg_1 - cg_2$ in $R(g_1, g_2)$ is such that $x + (-x) = (a + bg_1 + cg_2 + (-a - bg_1 - cg_2) = (a-a) + (b-b)g_1 + (c-c)g_2 = 0 + 0g_1 + 0g_2 = 0$, thus for every x in $R(g_1,g_2)$ we see $-x$ is in $R(g_1, g_2)$.

Further if $x = a + bg_1 + cg_2$ and $y = d + eg_1 + fg_2 \in R(g_1, g_2)$ then $x \times y = y \times x$ and $x \times y \in R(g_1, g_2)$. We see $(R(g_1, g_2), \times)$ is a semigroup in fact the semigroup is commutative with unit so is a monoid. Thus it is easily verified $(R(g_1, g_2), +, \times)$ is a ring, infact a commutative ring with unit and has nontrivial zero divisors for ag_1 and bg_2 in $R(g_1, g_2)$ are such that $ag_1 \times bg_2 = 0$, for all $a, b \in R$.

We define $(R(g_1, g_2), +, \times)$ as the special general ring of special dual like numbers.

We call it "special general" as $R(g_1, g_2)$ contains also elements of the form ag_1, bg_2 and c where a, b, c $\in$ R.

Example 3.2: Let $M = \{a + bg_1 + cg_2 \mid a, b, c \in R, g_1 = 7$ and $g_2 = 8, g_1, g_2 \in Z_{14}, g_1^2 = 7, g_2^2 = 8$ and $g_1 \times g_2 = g_2 \times g_1 = 0\}$ be the special general ring of three dimensional special dual like numbers.

In view of this we have the following theorem.

THEOREM 3.1: *Let $R(g_1, g_2)$ ($Q(g_1, g_2)$ or $Z(g_1, g_2)$) $= \{a + bg_1 + cg_2 \mid a, b, c \in R, g_1^2 = g_1, g_2^2 = g_2$ and $g_1g_2 = g_2g_1 = 0\}$. $\{R(g_1, g_2), +, \times\}$ is the special general ring of three dimensional special dual like numbers.*

The proof is direct and hence is left as an exercise to the reader.

Example 3.3: Let

$Z(g_1, g_2) = \{a + bg_1 + cg_2 \mid a, b, c \in Z, g_1 = 5; g_2 = 6, g_1, g_2 \in Z_{10}\}$ be the special general ring of three dimensional special dual like number ring.

Example 3.4: Let $Z(g_1, g_2) = \{a + bg_1 + cg_2 \mid a, b, c \in Z$ and $g_1 = (1\ 0\ 0\ 1\ 0\ 0\ 1)$ and $g_2 = (0\ 1\ 1\ 0\ 0\ 1\ 0)\}$.

We see $g_1^2 = (1\ 0\ 0\ 1\ 0\ 0\ 1) = g_1$ and $g_2^2 = (0\ 1\ 1\ 0\ 0\ 1\ 0) = g_2$ further $g_1\ g_2 = g_2g_1 = (0\ 0\ 0\ 0\ 0\ 0\ 0)$ we see $Z(g_1, g_2)$ is a special general ring of special dual like numbers.

Example 3.5: Let

$$M = \{a + bg_1 + cg_2 \mid a, b, c \in Q, g_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{ and}$$

$$g_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}; \ g_1^2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = g_1 \text{ and}$$

$$\left. g_2^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = g_2 \text{ with } g_1g_2 = g_2g_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

be the special general three dimensional ring of special dual like numbers.

We just show how product is performed.

$$\text{Let } x = 5 + 7 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$y = -2 \ -4 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ be in M.}$$

$$\text{To find } x \times y = -10 + (-14) \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$- 20 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} -28 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$-12 \times (0) + 40 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} + 42\ (0) + 24 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= -10 + (-62) \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} + 70 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \in \text{M.}$$

This is the way product on M is performed.

Example 3.6: Let

$$S = \{a + bg_1 + cg_2 \mid a, b, c \in R, \quad g_1 = \begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} \text{ and } g_2 = \begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix}\}$$

be a special general ring of special dual like numbers.

$$\text{Suppose } x = 3 + 2\begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix} \text{ and}$$

$$y = -3 - 2\begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} + 7\begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix} \text{ are in S,}$$

$$\text{then } x + y = 8 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in S.$$

$$\text{We find } x \times y = (3 + 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}) \times$$

$$(-3 - 2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix})$$

$$= -9 - 6\begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} - 37\begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix} - 6\begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} - 4\begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} + 14\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$+ 21\begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix} + 14\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix} + 7\begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix}$$

$$= -9 - 16\begin{bmatrix}1\\1\\0\\1\\1\\0\\1\end{bmatrix} + 25\begin{bmatrix}0\\0\\1\\0\\0\\1\\0\end{bmatrix} \text{ is in S.}$$

Thus (S, +, ×) is a special general ring of special dual like numbers.

Now we can as in case of dual numbers define general matrix ring of special dual like numbers. However the definition is a matter of routine.

Now we illustrate this situation only by examples.

Example 3.7: Let S = {(a_1, a_2, a_3, a_4, a_5, a_6) | $a_i = x_i + y_1g_1 + z_1g_2$ where x_i, y_i, $z_i \in Q$; $1 \leq i \leq 6$, $g_1 = 4$ and $g_2 = 3$, 3, 4 $\in Z_6$} be the general special ring of special dual like numbers.

We see (S, +) is an abelian group for if $x = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $y = (b_1, b_2, b_3, b_4, b_5, b_6)$ are in S then
$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6)$ is in S.

Consider $x \times y = (a_1, a_2, \ldots, a_6) \times (b_1, b_2, \ldots, b_6)$
$= (a_1b_1, a_2b_2, \ldots, a_6b_6)$, $x \times y \in S$. Thus (S, +, ×) is a special general ring of row matrix special dual like numbers.

Let P = {$a + bg_1 + cg_2$ | $a = (a_1, a_2, \ldots, a_6)$, $b = (b_1, b_2, \ldots, b_6)$ and $c = (c_1, c_2, \ldots, c_6)$ with $g_2 = 3$ and $g_1 = 4$, 3, 4 $\in Z_6$; $3^2 = 3$ (mod 6), $4^2 = 4$ (mod 6) and $3.4 = 4.3 \equiv 0$ (mod 6)} is also a special general ring of row matrices of special dual like numbers. Clearly P is isomorphic with S as rings.

Example 3.8: Let

$$M = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \text{ where } a_i = x_i + y_ig_i + zg_2,\ x_i, y_i, z_i \in Z;\ 1 \leq i \leq 5,$$

$g_1 = 7$ and $g_2 = 8$ with 7, 8 $\in Z_{14}$} be the special general ring of column matrices of special dual like numbers under the natural product $\times_n$.

$$\text{We see if } x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \text{ are in M, then}$$

$$x + y = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 \\ a_5 + b_5 \end{bmatrix} \text{ is in M.}$$

$$\text{We find } x \times_n y = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \times_n \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} a_1b_1 \\ a_2b_2 \\ a_3b_3 \\ a_4b_4 \\ a_5b_5 \end{bmatrix} \in M.$$

$$\text{Suppose } N = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} g_1 + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} g_2 \;\middle|\; x_i, y_i, z_i \in Z, \right.$$

$1 \leq i, j, k \leq 5$ with $g_1 = 7$ and $g_2 = 8$ in $Z_{14}\}$ is again a special general ring of column matrix special dual like numbers.

We see clearly M and N are isomorphic as rings under the natural product $\times_n$.

Example 3.9: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \text{ where } a_i = x_i + y_ig_1 + z_ig_2 \right.$$

with $x_i, y_i, z_i \in Q$; $1 \leq i \leq 12$; $g_1 = (7, 8, 7, 8, 0)$ and

$g_2 = (8, 7, 8, 7, 8)$ with $7, 8 \in Z_{14}\}$

be the special general ring of 3 × 4 matrices of special dual like numbers under natural product $\times_n$.

$$\text{Suppose } x = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ b_9 & b_{10} & b_{11} & b_{12} \end{bmatrix} \text{ are in S,}$$

$$\text{then } x + y = \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 & a_7+b_7 & a_8+b_8 \\ a_9+b_9 & a_{10}+b_{10} & a_{11}+b_{11} & a_{12}+b_{12} \end{bmatrix} \text{ is in S.}$$

$$\text{We find } x \times_n y = \begin{bmatrix} a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \\ a_5b_5 & a_6b_6 & a_7b_7 & a_8b_8 \\ a_9b_9 & a_{10}b_{10} & a_{11}b_{11} & a_{12}b_{12} \end{bmatrix} \in S.$$

Thus (S, +, $\times_n$) is the special matrix general ring of special dual like numbers.

Finally we give an example of the notion of special general square matrix special dual like number ring.

Example 3.10: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i = t_i + s_i g_1 + r_i g_2 \right.$$

where $t_i, s_i, r_i \in Q$, $1 \le i \le 16$, $g_1 = \begin{bmatrix} 13 \\ 14 \\ 0 \\ 13 \\ 14 \end{bmatrix}$ and $g_2 = \begin{bmatrix} 14 \\ 13 \\ 13 \\ 0 \\ 0 \end{bmatrix}$

with $13, 14 \in Z_{26}\}$

be the special general ring special dual like numbers of square matrices under the usual product × or the natural product $\times_n$. Clearly P is a three dimensional commutative ring under $\times_n$.

Now we just show how we can generate the idempotents so that $x = a + bg_1 + cg_2$ forms a three dimensional special dual like numbers.

We get these idempotents from various sources.

(i) From the idempotents of Z_n (n not a prime or a prime power) has atleast two non trivial idempotents.

(ii) From the standard basis of any vector space.

For if x = (1 0 0 0 0 … 0) and y = (0, 1, 0, …, 0) we see $x^2 = x$, $y^2 = y$ and xy = yx = (0, 0, …, 0).

This is true even if $x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$;

$$x \times_n x = x,\ y \times_n y = y \text{ and } x \times_n y = y \times_n x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Also if $x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then

$$x \times_n y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = y \times_n y,\ x \times_n x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and } y \times_n y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Finally if $x = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$

then also $x \times_n y = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} = y \times_n x$ and

$$x \times_n x = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \text{ and } y \times_n y = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

All these idempotents can contribute for three dimensional special dual like number.

(iii) We know if we have a normal operator T on a finite dimensional complex inner product space V or a selfadjoint operator on a finite dimensional real inner product space V.

Suppose $c_1, c_2, \ldots, c_k$ are distinct eigen values of T, W_j's the characteristic space associated with c_j and E_j the orthogonal projection of V on W_j. Then W_j is orthogonal to W_i ($i \neq j$). E_i's are such that $E_i^2 = E_i$, i = 1, 2, …, k so we can have special dual like numbers of higher dimension can be got from this set of projections.

(iv) If we take either the elements of a lattice or a semilattice we get idempotents. All the more if we take the atoms of a lattice say $a_1, \ldots, a_n$ then we always have $a_i \cap a_j = 0$ if $i \neq j$ and $a_i \cap a_i = a_i$; $1 \leq i, j \leq M$. By this method also we can get a collection of special dual like numbers.

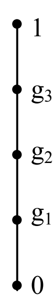
Finally we can construct matrices using these special dual like numbers to get any desired dimension of special dual like numbers.

Now we will illustrate them and describe by a n-dimensional special dual like numbers.

Let $x = a_1 + a_2g_1 + \ldots + a_ng_{n-1}$ be such that $a_i \in R$ (or Q or Z), $1 \leq i \leq n$ and g_j's are such that $g_j^2 = g_j$, $g_j \cdot g_i = g_k$ or 0 if $i \neq j$; $1 \leq i, k, j \leq n-1$. We see $x^2 = A_1 + A_2g_1 + \ldots + A_ng_{n-1}$ where $A_j \in R$ ($1 \leq j \leq n$).

We will first illustrate this situation by some examples.

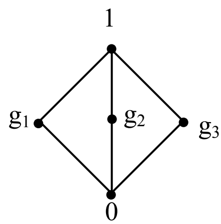
Example 3.11: Let $x = a_1 + a_2g_1 + a_3g_2 + a_4g_3$ where $a_i \in R$;



$1 \leq i \leq 4$ and g_1, g_2 and g_3 are marked in the diagram and $g_i \cap g_j = g_k$ or 0 if $i \neq j$ and $g_i \cap g_i = g_i$; $1 \leq i, j, k \leq 3$.

Of course we can take '$\cup$' as operation and still the compatibility is true.

Example 3.12: Suppose we take $x = a_1 + a_2g_1 + a_3g_2 + a_4g_3$ with $a_i \in Q$; $1 \leq i \leq 4$ and g_1, g_2 and g_3 from the lattice



we see we cannot claim x to be special dual like number of dimension three as this lattice is not distributive.

We so just define the following new concept.

DEFINITION 3.1: *Let F be the field or a commutative ring with unit. L be a distriblute lattice of finite order say n + 1.*

$$FL = \left\{ \sum_i a_i m_i \,\middle|\, a_i \in F \text{ and } m_i \in L;\ 0 \leq i \leq n+1 \right\} \ (L = \{0 =$$

$m_0, m_1, m_2, \ldots, m_{n+1} = 1\}$). We define + and $\times$ on FL as follows:

(1) For $x = \sum a_im_i$ and $y \sum b_im_i$ in FL; $x = y$ in and only if $a_i = b_i$ for $i = 0, \ldots, n+1$.

(2) $0.m_i = 0$, $i = 0, i, \ldots, n+1$ and $am_0 = 0$ for all $a \in F$.

(3) $x + y = \sum (a_i + b_i)\, m_i$ for all $x, y \in FL$.

(4) $x.1 = 1.x = x$ for $m_{n+1} = 1 \in L$ for all $x \in F$.

(5) $x \times y = \sum a_im_i \times \sum b_im_i$

$$= \sum a_ib_j\,(m_i \cap m_j)$$

$$= \sum_k a_k m_k$$

(or equivalently $\sum a_ib_j\,(m_i \cup m_j) = x \times y = \sum a_k\, m_k$).

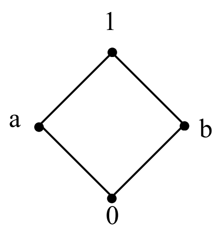
(6) $am_i = m_i a$ for all $a \in F$ and $m_i \in L$.
(7) $x \times (y + z) = x \times y + x \times z$ for all $x, y, z \in FL$.

Thus FL is a ring, which is defined as a ring lattice.

We see the ring lattice is a n-dimensional general ring of special dual like numbers.

We will illustrate this situation by some simple examples.

Example 3.13: Let L =



be a distribute lattice. Q be the ring of rational. QL be the lattice ring.

QL = $\{m_0 + m_1a + m_2b \mid m_0, m_1, m_2 \in Q$ and $a, b \in L\}$.

We just show how product is performed on QL.

Take $x = 5 - 3a + 8b$ and $y = -10 + 8a - 7b$ in QL.

$x + y = -5 + 5a + b \in QL$.
$x \times y = (5 - 3a + 8b)(-10 + 8a - 7b)$
$= -50 + 30a - 80b + 40a - 24a + 8 \times 8 (b \cap a)$
$- 35 b + 21 (a \cap b) - 56b$
$= -50 + 46a - 91b \in QL$.

Thus QL is a three dimensional general ring of special dual like numbers.

Suppose we take '$\cup$' as the operation on QL.

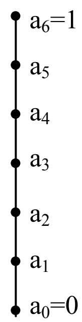
$x \times y = (5 - 3a + 8b)(-10 + 8a - 7b)$

$= -50 + 30a - 80b + 40a - 42a + 8 \times 8 (b \cup a) - 35b + 2 (a \cup b) - 56b$

$= -50 + 46a - 91b + 64 + 21$

$= 35 + 46a - 91b \in QL.$

Example 3.14: Let Z be the ring of integers. L be the chain lattice given by



$ZL = \left\{ \sum_{i=0}^{6} a_i m_i \,\middle|\, m_i \in Z \text{ and } a_i \in L; 0 \le i \le 6 \right\}$ be the lattice ring.

ZL is a 5-dimensional special general ring of special dual like numbers.

Suppose $x = m_1 + m_2 a_1 + m_3 a_2 + m_4 a_3 + m_5 a_4 + m_6 a_5$ and

$y = n_1 + n_2 a_1 + n_3 a_2 + n_4 a_3 + n_5 a_4 + n_6 a_5$ are in ZL, then we can find xy and x + y.

Suppose $y = -7 - 5a_2 + 3a_4 + 6a_5$ and $x = 3 + 4a_1 + 5a_2 - 8a_3$ are in ZL.

$x + y = -4 + 4a_1 + 0 - 8a_3 + 3a_4 + 6a_5$ and

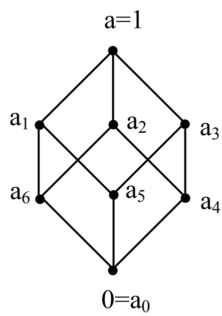
$x \times y = (3 + 4a_1 + 5a_2 - 8a_3)(-7 - 5a_2 + 3a_4 + 6a_5)$

$= -21 - 28a_1 - 35a_2 + 56 a_3 - 15a_2 - 20a_1 - 25a_2 + 40a_2 + 9a_4 + 12a_1 + 15a_2 - 24a_3 + 18a_5 + 24a_1 + 30a_2 - 48a_3$

$= -21 - 12a_1 + 10a_2 - 16a_3 + 9a_4 + 18a_5 \in ZL.$

Thus ZL is a six dimensional general ring of special dual like numbers.

Example 3.15: Let Z be the ring of integers. L be a lattice given by the following diagram.



L is a distribute lattice. ZL be the lattice ring given by ZL = $\{m_1 + m_2a_1 + \ldots + m_6a_6 \mid a_j \in L; m_i \in Z; 1 \leq i \leq 6, 1 \leq j \leq 6\}$.

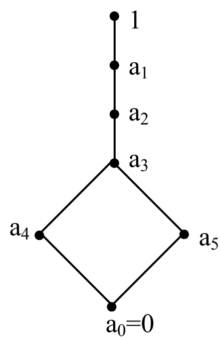
Take $x = 3 + 4a_4 + 5a_6$ and $y = 4 - 2a_2 + 3a_5$ we find x + y and x × y (where product on L is taken as '$\cup$'.

$x + y = 7 - 2a_2 + 4a_4 + 3a_5 + 5a_6.$
$x \times y = (3 + 4a_4 + 5a_6) \times (4 - 2a_2 + 3a_5)$
$= 12 + 16a_4 + 20a_6 - 6a_2 - 8a_2 - 10a_2 + 9a_5 + 12a_3 + 15a_1$
$= 12 + 15a_1 - 24a_2 + 12a_3 + 16a_4 + 9a_5 + 20a_6 \in ZL.$

Suppose we replace '$\cup$' by '$\cap$' on ZL then x × y;
$x \times y = (3 + 4a_4 + 5a_6)(4 - 2a_2 + 3a_5)$
$= 12 + 16a_4 + 20a_6 - 6a_2 - 8a_4 \cap a_2 - 10a_6 \cap a_2 +$
$9a_5 + 12a_5 \cap a_4 + 15a_6 \cap a_4$
$= 12 + 16a_4 + 20a_6 - 6a_2 - 8a_4 - 10a_6 + 9a_5 +$
$12 \times 0 + 15 \times 0.$
$= 12 + 8a_4 + 10a_6 + 9a_5 - 6a_2 \in ZL.$

Clearly x × y ≠ x ⊗ y for we see × is under '∪' and ⊗ is under '∩'.

Example 3.16: Let R be the field of reals. L =



be a lattice. RL be the lattice ring RL is a 5-dimensional general ring of special and like numbers.

Thus lattices help in building special dual like number general ring. However we get two types of general rings of special dual like number rings depending on the operation '∪' or '∩'.

Example 3.17: Let F be a field. M = {(0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0) (0, 0, 1, 0, 0, 0), (0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)} be the semigroup under product. FM = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid g_1$ = (1, 0, 0, 0, 0, 0), g_2 = (0, 1, 0, 0, 0, 0), g_3 = (0, 0, 1, 0, 0, 0), g_4 = (0, 0, 0, 1, 0, 0), g_5 = (0, 0, 0, 0, 1, 0) and g_6 = (0, 0, 0, 0, 0, 1) where $g_i^2 = g_i$, $1 \leq i \leq 6\}$ be the seven dimensional general ring of special dual like numbers.

Example 3.18: Let F = Q be the field.
S = {(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0)} be the idempotent five dimensional general ring of special dual like numbers.

Example 3.19: Let F = R be the field.

S = {(0, 0, …, 0), (1, 0, …, 0) … (0, 0, …, 0, 1)} be the idempotent semigroup of order n + 1. Clearly FS the semigroup ring is a n + 1 dimensional general ring of special dual like numbers.

Example 3.20: Let V be a vector space over a field R. $W_1, W_2, \ldots, W_t$ be t vector subspaces of V over R such that

$V = W_1 \oplus W_2 \oplus \ldots \oplus W_t$ is a direct sum. Suppose $E_1, E_2, \ldots, E_t$ be t projection operator on $W_1, W_2, \ldots, W_t$ respectively. I be the identity operator.

Now S = $\{a_1 + a_2E_1 + a_3E_2 + \ldots. + a_{t+1} E_t \mid a_i \in R; 1 \leq i \leq t + 1;$ F_j is a projection of V onto W_j; $1 \leq j \leq t\}$; S is a general t + 1 dimensional ring of special dual like (operators) numbers.

In this way we get any desired dimensional special dual like operator general rings.

Finally show how we construct special dual like rings using idempotents in Z_n.

Example 3.21: Let Z_n be the ring of integers. S = $\{g_1, g_2, \ldots, g_t, 0\}$ be idempotents of S such that $\{m_1 + m_2g_1 + m_3g_2 + \ldots + m_{t+1}\ g_t \mid m_i \in R;\ 1 \leq i \leq t+1;\ g_j \in S;\ 1 \leq j \leq t\}$; P is a t + 1 dimensional general ring of special dual like numbers.

Example 3.22: Let Z_n be the ring of modulo integers. S = $\{0, g_1, g_2, g_3, g_4\} \subseteq Z_n$ be idempotents such that $g_i^2 = g_i$; $1 \leq i \leq 4$; $g_ig_j = 0$ or g_k; $1 \leq i, j, k \leq 4$.

$$\text{Consider } P = \left\{\begin{bmatrix}0\\g_1\\g_2\end{bmatrix}, \begin{bmatrix}0\\g_2\\0\end{bmatrix}, \begin{bmatrix}g_1\\0\\0\end{bmatrix}, \begin{bmatrix}g_2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\g_1\end{bmatrix} \middle|\ g_1.g_2 = 0\right\}.$$

Suppose

$$B = \left\{ a_1 + a_2 \begin{bmatrix} 0 \\ 0 \\ g_1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ g_1 \\ g_2 \end{bmatrix} + a_4 \begin{bmatrix} 0 \\ g_2 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} g_1 \\ 0 \\ 0 \end{bmatrix} + a_6 \begin{bmatrix} g_2 \\ 0 \\ 0 \end{bmatrix} \right| a_i \in R,$$

$1 \leq i \leq 6\}$. B is a 6-dimensional special dual like number general ring.

We can construct idempotent semigroup or matrices using the idempotents in Z_n. Using these idempotent matrices we can build any desired dimensional general ring of special dual like numbers.

Now having seen methods of constructing different types of special dual like numbers of desired dimension. Now we can also construct t-dimensional special semiring semifield of special dual like numbers.

We illustrate this only by examples.

Example 3.23: Let M = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid a_i \in Z^+, 1 \leq i \leq 5, g_1 = (1, 0, 0, 0, 0), g_2 = (0, 1, 0, 0, 0), g_3 = (0, 0, 1, 0, 0), g_4 = (0, 0, 0, 1, 0)$ and $g_5 = (0, 0, 0, 0, 1)\} \cup \{0\}$ be the 6 dimensional general semifield of special dual like numbers.

Example 3.24: Let

$$S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in Z^+, 1 \leq i \leq 4;$$

$$g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } g_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Bigg\} \cup \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

be the special dual like number semifield of dimension five.

Example 3.25: Let

$$M = \{a_1 + a_2g_1 + a_3g_2 + \ldots + a_7g_8 \mid a_i \in Q^+; 1 \le i \le 9;$$

$$g_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$g_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, g_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, g_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$g_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } g_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$g_i \times_n g_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ if } i \ne j;\ g_i^2 = g_i$$

$$\text{for } i = 1, 2, \ldots, 8\} \cup \{0\}$$

be the special semifield of special dual like numbers of dimension of nine.

Example 3.26: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} g_1 + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} g_2 + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} g_3 + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} g_4 \,\middle|\, a_i, b_j, c_k, \right.$$

$$d_t, e_s \in R^+; 1 \le i, j, k, t, s \le 3; g_1 = (4, 3, 0), g_2 = (3, 0, 0),$$

$$\left. g_3 = (0, 0, 4) \text{ and } g_4 = (0, 4, 3), 4, 3 \in Z_6 \right\}$$

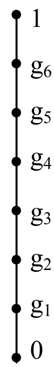
be the special five dimensional semifield of special dual like numbers.

Example 3.27: Let

$$P = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid a_i \in R^+,$$

$$1 \leq i \leq 7, g_j \in L; 1 \leq j \leq 6\} \cup \{0\};$$

where L is a chain lattice given below:



Clearly P is a seven dimensional semifield of special dual like numbers.

We see every distributive lattice paves way for special dual like numbers.

However modular lattices that is lattices which are not distributive, does not result in special dual like numbers on which we can define some algebraic structure on them.

Another point to be noted is lattices and Boolean algebras do not in any way help in constructing dual numbers, they are helpful only in building special dual like numbers.

We give examples of semirings and S-semirings of special dual like numbers.

Example 3.28: Let M = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_3g_2 + \right.$ x_4g_3 where $x_j \in Q^+ \cup \{0\}$, $g_1 = (3, 4, 0, 0)$, $g_2 = (0, 3, 0, 0)$, $g_3 = (3, 4, 0, 0)$ with $3, 4 \in Z_6$; $1 \leq i \leq 6$ and $1 \leq j \leq 3\}$ be the semiring of special dual like number. Clearly M is not a semifield for we see in M we have elements x, y ∈ M;

$$x \times_n y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = y \times_n x.$$

$$\text{Consider } N = \left\{\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \end{bmatrix} g_1 + \right.$$

$$\left. \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix} g_2 + \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{bmatrix} g_3 \right|$$

$x_i, y_j, z_k, s_r \in Q^+ \cup \{0\}$; $g_1 = (3, 4, 0, 0)$, $g_2 = (0, 3, 0, 0)$, $g_3 = (4, 0, 3, 4)$; $3, 4 \in Z_6$; $1 \leq i, j, k, r \leq 6\}$ be the special semiring of special dual like numbers.

We see M and N are isomorphic as semirings.
We define $\eta : M \to N$ as follows:
$\eta(A) =$

$$\begin{bmatrix} x_1 + y_1g_1 + z_1g_2 + s_1g_3 & x_2 + y_2g_1 + z_2g_2 + s_2g_3 & x_3 + y_3g_1 + z_3g_2 + s_3g_3 \\ x_4 + y_4g_1 + z_4g_2 + s_4g_3 & x_5 + y_5g_1 + z_5g_2 + s_5g_3 & x_6 + y_6g_1 + z_6g_2 + s_6g_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \end{bmatrix} g_1 + \begin{bmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{bmatrix} g_2 + \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{bmatrix} g_3$$

is a one to one onto map. Infact it is easily verified η is an isomorphisms of semirings. This result is true for any m × n matrix of semirings with entries from any t-dimensional special dual like numbers. We denote by $R(g_1, g_2) = \{a_1 + a_2g_1 + a_3g_2 \mid a_i \in R; 1 \le i \le 3,\ g_1^2 = g_1,\ g_2^2 = g_2 \text{ and } g_1g_2 = g_2g_1 = 0\}$

$Q(g_1, g_2, g_3) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_i \in Q; 1 \le i \le 4; g_j^2 = g_j, 1 \le k, j \le 3; g_j\, g_k = g_k\, g_j = (0)\}$. On similar lines we have a t-dimensional special dual like number collection which is denoted by

$R(g_1, g_2, \ldots, g_{t-1}) = \{a_1 + a_2g_1 + a_3g_2 + \ldots + a_tg_{t-1} \mid a_i \in R, 1 \le i \le t\, ;\ g_k^2 = g_k \text{ and } g_j.\, g_k = (0) = g_k\, g_j; 1 \le j, k \le t-1\}$. R can be replaced by Q or Z still the results hold good. In all these cases we can say $R(g_1) \subseteq R(g_1, g_2) \subseteq R(g_1, g_2, g_3) \subseteq \ldots \subseteq R(g_1, g_2, \ldots, g_{t-1})$.

However if we replace R by R^+ we see this chain is not possible and every element in $R^+(g_1, g_2, \ldots, g_{t-1})$ is of dimension t and t alone. However if R^+ is replaced by $R^+ \cup \{0\}$ then we see the chain relation is possible. When the chain relation is not possible the set $R^+(g_1, g_2, \ldots, g_{t-1}) \cup \{0\}$ is a semifield of dimension t.

Example 3.29: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle|\ a_i = x_i + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + \right.$$

$x_6g_5 + x_7g_6 + x_8g_7$ where $1 \le i \le 4$; $x_j \in R^+$; $1 \le j \le 8$ and

$g_1 = (1, 0, \ldots, 0)$, $g_2 = (0, 1, 0, \ldots, 0)$, $g_3 = (0, 0, 1, 0, \ldots, 0)$,

$g_4 = (0, 0, 0, 1, 0, 0, 0)$, $g_5 = (0, 0, 0, 0, 1, 0, 0)$,

$$\left. g_6 = (0, 0, 0, 0, 0, 1, 0) \text{ and } g_7 = (0, 0, \ldots, 0, 1)\right\} \cup \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

be a semifield of special dual like numbers under the natural product $\times_n$.

$N = \{A_1 + A_2g_1 + \ldots + A_8g_7 \mid$ where $A_i \in \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$; $x_j \in R^+$; $1 \le i \le 8$; $1 \le j \le 4$. $g_1 = (1, 0, \ldots, 0), \ldots, g_7 = (0, 0, \ldots, 0, 1)\} \cup \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ be the semifield under $\times_n$ of special dual like numbers. Clearly M is isomorphic to N as semifields.

If in M and N instead of using R^+ if we use $R^+ \cup \{0\}$ we get semirings under natural product $\times_n$ as well as under the usual product $\times$.

Thus we can study M or N and get the properties of both as they are isomorphic.

Example 3.30: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} d_1 + d_2g_1 + d_3g_2 + d_4g_3 + d_5g_4 \\ c_1 + c_2g_1 + c_3g_2 + c_4g_3 + c_5g_4 \\ e_1 + e_2g_1 + e_3g_2 + e_4g_3 + e_5g_4 \end{bmatrix} \right.$$ with d_k, c_j,

$e_p \in Q^+ \cup \{0\}$ $1 \le j, k, p \le 5$; and $g_1 = (5, 6, 0)$ $g_2 = (0, 0, 5)$, $g_3 = (0, 0, 6)$, $g_4 = (6, 5, 0)$ with $6, 5 \in Z_{10}\}$ be the general semiring of five dimensional special dual like numbers. Clearly S is only a semiring and not a semifield.

$$P = \left\{ \sum_{i=0}^{\infty} \begin{bmatrix} d_1^i \\ c_1^i \\ e_1^i \end{bmatrix} x^i + \sum_{i=0}^{\infty} \begin{bmatrix} d_2^i \\ c_2^i \\ e_2^i \end{bmatrix} g_1 x^i + \sum_{i=0}^{\infty} \begin{bmatrix} d_3^i \\ c_3^i \\ e_3^i \end{bmatrix} g_2 x^i + \right.$$

$$\left. \sum_{i=0}^{\infty} \begin{bmatrix} d_4^i \\ c_4^i \\ e_4^i \end{bmatrix} g_3 x^i + \sum_{i=0}^{\infty} \begin{bmatrix} d_5^i \\ c_5^i \\ e_5^i \end{bmatrix} g_4 x^i \;\middle|\; d_j^i, c_t^i, e_p^i \in Q^+ \cup \{0\}, 1 \le j \le 5; \right.$$

$1 \le t \le 5$, $1 \le p \le 5$ with $g_1 = (5, 6, 0)$, $g_2 = (0, 0, 5)$,

$g_3 = (0, 0, 6)$ and $g_4 = (6, 5, 0)\}$

is a general semiring of five dimension special dual like numbers and S and P are isomorphic as semirings.

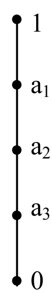
Interested reader can study subsemirings, semiideals and other related properties of semirings.

We can also use lattices to get any desired dimensional special semiring of special dual like numbers. Thus lattices play a major role of getting special dual like numbers.

Further for a given lattice we get two distinct classes of general special semiring of t-dimensional special dual like numbers.

We will illustrate this by an example.

Example 3.31: Let L be the lattice given by the following diagram.



Clearly $a_i \cap a_i = a_i \cup a_i = a_i$, $a_1 \cap a_2 = a_2$, $a_1 \cup a_2 = a_1$, $a_1 \cap a_3 = a_3$, $a_1 \cup a_3 = a_1$ $a_2 \cap a_3 = a_3$, $a_2 \cup a_3 = a_2$.

Now let $S = \{x_1 + x_2a_1 + x_3a_2 + x_4a_3 \mid x_i \in Q^+ \cup \{0\};$
$1 \leq i \leq 4,\ 1, a_1, a_2, a_3 \in L\}$.

Consider $x = 3 + 2a_1 + 4a_2 + 5a_3$ and $y = 8 + 4a_1 + 6a_2 + a_3$ in S. $x + y = 11 + 6a_1 + 10a_2 + 6a_3$.

$$x \times y = (3 + 2a_1 + 4a_2 + 5a_3)\,(8 + 4a_1 + 6a_2 + a_3)$$
$$= 24 + 16a_1 + 32a_2 + 40a_3 + 12a_1 + 8a_1 + 16a_2 + 20a_3$$
$$+ 18a_2 + 12a_2 + 24a_2 + 30a_3 + 3a_3 + 2a_3 + 4a_3 + 5a_3$$

$= 24 + 36a_1 + 102\ a_2 + 104a_3$ … I

(operation under $\cap$)

Now $x \times y = 24 + 16 + 32 + 40 + 12 + 8a_1 + 16a_1 + 20a_1 + 18 + 12a_1 + 24a_2 + 30a_2 + 3 + 2a_1 + 4a_2 + 5a_3$

$= 145 + 58a_1 + 58a_2 + 5a_3$ …II

(operation under $\cup$)

Clearly I and II are not equal so for a given lattice we can get two distinct general special semiring of four dimensional special dual like numbers.

Thus lattices play a major role in building special dual like number.

We can also build matrices with lattice entries and use natural product to get special dual like numbers.

Now we proceed onto study the vector spaces and semivector spaces of t-dimensional special dual like numbers.

We also denote them by simple examples.

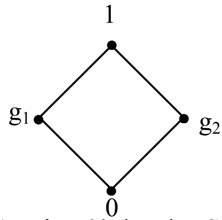
Example 3.32: Let S = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid g_1 = (0, 0, 4), g_2 = (4, 0, 0), g_3 = (3, 0, 0), g_4 = (0, 4, 3)$ and $g_5 = (0, 3, 0)$ where $4, 3 \in Z_6$; $a_i \in Q\ 1 \leq i \leq 6\}$ be a special vector space of special dual like numbers over the field Q.

We see if T is a linear operator on S then to find the eigen values associated with T.

The eigen values will be rationals. On the other hand we use the fact $Q(g_1, g_2, \ldots, g_t)$ is a Smarandache ring and study the Smarandache vector space of special dual like numbers over the general S-ring of special dual like numbers, we can get dual numbers as eigen values.

We will illustrate this situation by some simple examples.

Example 3.33: Let S = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle|\ a_i = x_1 + x_2g_1 + x_3g_2 \right.$ where g_1 and g_2 are the elements of the lattice L

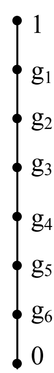


$1 \leq i \leq 4$; $x_j \in Q$; $1 \leq j \leq 3\}$ be the Smarandache special vector space of special dual like numbers over the Smarandache ring.

$P = \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Q;\ 1 \leq i \leq 3;\ g_1^2 = g_1,$ $g_1 \cap g_2 = g_2 \cap g_1 = 0$ and $g_2^2 = g_2;\ g_1, g_2 \in L\}$.

Clearly eigen values of any linear operator can also be special dual like numbers. So by using the Smarandache vector spaces of special dual like numbers we can get the eigen values to be special dual like numbers. This is one of the advantages of using S-vector spaces over S-rings which are general special dual like rings.

Example 3.34: Let S = $\{(a_1, a_2, a_3, a_4)$ where $a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_5 + x_5g_4 + x_6g_5 + x_7g_6 \mid x_i \in R;\ g_j \in L$ where L



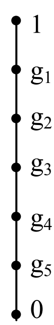
$1 \leq j \leq 6$, $1 \leq i \leq 7\}$ be a S-vector space of special dual like numbers over the S ring
$R(g_1, g_2, g_3, g_4, g_5, g_6) = \{x_1 + x_2g_1 + x_3g_2 + \ldots + x_7g_6 \mid g_i \in L;$ $1 \leq i \leq 6$, $x_j \in R$; $1 \leq j \leq 7\}$ of special dual like numbers. If T is a linear operator on S then the eigen values related with T can be specal dual like numbers from $R(g_1, g_2, \ldots, g_6)$.

Similarly the eigen vectors related with any linear operator can be special dual like numbers.

Now we proceed onto study linear functional of a vector space of special dual like numbers and S-vector space of special dual like numbers.

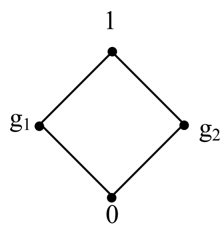
Example 3.35: Let V = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right|$ $a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 +$

$x_5g_4 + x_6g_5$; $g_j \in L$ where L is a lattice given by



$1 \leq j \leq 5$, $x_i \in Q$, $1 \leq i \leq 6\}$ be a S-vector space of special dual like numbers over the S-ring, $Q(g_1, g_2, g_3, g_4, g_5) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 \mid x_i \in Q; 1 \leq i \leq 6; g_j \in L; 1 \leq j \leq 5\}$

Example 3.36: Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_7 \\ a_4 & a_5 & a_6 & a_8 \end{bmatrix} \right|$ $a_i = x_1 + x_2g_1 + x_3g_2$; $1 \leq i \leq 8$, $x_j \in R$; $1 \leq j \leq 3$, $g_1, g_2 \in L$;

L = 

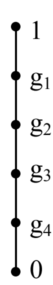
$\Big\}$

be the S-vector space of special dual like numbers over the S-ring $R(g_1, g_2) = \{x_1 + x_2g_1 + x_3g_2;\ x_i \in R;\ g_1, g_2 \in L,\ 1 \leq i \leq 3\}$.

We see V is a S-linear algebra under the natural product $\times_n$ over the S-ring, $R(g_1, g_2)$ and for any S-linear operator on V we can have the eigen vectors to be special dual like numbers.

Now having seen examples of S-linear algebras, S-linear operators T and eigen vectors associated with T are special dual like numbers we proceed onto give examples of special n-dimensional semivector spaces / semilinear algebras of special dual like numbers and strong special n-dimensional semivector spaces / semilinear algebras of special dual like numbers.

Example 3.37: Let $S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right|$ $a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$, $1 \leq i \leq 4$, $x_j \in R^+ \cup \{0\}$; $1 \leq j \leq 5$ and $g_p \in L$ where

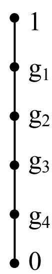
L = 

$1 \leq p \leq 4\}$ be the semivector space of special dual like numbers over the semifield $R^+ \cup \{0\}$. The eigen values of S associated with any linear operator is real and the eigen vectors are from $(R^+ \cup \{0\})$ (g_1, g_2, g_3, g_4).

Example 3.38: Let S = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \right|$ $a_i = x_1 + x_2g_1 + x_3g_2$

$+ x_4g_3 + x_5g_4$, with $x_k \in Q^+ \cup \{0\}$; $g_j \in L$ where

L =

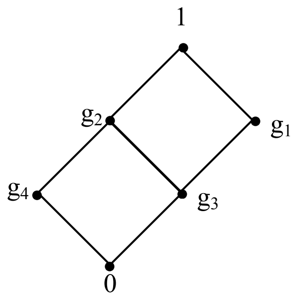


$1 \leq i \leq 18$, $1 \leq k \leq 5$ and $1 \leq j \leq 4\}$ be the strong semivector space of special dual like numbers over the semifield $R^+(g_1, g_2, g_3, g_4) = \{x_1+x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid g_j \in L, 1 \leq j \leq 4, x_i \in R^+, 1 \leq i \leq 5\} \cup \{0\}$. The eigen values of S related with any linear operator on T can be special dual like numbers.

Example 3.39: Let

P = $\left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \right|$ $a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 +$

x_5g_4; $1 \leq i \leq 20$, $x_j \in Q^+ \cup \{0\}$; $1 \leq j \leq 5$ and $g_i \in L$;

L = 

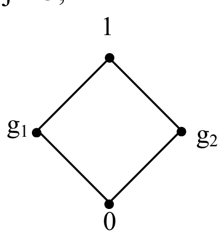
$g_k \in L$; $1 \leq k \leq 4\}$ be a strong semivector space over the semifield $Q^+(g_1, g_2, g_3, g_4) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4, x_i \in Q^+, 1 \leq i \leq 5\}$, $g_j \in L$; $1 \leq j \leq 4\} \cup \{0\}$.

Any linear operator T has its associated eigen values to be special dual like numbers.

Further if $f : P \rightarrow Q^+(g_1, g_2, g_3, g_4) \cup \{0\}$; then f also has for any $A \in P$; f (A) to be a special dual like numbers.

Finally we give examples of them.

Example 3.40: Let $M = \{(a_1, a_2, a_3) \mid a_i = x_1 + x_2g_1 + x_3g_2; 1 \leq i \leq 3\ x_j \in Q^+ \cup \{0\}; 1 \leq j \leq 3;$

$g_1, g_2 \in L =$  $\}$

be a strong semivector space over the semifield

$Q^+(g_1, g_2) \cup \{0\} = \{x_1 + x_2g_1 + x_3g_2\} \cup \{0\}$ where $x_i \in Q^+$ and $g_j \in L$, $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

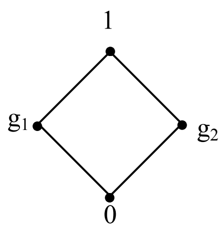
Define $f : M \rightarrow Q^+(g_1, g_2) \cup \{0\}$ as

$f((a_1, a_2, a_3)) = f(x_1 + x_2g_1 + x_3g_2, y_1 + y_2g_1 + y_3g_2, z_1+z_2g_1 + z_3g_2)$
$= x_1 + y_1 + z_1 + (x_2+y_2+z_2)g_1 + (x_3+y_3+z_3)g_2 \in Q^+(g_1,g_2) \cup \{0\}$ if $x_i, y_j, z_k \in Q^+$; $1 \leq i, j, k \leq 3$ and 0 if even one of x_i, y_j or z_k is zero.

f is a semilinear functional on M.

Example 3.41: Let $S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i = x_1 + x_2\, g_1 + x_3g_2;\right.$ $1 \leq i \leq 4$, $x_1, x_2, x_3 \in Z_7$;

$g_1, g_2 \in L =$

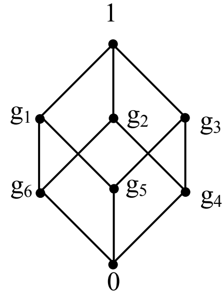


$\}$

be the special vector space of special dual like numbers.

Example 3.42: Let $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + \right.$ $x_5g_4 + x_6g_5 + x_7g_6$; $1 \leq i \leq 6$,

L =



;

$x_k \in Z_{11}$; $1 \leq k \leq 7\}$ be the special vector space of special dual like numbers over the field Z_{11}.

$$\text{Define } f : S \to Z_{11} \text{ by } f\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right) = x_1 + y_1 + z_1 + d_1 + e_1 + f_1$$

(mod 11);

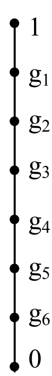
where

$$a_1 = x_1 + x_2g_1 + \ldots + x_7g_6$$
$$a_2 = y_1 + y_2g_1 + \ldots + y_7g_6$$
$$a_3 = z_1 + z_2g_1 + \ldots + z_7g_6$$
$$a_4 = d_1 + d_2g_1 + \ldots + d_7g_6$$
$$a_5 = e_1 + e_2g_1 + \ldots + e_7g_6$$
$$a_6 = f_1 + f_2g_1 + \ldots + f_7g_6 \,;$$

f is a linear functional on S.

Example 3.43: Let $S = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1\right.$ $+\ldots+ x_7g_6$; $1 \leq i \leq 30$, $x_j \in Z_{37}$; $1 \leq j \leq 7$

and $g_k \in L$ =



$1 \leq k \leq 6\}$ be special vector space of dual like numbers over the field Z_{37}. Clearly S has only finite number of elements. If T is any linear operator then the eigen vector associated with T are special dual like numbers.

Example 3.44: Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 \text{ where } 1 \leq i \leq 3, x_j \in Z_5; 1 \leq j \leq 6\right.$

and $g_k \in L$ =

$1 \leq k \leq 5\}$ be a special vector space of special dual like numbers over the field Z_5.

M is also finite dimensional; M under the natural product $\times_n$ is a special linear algebra of special dual like numbers over Z_5.

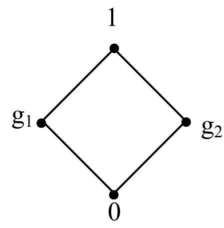
Now we give example Smarandache special vector spaces / linear algebras of special dual like numbers over the S-ring $Z_p(g_1, g_2, \ldots, g_t)$; where $Z_p(g_1, g_2, \ldots, g_t) = \{x_1 + x_2g_1 + \ldots + x_{t+1}g_t \mid x_i \in Z_p; 1 \leq i \leq t+1$ and $g_j \in L$; L is distributive lattice, $1 \leq j \leq t$; p a prime$\}$.

We give a few examples. The main property enjoyed by these Smarandache vector spaces are that they have finite number of elements in them and the eigen values can be special dual like numbers from $Z_p(g_1, \ldots, g_t)$.

We will illustrate this situation by some examples.

Example 3.45: Let S = $\{(a_1, a_2, a_3) \mid a_i = x_1 + x_2g_1 + x_3g_2$ where $1 \leq i \leq 3$; $x_j \in Z_3$, $1 \leq j \leq 3$ and

$g_1, g_2 \in L =$



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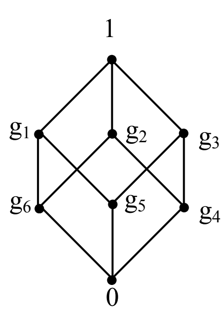
be a Smarandache special vector space of special dual like numbers over the S-ring

$$Z_3(g_1, g_2) = \{x_1 + x_2g_1 + x_3g_2 \mid g_1, g_2 \in L, x_i \in Z_3\ 1 \leq i \leq 3\}.$$

Clearly the eigen values in general of T of S (T : S → S) can also be special dual like numbers from $Z_3(g_1, g_2)$.

Example 3.46: Let S = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix}\right.$ where $a_i = x_1 + x_2g_1 + \ldots +$ x_7g_6; $1 \leq i \leq 10$, $g_j \in L$, $1 \leq j \leq 6$ and $x_k \in Z_7$; $1 \leq k \leq 7$, where

L =



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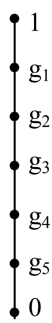
be the Smarandache special dual like number vector space over the S-ring $Z_7(g_1, g_2, \ldots, g_6) = \{x_1 + x_2g_1 + \ldots + x_7g_6 \mid g_i \in L, 1 \leq i \leq 6$ and $x_j \in Z_7; 1 \leq j \leq 7\}$.

Clearly $Z_7 \subseteq Z_7(g_1) \subseteq Z_7(g_1, g_2) \subseteq Z_7(g_1, g_2, g_3) \subseteq \ldots \subseteq Z_7(g_1, g_2, \ldots, g_6)$.

All $Z_7(g_1, g_2, \ldots, g_t)$; $1 \leq t \leq 6$ is also a S-ring for Z_7; the field is properly contained in them.

The eigen values related with a linear operator T on S can also be a special dual like number.

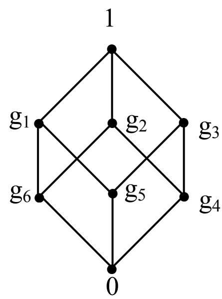
Example 3.47: Let $S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right.$ where $a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5$ where $x_j \in Z_{13}$, $1 \leq j \leq 6$; $1 \leq i \leq 9$ and

$g_k \in L =$ 

$1 \leq k \leq 5\}$ be is Smarandache special vector space of special dual like numbers over the S-ring; $Z_{13}(g_1, g_2, g_4) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_4$ where the operation on g_j's are intersection and g_1, g_2, g_4 are in L; $x_j \in Z_{13}$, $1 \leq j \leq 4\}$, Here also for any linear operator on S we can have the eigen values to be special dual like numbers from $Z_{13}(g_1, g_2, g_4)$.

Finally we give examples of polynomial special dual like number vector spaces.

Example 3.48: Let S = $\left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{24} \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 \right.$ $+\dots+ x_7g_6$; with $x_j \in Z_{11}$; $1 \leq j \leq 7$, $1 \leq i \leq 24$ and $g_k \in L$;

L = 

$1 \leq k \leq 6\}$ be a special vector space of special dual like numbers over the field Z_{11}.

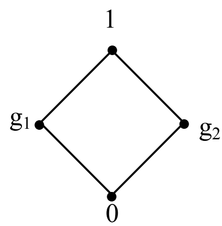
The eigen values of any linear operator on S has only elements from Z_{11}, however the eigen vectors of T can be special dual like numbers.

However if S is defined over the S-ring, $Z_{11}(g_1, g_2, \dots, g_6)$ with $g_i \in L$ then S is a Smarandache special vector space over the S-ring, Z_{11} $(g_1, g_2, \dots, g_6)$ and the eigen values associated with a linear operator on S can be special dual like numbers.

Thus we see the possibility of getting eigen values of special dual like numbers will certainly find nice applications. Finally we give examples of Smarandache vector spaces / linear algebras over the S-ring of special dual like number where the S-rings are $Z_n(g_1, \dots, g_t)$; n not a prime but a composite number.

Example 3.49: Let V = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i = x_1 + x_2\, g_1 + x_3g_2 \text{ with} \right.$

$x_j \in Z_{12}$; $1 \leq i \leq 4$; $1 \leq j \leq 3$ and $g_1, g_2 \in L =$



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be the strong Smarandache special dual like number vector space over the S-ring

$Z_{12}(g_1, g_2) = \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Z_{12};\ g_1, g_2 \in L;\ 1 \leq i \leq 3\}$.

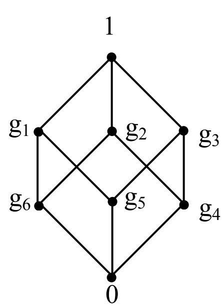
Example 3.50: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{bmatrix} \middle| \ a_i = x_1 + x_2g_1 + x_3g_2 + \right.$$

$x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6$ with $1 \leq i \leq 20$ and $x_j \in Z_{24}$,

$1 \leq j \leq 7$ and

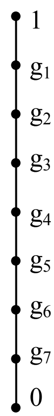
$g_p \in L =$



$1 \leq p \leq 6\}$ be the Smarandache special vector space of special dual like numbers over the S-ring Z_{24}. Clearly M is not a strong Smarandache vector space over a S-ring.

Example 3.51: Let P = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{16} \end{bmatrix} \right| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 +$ $x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7$ with $1 \leq i \leq 16$; $x_j \in Z_{30}$, $1 \leq j \leq 8$ and

$g_p \in L =$



$1 \leq p \leq 7\}$ be a strong Smarandache special dual like number vector space over the S-ring.

$Z_{30}(g_1, \ldots, g_7) = \{x_1 + x_2g_1 + \ldots + x_8g_7 \mid x_i \in Z_{30}, 1 \leq i \leq 8$ and $g_j \in L; 1 \leq j \leq 7\}$.

This P has eigen vaues which can be special dual like numbers for any associated linear operator T of P. Also T can have eigen vectors which can be special dual like numbers.

Study of these properties using strong Smarandache special dual like numbers using $Z_n(g_1,\ldots,g_t)$ can lead to several applications and the S-ring $Z_n(g_1,\ldots,g_t)$ can be so chosen that $Z_n(g_1,\ldots,g_t)$ contains a field as a subset of desired quality.

Chapter Four

SPECIAL DUAL LIKE NEUTROSOPHIC NUMBERS

The concept of neutrosophy and the indeterminate I, was introduced and studied by in [11].

Recently in 2006 neutrosophic rings was introduced and studied [23]. In this chapter we study the notion of neutrosophic special dual like numbers.

Consider $S = \langle Q \cup I \rangle = \{a + bI \mid a, b \in Q\}$; S is a ring S is a general special dual like number ring.

Suppose $T = \langle R \cup I \rangle = \{a + bI \mid a, b \in R, I^2 = I\}$; T is a general neutrosophic ring of special dual like numbers.

Let $F = \langle Z \cup I \rangle = \{a + bI \mid a, b \in Z; I^2 = I\}$; F is a general neutrosophic ring of special dual like numbers.

Like $S = \langle Z_n \cup I \rangle = \{a + bI \mid a, b \in Z_n, I^2 = I\}$ is a general neutrosophic ring of special dual like numbers.

Example 4.1: Let S = $\{\langle Z_{12} \cup I\rangle\} = \{a + bI \mid a, b \in Z_{12}, I^2 = I\}$ be the general neutrosophic ring of special dual like numbers of finite order.

Example 4.2: Let T = $\{\langle 5Z \cup I\rangle\} = \{a + bI \mid a, b \in 5Z, I^2 = I\}$ be the general neutrosophic ring of special dual like numbers of infinite order.

Example 4.3: Let M = $\{\langle R \cup I\rangle\} = \{a + bI \mid a, b \in R, I^2 = I\}$ be the general neutrosophic ring of special dual like numbers.

Example 4.4: Let M = $\{\langle Z_{39} \cup I\rangle\} = \{a + bI \mid a, b \in Z_{39}, I^2 = I\}$ be the general neutrosophic ring of special dual like numbers.

Clearly we have to use the term only general ring as M contains Z_{39} as a subring as well as $Z_{39}I \subseteq M$ as a neutrosophic subring which is also an ideal, that is every element is not of the form a + bI, both a and b not zero.

A ring which has special dual like numbers as well as other elements will be known as the general neutrosophic ring of special dual like numbers.

Example 4.5: Let S = $\{\langle Z_5 \cup I\rangle\} = \{a + bI \mid a, b \in Z_5, I^2 = I\}$ be the general neutrosophic ring of special dual like numbers of dimension two. Clearly S is a Smarandache ring. $Z_5I \subseteq S$ is an ideal of S. $Z_5 \subseteq S$ is only a subring of S which is not an ideal. Clearly S is a finite ring characteristic five.

Example 4.6: S = $\{\langle Z \cup I\rangle\} = \{a + bI \mid a, b \in Z, I^2 = I\}$ be the general neutrosophic ring of special dual like numbers.

S has ideals and subrings which are not ideals. Clearly S is of infinite order and of dimension two.

Now we build matrices and polynomials using general neutrosophic ring of special ring of special dual like numbers.

Consider A = $\left\{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \middle| x_i \in \langle Z \cup I\rangle; i = 1, 2, 3, 4\right\}$;

A is a non commutative general neutrosophic matrix ring of special dual like numbers under the usual product ×.

Infact A has zero divisors, units, idempotents, ideals and subrings which are not ideals.

If on A we define the natural product $\times_n$ then A is a commutative neutrosophic with zero divisors, units and ideals.

For $\begin{bmatrix} 0 & x_1 \\ 0 & x_2 \end{bmatrix} \times_n \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $x_i \in \langle Z \cup I\rangle$; $1 \leq i \leq 2$.

We can have general neutrosophic row matrix ring of special dual like numbers.

Consider
B={(a_1, a_2, …, a_{10}) | a_i = a+ bI with a, b ∈ Q and I^2=I; 1≤i≤ 10}; B is a general neutrosophic row matrix ring of special dual like numbers. B has zero divisors, units and idempotents.

Let C = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_i \in \langle Q \cup I\rangle;\ 1 \leq i \leq n\right\}$; C is a general neutrosophic column matrix ring of special dual like numbers under the natural product $\times_n$.

$$\text{If } x = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ I \\ I \end{bmatrix} \in C \text{ we see } x^2 = x \text{ and so on.}$$

However we cannot define usual product × on C.

Finally consider

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{45} \\ a_{41} & a_{47} & \ldots & a_{60} \end{bmatrix} \middle| \; a_i = x + yI \in \langle R \cup I \rangle; \right.$$

$$x, y \in R; I^2 = I \;\; 1 \leq i \leq 6\};$$

P is a general neutrosophic 4 × 15 matrix ring of special dual like numbers under the natural product $\times_n$.

P has zero divisors, units and idempotents.

Further

$$I_{4 \times 15} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \end{bmatrix}$$

is the unit (i.e., the identity element of P with respect to the natural product $\times_n$.

Now we will give more examples of this situation.

Example 4.7: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in \langle Z_6 \cup I \rangle; 1 \leq i \leq 9; I^2 = I \right\}$$

be the general neutrosophic square matrix ring of special dual like numbers.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is the identity with respect to natural product $\times_n$.

If on S we define the usual product × then S has $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to be the unit. (S, +, $\times_n$) is a commutative ring where as (S, +, ×) is a non commutative ring.

S has units, zero divisors, ideals and subrings which are not ideals. Further S has only finite number of elements in it.

$X = \begin{bmatrix} I & 0 & 0 \\ I & I & 0 \\ 0 & I & I \end{bmatrix}$ is an idempotent under natural product $\times_n$ and X is not an idempotent under the usual product ×.

Example 4.8: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in \langle Z_3 \cup I \rangle; 1 \leq i \leq 16 \right\}$$

be the general neutrosophic matrix ring of special dual like numbers under the natural product $\times_n$. P has zero divisors, units, idempotents, ideal and subrings which are not ideals.

$$x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \text{ is the unit, } y = \begin{bmatrix} 0 & I \\ 1 & 0 \\ 0 & I \\ \vdots & \vdots \\ I & 0 \end{bmatrix} \text{ is an idempotent.}$$

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \right| a_i \in \langle Z_3 \cup I \rangle; 1 \leq i \leq 4 \right\} \subseteq P$$

is a subring as well as ideal of P.

$$x = \begin{bmatrix} a_1 & 0 \\ a_2 & 0 \\ \vdots & \vdots \\ a_{16} & 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} 0 & b_1 \\ 0 & b_2 \\ \vdots & \vdots \\ 0 & b_{16} \end{bmatrix}$$

in P are such that $x \times_n y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$ is a zero divisor in P.

P has only finite number of elements in it.

Example 4.9: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{bmatrix} \middle| a_i \in \langle R \cup I \rangle; 1 \leq i \leq 10 \right\}$$

be the general neutrosophic 2 × 5 matrix ring of special dual like numbers under the natural product $\times_n$. S is of infinite order.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{bmatrix} \middle| a_i \in \langle Z \cup I \rangle; 1 \leq i \leq 10 \right\} \subseteq S$$

is only a subring which is not an ideal of S.

S has zero divisors, units, idempotents.

Clearly $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \in S$ is the unit in S.

$\begin{bmatrix} I & I & I & I & I \\ I & I & I & I & I \end{bmatrix}$ in S is an idempotent;

$y = \begin{bmatrix} I & 0 & 1 & I & 0 \\ 0 & I & 1 & I & I \end{bmatrix} \in S$ is also an idempotent of S.

Example 4.10: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| a_i \in \langle Z_4 \cup I \rangle; 1 \leq i \leq 12 \right\}$$

be the general 4 × 3 matrix neutrosophic special dual like number ring of finite order. P is commutative. P has units, idempotents, and zero divisors.

$$I_{4\times3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in P \text{ is the unit of P.}$$

Now we proceed onto study neutrosophic general polynomial ring of special dual like elements of dimension two.

Example 4.11: Let

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z \cup I\rangle; I^2 = I \right\}$$

be the general neutrosophic polynomial ring of special dual like numbers. P has ideals and subrings which are not ideals.

Example 4.12: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z_8 \cup I\rangle; I^2 = I \right\}$$

be the general neutrosophic polynomial ring of special dual like numbers. S has zero divisors and ideals.

Example 4.13: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle R \cup I\rangle; I^2 = I \right\}$$

be the general neutrosophic polynomial ring of special dual like numbers. S has subrings which are not ideals.

For take

$$P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z \cup I \rangle;\ I^2 = I \right\} \subseteq S;$$

P is only a subring of S and is not an ideal of S.

Example 4.14: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z_7 \cup I \rangle \right\}$$

be the general neutrosophic polynomial ring of special dual like numbers.

Example 4.15: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle R \cup I \rangle;\ I^2 = I \right\}$$

be the general neutrosophic polynomial of special dual like numbers can S have irreducible polynomials.

Now having seen polynomial general neutrosophic ring of special dual like numbers, we now proceed onto give a different representation for the general ring of matrix neutrosophic special dual like numbers.

Example 4.16: Let
$M = \{(x_1, x_2, x_3) + (y_1, y_2, y_3)I \mid I^2 = I,\ x_i, y_j \in R;\ 1 \leq i, j \leq 3\}$ be the neutrosophic general ring of special dual like numbers.

Consider $N = \{(a_1, a_2, a_3) \mid a_1 = x_1 + y_1I;\ a_2 = x_2 + y_2I$ and $a_3 = x_3 + y_3I,\ x_i, y_j \in R;\ 1 \leq i, j \leq 3,\ I^2 = I\}$; N is also a neutrosophic general ring of special dual like numbers.

Clearly N and M are isomorphic as rings, for define $\eta : M \to N$ by $\eta((x_1, x_2, x_3) + (y_1, y_2, y_3)I)$

$= (x_1 + y_1I, x_2 + y_2I, x_3 + y_3I).$

It is easily verified η is a ring isomorphism.

By considering $\phi : N \to M$ given by $\phi (x_1 + y_1I, x_2+y_2I, x_3+y_3I)$

$= (x_1, x_2, x_3) + (y_1, y_2, y_3)I$ we see φ is an isomorphism from N to M.

Thus N and M are isomorphic, that is we say M and N are isomorphically equivalent so we can take M is place of N and vice versa. Hence we can work with a m × n matrix with entries from $\langle Z \cup I\rangle$ ($\langle R \cup I\rangle$ or $\langle Q \cup I\rangle$ or $\langle Z_n \cup I\rangle$) or A + BI where A and B are m × n matrices with entries from Z (or R or Q or Z_n).

We will only illustrate this situation by some examples.

Example 4.17: Let

$$M = \left\{\left.\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}\right| a_i \in \langle Z_{25} \cup I\rangle; 1 \leq i \leq 9; I^2 = I\right\}$$

be the general neutrosophic ring of 3 × 3 matirces of special dual like numbers.

Take

$$N = \left\{\left.\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{pmatrix} I \right| x_i, y_j \in Z_5; \right.$$

$$\left. 1 \leq i , j \leq 9; I^2 = I\right\}$$

be the general ring of neutrosophic matrix special dual like numbers.

We see $\eta : M \to N$ defined by

$$\eta\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} = \eta\begin{pmatrix} x_1+y_1I & x_2+y_2I & x_3+y_3I \\ x_4+y_4I & x_5+y_5I & x_6+y_6I \\ x_7+y_7I & x_8+y_8I & x_9+y_9I \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{pmatrix} I \in N.$$

Clearly η is a ring isomorphism.

Consider $\phi : N \to M$ given by

$$\phi\left(\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{pmatrix} I\right)$$

$$= \begin{pmatrix} x_1+y_1I & x_2+y_2I & x_3+y_3I \\ x_4+y_4I & x_5+y_5I & x_6+y_6I \\ x_7+y_7I & x_8+y_8I & x_9+y_9I \end{pmatrix}.$$

ϕ is again a ring isomorphism thus $N \cong M$ and $M \cong N$. So we say M can be replaced by N and vice versa.

Example 4.18: Let

$$S = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} \right] \middle| \; a_i = x_i + y_iI;\ x_i, y_i \in Z_{11};\ 1 \leq i \leq 6;\ I^2 = I, \right.$$

that is $a_i \in \langle Z_{11} \cup I\rangle\}$ be the general ring of neutrosophic column matrix of special dual like elements.

Take

$$P = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \right] + \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{array} \right] I \;\middle|\; x_i, y_i \in Z_{11};\ 1 \leq i, j \leq 6;\ I^2 = I \right\}$$

be the general ring of column matrix coefficient neutrosophic special dual like number.

Clearly $\eta : S \to P$ defined by

$$\eta\left(\left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{array} \right] \right) = \eta\left(\left[\begin{array}{c} x_1 + y_1I \\ x_2 + y_2I \\ x_3 + y_3I \\ x_4 + y_4I \\ x_5 + y_5I \\ x_6 + y_6I \end{array} \right] \right) = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \right] + \left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{array} \right] I \ \in P,$$

η is a ring isomorphism that is $S \cong P$.

Similarly $\phi : P \to S$ can be defined such that;

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} I\right) = \left(\begin{bmatrix} x_1 + y_1 I \\ x_2 + y_2 I \\ x_3 + y_3 I \\ x_4 + y_4 I \\ x_5 + y_5 I \\ x_6 + y_6 I \end{bmatrix}\right) \in S;$$

thus ϕ is an isomorphism of rings and $P \cong S$. Thus as per need S can be replaced by P and vice versa.

Finally it is a matter of routine to check if

$$M = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{pmatrix} \middle| \; a_i \in \langle Q \cup I \rangle; 1 \le i \le 30 \right\}$$

be the general ring of neutrosophic matrix of special dual like numbers and if

$$N = \left\{ \begin{pmatrix} x_1 & x_2 & \ldots & x_{10} \\ x_{11} & x_{12} & \ldots & x_{20} \\ x_{21} & x_{22} & \ldots & x_{30} \end{pmatrix} + \begin{pmatrix} y_1 & y_2 & \ldots & y_{10} \\ y_{11} & y_{12} & \ldots & y_{20} \\ y_{21} & y_{22} & \ldots & y_{30} \end{pmatrix} I \right.$$

$$\left. \text{where } x_i, y_j \in Q, 1 \le i, j \le 30, I^2 = I \right\}$$

be the general neutrosophic matrix ring of special dual like numbers then M is isomorphic with N. Hence we can use M in place of N or vice versa as per the situation.

Now finally we show the same is true for polynomial rings with matrix coefficients.

For if $p(x) = \sum_{i=0}^{\infty} a_i x^i$ with $a_i = x_i + y_iI$; $0 \le i \le n$ then

$$p(x) = \sum_{i=0}^{n} x_i x^i + \sum_{i=0}^{n} y_i I x^i = \sum_{i=0}^{n} x_i x^i + \left(\sum_{i=0}^{n} y_i x^i\right) I$$

for $x_i, y_i \in Q$ (or Z or R or Z_n).

Similarly if

$$p(x) = \sum_{i=0}^{n} a_i x^i \text{ with } a_i = \begin{pmatrix} x_1 + y_1 I \\ x_2 + y_2 I \\ x_3 + y_3 I \\ x_4 + y_4 I \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} I$$

for $x_j, y_k \in Q$ (or Z or R or Z_n); $1 \le j, k \le 4$; $0 \le i \le n$.

$$\text{Thus } p(x) = \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} x^i + \sum_{i=0}^{\infty} \left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} I \right) x^i$$

$$= \sum_{i=0}^{\infty} \begin{pmatrix} x_1^i \\ x_2^i \\ x_3^i \\ x_4^i \end{pmatrix} x^i + \left(\sum_{i=0}^{\infty} \begin{pmatrix} y_1^i \\ y_2^i \\ y_3^i \\ y_4^i \end{pmatrix} x^i \right) I .$$

Similar results hold good for row neutrosophic matrices, rectangular neutrosophic matrices or square neutrosophic matrices as coefficient of the polynomials. Hence as per need we can replace one polynomial ring by its equivalent polynomial ring and vice versa.

All properties of rings can be derived for general neutrosophic rings of special dual like numbers. This is left as an exercise to the student as it can be realized as a matter of routine. Now we can also build using the neutrosophic dual like numbers a + bI (a, b $\in$ R or Q or Z or Z_n) vector spaces.

Let
$V = \{(a_1, a_2, \ldots, a_{15}) \mid a_i = x_i + y_iI;\ 1 \le i \le 15,\ I^2 = I,\ x_i, y_i \in Q\}$ be the general neutrosophic vector space of special dual like numbers over the field Q.

We see V is also a general neutrosophic linear algebra of special dual like numbers.

This definition and the properties are a matter of routine hence left as an exercise to the reader. So we provide only some examples of them.

Example 4.19: Let

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{pmatrix} \middle| \ a_i \in \langle Q \cup I \rangle;\ 1 \le i \le 15 \right\}$$

be a general neutrosophic vector space of special dual like numbers over the field Q. Infact using the natural product $\times_n$ of matrices. V is a linear algebra of neutrosophic special dual like numbers.

Example 4.20: Let

$$W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{pmatrix} \middle| \ a_i \in \langle Z_{19} \cup I \rangle;\ 1 \le i \le 15 \right\}$$

be the general neutrosophic vector space of special dual like numbers over the field Z_{19}.

Example 4.21: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in \langle R \cup I \rangle; 1 \leq i \leq 9 \right\}$$

be the general neutrosophic Smarandache vector space of special dual like numbers over the Smarandache ring $\langle R \cup I \rangle$.

The eigen values and eigen vectors associated with P can be special dual like numbers from $\langle R \cup I \rangle$.

All other properties like basis, dimension, subspaces, direct sum, pseudo direct sum, linear transformation and linear operator can be found in case of general neutrosophic vector spaces of special dual like numbers which is a matter of routine and hence is left as an exercise to the reader.

Now we can also define neutrosophic general semiring / semifield of special dual like numbers and also the concept of general neutrosophic vector spaces of special dual like numbers.

We only illustrate them by some examples as they are direct and hence left for the reader as an exercise.

Example 4.22: Let M = $\{(a_1, a_2, a_3) \mid a_i = x_i + y_iI$ where $a_i \in \langle R^+ \cup \{0\} \cup I \rangle$, $1 \leq i \leq 3$, $I^2 = I\}$ be the general semiring of neutrosophic special dual like numbers.

Clearly M is not a semifield as M has zero divisors, however M is a strict semiring.

Example 4.23: Let

$$W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \middle| \; a_i = x_i + y_iI,\ x_i, y_i \in Z^+ \cup \{0\},\ I^2 = I,\ 1 \le i \le 5 \right\}$$

be the general neutrosophic semiring of special dual like numbers under the natural product $\times_n$. Clearly W is not a semifield.

Example 4.24: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i = x_i + y_iI,\ x_i, y_i \in Q^+ \cup \{0\},\ 1 \le i \le 4 \right\}$$

be the general neutrosophic non commutation semiring of special dual like numbers. T is not a general neutrosophic semifield.

Example 4.25: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{48} \end{bmatrix} \middle| \; a_i = x_i + y_iI, \right.$$

$$\left. x_i, y_i \in Z^+ \cup \{0\},\ 1 \le i \le 48 \right\}$$

be the general neutrosophic special dual like number semiring under natural product. S has zero divisors, so is not a semifield.

Example 4.26: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i = x_i + y_iI, x_i, y_i \in Z^+, \right.$$

$$\left. 1 \leq i \leq 12 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

be the general special neutrosophic semivector space over the semifield $Z^+ \cup \{0\}$ of special dual like numbers.

Clearly M under the $\times_n$ is a linear algebra.

Also M is a semifield.

Example 4.27: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i = x_i + y_iI, x_i, y_i \in Q^+, \right.$$

$$\left. 1 \leq i \leq 9, I^2 = I \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

be the semifield of general neutrosophic special dual like numbers only under $\times_n$, under usual product $\times$, T is only a semidivision ring.

Example 4.28: Let W = {(a_1, a_2, a_3, a_4) | $a_i = x_i + y_iI$, $x_i, y_i \in R^+$; $1 \leq i \leq 4$} ∪ {(0, 0, 0, 0)} be a semifield of general neutrosophic special dual like numbers.

Example 4.29: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \end{bmatrix} \middle| \; a_i = x_i + y_iI, \right.$$

$$\left. x_i, y_i \in \langle R^+ \cup \{0\} \cup I \rangle,\ 1 \leq i \leq 20 \right\}$$

be the semiring of neutrosophic special dual like numbers under natural product $\times_n$. V is not a semifield however V is a general neutrosophic semilinear algebra of special dual like numbers over the semifield $R^+ \cup \{0\}$.

Infact V is a strong Smarandache semilinear algbera of neutrosophic special dual like numbers over the Smarandache general neutrosophic ring of special dual like numbers.

Example 4.30: Let

$$B = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix} \middle| \; a_i = x_i + y_iI, \right.$$

$$\left. x_i, y_i \in R^+ \cup \{0\},\ 1 \leq i \leq 12 \right\}$$

be the general semilinear algebra of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

All properties related with semivector spaces / semilinear algebras of special dual like numbers over the semifield like basis, dimension, semilinear transformation, semilinear operator, semilinear functions, direct sum of semivector subspaces and pseudo direct sum of semivector spaces can be derived in case of these new structure. As it is direct it is considered as a matter of routine and hence is left as an exercise to the reader.

Now can we have higher dimensional neutrosophic special dual like numbers. We construct them in the following.

Let

$R(g_1, g_2) = \{x_1 + x_2g_1 + x_3g_2 \mid g_1 = (I, I, I)$ and $g_2 = (I, 0, I)\}$ is a three dimensional neutrosophic special dual like number.

For if $a = 3 + 4(I, I, I) + 2(I, 0, I)$

and $b = -1 + 3(I, I, I) - 7(I, 0, I)$ are in $R(g_1, g_2)$ then

$a + b = 2 + 7(I, I, I) - 5(I, 0, I)$

and $a \times b = -3 - 4(I, I, I) - 2(I, 0, I) + 9(I, I, I) + 12(I, I, I) + 6(I, 0, I) - 21(I, 0, I) - 28(I, 0, I) - 14(I, 0, I)$

$= -3 + 17(I, I, I) - 49(I, 0, I) \in R(g_1, g_2)$.

It is easily verified $R(g_1, g_2)$ is a general ring of neutrosophic special dual like numbers of dimension three.

Likewise we can build many three dimensional neutrosophic special dual like numbers.

For $Q(g_1, g_2) = \{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Q; 1 \leq i \leq 3$,

$$g_1 = \begin{bmatrix} I & I \\ I & 0 \\ 0 & I \end{bmatrix} \text{ and } g_2 = \begin{bmatrix} I & I \\ I & I \\ I & I \end{bmatrix}, I^2 = I\}; (Q(g_1, g_2), \times_n, +)$$

is a general neutrosophic using of special dual like numbers.

Example 4.31: Let $W = \{Z(g_1, g_2)\} = \{a_1 + a_2g_1 + a_3g_2) \mid a_i \in Z; 1 \leq i \leq 3, g_1 = (I, I, I, I, I)$ and $g_2 = (0, I, 0, I, 0)\}$ be a three dimensional special dual like number general neutrosophic ring.

Example 4.32: Let M = {Q(g_1, g_2)} = {$a_1 + a_2g_1 + a_3g_2$ where $a_i \in Q$; $1 \leq i \leq 3$; $g_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and $g_2 = \begin{bmatrix} I & I \\ I & I \end{bmatrix}$}; be a three dimensional neutrosophic special dual like number ring where $g_1 \times_n g_2 = g_1$. Clearly M under the usual product is also M is a three dimensional neutrosophic special dual like number ring of $g_1 \times g_2 = g_2$.

However both rings are different.

In this matter we can define any desired dimensional neutrosophic special dual like numbers; we give only examples of them.

Example 4.33: Let Z(g_1, g_2, g_3) = {$a_1 + a_2g_1 + a_3g_2 + a_4g_3$ | $a_i \in Z$, $1 \leq i \leq 4$ with g_1 = (I, I, I, I, I, I), g_2 = (I, 0, I, 0, I, 0) and g_3 = (0, I, 0, I, 0, I) where $g_i^2 = g_i$, i = 1, 2, 3; g_1g_2 = (I, 0, I, 0, I, 0), g_2g_3 = (0, 0, 0, 0, 0, 0) and g_1g_3 = (0, I, 0, I, 0, I)} be a four dimensional neutrosophic general special dual like number ring.

Example 4.34: Let Z(g_1, g_2, g_3, g_4, g_5) = {$a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5$ | $a_i \in Z$; $1 \leq i \leq 6$; g_1 = (I, 0, 0, 0, 0), g_2 = (0, I, 0, 0, 0) g_3 = (0, 0, I, 0, 0), g_4 = (0, 0, 0, I, 0) and g_5 = (0, 0, 0, 0, I)} be the general neutrosophic ring of six dimensional special dual like numbers.

Example 4.35: Let

$$Z_7 (g_1, g_2, g_3, g_4, g_5, g_7, g_8) = \{a_1 + a_2g_1 + \ldots + a_9g_8,\ a_j \in Z_7;$$

$$1 \leq j \leq 9,\ g_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ g_4 = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix},\ g_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix},$$

$$g_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, g_7 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \text{ and}$$

$$g_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\Bigg\}$$

be the nine dimensional general neutrosophic ring of special dual like numbers of finite order.

Thus we can construct any n-dimensional neutrosophic ring of special dual like numbers.

We can also have semirings / semifield of neutrosophic special dual like numbers of desired dimension.

We will only illustrate this situation by some examples.

Example 4.36: Let

$$S = \{a_1 + a_2g_1 + a{=}g_2 + a_4g_3 + a_5g_4 \mid a_i \in R^+ \cup \{0\};$$

$$1 \leq i \leq 5,\ g_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$$

$$\text{and } g_4 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\Bigg\}$$

be a five dimensional neutrosophic dual like number semiring. Clearly S is only a semiring and not a semifield.

Example 4.37: Let

$$S = \{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7 \mid$$

$$a_i \in Z^+ \cup \{0\};\ 1 \le i \le 8,\ g_1 = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ g_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ g_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ g_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix},\ g_7 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \\ 0 \end{bmatrix},\ g_8 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}\Bigg\}$$

be the eight dimensional neutrosophic special dual like number semiring. Clearly S is not a semifield.

Now having seen examples of any higher dimensional neutrosophic special dual like numbers we can as a matter of routine construct semivector spaces and vector spaces of higher dimensional neutrosophic special dual like numbers.

Example 4.38: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle|\ a_i = x_1 + x_2 g_1 + x_3 g_2 + x_4 g_3,\ 1 \le i \le 4,\ x_i \in Q, \right.$$

$$\left. g_1 = (I, 0, 0),\ g_2 = (0, I, 0) \text{ and } g_3 = (0, 0, I) \right\}$$

be a special neutrosophic four dimensional vector space of special dual like numbers over the field Q.

Example 4.39: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \right| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 +$$

$$x_5g_4 + x_6g_5;\ 1 \leq i \leq 12,\ x_j \in Z_{19},\ 1 \leq j \leq 6,$$

$$g_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_4 = \begin{bmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left. g_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix} \right\}$$

be the general neutrosophic for six dimensional vector space of special dual like numbers over the field Z_{19}. T is a finite order.

Example 4.40: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \right| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 +$$

$$x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7 + x_9g_8 \text{ where } 1 \leq i \leq 8,\ x_j \in Q^+ \cup \{0\},$$

$$1 \leq j \leq 9 \text{ with } g_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$g_3 = \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, g_4 = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, g_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left. g_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \; g_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \text{ and } g_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \right\}$$

be a general neutrosophic nine dimensional semivector space of special dual like numbers over the semifield $Q^+ \cup \{0\}$.

Example 4.41: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{bmatrix} \right| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 +$$

$$x_5g_4 + x_6g_5 + x_7g_6; \;\; 1 \leq i \leq 10,$$

$$x_j \in x_{24}; \; 1 \leq j \leq 7; \; g_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left. g_4 = \begin{bmatrix} 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, g_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix} \text{ and } g_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \right\}$$

be a general neutrosophic seven dimensional Smarandache vector space over the S-ring Z_{24} of special dual like numbers.

Example 4.42: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \right.$$

with $g_1 = (0, 0, 0, I)$, $g_2 = (0, 0, I, 0)$ $g_3 = (0, I, 0, 0)$ and

$g_4 = (I, 0, 0, 0)$, $x_j \in \langle Q^+ \cup \{0\} \cup I\rangle$ $1 \leq j \leq 4$, $1 \leq i \leq 9\}$

be a general neutrosophic strong semivector space of special dual like numbers over the semifield $\langle Q^+ \cup \{0\} \cup I\rangle$.

Clearly under the natural product $\times_n$; M is a strong semilinear algebra over the $\langle Q^+ \cup \{0\} \cup I\rangle$. Likewise with usual product $\times$, M is a strong non commutative semilinear algebra over $\langle Q^+ \cup \{0\} \cup I\rangle$.

Thus working with properties of these structures is considered as a matter of routine and this task is left as an exercise to the reader.

Chapter Five

MIXED DUAL NUMBERS

In this chapter we proceed onto define the new notion of mixed dual numbers. We say $x = a_1 + a_2g_1 + a_3g_2$ is a mixed dual number if $g_1^2 = g_1$ and $g_2^2 = 0$ with $g_1g_2 = g_2\, g_1 = g_1$ (or g_2 or 0 where g_1, g_2 are known as the new elements and $a_1, a_2, a_3 \in$ R (or Q or Z or Z_n).

First we will illustrate this situation by some examples.

Example 5.1: Let S = $\{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Q,\ 1 \leq i \leq 3, g_1 = 4, g_2 = 6;\ 4, 6 \in Z_{12};\ g_1^2 = g_1 \pmod{12}$ and $g_2^2 = 0 \pmod{12}\}$ be a mixed dual number collection.

Example 5.2: Let T = $\{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Z,\ 1 \leq i \leq 3, g_1 = 9$ and $g_2 = 6$ in Z_{12} with $g_1^2 = g_1 \pmod{12}$ and $g_2^2 = 0 \pmod{12}$ $g_1g_2 = 9 \times 6 = 54 \equiv 6 \pmod{12}\}$ be the mixed dual number.

Mixed dual numbers should have minimum dimension to be three.

Example 5.3: Let S = $\{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Q, 1 \leq i \leq 3, g_1 = 5 \text{ and } g_2 = 10 \in Z_{20}, g_1^2 = 5 \pmod{20} \text{and } g_2^2 = 0 \pmod{20}\}$ be the mixed dual number.

Consider $x = 5 + 3g_1 + 2g_2$ and $y = 3 - 4g_1 + 5g_2$ in S.

$$x + y = 8 - g_1 + 7g_2 \in S.$$

$$x \times y = (5 + 3g_1 + 2g_2) \times (3–4g + 5g_2)$$

$$= 15 + 9g_1 + 6g_2 - 20g_1 - 12g_1 - 8g_2 + 25g_2 + 15g_2 + 0$$

$$= 15 - 23g_1 + 32g_2 \in S.$$

Example 5.4: Let P = $\{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Z, 1 \leq i \leq 3, g_1 = 21$ and $g_2 = 14$ in Z_{28}. Clearly $g_1^2 = g_1 \pmod{28}$ and $g_2^2 = 0 \pmod{28}$ $g_1g_2 = g_1 = g_2g_1 \pmod{28}\}$. P is a mixed dual number.

Example 5.5: Let W = $\{a_1 + a_2g_1 + a_3g_2 \mid a_i \in Z, 1 \leq i \leq 3, g_1 = 9, g_2 = 12 \in Z_{36}$ are new elements such that $g_1^2 = g_1$ (mod 26) and $g_2^2 = 144 = 0 \pmod{36}$ and $g_1g_2 = g_2g_1 = 0 \pmod{36}\}$; W is a mixed dual number.

Take $x = -2 + g_1 + g_2$ and $y = 5 + 7g_1 + 10g_2$ in W.

$x + y = 3 + 8g_1 + 11g_2$.

$$x \times y = (-2 + g_1 + g_2) \times (5 + 7g_1 + 10g_2)$$

$$= -10 - 14g_1 - 20g_2 + 5g_1 + 7g_1 + 0 + 5g_2 + 0 + 0$$

$$= -10 - 2g_1 - 15g_2 \in W.$$

We wish to give structures on these mixed dual numbers.

Let S = {a + bg_1 + cg_2 | a, b, c ∈ C or Z or Q or R or Z_n; $g_1^2 = g_1$ and $g_2^2 = 0$, $g_1g_2 = g_2g_1 = g_1$ or g_2 or 0} be the collection of mixed dual numbers.

S is a general ring of mixed dual numbers denoted by $C(g_1, g_2)$ or $Z(g_1, g_2)$ or $R(g_1, g_2)$ or $Q(g_1, g_2)$ or $Z(g_1, g_2)$.

Clearly $C(g_1) \subseteq C(g_1, g_2)$ and $C(g_1)$ is a two dimensional special dual like number.

Also $C(g_2) \subseteq C(g_1, g_2)$ and $C(g_2)$ is a two dimensional dual number. $C \subseteq C(g_1, g_2)$. The same result is true if C is replaced by R or Z or Q or Z_n.

We will illustrate this situation by some examples.

Example 5.6: Let S = {a + bg_1 + cg_2 | a, b, c ∈ Q; $g_1 = 16$ and $g_2 = 20$ in Z_{40}, $g_1^2 = 16 = g_1$ (mod 40) and $g_2^2 = 0$ (mod 40), $g_1g_2 = g_2g_1 = 320 \equiv 0$ (mod 40)} be a three dimensional mixed dual numbers. (S, +, ×) is a general ring of three dimensional mixed dual numbers.

Example 5.7: Let P = {a + bg_1 + cg_2 | a, b, c ∈ Z; $g_1 = 22$ and $g_2 = 33 \in Z_{44}$, $g_1^2 = 0$ (mod 44) and $g_2^2 = 33$ (mod 44), $g_1g_2 = g_2g_1 = 22$ (mod 44)} be the three dimensional mixed dual number general ring.

Example 5.8: Let T = {a + bg_1 + cg_2 | a, b, c ∈ Q, $g_1 = 12$, $g_2 = 16 \in Z_{48}$ $g_1^2 = 12^2 = 0$ (mod 48) and $g_2^2 = 16$ (mod 28), $g_1g_2 = g_2g_1 = 0$ (mod 48)} be a three dimensional general ring of mixed dual numbers.

Example 5.9: Let M = {a + bg_1 + cg_2 | a, b, c ∈ Z_7, $g_1 = 13$, $g_2 = 26 \in Z_{52}$, $g_1^2 = g_1$ (mod 52) and $g_2^2 = 0$ (mod 52), $g_1g_2 = g_2g_1 = 26$ (mod 52)} be the three dimensional general ring of mixed dual numbers.

Example 5.10: Let M = {a + bg_1 + cg_2 | a, b, c ∈ Z; g_1 = 30 and g_2 = 40 ∈ Z_{60}, g_1^2 = 0 (mod 60) and g_2^2 = 40 (mod 60), g_1g_2 = g_1g_2 = 0 (mod 60} be a general ring of mixed dual numbers.

Example 5.11: Let M = {a + bg_1 + cg_2 | a, b, c ∈ R; g_1 = 34 and g_2 = 17 ∈ Z_{68} we see g_1^2 = 0 (mod 68) and g_2^2 = 17 (mod 68)} be the general ring of mixed dual numbers.

Clearly $g_1g_2 = g_1 = g_2g_1$ (mod 68); we have several subrings of mixed dual numbers.

Example 5.12: Let M = {a + bg_1 + cg_2 | a, b, c ∈ Z_{20}; g_1 = 36 and g_2 = 48 ∈ Z_{72} such that g_2^2 = 0 (mod 72), g_1^2 = 0 (mod 72), g_1g_2 = g_2g_1 = 0 (mod 72)}, M is a three dimensional dual number general ring.

Now we proceed onto study the mixed dual numbers generated from Z_n, where n = 4m, m any composite number.

THEOREM 5.1: *Let Z_{4m} be the ring, m any composite number. Z_{4m} has element g_1, g_2 such that $g_1^2 = g_1$ (mod 4m) and $g_2^2 = 0$ (mod 4m), $g_1g_2 = g_2g_1 = 0$ or g_1 or g_2 (mod 4m). Thus g_1, g_2 contribute to mixed dual number.*

The proof is direct by exploiting number theoretic methods hence left as an exercise to the reader.

Example 5.13: Let S = {$a_1 + a_2g_1 + a_3g_2 + a_4g_3$ | a_i ∈ Q; 1 ≤ i ≤ 4, g_1 = 4, g_2 = 6 and g_3 = 9 ∈ Z_{12}; 9^2 = 9 (mod 12), 6^2 ≡ 0 (mod 12) 4^2 = 4 (mod 12) 6 × 9 ≡ 6 (mod 12), 4 × 6 ≡ 0 (mod 12), 4 × 9 = 0 (mod 12)}. S is a four dimensional mixed number.

Let x = 5 + 3g_1 + 2g_2 – 4g_3 and y = 6 – g_1 + 5g_2 + g_3 ∈ S,

$$x + y = 11 + 2g_1 + 7g_2 - 3g_3 \in S.$$

$$x \times y = (5 + 3g_1 + 2g_2 - 4g_3) \times 16 - g_1 + 5g_2 + g_3)$$
$$= 30 + 18g_1 + 12g_2 - 24g_3 - 5g_1 - 3g_1 + 0 + 0 +$$
$$25g_2 + 0 + 0 - 20g_2 + 5g_3 + 3 \times 0 + 2g_2 - 4g_3$$

$$= 30 + 10g_1 + 19g_2 - 23g_3 \in S.$$

We can have higher dimensional mixed dual number also.

Example 5.14: Let P = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 \mid a_i \in R, 1 \leq i \leq 4, g_1 = 16, g_2 = 20$ and $g_3 = 25 \in Z_{40}$, $g_1^2 = 16 \pmod{40}$, $g_2^2 = 0 \pmod{40}$ and $g_3^2 = 25 \pmod{40}$, $g_1g_2 = 16 \times 20 \equiv 0 \pmod{40}$, $g_1 \times g_2 = 0 \pmod{40}$, $g_2 \times g_3 = 20 \times 25 \equiv 20 \pmod{40}\}$ be a four dimensional mixed dual number.

Example 5.15: Let S = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in R, 1 \leq i \leq 5, g_1 = 16, g_2 = 20, g_3 = 40, g_4 = 60 \in Z_{80}$, $g_1^2 = g_1 = 16 \pmod{80}$, $g_2^2 = 20^2 = 0 \pmod{80}$ and $g_3^2 = 40^2 = 0 \pmod{80}$ and $g_4^2 = 60^2 = 0 \pmod{80}$. $g_1g_2 = 0 \pmod{80}$, $g_2g_3 = 0 \pmod{80}$, $g_1g_3 = 0 \pmod{80}$, $g_3g_4 = 0 \pmod{80}$, $g_1g_4 = 0 \pmod{80}$, $g_2g_4 = 0 \pmod{80}\}$ be a five dimensional mixed dual number.

Example 5.16: Let P = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in Z_{19}, 1 \leq i \leq 5, g_1 = 12, g_2 = 16, g_3 = 24$ and $g_4 = 36 \in Z_{48}$, $g_1^2 = 12^2 = 0 \pmod{48}$, $g_2^2 = 16^2 = 16 \pmod{48}$, $g_3^2 = 24^2 = 0 \pmod{48}$ and $g_4^2 = 36^2 = 0 \pmod{48}$, $12.16\ g_1g_2 = 0 \pmod{48}$, $g_1.g_3 = 12.24 \equiv 0 \pmod{48}$, $g_1g_4 = 12.36 \equiv 0 \pmod{48}$, $g_2g_3 = 0 \pmod{48}$, $g_2.g_4 \equiv 0 \pmod{48}$ and $g_3.g_4 = 0 \pmod{48}\}$ be a five dimensional mixed dual number.

Example 5.17: Let us consider S = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid a_i \in Q, 1 \leq i \leq 6, g_1 = 16$, $g_1^2 = 16 \pmod{120}$, $g_2 = 25$, $g_2^2 = 25 \pmod{120}$, $g_3 = 40$, $g_3^2 = 40 \pmod{120}$ $g_4 = 60$ $g_4^2 = 0 \pmod{120}$, $g_5 = 96$, $g_5^2 = 96 \pmod{120}$ belong to $Z_{120}\}$. S is a general ring of 6 dimensional mixed dual numbers.

Clearly $g_1\, g_2 \equiv g_3 = 40 \pmod{120}$.

$$g_1 g_3 = g_3 \pmod{120}$$
$$g_1 \times g_4 = 0 \pmod{120}$$
$$g_1 \times g_5 = g_5 \pmod{120}$$
$$g_2 \times g_3 = g_3 \pmod{120}$$
$$g_2 \times g_4 = g_4 \pmod{120}$$
$$g_2 \times g_5 = 0 \pmod{120}$$
$$g_3 \times g_4 = 0 \pmod{120}$$
$$g_3 \times g_5 = 0 \pmod{120} \text{ and}$$
$$g_4 \times g_5 = 0 \pmod{120}.$$

Thus P = $\{0, g_1, g_2, g_3, g_4, g_5) \subseteq Z_{120}$ is a semigroup under product and is defined as the mixed dual number component semigroup of Z_{120}.

Example 5.18: Let S = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7$ with $a_i \in Q$, $1 \leq i \leq 9$, $g_1 = 16$, $g_2 = 60$, $g_3 = 96$, $g_4 = 120$, $g_5 = 160$, $g_6 = 180$ and $g_7 = 225$ in $Z_{240}\}$ be a general ring of mixed dual numbers of dimension eight.

$$g_1^2 = 16^2 = 16 \pmod{240},$$
$$g_2^2 = 60^2 = 0 \pmod{240},$$
$$g_3^2 = 96^2 = 96 \pmod{240},$$
$$g_4^2 = 120^2 = 0 \pmod{240},$$
$$g_5^2 = 160^2 = 160 \pmod{240},$$
$$g_6^2 = 180^2 = 0 \pmod{240},$$
$$\text{and } g_7^2 = 225^2 = 225 \pmod{240}.$$

$$g_1g_2 = 16 \times 60 = 0 \pmod{240},$$
$$g_1g_3 = 16 \times 96 = 96 \pmod{240},$$
$$g_1g_4 = 16 \times 120 = 0 \pmod{240},$$
$$g_1g_5 = 16 \times 160 = 160 \pmod{240},$$
$$g_1 \times g_6 = 16 \times 180 = 0 \pmod{240},$$
$$g_1 \times g_7 = 16 \times 225 = 0 \pmod{240},$$
$$g_2 \times g_3 = 60 \times 96 = 0 \pmod{240},$$
$$g_2 \times g_4 = 60 \times 120 = 0 \pmod{240},$$

$g_2 \times g_5 = 60 \times 160 = 0 \pmod{240}$,
$g_2 \times g_6 = 60 \times 180 = 0 \pmod{240}$,
$g_2 \times g_7 = 60 \times 225 = 60 \pmod{240}$,
$g_3 \times g_4 = 96 \times 120 = 0 \pmod{240}$,
$g_3 \times g_5 = 96 \times 160 = 0 \pmod{240}$,
$g_3 \times g_6 = 96 \times 180 = 0 \pmod{240}$,
$g_3 \times g_7 = 96 \times 225 = 0 \pmod{240}$,
$g_4 \times g_5 = 120 \times 160 = 0 \pmod{240}$,
$g_4 \times g_6 = 120 \times 180 = 0 \pmod{240}$,
$g_4 \times g_7 = 120 \times 225 = 120 \pmod{240}$,
$g_5 \times g_6 = 160 \times 180 = 0 \pmod{240}$,
$g_5 \times g_7 = 160 \times 225 = 0 \pmod{240}$ and
$g_6 \times g_7 = 180 \times 225 = 0 \pmod{240}$.

Thus $P = \{0, g_1, g_2, \ldots, g_7\} \subseteq Z_{240}$ is a mixed dual number semigroup component of Z_{240}.

In view of this we propose the following problem.

If Z_n ($n = p_1p_2 \ldots p_t$ each p_i's distinct). Find the cardinality of the mixed dual component semigroup of Z_n.

Now having seen examples of mixed dual general ring of n-dimension we just proceed to give methods of construction of such rings of any desired dimension. We give a method of constructing any desired dimensional general ring of mixed dual number component semigroup of Z_n.

Suppose $S = \{0, g_1, \ldots, g_t \mid g_1, \ldots, g_k$ are nil potentelements of order two and $g_{k+1}, \ldots, g_t$ are idempotents we take m tuples $x_1, \ldots, x_m$ with x_j's either all idempotents or all nilpotents of order two in such a way $x_i, x_j = x_i$ if $i = j$ in case x_i is an idempotent tuple $x_ix_j = 0$ if $i = j$ in case x_j's are nilpotent of order two $x_i \times x_j = x_k$, x_k is either nilpotent of order two or idempotent if $i \neq j\}$.

That is if $x_i = (g_1, \ldots, g_r)$ and $x_j = (g_s, \ldots, g_p)$, $1 \leq r, s, p \leq t$ then $x_ix_j = x_k = \{g_q, g_s, \ldots, g_l)$ is such that every component in x_k

is either nilpotent of order two or idempotent 'or' used in the mutually exclusive sense, $1 \le p, s, \ldots, l \le t$.

We will illustrate this situation by some examples.

Example 5.19: Let $P = \{g_1, g_2, \ldots, g_7, 0\} \subseteq Z_{240}$ (given in example 5.18).

Consider $x_1 = (0, 16, 0, 0, 0)$, $x_2 = (16, 0, 0, 0, 0)$, $x_3 = (0, 0, 16, 0, 0)$, $x_4 = (0, 0, 0, 16, 0)$, $x_5 = (0, 0, 0, 0, 16)$, $x_6 = (120, 0, 0, 0, 0, 0)$, $x_7 = (0, 120, 0, 0, 0)$, $x_8 = (0, 0, 120, 0, 0)$, $x_9 = (0, 0, 0, 120, 0)$, $x_{10} = (0, 0, 0, 0, 120)$, $x_{11} = (60, 0, 0, 0, 0)$, $x_{12} = (0, 60, 0, 0)$, $x_{13} = (0, 0, 60, 0, 0)$, $x_{14} = (0, 0, 0, 60, 0)$ and $x_{15} = (0, 0, 0, 0, 60)$.

Using $S = \{x_1, x_2, \ldots, x_{15}, (0, 0, \ldots, 0)\}$ we can construct a 16 dimensional general ring of mixed dual numbers.

We can also instead of row matrices use the column matrices like

$$x_1 = \begin{bmatrix} 96 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 96 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 96 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 96 \\ 0 \\ 0 \end{bmatrix}, \quad x_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 96 \\ 0 \end{bmatrix},$$

$$x_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 96 \end{bmatrix}, x_7 = \begin{bmatrix} 180 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_8 = \begin{bmatrix} 0 \\ 180 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_9 = \begin{bmatrix} 0 \\ 0 \\ 180 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 180 \\ 0 \\ 0 \end{bmatrix},$$

$$x_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 180 \\ 0 \end{bmatrix}, x_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 180 \end{bmatrix}, x_{13} = \begin{bmatrix} 120 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_{14} = \begin{bmatrix} 0 \\ 120 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_{15} = \begin{bmatrix} 0 \\ 0 \\ 120 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$x_{16} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 120 \\ 0 \\ 0 \end{bmatrix}, x_{17} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 120 \\ 0 \end{bmatrix} \text{ and } x_{18} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 120 \end{bmatrix}.$$

Using $P = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_1, x_2, \ldots, x_{18} \right\}$

we can construct a general ring of eighteen dimensional mixed dual numbers where

$$x_i \times_n x_j = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x \\ 0 \end{bmatrix} \text{ under the natural product } \times_n.$$

Finally we can find

$$x_1 = \begin{bmatrix} 120 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 & 120 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ldots, x_n$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 96 \end{bmatrix}$$

using natural product $\times_n$ we can find a n dimensional general ring of mixed dual numbers.

Thus we mainly get mixed dual numbers Z_n.

However we are not aware of getting mixed dual numbers by any other way. We feel if we can find linear operator in Hom(V,V) such that T_i o $T_i = T_i$ or O_T, zero operator and if $T_i^2 = T_i$ and $T_j^2 = 0$ then T_i o $T_j = T_j$ o $T_i = 0$ or T_k where T_k is again an idempotent operator or a nilpotent operator of order two. This task is left as an open problem to the reader.

Now having introduced the concept of mixed dual numbers, we proceed onto introduce the notion of fuzzy special dual like numbers and fuzzy mixed dual numbers.

Let [0, 1] be the fuzzy interval.

Let g_1 be a new element such that $g_1^2 = g_1$ we call $x = a + bg_1$ with a, b $\in$ [0, 1] to be a fuzzy special dual like number of dimension two. Clearly if $x = a + bg_1$ and $y = c + dg_1$, a, b, c, d $\in$ [0, 1] are two fuzzy special dual like numbers then x + y and x $\times$ y in general need not be again a fuzzy dual like number for a + c and bc + ad + bd may or may not be in [0, 1], we over come this problem by defining min or max of x, y.

For if $x = 0.03 + 0.4g_1$ and $y = 0.1 + 0.7g_1$ the min (x, y) = $0.03 + 0.4g$, and max (x, y) = $0.1+ 0.7g_1$.

Thus if S = {a + bg_1 | a, b ∈ [0, 1] and $g_1^2 = g_1$ is a new element}, then {S, min} and {S, max} are general semigroups of dimension two of special dual like number.

We will first illustrate this situation by some examples.

Example 5.20: Let A = {a + bg | a, b ∈ [0, 1] g = 4 ∈ Z_6} be the general semigroup of fuzzy special dual like numbers under min or max operation of dimension two.

Example 5.21: Let W = {x + yg | x, y ∈ [0, 1], g = 4 ∈ Z_{12}} be the general semigroup of fuzzy special dual like number under max operation of dimension two.

Example 5.22: Let

$$M = \{x + yg \mid x, y \in [0, 1] \text{ and } g = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \\ 4 \end{bmatrix} 3, 4 \in Z_6\}$$

be the general fuzzy semigroup of special dual like number under max operation of dimension two.

Example 5.23: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ with } x_i, y_i \in [0, 1]; 1 \leq i \leq 4, \; g = 7 \in Z_{14} \right\}$$

be the general fuzzy semigroup of special dual like number under max operation.

Example 5.24: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ with} \right.$$

$$\left. x_i, y_i \in [0, 1];\ 1 \leq i \leq 12,\ g = (7, 8, 8, 7, 8)\ 7, 8 \in Z_{14} \right\}$$

be the general fuzzy semigroup of special dual like number under max operation.

Example 5.25: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ with } x_i, y_i \in [0, 1]; \right.$$

$$\left. 1 \leq i \leq 18,\ g = \begin{bmatrix} 11 & 12 & 0 \\ 12 & 11 & 12 \\ 12 & 11 & 0 \\ 11 & 12 & 11 \end{bmatrix} 11, 12 \in Z_{22} \right\}$$

be the general fuzzy semigroup of special dual like number under min operation of dimension two.

Now we proceed onto give examples of higher dimension general fuzzy semigroup of special dual like number.

Example 5.26: Let

$$M = \{a + bg_1 + cg_2 + dg_3 \mid a, b, c, d \in [0, 1];$$

$$g_1 = \begin{bmatrix} 11 & 12 & 11 \\ 0 & 11 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 12 & 0 & 11 \\ 11 & 0 & 12 \end{bmatrix} \text{ and } g_3 = \begin{bmatrix} 11 & 12 & 0 \\ 11 & 0 & 12 \end{bmatrix}\Bigg\}$$

be the general fuzzy semigroup of special dual like number of dimension four.

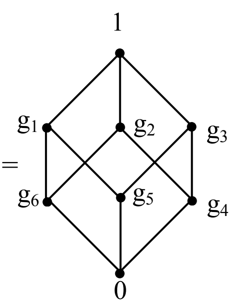
Example 5.27: Let T = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid a_i \in [0, 1]; 1 \leq i \leq 6; g_1 = (13, 0, 0, 14), g_2 = (0, 13, 0, 0), g_3 = (0, 0, 0, 14), g_4 = (0, 0, 13, 0)$ and $g_5 = (13, 0, 0, 0)$ are idempotents $13, 14 \in Z_{26}\}$ be the general fuzzy semigroup of special dual like number of dimension six.

Example 5.28: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$$

$$+ x_6g_5 + x_7g_6, 1 \leq i \leq 15, x_j \in [0, 1], 1 \leq j \leq 7,$$

$g_p \in L =$



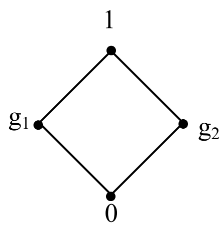
$1 \leq p \leq 6\}$ be the general fuzzy semigroup of special dual like numbers of dimension seven under max (min) operation.

Example 5.29: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \end{bmatrix} \middle| \; a_i = x_1 + x_2 g_1 + x_3 g_2, \right.$$

$$1 \le i \le 16, x_s \in [0, 1], 1 \le j \le 3,$$

$g_1, g_2 \in L =$



$\left. \right\}$

be the general fuzzy semigroup of special dual like numbers of dimension three under max operation.

Now we proceed onto give examples of general fuzzy semigroup of mixed dual numbers.

Example 5.30: Let
$M = \{a_1 + b_1 g_1 + c_1 g_1 \mid a, b, c \in [0, 1]\ g_1 = 6$ and $g_2 = 4 \in Z_{12}\}$ be the general fuzzy semigroup of mixed dual number of dimension three. $g_1^2 = 6^2 = 0 \pmod{12}$ and $g_2^2 = 4 = g_2 \pmod{12}$. Finally $g_1 g_2 = g_2 g_1 = 0 \pmod{12}$.

Example 5.31: Let $S = \{x_1 + x_2 g_1 + x_3 g_2 + x_4 g_3 \mid x_i \in [0, 1]$ $1 \le i \le 4$; $g_1 = 6$, $g_2 = 4$, $g_3 = 9 \in Z_{12}$; $g_1^2 = 6^2 = 0 \pmod{12}$ and $g_2^2 = 4^2 = g_2 \pmod{12}$, $g_3^2 = 9 = g_3 \pmod{12}$, $g_1 g_2 = 0 \pmod{12}$ $g_1 g_3 = 69 = 54 = 6 \pmod{12}$, $g_2 g_3 = 4.9 = 36 = 0 \pmod{12}\}$ be the general fuzzy semigroup of mixed dual number of dimension four under min or max operation.

Example 5.32: Let
$P = \{(a_1, a_2, a_3) \mid a_i \in \langle[0, 1] \cup [0, I]\rangle; 1 \leq i \leq 3\}$ be a general fuzzy semigroup of neutrosophic special dual like numbers under min or max operation.

Example 5.33: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{19} & a_{20} \end{bmatrix} \middle| \; a_i \in \langle[0, I] \cup [0, 1]\rangle; 1 \leq i \leq 20 \right\}$$

be the general fuzzy semigroup of neutrosophic special dual like numbers under min or max operation.

Example 5.34: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{32} \end{bmatrix} \middle| \; a_i \in \langle[0, 1] \cup [0, I]\rangle; 1 \leq i \leq 32 \right\}$$

be the general fuzzy semigroup of neutrosophic special dual like numbers under min or max operation.

Thus fuzzy neutrosophic numbers under min or max operation are special dual like numbers.

Finally we see as in case of dual numbers we can in case of special dual like numbers and mixed dual numbers define the notion of natural class of intervals and operations on them to obtain nice algebraic structures.

Example 5.35: Let

$$R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_i \in \langle[0, I] \cup [0, 1]\rangle; 1 \leq i \leq 6 \right\}$$

be the general fuzzy semigroup of neutrosophic special dual like numbers under min or max operation.

Now we give examples of mixed dual numbers.

Example 5.36: Let M = $\{(a_1, a_2, a_3, a_4) \mid a_i = x_1 + x_2g_1 + x_3g_2, x_j \in [0, 1], 1 \leq i \leq 4, 1 \leq j \leq 3$ $g_1 = 6$ and $g_2 = 4 \in Z_{12}$; $g_1^2 = 0$ and $g_2^2 = 12$, $g_1g_2 = 0 \pmod{12}\}$ be the general fuzzy semigroup of mixed dual numbers under min or max operation.

Example 5.37: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \text{ with} \right.$$

$x_j \in [0, 1], 1 \leq i \leq 30, 1 \leq j \leq 4$; $g_1 = 6$ and

$g_2 = 4$ and $g_3 = 9 \in Z_{12}\}$

be the general fuzzy semigroup of mixed dual number of dimension four under max or min.

Example 5.38: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + \right.$$

$x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7$, $1 \leq i \leq 20$ with $x_j \in [0, 1]$, $1 \leq j \leq 8$;

$g_1 = 16$ and $g_2 = 60$ and $g_3 = 96$, $g_4 = 120$, $g_5 = 160$,

$g_6 = 180$ and $g_7 = 225 \in Z_{240}\}$

be the general fuzzy semigroup of mixed dual number under max or min of dimension 8.

Finally just indicate how mixed dual number vector spaces, semivector spaces can be constructed through examples.

Example 5.39: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2 \text{ where } g_1 = 6, \right.$$

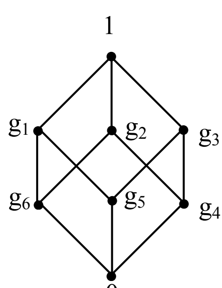
$$g_2 = 4 \in Z_{12},\ x_j \in Q;\ 1 \leq i \leq 8,\ 1 \leq j \leq 3;\ g_1^2 = 0 \ (\text{mod } 12),$$

$$\left. g_2^2 = 4 \ (\text{mod } 12) \text{ and } g_1g_2 = 0 \ (\text{mod } 12) \right\}$$

be the general vector space of mixed dual numbers over the field Q. Infact M is a general linear algebra of mixed dual numbers over Q under the natural product $\times_n$.

Example 5.40: Let P ={$(a_1, a_2, \ldots, a_{15})$ | $a_i = x_1 + x_2g_1 + x_3g_2 + 4g_3 + x_5g_4 + x_6g_5 + x_7g_6$ with $x_j \in Q$; $1 \leq i \leq 15$, $1 \leq j \leq 7$;

$g_p \in L=$



$1 \leq p \leq 6$ be a vector space / linear algebra of special dual like numbers over the field Q}.

Example 5.41: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_3g_2 + 4g_3 + x_5g_4 + x_6g_5 \right.$$

$$+ x_7g_6 + x_8g_7 \text{ with } 1 \leq i \leq 4,\ x_j \in R;\ 1 \leq j \leq 8;\ g_1 = 16,$$

$$g_2 = 60,\ g_3 = 96,\ g_4 = 120,\ g_5 = 160,\ g_6 = 180 \text{ and}$$

$$g_7 = 225 \text{ in } Z_{240};\ g_2^2 = 0,\ g_1^2 = 16,\ g_3^2 = 96,\ g_4^2 = 0,$$

$$\left. g_5^2 = 160,\ g_6^2 = 0 \text{ and } g_7 = 225 \right\}$$

be the general vector space of mixed dual numbers over the field R (or Q). S is a non commutative linear algebra of mixed dual numbers over R (or Q) under usual product × and under $\times_n$; S is a commutative linear algebra of mixed dual numbers over the field.

Study of basis, linear transformation, linear operator, linear functionals, subspaces, dimension, direct sum, pseudo direct sum, eigen values and eigen vectors are a matter of routine hence the reader is expected to derive / describe / define them with appropriate modifications.

Example 5.42: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{21} & a_{22} \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3,\ 1 \leq i \leq 22, \right.$$

$$x_j \in Q;\ 1 \leq j \leq 4;\ g_1 = 4,\ g_2 = 6 \text{ and } g_3 = 9 \text{ in } Z_{12} \text{ and}$$

$$\left. Q(g_1, g_2, g_3) = x_1 + x_2g_1 + x_3g_2 + x_4g_4 = a_i \right\}$$

be a Smarandache general vector space of mixed dual numbers over the Smarandache general ring of mixed dual numbers $Q(g_1, g_2, g_3)$.

Clearly M is a S-linear algebra over the S-ring, $Q(g_1, g_2, g_3)$ under the natural product $\times_n$. Further in general the eigen values and eigen vectors can be mixed dual numbers.

Example 5.43: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{24} \end{bmatrix} \middle| \, a_i \in Q(g_1, g_2, g_3, g_4, g_5, g_6, g_7), \right.$$

$1 \leq i \leq 24$, $p = g_1 = 16$, $g_2 = 60$, $g_3 = 96$, $g_4 = 120$, $g_5 = 160$,

$g_6 = 180$, $g_7 = 225\} \subseteq Z_{240}$ and $a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3$

$+ \dots + x_8g_7$; $1 \leq i \leq 8$, $x_j \in Q$; $1 \leq p \leq 7\}$

be the Smarandache general vector space (S-linear algebra under natural product $\times$) of mixed dual numbers over the S-ring, $Q(g_1, g_2, \dots, g_7)$.

We now proceed onto give examples of semivector space of mixed dual numbers.

Example 5.44: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_2g_1 + x_3g_2; \right.$$

$x_j \in Q^+ \cup \{0\}$, $1 \leq i \leq 8$, $1 \leq j \leq 3$, $g_1 = 12$ and $g_2 = 16$ with

$g_1^2 = 0 \pmod{48}$, $g_2^2 = 16 \pmod{48}$ in $Z_{48}\}$

be the general semivector space of mixed dual numbers over the semifield $Z^+ \cup \{0\}$.

Example 5.45: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_2g_1 + \ldots + x_8g_7 \right.$$

$$\text{with } 1 \leq i \leq 30,\ x_j \in Z^+ \cup \{0\},\ 1 \leq j \leq 8;$$

$$\left. T = \{0, g_1, g_2, \ldots, g_7\} \subseteq Z_{240} \right\}$$

be the general semivector space of mixed dual number over the semifield $Z^+ \cup \{0\}$.

Example 5.46: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_2g_1 + x_3g_2 + x_4g_3 \right.$$

$$\text{with } 1 \leq i \leq 9,\ x_j \in Q^+ \cup \{0\},\ 1 \leq j \leq 4,\ g_1 = 6,\ g_2 = 4$$

$$\left. \text{and } g_3 = 9 \in Z_{12} \right\}$$

be a general S-semivector space of mixed dual numbers over the Smarandache semiring.

$P = \{(Q^+ \cup \{0\})\ (g_1, g_2, g_3) = x_1 + x_2g_1 + x_3g_2 + x_4g_3$ with $x_j \in Q^+ \cup \{0\}$, $g_1, g_2, g_3 \in Z_{12}$ $g_1 = 6$, $g_2 = 4$ and $g_3 = 9\}$. In this case M is a Smarandache semilinear algebra over P. Further the eigen values and eigen vectors associated with any $T : M \to M$ can be mixed dual numbers.

Example 5.47: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + \ldots + x_8g_7 \text{ with } 1 \leq i \leq 10, \right.$$

$$x_j \in Z_{11}, 1 \leq j \leq 8, g_1 = 16, g_2 = 60, g_3 = 96, g_4 = 120,$$

$$\left. g_6 = 160 \text{ and } g_7 = 225 \in Z_{240} \right\}$$

be the vector space of mixed dual numbers over the field Z_{11}. S is not only finite dimensional but S has only finite number of elements in it.

Example 5.48: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{30} \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \right.$$

$$\text{with } 1 \leq i \leq 30, x_j \in Z_{25}, 1 \leq j \leq 4, g_1 = 6, g_2 = 4 \text{ and}$$

$$\left. g_3 = 9 \text{ in } Z_{12} \right\}$$

be the Smarandache general vector space of mixed dual numbers over the S-ring Z_{25}.

For all these semivector spaces, semilinear algebras and finite vector spaces of mixed dual numbers we can derive all properties with no difficulty. Thus this task is left as an exercise to the reader.

Now we indicate how intervals of special dual like numbers and mixed dual like numbers are constructed and the algebraic structures defined on them.

Let $N_o(S) = \{(a_i, a_j) \mid a_i, a_j \in S = \{x_1 + x_2g_1$ with $x_1, x_2 \in Q$ (or Z or Z_n or R or C) $g_1^2 = g_1$ is a new element}} be the natural class of open intervals with special dual like numbers.

Similarly we can define closed intervals, open-closed intervals and closed-open intervals of special dual like numbers of any dimension.

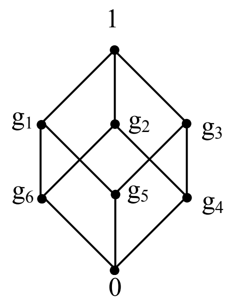
We will illustrate this situation first by some examples.

Example 5.49: Let $M = \{[a, b] \mid a, b \in Q(g_1, g_2, g_3) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_i \in Q, 1 \leq i \leq 4, g_1 = 6, g_2 = 9$ and $g_3 = 4 \in Z_{12}\}\}$ be the closed interval general ring of mixed dual numbers.

Example 5.50: Let $P = \{(a, b] \mid a, b \in \langle R \cup I\rangle\}$ be the open-closed intervals general ring of neutrosophic special dual like numbers.

Example 5.51: Let $W = \{[a, b) \mid a, b \in S = \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Q; 1 \leq i \leq 3, g_1 = 10, g_2 = 6 \in Z_{30}\}\}$ be the general ring of closed-open interval special dual like numbers of dimension three.

Example 5.52: Let $T = \{(a, b) \mid a, b \in S = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 \mid x_j \in R, 1 \leq j \leq 7,$

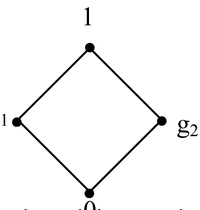
$g_k \in L=$ 

$1 \leq k \leq 6\}\}$ be the seven dimensional open interval general ring of special dual like numbers.

Example 5.53: Let P = {[a, b) | a, b $\in$ S = {$x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$ where $g_1 = 16$, $g_2 = 96$, $g_3 = 160$, $g_4 = 225 \in Z_{240}$, $x_i \in Z$, $1 \leq i \leq 5$}} be the closed-open interval general ring of special dual like numbers.

Example 5.54: Let M = {($a_1, a_2, \ldots, a_n$) | $a_i = [x_i, y_i]$ where x_i, $y_i \in$ S = {$x_1 + x_2g_1 + x_3g_2$ | $x_1, x_2, x_3 \in Q$,

$g_1, g_2 \in L =$



$1 \leq i \leq n$}} be the closed interval row matrix general ring of special dual like numbers.

Example 5.55: Let

$$P = \left\{ \left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{array} \right] \middle| a_i = (c_i, d_i] \text{ with } c_i, d_i \in S = \{x_1 + x_2g_1 + \right.$$

$$x_3g_2 + x_4g_3 + x_5g_4 \mid x_j \in Q, 1 \leq j \leq 4, g_1 = 16, g_2 = 96,$$

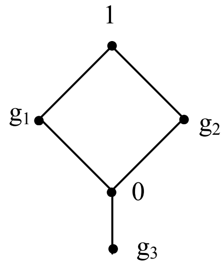
$$\left. g_3 = 160 \text{ and } g_4 = 225 \in Z_{240}\} \; 1 \leq i \leq 12 \right\}$$

be the open-closed interval column matrix general ring of special dual like numbers.

Example 5.56: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i = [c, d] \text{ with } c, d \in S \right.$$

$$= \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_t \in Q, 1 \leq t \leq 4,$$

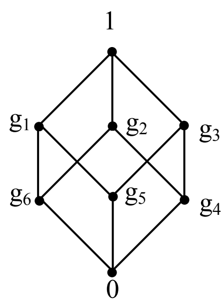
$g_j \in L =$ 

$1 \leq j \leq 3\}\}$ be the closed interval square matrix general non commutative ring of special dual like numbers.

Example 5.57: Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (d_i, c_i],\ d_i, c_i \in P = \{x_1 + x_2g_1 + x_3g_2 + \right.$$

$$x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 \text{ where } x_j \in Q,\ 1 \leq j \leq 7 \text{ and}$$

$g_p \in L =$  ;

$1 \leq p \leq 6\}\}$ be the open-closed interval general polynomial ring of special dual like numbers.

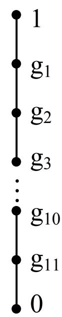
These interval rings has zero divisors, units, idempotents, subrings and ideals. All properties can be derived which is a matter of routine.

Example 5.58: Let

$$T = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = (c, d],\ c, d \in S = \right.$$

$\{x_1 + x_2g_1 + \ldots + x_{12}g_{11} \mid x_j \in Z,$

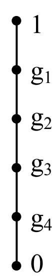
$g_p \in L =$



$1 \leq j \leq 12$ and $1 \leq p \leq 11\}\}$ be the closed-open interval coefficient polynomial general ring of special dual like numbers.

Example 5.59: Let S = {[a, b) | a, b ∈ P = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_i \in Q^+ \cup \{0\}, 1 \leq i \leq 5,$

$g_j \in L =$



$1 \leq j \leq 4\}\}$ be the closed open-interval general semiring of special dual like numbers.

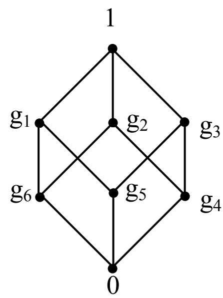
Clearly S is not a semifield.

Example 5.60: Let S = {(a, b) | a, b ∈ P = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_i \in Z^+ \cup \{0\}$, $g_1 = 16$, $g_2 = 120$, $g_3 = 96$ and $g_4 =$

$225 \in Z_{240},\ 1 \leq i \leq 5\}\}$ be the open interval general semiring of special dual like numbers.

Example 5.61: Let $S = \{(a_1, a_2, a_3, a_4) \mid a_i \in P = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 \mid x_i \in Q^+ \cup \{0\}, 1 \leq j \leq 6,$

$g_p \in L =$



where $1 \leq p \leq 6\}\}$ be the interval row matrix general semiring of special dual like numbers.

Clearly M is not a semifield only a smarandache semiring.

Example 5.62: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i = [c, d];\ c, d \in S = \{x_1 + x_2g_1 + x_3g_2 \mid x_j \in \right.$$

$Z^+ \cup \{0\},\ 1 \leq j \leq 3$ and $g_1 = 6,\ g_2 = 10 \in Z_{30}\},\ 1 \leq i \leq 9\}$

be the column interval matrix semiring of special dual like numbers.

Example 5.63: Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right|$ where $a_j = [c, d]$; c, d $\in$ P = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + \ldots + x_{16}g_{15}$ where g_t be elements of a chain lattice with 17 elements $x_i \in Z^+ \cup \{0\}$; $1 \leq i \leq 16$, $1 \leq t \leq 15\}$, $1 \leq j \leq 9\}$ be a closed square interval matrix general semiring of special dual like numbers. W is a non commutative semiring under usual product $\times$ of matrices where as a commutative ring under the natural product $\times_n$ of matrices.

Example 5.64: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix}\right| \quad a_i = (c, d];\ c, d \in S = \{x_1 + x_2g_1 +$$

$$x_3g_2 + x_4g_3 + x_5g_4 \mid x_j \in R^+ \cup \{0\};\ 1 \leq j \leq 5,\ g_1 = 16,$$

$$g_2 = 96,\ g_3 = 160 \text{ and } g_4 = 225 \in Z_{240}\};\ 1 \leq i \leq 30\}$$

be the rectangular matrix of open-closed interval general semiring of special dual like numbers. Clearly the usual product of matrices cannot be defined on M. M is not a semifield has zero divisors.

Example 5.65: Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \right|$ $a_i = (c, d]$, c, d $\in$ P = $\{x_1 + x_2g_1 + \ldots + x_{18}g_{17} \mid x_j \in Z^+ \cup \{0\}$, $1 \leq j \leq 18$ and g_p are elements of chain lattice of order 19, $1 \leq p \leq 17\}\}$ be the closed interval coefficient polynomial semiring of special dual like numbers.

Example 5.66: Let M = $\{(a_1, a_2, a_3, \ldots, a_{10}) \mid a_i = [c, d)$; c, d $\in$ S = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_j \in Q$, $1 \leq j \leq 5$, $g_1 = 16$ $g_2 = 96$, $g_3 = 160$, and $g_4 = 225 \in Z_{240}\}$, $1 \leq i \leq 10\}$ be the interval row matrix general vector space of special dual like numbers over the field Q.

Likewise we can define interval column matrix general vector space / linear algebra of special dual like numbers, interval rectangular matrix general vector space / linear algebra of special dual like numbers and interval matrix general vector space/ linear algebra of special dual like numbers.

The reader is expected to give examples of all these cases.

Example 5.67: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i = (c, d]; c, d \in S = \{x_1 + x_2g_1 + x_3g_2 \mid x_j \in Z^+ \cup \{0\}; 1 \leq j \leq 3, g_1 = 6, g_2 = 10 \in Z_{30}\} \right\}$$

be the closed open interval general semivector space of special dual like numbers over the semifield $Z^+ \cup \{0\}$.

Likewise semivector spaces of row matrices, column matrices and square matrices with interval entries can be constructed. This task is also left to the reader.

Example 5.68: Let $T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \; a_i = (c, d]; c, d \in \{x_1 + x_2g_1 + \ldots + x_{20}g_{19} \mid x_j \in Z_{150}, 1\ 1 \leq j \leq 20, g_p \in L, \text{L a chain lattice of order 21}, 1 \leq p \leq 19\}, 1 \leq i \leq 18 \right\}$ be a Smarandache vector space rectangular matrix of intervals of special dual like numbers over the S-ring Z_{150}.

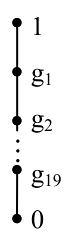
Example 5.69: Let P = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right|$ where a_j = [c, d]; c, d

$\in$ S ={$x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$ | $x_j \in Z_{19}$, $1 \leq j \leq 5$, $g_1 = 16$, $g_2 = 96$, $g_3 = 160$ and $g_4 = 225 \in Z_{240}$}, $1 \leq i \leq 9$} be a square matrix with closed intervals entries. P is a general vector space of special dual like numbers over the field Z_{19}.

Now we can also construct intervals of mixed dual numbers. This is also considered as a matter of routine. So we only give some examples so that interested reader can work in this direction.

Example 5.70: Let W = {[a, b] | a, b $\in$ P = { $x_1 + x_2g_1 + x_3g_2 + \ldots + x_8g_7$ | $x_i \in Q$, $g_1 = 16$, $g_2 = 60$, $g_3 = 96$, $g_4 = 160$, $g_5 = 180$ and $g_6 = 120$ and $g_7 = 225 \in Z_{240}$}}. W is a general ring of natural class of closed intervals of mixed dual numbers.

Example 5.71: Let S = {(a, b] | a, b $\in$ P = {$x_1 + x_2g_1 + \ldots + x_{20}g_{19}$ | $x_i \in R$, $1 \leq i \leq 20$, $g_p \in L$, L a chain lattice of order 21,



$1 \leq p \leq 19$}} be the general ring of open-closed intervals of special dual numbers.

Using chain lattices or distributive lattices one cannot construct mixed dual numbers.

Example 5.72: Let M = {(a, b) | a, b ∈ S = {$x_1 + x_2g_1 + x_3g_2 + x_4g_3$ | $x_i \in Q$, $1 \leq i \leq 4$ and $g_1 = 6$, $g_2 = 4$ and $g_3 = 9 \in Z_{12}$}} be the general ring of open intervals of mixed dual numbers.

Example 5.73: Let S = {(a, b] | a, b ∈ P = {$x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7$ | $x_i \in Z_{200}$, $1 \leq i \leq 8$ and $g_1 = 16$, $g_2 = 60$, $g_3 = 120$, $g_4 = 96$, $g_5 = 160$, $g_7 = 180$ and $g_6 = 225 \in Z_{240}$}} be the open-closed interval general ring of mixed dual numbers.

Example 5.74: Let S = {(a, b) | a, b ∈ P = {$x_1 + x_2g_1 + x_3g_2$ | $x_i \in Z$, $1 \leq i \leq 3$, $g_1 = 6$ and $g_2 = 4 \in Z_{12}$}} be the open interval general ring of mixed dual numbers.

Let x = $(3+5g_1+g_2, 7g_2+5)$ and y = $(-7+8g_2, 5+g_1+3g_2) \in S$.

Now $x + y = (-4 + 5g_1 + 9g_2, 10+g_1 + 10g_2) \in S$;

$x \times y = ((3 + 5g_1 + g_2) \times (-7 + 8g_2), (7g_2+5)(5 + g_1 + 3g_2)$

$= (-21 - 35g_1 - 7g_2 + 24g_2 + 40g_1g_2 + 8g_2^2 + 35g_2 + 25 + 7g_1g_2 + 5g_1 + 21g_2^2 + 15g_2)$

$= (-21 - 35g_1 + 25g_2, 25 + 5g_1 + 71g_2)$
($\because g_2^2 = g_2$ and $g_1g_2 = 0$).

$x \times y \in S$. This is the way operations '+' and '×' are performed on S

Example 5.75: Let $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right|$ $a_i = (c, d)$; c, d ∈ P = {$x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$ | $x_i \in Z_7$; $1 \leq i \leq 5$, $g_1 = 12$, $g_2 = 16$, $g_3 = 24$, $g_4 = 36 \in Z_{48}$}, $1 \leq j \leq 4$} be the general ring of open interval matrices of mixed dual number.

Clearly cardinality of S is finite.

Example 5.76: Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle| a_j = [c, d);\right.$

c, d ∈ S = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_i \in Q; 1 \leq i \leq 4, g_1 = (6, 6, 6), g_2 = (4, 4, 4), g_3 = (9, 9, 9), 4, 6, 9 \in Z_{12}\}$ $1 \leq j \leq 10\}$ be the closed - open interval matrix ring of mixed dual number.

Example 5.77: Let W = $\{(a_1, a_2) \mid a_j = (c, d]; c, d \in P = \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Z_5; 1 \leq i \leq 3$ and $g_1 = 14$ and $g_2 = 21 \in Z_{28}\}, 1 \leq j \leq 2\}$ be the open-closed interval general ring of mixed dual numbers.

Example 5.78: Let T = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_j = [c, d]; 1 \leq i \leq\right.$

30, c, d ∈ P = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + \ldots + x_8g_7 \mid x_i \in R; 1 \leq j \leq 8, g_1 = 16, g_2 = 60, g_3 = 96, g_4 = 120, g_5 = 160, g_6 = 180$ and $g_7 = 225 \in Z_{240}\}, 1 \leq i \leq 30\}$ be the closed-open interval general ring of 10 × 3 matrices of mixed dual numbers.

Example 5.79: Let L = $\{(a_1, a_2, a_3) \mid a_j = (c, d]; c, d \in \{x_1 + x_2g_1 + x_3g_2 \mid x_i \in Z_6; 1 \leq i \leq 3, g_1 = 6, g_2 = 4 \in Z_{12}\}; 1 \leq j \leq 3\}$ be the open-closed interval general ring of mixed dual numbers.

Let x = $((3 + 2g_1 + g_1 + 3g_2], (4 + 5g_2, g_1 + 4], (3g_1 + g_2, 3g_2 + 4g_1 + 1])$ and

y = $((2 + g_1, g_2 + 4], (3g_1 + g_2, g_2], (0, 4g_1])$ be in L.

x + y = $[(5 + 3g_1 + g_2, g_1 + 4g_2 + 4], (4 + 3g_1, g_1 + g_2 + 4], (3g_1 + g_2, 1+ 3g_2 + 3g_1]) \in L$

$$x \times y = ((3 + 2g_1 + g_2, g_1 + 3g_2] \times (2 + g_1, g_2 + 4], (4 + 5g_2, g_1 + 4] (3g_1 + g_2, g_2], (3g_1 + g_2, 3g_2 + 4g_1 + 1] (0, 4g_1])$$

$$= ((6 + 3g_1 + 2g_1^2 + 2g_1 + g1g_2, g_1g_2 + 4g_1 + 3g_2^2 + 12g_2], (12g_1 + 42 + 15g_1g_2 + 5g_2^2, g_1g_2 + 4g_2], (0, 12g_1g_2 + 16g_1^2 + 4g_1])$$

$$= (6 + 3g_1, 4g_1 + 3g_2], (2g_1 + 3g_2, 4g_2], (0, 4g_1]) \in L.$$

Thus L is a ring.

Example 5.80: Let S = {[a, b] | a, b ∈ P = {$x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$ | $x_i \in Z^+ \cup \{0\}$, $1 \leq i \leq 5$, $g_1 = 12$, $g_2 = 16$, $g_3 = 24$, $g_4 = 36 \in Z_{48}$}} be the closed interval general semiring of mixed dual numbers.

Example 5.81: Let M = {(a_1, a_2) | a_i = [c, d), c, d ∈ S = {$x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5$ | $x_j \in Z^+ \cup \{0\}$, $1 \leq j \leq 6$, $g_p \in L$ = a chain lattice of order seven $1 \leq p \leq 5$}, $1 \leq i \leq 2$} be the closed open interval general semiring of mixed dual numbers.

Example 5.82: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \end{bmatrix} \middle| \, a_i = [c, d), c, d \in S = \{x_1 + x_2g_1 + \right.$$

$$x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7 \text{ with } x_j \in Q^+ \cup \{0\},$$

$$1 \leq j \leq 8, g_1 = 16, g_2 = 60, g_3 = 96, g_4 = 120, g_5 = 160,$$

$$\left. g_6 = 180, \text{ and } g_7 = 225 \in Z_{240}\}, 1 \leq i \leq 16 \right\}$$

be the open-closed interval rectangular matrix of semiring of mixed dual numbers. Clearly T is not a semfield of mixed dual numbers.

Now we see we can build as in case of special dual like numbers in case of mixed dual numbers also vector spaces and semivector spaces / linear algebra of intervals. This work is left for the reader, however we give problems in this regard in the last chapter of this book.

Finally we can have fuzzy interval mixed dual numbers and fuzzy interval special dual like numbers and they are fuzzy semigroups under max or min operations.

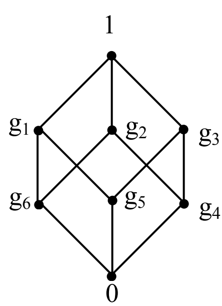
We will illustrate this situation by some examples.

Example 5.83: Let S = $\{[a, b) \mid a = x_1 + x_2g_1 + x_3g_2$ and $b = y_1 + y_2g_1 + y_3g_2$ where $x_i, y_j \in [0, 1]$, $1 \leq i, j \leq 3$, $g_1 = 6$ and $g_2 = 4 \in Z_{12}\}$ be the closed-open interval fuzzy semigroup of mixed dual number under max or min operation.

Example 5.84: Let
M = $\{(a, b) \mid a = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6$ and

$B = y_1 + y_2g_1 + \ldots + y_7g_6$ where $x_i, y_j \in [0, 1]$, $1 \leq i, j \leq 7$ and

$g_p \in L =$



$1 \leq p \leq 6\}\}$ be the open interval fuzzy semigroup of special dual like numbers under min (or max) operator.

Example 5.85: Let S = $\{(a_1, a_2, a_3, a_4) \mid a_i = [c, d], c, d \in P = \{x_1 + x_2g_1 + \ldots + x_9g_8 \mid x_j \in [0, 1], 1 \le j \le 9$ and $g_p \in L$; L a chain lattice of order 10 given by $L = \{1 > g_1 > g_2 > \ldots > g_8 > 0\}$, $1 \le p \le 8\}, 1 \le i \le 4\}$ be the closed interval general fuzzy semigroup of special dual like numbers under min or max operation.

Example 5.86: Let W = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| a_i = [c, d), c, d \in S = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_j \in [0, 1], 1 \le j \le 4 \text{ and } g_1 = 6, g_2 = 4 \text{ and } g_3 = 9 \in Z_{12}\}; 1 \le i \le 10\right\}$ be the closed open interval general fuzzy semigroup of mixed dual numbers for in $x \times y = \min \{x, y\}$, we take $\min \{x_1, y_1\} + \min \{x_2, y_2\} g_1^2 + \min \{x_3y_3\} g_2^2 + \ldots + \min \{x_2, y_3\}g_1 \times g_2$ and so on be it min or max operation we take only g_ig_j (product modulo 12), $1 \le i, j \le 3$.

Example 5.87: Let P = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \end{bmatrix} \middle| a_i = (c, d], c, d \in \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 + x_8g_{7\,v} \text{ with } x_j \in [0, 1], 1 \le j \le 8 \text{ and } g_1 = 16, g_2 = 60, g_3 = 120, g_4 = 96, g_5 = 180, g_7 = 160 \text{ and } g_8 = 225 \in Z_{240}\}, 1 \le i \le 36\right\}$ be the open closed interval fuzzy semigroup of mixed dual numbers.

Interested reader can construct more examples; derive related properties as most of the results involved can be derived as a matter of routine.

Chapter Six

APPLICATIONS OF SPECIAL DUAL LIKE NUMBERS AND MIXED DUAL NUMBERS

Only in this book the notion of special dual like number is defined. In a dual number a + bg_1 we have $g_1^2 = 0$; a and b reals and in special dual like number a + bg we have $g^2 = g$; a and b reals. Certainly special dual like numbers will find appropriate applications once this concept becomes popular among researchers. For we have the neutrosophic ring $\langle R \cup I\rangle$ or $\langle Q \cup I\rangle$ or $\langle Z \cup I\rangle$ or $\langle Z_n \cup I\rangle$ happens to be special ring. Thus where ever neutrosophic concepts are applied certainly the special dual like number concept can be used. We view I only as an idempotent of course not as an indeterminate.

Since to generate special dual like numbers distributive lattices are used certainly these concepts will find suitable applications. Further we also make use of the modulo integers in the construction of special dual like numbers. Keeping all these in mind, researchers would find several applications of this new number.

Finally the notion of mixed dual numbers exploits both the concept of special dual like numbers and dual numbers, so basically the least dimension of mixed dual numbers are three.

For if x = a + bg_1 + cg_2 g_1 and g_2 two new elements such that $g_1^2 = 0$, $g_2^2 = g_2$ and $g_1g_2 = g_2g_1 = 0$ or g_1 or g_2 and a, b, c are reals then we define, x to be a mixed dual number.

It is pertinent to mention here we cannot use lattices to construct mixed dual numbers.

The only concrete structure from where we get mixed dual numbers are from Z_n, n not a prime n = 4m. So we think this new numbers will also find applications only when this concept becomes popular and more research in this direction are taken up by researchers. Also this study forces more research on the modulo integers Z_n, n a composite number.

Chapter Seven

SUGGESTED PROBLEMS

In this chapter we suggest 145 number of problems of which some are simple exercise and some of them are difficult or can be treated as research problems.

1. Discuss some properties of special dual number like rings.

2. Is M = {a + bg | a, b $\in$ Q; g = 10 $\in$ Z_{15}} be a semigroup under $\times$. Enumerate a few interesting properties associated with it.

3. Let S = $\left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}\right| a_i = x_i + y_i g \text{ with } x_i, y_i \in Q; 1 \le i \le 8,\right.$

   $g = \begin{pmatrix} 3 & 4 & 4 & 3 \\ 4 & 3 & 3 & 4 \end{pmatrix}$, 3, 4 $\in$ Z_6} be the ring of special dual like numbers under natural product $\times_n$.

(i) Find subrings in S which are not ideals of S.
(ii) Find ideals of S.
(iii) Find zero divisors in S.
(iv) Show ideals of S form a modular lattice.

4. Show if $S = \{a + bg \mid a, b \in R, g^2 = g = 3 \in Z_6\}$ is the special dual like number ring then any $x \times y$ in S need not in general be of the form a + bg; b, a ∈ R.

(i) Can S have zero divisors?
(ii) Can a + bg have inverse? ($a, b \in R \setminus \{0\}$).
(iii) Can $x = a + bg \in S$ be an idempotent? (with $a, b \in R \setminus \{0\}$).

5. Enumerate the special properties enjoyed by $Z_n(g)$.

6. Let $S = \{a + bg \mid a, b \in Z_7\ g = 11 \in Z_{22}\}$ be the special dual like number ring.

(i) Find subrings of S which are not ideals? (is it possible).
(ii) Find the cardinality of S.
(iii) Does S contain subring?
(iv) Can S have zero divisor or idempotents?

7. Is (Q (g), +, ×) where $g = 9 \in Z_{12}$ an integral domain?

8. Let $Z(g) = \left\{a + bg \mid a, b \in Z \text{ and } g = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\right\}$ be a special dual like number ring.

(i) Can Z(g) have idempotents?
(ii) Can Z(g) have ideals?

9. Let $S = Z_p(g) = \{a + bg \mid a, b \in Z_p, g^2 = g\}$ be the special dual like number ring (p a prime).
   (i) Can S have subrings which are not ideals?
   (ii) Can S have zero divisors?
   (iii) Can $a + bg$, $a, b \in Z_p \setminus \{0\}$ have inverse?
   (iv) Can S have idempotents of the form $a + bg$, $a, b \in Z_p \setminus \{0\}$?

10. Find the orthogonal subrings of S given in problem 9.

11. Let $S = \{(x_1, x_2, x_3, x_4, x_5) + (y_1, y_2, y_3, y_4, y_5)g \mid x_i, y_i \in Q, 1 \leq i \leq 5, g = 4 \in Z_6\}$ be the special dual like number ring.
    (i) Prove S have zero divisors.
    (ii) Can S have idempotents?
    (iii) Find ideal of S.

12. Let $M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in Q; 1 \leq i \leq 10, g = (3, 4, 3, 4, 4, 3, 4) \text{ with } 3, 4 \in Z_6\right\}$ be a special dual like ring under the natural product $\times_n$.

    (i) Do the zero divisors of M form an ideal?
    (ii) Does M contain a subring which is not an ideal?

13. Let $M = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in Z_{31}; 1 \leq i \leq 16, g = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 3 & 4 \end{bmatrix}, 3, 4 \in Z_6\right\}$ be a special dual like ring under the natural product $\times$.

(i) Show M is non commutative.
(ii) Find zero divisors of M.
(iii) What is the cardinality of M?
(iv) Is M a S-ring?

14. In M in problem 13 is under natural product $\times_n$ distinguish the special features of M under $\times_n$ and under $\times$.

15. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in\right.$ Z_3; $1 \leq i \leq 30$, $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $g \times_n g = g\}$ be the special dual like number ring under the natural product $\times_n$.

(i) Find the number of elements in P.
(ii) Find subrings which are not ideals in P.

16. Describe some of the special features enjoyed by special dual like number vector spaces V over the field Q or R.

17. Let V = $\{a + bg \mid a, b \in R, g^2 = g$, g the new element$\}$ be the special dual like number vector space over the field R.
(i) Find a basis of V over R.
(ii) Write V as a direct sum of subspaces.
(iii) Find L(V,R) = {all linear functional from V to R}. What is the algebraic structure enjoyed by L(V,R)?

18. Let W = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i = x_i + y_i g \text{ where } x_i, y_i \in Q; 1 \leq i \leq\right.$ 4, $g = 7 \in Z_{14}\}$ be the special dual like number vector space over the field Q.

(i) Is W a linear algebra under usual matrix product?
(ii) Find a basis of W over Q as a vector space as well as a linear algebra.

(iii) Is the dimension of W the same as a vector space or as a linear algebra?
(iv) Write W as a pseudo direct sum of subspaces.

19. Let P = $\left\{\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \\ b_7 & b_8 \end{bmatrix}\right) g \;\middle|\; a_i, b_j \in Z_7, 1 \leq i, j \leq 8;\right.$

$g = 13 \in Z_{26}\}$ be a linear algebra of special dual numbers under the natural product $\times_n$ over Z_7.

(i) Find Hom (P, P).
(ii) Find a basis of P over Z_7.
(iii) Find the number of elements in P.

20. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_8 \\ a_9 & a_{10} & \ldots & a_{16} \\ a_{17} & a_{18} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{32} \end{bmatrix} \;\middle|\; a_i = x_i + y_i g \text{ with } x_i, y_i \in\right.$

Z_{11}, $1 \leq i \leq 32$; g = $\begin{bmatrix} 4 \\ 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}$ 4, 3 $\in Z_6\}$ be a special dual like number vector space over the field 11.

(i) Find dimension of M over Z_{11}.
(ii) Find the number of elements in M.
(iii) If on M we define the natural product $\times_n$, what is the dimension of M as a linear algebra over Z_{11}?
(iv) Find L (M, Z_{11}).

21. Let
$S = \{a + bg \mid a, b \in Z^+ \cup \{0\}, g = (13, 14), 13, 14 \in Z_{26}\}$
be the semiring.
(i) Can S be a semifield?
(ii) Is S a strict semiring?
(iii) Can S have zero divisors?

22. Let $M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| a_i = x_i + y_ig \text{ where } x_i, y_i \in Q^+ \cup \{0\}; 1 \leq\right.$
$i \leq 4, g = (17, 18, 17, 18), 17, 18 \in Z_{34}\}$ be the semiring.
(i) Does M contain subsemirings which are not ideals?
(ii) Can $T = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| a_i = x_i + y_ig \text{ with } x_i, y_i \in Q^+ \cup \{0\};\right.$
$1 \leq i \leq 2, g = (17, 18, 17, 18), 17, 18 \in Z_{34}\} \subseteq M$ be a semiideal of M?
(iii) Suppose $W = \left\{\begin{bmatrix} 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| a_i = x_i + y_ig \text{ with } x_i, y_i \in\right.$
$Q^+ \cup \{0\}; 1 \leq i \leq 2, g = (17, 18, 17, 18), 17, 18 \in Z_{34}\} \subseteq M$; can W be a semiideal such that T and W are orthogonal?

23. Give an example of a general semifield of special dual like numbers.

24. Let $P = \{a + bg \mid a, b \in Z^+, g = \begin{pmatrix} 5 & 6 & 5 & 6 \\ 6 & 5 & 6 & 5 \end{pmatrix}; 5, 6 \in$
$Z_{10}\} \cup \{0\}$ be the semified of special dual like numbers.

(i) Can P have subsemifields?
(ii) Can P have subsemirings?

25. Let M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| a_i = x_i + y_i g \text{ where } x_i, y_i \in Q^+ ; g = \begin{bmatrix} 13 \\ 14 \\ 13 \end{bmatrix};\right.$ $13, 16 \in Z_{26}\ 1 \leq i \leq 3\} \cup \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ be the semiring of special dual like numbers under natural product $\times_n$.

(i) Can M be a semifield?
(ii) Can M have semiideals?
(iii) Can M have subsemirings?

26. Give an example of a general semiring of special dual like numbers which is not a semifield.

27. Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| a_i = x_i + y_i g \text{ with } x_i, y_i \in R^+; 1 \leq i \leq 4, g = (5, 6), 5, 6 \in Z_{10}\right\} \cup \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ be the general semiring of special dual lime numbers under the usual product ×.

(i) Can M be a semifield?
(ii) Is M a S-semiring?
(iii) Can M have right semiideals which are not left semiideals?

28. Suppose M in problem (27) is under natural product $\times_n$ what can we say about M?

29. Let P = $\{x + yg \mid x, y \in Q^+, g = (1\ 0\ 0\ 1\ 1\ 1\ 0)\} \cup \{0\}$ be the semifield of special dual like numbers. Study the special features enjoyed by P.

30. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{12} \\ a_{13} & a_{14} & a_{15} & \dots & a_{24} \\ a_{25} & a_{26} & a_{27} & \dots & a_{36} \end{bmatrix}\right|$ with $a_i = x_i + y_ig$

where $x_i, y_i \in Z^+$; $1 \leq i \leq 36$, $g = 3 \in Z_6\} \cup$

$\left\{\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}\right\}$ be a semiring of special dual like numbers.

(i) Can $Z^+ \subseteq S$? Justify.
(ii) Can $Z^+g \subseteq S$? Justify.
(iii) Can S have subsemifield?

31. Find all the idempotents of Z_{46}.

32. Find all the idmepotents of Z_{12}.

(i) Are the idempotents in Z_{12} orthogonal?
(ii) Do the set of idempotents of Z_{12} form a semigroup under product?

33. Find all the idempotents of Z_{30}.
(i) How many idempotents does Z_{30} contain?
(ii) Do the set with 0 form a semigroup under product?

34. Find the number of idempotents in Z_{105}.

35. Let Z_n be such that $n = p_1\ p_2 \dots p_t$; $t < n$ and each p_i is a prime and $p_i \neq p_j$ if $i \neq j$.
(i) Find all the idempotents in Z_n.

(ii) What is the order of the semigroup of idempotents of Z_n with zero?
(iii) Are the idempotents of Z_n orthogonal?

36. Let Z_{4900} be the ring of modulo integers. Find the number of idempotents in Z_{4900}.

(i) Hence or otherwise find the number of idempotents in $Z_{p_1^2, p_2^2, p_3^2}$ each p_i is a distinct prime; i = 1, 2, 3.
(ii) Further if $Z_{p_1^{n_1}, p_2^{n_2}, \ldots, p_t^{n_t}}$ be the ring of integers $p_i \neq p_j$ if $i \neq j$ are distinct primes; $n_i \geq 2$; $1 \leq i \leq t$. Find the number of idempotents in $Z_{p_1^{n_1}, p_2^{n_2}, \ldots, p_t^{n_t}}$.

37. Prove Z_p, p a prime cannot have idempotents, other than 0 and 1.

38. Prove using 5, 6, 0 of Z_{10} we can build infinitely many idempotents which can be used to construct special dual like numbers.

39. Study the special dual like number semivector space / semilinear algebra.

40. Let $V = \{(a_1, a_2, \ldots, a_5) \mid a_i = x_i + y_i g$ where $x_i, y_i \in Z^+$; $1 \leq i \leq 5$, $g = 7 \in Z_{14}\} \cup \{(0, 0, \ldots, 0)\}$ be a semivector space over the semifield $F = \{a + bg \mid a, b \in Z^+\} \cup \{0\}$. ($g = 7 \in Z_{14}$).

(i) Find a basis for V.
(ii) Is V finite dimensional over F?
(iii) If F is replaced by $Z^+ \cup \{0\}$; will V be finite dimensional?
(iv) Is V a semilinear algebra over F?
(v) What is dimension of V as a semilinear algebra?
(vi) Write V as a direct sum of semivector spaces.

41. Can Z_{p^2} have idempotents, p a prime?

42. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} \middle| a_i = x_i + y_i g \,;\, g = 10 \in Z_{30},\; x_i, y_i \in \right.$

$Z^+; 1 \leq i \leq 8\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ be the semivector space of

special dual like numbers over the semifield
$F = \{a+bg \mid a, b \in Z^+, g = 10 \in Z_{10}\} \cup \{0\}$.

(i) Find a basis of S over F.
(ii) Can S be made into a semilinear algebra?
(iii) Study the special features enjoyed by S.

43. Find the algebraic structure enjoyed by $Hom_F(S, S)$, S given in problem 42.

44. Find the properties enjoyed by
L(S, F) = {all linear functional from S to F}, S given in problem (42).

45. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_{10} \\ a_{11} & a_{12} & a_{13} & \ldots & a_{20} \end{bmatrix} \middle| a_i = x_i + y_i g \,;\, g = 17 \right.$

$\in Z_{34},\; x_i, y_i \in Q^+; 1 \leq i \leq 20\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \right\}$ be

the semivector space over the semifield $S = \{a + bg \mid a, b \in Q^+\} \cup \{0\}$ of special dual like numbes.

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \; a_i = x_i + y_i g \; ; \; g = 17 \in Z_{38}, \; x_i, y_i \in \right.$$

$$\left. Q^+; 1 \le i \le 21 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right\} \text{ be the semivector space}$$

over the semifield $S = \{a + bg \mid a, b \in Q^+\} \cup \{0\}$.

(i) Find Hom (M, P).
(ii) Study the algebraic structure enjoyed by Hom(M, P).
(iii) Study the properties of Hom(M, M) and Hom(P, P) and compare them.
(iv) Study L(M, S) and L(P, S) and compare them.
(v) What will be the change if S is replaced by $Z^+ \cup \{0\}$?
(vi) Study (i), (ii) and (iii) when S is replaced by $Z^+ \cup \{0\}$.

46. Let

$$S = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = x_i + y_i g \; ; \; g = 4 \in Z_6, \; x_i, y_i \in Z^+ \cup \{0\} \right\}$$

be the semivector space of special dual like numbers over the semifield
$F = \{a + bg \mid a, b \in Z^+; 4 = g \in Z_6\} \cup \{0\}$.

(i) Find dimension of S over F.
(ii) Find a basis of S over F.
(iii) Find $Hom_F(S, S)$
(iv) Find L(S, F).

47. Determine some interesting features enjoyed by special set vector spaces of special dual like numbers.

48. Let M = $\left\{(a_1, a_2), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| a_i = x_i + y_i g ; g = (4,\right.$

3, 4, 3), 4, 3 $\in Z_6$, $x_i, y_i \in Q$; $1 \le i \le 4\}$ be the special set vector space of special dual like numbers over the set 3Z $\cup$ 5Z.

(i) Find Hom (V, V).
(ii) Find L(V, 3Z $\cup$ 5Z).

49. Let T = $\{a + bg_1, c + dg_2, e + fg_3 \mid a, b, c, d, e, f \in Q; g_1 =$

$(7, 8, 7, 8)$, $g_2 = \begin{bmatrix} 5 \\ 6 \\ 5 \\ 6 \end{bmatrix}$ and $g_3 = \begin{bmatrix} 13 & 14 \\ 0 & 13 \end{bmatrix}$, $7, 8 \in Z_{14}$, $5, 6 \in$

Z_{10} and $13, 14 \in Z_{26}\}$ be a special set vector space of special dual like numbers over the set S = 3Z $\cup$ 7Z $\cup$ 11Z.

(i) Find set special vector subspaces of T over S.
(ii) Write T as a direct sum of set special vector subspaces over S.
(iii) Find $\text{Hom}_S(T, T)$.
(iv) Find L(T, S).

50. Let W = $\left\{a + bg_1, \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| a, b \in Z^+ \cup \{0\},\right.$

$g_1 = \begin{bmatrix} 11 & 11 \\ 12 & 12 \end{bmatrix}$, $11, 12 \in Z_{22}$, $a_i = x_i + y_i\, g_2$ with $x_i, y_i \in$

$Q^+ \cup \{0\}$ $1 \leq i \leq 4$, $g_2 = \begin{bmatrix} 7 & 8 & 7 & 8 \\ 8 & 7 & 8 & 7 \end{bmatrix}$, $7, 8 \in Z_{14}\}$ be a

special set semivector space over the set $S = 3Z^+ \cup 5Z^+ \cup \{0\}$ of special dual like numbers.

(i) Find $Hom_S(W, W)$.
(ii) Find $L(W, S)$.
(iii) Can W have a basis?
(iv) Write W as a pseudo direct sum of special set semivector subspaces of W over S.

51. Let $V = \left\{(a_1, a_2, a_3, a_4), \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_i + y_i g\right.$

$x_i, y_i \in R^+ \cup \{0\}, g = 4 \in Z_6\}$ $1 \leq i \leq 9\}$ and

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix} \middle| a_i = x_i + y_i g, \right.$$

$g = 4 \in Z_6$, $x_i, y_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 10\}$ be special set semivector spaces of special dual like numbers over the set $S = 3Z^+ \cup 5Z^+ \cup \{0\}$.

(i) Find $Hom_S(V, M)$.
(ii) Study Hom (V, V) and Hom (M, M) and compare them.
(iii) Study L(V, S) and L(M, S) and compare them.

52. Prove M = {A + Bg | A and B are m × n matrices with

entries from Q and $g = \begin{bmatrix} 4 & 0 & 4 & 3 \\ 3 & 4 & 0 & 4 \\ 4 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{bmatrix}$, 3, 4 ∈ Z_6} and S

= {$(a_{ij})_{m \times n}$ where $a_{ij} = c_{ij} + d_{ij}$ g where $c_{ij}, d_{ij} \in Q$; $1 \leq i \leq m$

and $1 \leq j \leq n$, $g = \begin{bmatrix} 4 & 0 & 4 & 3 \\ 3 & 4 & 0 & 4 \\ 4 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{bmatrix}$ 3, 4 ∈ Z_6} as general

ring of special dual like numbers are isomorphic.

(i) If M and S are taken as vector spaces of special dual like numbers over the field Q are they isomorphic?

53. Is it possible to get any n-dimensional special dual like numbers; n arbitrarty positive integer?

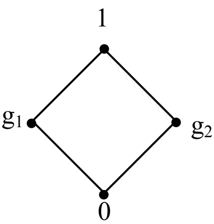
54. Find some special properties by n-dimensional special dual like numbers.

55. What is the significance of using lattices in the construction of special dual like numbers?

56. Give some applications of n-dimensional special dual like numbers?

57. What is the advantage of using n-dimensional special dual like numbers instead of dual numbers?

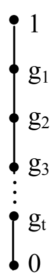
58. Prove $C(g_1, g_2) = \{a + bg_1 + cg_2 \mid a, b, c \in$ C(complex numbers)

$g_1, g_2 \in L =$ 

$\}$

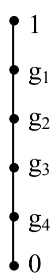
is a general ring of special dual like numbers of dimension three.

59. Study some special features enjoyed by $C(g_1, g_2, \ldots, g_t) = \{x_1 + x_2g_1 + \ldots + x_{t+1}g_t \mid x_j \in L; 1 \leq j \leq t + 1.\ g_k \in L =$



$1 \leq k \leq t\}$

60. Study the 5×3 matrices with entries from $C(g_1, g_2, g_3, g_4)$ where $g_i \in L =$

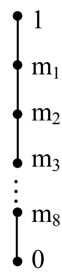


$1 \leq i \leq 4$,

61. Obtain some interesting properties about lattice ring RL where L is a distributive lattice of finite order n and R a

commutative ring with unit. Show RL is a (n–1) special dual like number ring.

62. Let ZL = $\left\{ a + \sum_i a_i m_i \,\middle|\, a, a_i \in Z, m_i \in L = \right.$



$1 \leq i \leq 8\}$ be the lattice ring.

(i) What is dimension of ZL as a special dual like number ring?

(ii) Can ZL have ideals of lesser dimension?

(iii) Can ZL have 4-dimension special dual like ring?

(iv) Can ZL have zero divisor?

(v) Is ZL an integral domain?

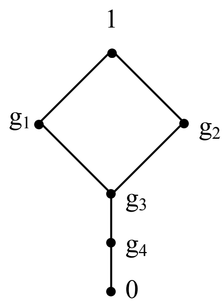
63. Let Z_{84} be the ring of integers. Find all idempotents of Z_{84}. Is that collection a semigroup under multiplication modulo 84?

64. Give an example of a 8-dimensional general ring of special dual like numbers.

65. Give an example of a 5-dimensional general semiring of special dual like numbers.

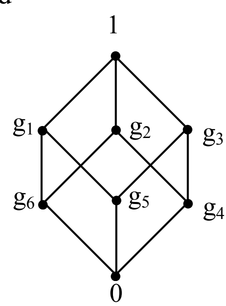
66. Give an example of a finite 5- dimensional general ring of special dual like numbers.

67. Is Z_8 $(g_1, g_2, g_3, g_4) = \{ a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in Z_8, 1 \leq i \leq 5, g_j \in L,$



$1 \leq j \leq 4\}$ a general 5-dimensional special dual like number ring?

68. Let $S = \{ a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid a_i \in Z_{25}, 1 \leq i \leq 7$, and

$g_j \in L =$ 

$1 \leq j \leq 6\}$ be the general seven dimensional special dual like number ring.

(i) Find the number of elements in S.
(ii) Can S have ideals which are 3-dimensional?
(iii) Can S have 2- dimensional subring?
(iv) Can S have zero divisors?
(v) Can S have units?

69. Let $P = \{ a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 \mid a_i \in Z_7, 1\leq i \leq 5$, $g_1 = (1, 0, 0, 0)$, $g_2 = (0, 1, 0, 0)$ $g_3 = (0, 0, 1, 0)$ and $g_4 = (0, 0, 0, 1)\}$ be the special dual like number general ring.
(i) Prove P is a S-ring.
(ii) Can P have zero divisors?

(iii) Give examples of subrings which are not ideals.

70. Let M = { $a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 \mid a_i \in Z_3$;

$$1 \leq i \leq 3;\ g_1 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 4 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}\ g_3 = \begin{bmatrix} 0 & 4 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$g_4 = \begin{bmatrix} 0 & 3 \\ 4 & 0 \\ 0 & 0 \end{bmatrix},\ g_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 3 \end{bmatrix}$$ where 4, 3 $\in Z_6$} be the special dual like number ring.

(i) Find the number of elements in M
(ii) Can M have zero divisors?
(iii) Can $a_1+a_2g_1$ ($a_1, a_2 \in Z_3 \setminus \{0\}$) be an idempotent in M?
(iv) Can x in M have x^{-1} such that $xx^{-1} = 1$ ($x \notin Z_3$) ?

71. Let S = { $a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 \mid a_i \in$

$$Z^+ \cup \{0\},\ 1 \leq i \leq 6,\ g_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ g_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix},\ g_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$g_4 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix},\ g_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix};$$ 2 $\in Z_4$} be a general semiring of special dual like numbers.

(i) Is S a S-semiring?
(ii) Can S have zero divisors?
(iii) Is S a semifield?

72. Let S = {$a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6 + a_8g_7$ | $a_i \in Z^+$, $1 \leq i \leq 8$;

$$g_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} g_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$g_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, g_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } g_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

under natural product $\times_n$, g_i's are idempotents and $g_j \times_n g_k$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } j \neq k\} \cup \{0\} \text{ be a general semiring of special}$$

dual like numbers.

(i) Is S a semifield?
(ii) Can S have semiideals?
(iii) Can S have subsemifields?
(iv) Is S a S-semiring?

73. Let P = $\left\{ \sum_{i=0}^{\infty} a_i x^i \right|$ a_i = (0, 0, …, 1, 0, …, 0), a_1 = (1, 0, … 0) and a_2 = (0, 1, 0, 0, …0) of 9 tuples} be the polynomials with idempotent coefficient

(i) Prove (P, +) is not a semigroup.

(ii) Is (P, ×) a semigroup?
(iii) Can the semigroup (P, x) have ideals?
(iv) Can P have zero divisors?

74. Let $S = \left\{\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \\ g_5 & g_6 \\ g_7 & g_8 \end{pmatrix} \middle| g_i \in \{0, 3, 4\} \subseteq Z_6, 1 \leq i \leq 8\right\}$ be a semigroup under natural product $\times_n$.

(i) Prove S is finite?
(ii) Find ideals in S.
(iii) Find zero divisors in S.
(iv) Can S have subsemigroups which are not ideals?

75. Let S =

$$\left\{\left[\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{matrix}\right] + \left[\begin{matrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \end{matrix}\right] g_1 + \left[\begin{matrix} z_1 & z_2 & z_3 & z_4 \\ z_5 & z_6 & z_7 & z_8 \end{matrix}\right] g_2\right.$$

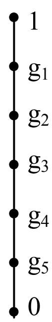
$$\left.\left[\begin{matrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \end{matrix}\right] g_3 + \left[\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{matrix}\right] g_4 + \left[\begin{matrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \end{matrix}\right] g_5 \right|$$

$a_i, x_j, y_k, z_p, c_t, d_s \in Q^+$, $1 \leq i, j, k, p, t, s, \leq 8$, $g_j \in L$ where L is

$1 \leq i \leq 5\} \cup \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}$ be the semiring of special dual like numbers.

(i) Is S a semifield?
(ii) Can S have subsemifields?
(iii) Can S have zero divisors?

76. Let M = $\left\{\begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_5 & A_6 & A_7 & A_8 \end{bmatrix}\right|$
$A_i = x_1^i + x_2^i g_1 + \ldots + x_6^i g_5$; $1 \leq i \leq 8$, $x_j^1 \in Q^+$, $1 \leq j \leq 6$ and $g_k \in L$ =



$1 \leq k \leq 5\} \cup \left\{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\}$ be the general semiring of special dual like numbers.

(i) Is M a semifield?
(ii) Can M have zero divisors?
(iii) Is M a strict semiring?
(iv) Can M be isomorphic to S given in problem (75)

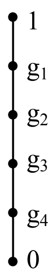
77. Show if idempotents are taken form distributive lattice of order 9 and if
S = $\{x_1 + x_2g_1 + \ldots + x_9g_7 \mid x_i \in Q; 1 \leq i \leq 8, g_j \in L, 1 \leq j \leq 7\}$ be the general ring of special dual like numbers then

$x \times y$ under the operation $\cap$ of g_i and g_j is different from $x \times y$ under the operation '$\cup$' of g_i and g_j.

78. Verify problem 77 if Q is replaced by $Q^+ \cup \{0\}$.

79. Obtain some interesting properties enjoyed by vector space of special dual like numbers over a field F.

80. Let $V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4;\ 1 \leq i \leq 9;\ x_j \in Z_{11},\ 1 \leq j \leq 5 \text{ and}\right.$

$g_k \in L =$  $; 1 \leq k \leq 4\}$

be a special vector space of dual like numbers over the field Z_{11}.

(i) Find the number of elements in V.
(ii) What is the basis of V over Z_{11}?
(iii) Write V as a direct sum of subspaces.
(iv) What is the algebraic structure enjoyed by $\text{Hom}_{Z_{11}}(V,V)$?
(v) If $T : V \to V$ is such that $T\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\right) = \begin{bmatrix} 0 & a_2 & 0 \\ a_1 & 0 & a_3 \\ 0 & a_4 & 0 \end{bmatrix}$; find the eigen values of T and eigen vectors of T.

81. Let $V = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_8 \end{array}\right] \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \,;\; 1 \leq i \leq 8, \right.$

$x_j \in Q$, $g_1 = (3, 0, 4)$, $g_2= (0, 3, 0)$ and $g_3 = (0, 4, 0)$; $3, 4 \in Z_6$, $1 \leq j \leq 4\}$ be a special vector of special dual like numbers over the field Q.

(i) Find a basis of V over Q.
(ii) Write V as a pseudo direct sum.
(iii) Suppose $W_1 = \left\{ \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ 0 \end{array}\right] \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \,; \right.$

$1 \leq i \leq 2$, $g_1 = (3\ 0\ 4)$, $g_2\ (0\ 3\ 0)$, $g_3 = (0, 4, 0)$, $3, 4$

$\in Z_6$, $1 \leq j \leq 4\} \subseteq V$, $W_2 = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \middle| \; a_1, a_2 \in \right.$

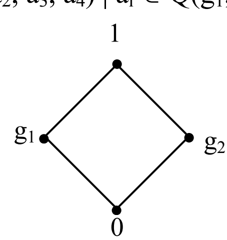
$$Q(g_1, g_2)\} \subseteq V, \; W_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in Q(g_1, g_2) \right\}$$

$$\subseteq V \text{ and } W_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_1 \\ a_2 \end{bmatrix} \middle| \; a_1, a_2 \in Q(g_1, g_2) \right\} \subseteq V \text{ are}$$

subspaces of V. Find projections E_1, E_2, E_3 and E_4 of V on W_1, W_2, W_3 and W_4 respectively and show projection contribute to special dual like numbers. Verify spectral theorem E_1, E_2, E_3 and E_4 by suitable and appropriate operations on V.

82. Let $V = \{(a_1, a_2, a_3, a_4) \mid a_i \in Q(g_1, g_2); 1 \leq i \leq 4,$

$g_1, g_2 \in L =$



$\}$

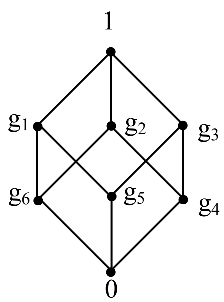
be a Smarandache special vector space of special dual like numbers over the S-ring $Q(g_1, g_2)$.

(i) Find a basis of S over $Q(g_1, g_2)$.
(ii) Write S as a direct sum of subspaces.
(iii) Find Hom(S, S).
(iv) Find $L(S, Q(g_1, g_2))$.
(v) Show eigen values can also be special dual like numbers.

83. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \end{bmatrix} \middle| \, a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 \right.$

$+ x_5g_4 + x_6g_5 + x_7g_6$, $1 \leq i \leq 12$ $a_i \in Z_{12}(g_1, g_2, \ldots, g_6)$ where

$g_j \in L=$



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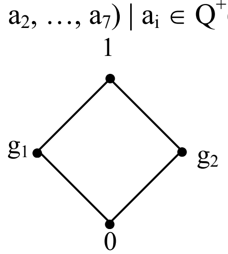
be a Smarandache vector space of special dual like numbers over the S-ring; $Z_{12}(g_1, g_2, \ldots, g_6)$,

(i) Find the number of elements in M.
(ii) Find a basis of M over $Z_{12}(g_1, \ldots, g_6)$.
(iii) Write M as a direct sum.
(iv) Find Hom (M, M).
(v) Find L $(M, Z_{12}, (g_1, g_2, \ldots, g_6))$.

83. Let $V = \{(a_1, a_2, \ldots, a_7) \mid a_i \in Q^+(g_1, g_2) \cup \{0\}, 1 \leq i \leq 7\}$,

$g_1, g_2 \in L=$



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be the special dual like number semivector space over the semifield $Q^+ \cup \{0\}$.

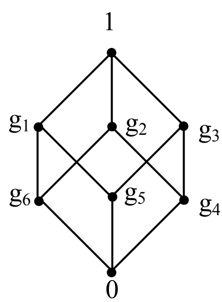
(i) Find a basis of V over $Q^+ \cup \{0\}$.

(ii) Study the algebraic structure enjoyed by Hom(V, V).

(iii) Study the set L (V, $Q^+ \cup \{0\}$) if f : V $\rightarrow Q^+ \cup \{0\}$ is given by f (a_1, a_2, … a_7) = $(x_1^1 + x_1^2 + \ldots + x_1^7)$ where $a_i = x_1^i + x_2^i g_1 + x_3^i g_2$; $1 \leq i \leq 7$.
Does $f \in$ L (V, $Q^+ \cup \{0\}$)?

84. Let S = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in Z^+(g_1, g_2, \ldots, g_6),\ 1 \leq i \leq 12\right.$

$g_j \in$ L=



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$1 \leq j \leq 6\}$ be a strong special semibivector space of special dual like numbers over the semifield $Z^+(g_1, g_2, \ldots, g_6) \cup \{0\}$.

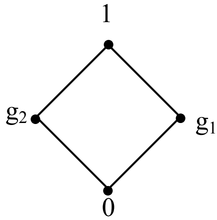
(i) Find a basis of S over $Z^+(g_1, g_2, \ldots, g_6) \cup \{0\}$.

(ii) Find Hom(S, S). For at least one T $\in$ Hom (S, S). find eigen values and eigen vectors associated with T.

(iii) Write S as a direct sum of special semivector subspaces of special dual like numbers.

(iv) Can S be made into a semilinear algebra by defining $\times_n$, the natural product?

86. Let $S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Z_7(g_1, g_2) \text{ where } g_1, g_2 \in L \right.$



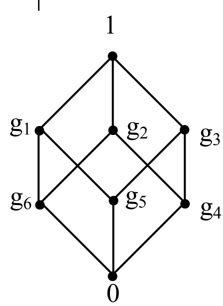
$\left. \vphantom{\sum} \right\}$

be a vector space special dual like numbers over the field Z_7.

(i) Find dimension of V over Z_7.
(ii) Can V be written as a direct sum?
(iii) Find Hom(V,V).
(iv) Study the structure of $L(V, Z_7)$.

87. What happens if in problem (86) Z_7 is replaced by the S-ring, $Z_7(g_1, g_2)$, that is V is a Smarandache vector space of special dual like numbers over the S-ring $Z_7(g_1, g_2)$.

88. Let $P = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Q\,(g_1, g_2, \ldots, g_6) \text{ where } g_j \in L = \right.$

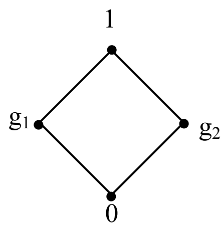


$1 \leq j \leq 6\}$ be a special vector space of special dual like numbers over the field Q.

(i) Find a basis of P over Q.
(ii) What is the dimension of P over Q?
(iii) Can P be a linear algebra ?

89. Study P in problem 88 as a S-vector space of special dual like numbers over the S-ring $Q(g_1, g_2, \ldots, g_6)$.

90. Let $S = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in Q^+(g_1, g_2) \cup \{0\}, g_1, g_2, \in L = \right.$



$\left.\right\}$

be a semivector space of special dual like numbers over the semifield $Q^+ \cup \{0\}$.

(i) Find a basis of V over S.
(ii) Write S as a direct sum of semivector subspaces.
(iii) If S is a linear algebra can S be written as a direct sum of semilinear algebras?
(iv) Study the algebraic structure enjoyed by Hom(S, S).
(v) Is $\langle Z_{20} \cup I \rangle$, a general neutrosophic ring of special dual like number?
(vi) Characteristize some of the special features of special dual like numbers.

93. Can Z_{56} have idempotents so that $a + bg_1$, $g_1 \in Z_{56} \setminus \{0, 1\}$ is an idempotent contributing to special dual like numbers?

94. Does Z_n for any n have a subset S such that S is an idempotent semigroup of Z_n?

95. Find all the idempotent in Z_{48}.

96. Is 0, 16, 96, 160 and 225 alone are idempotents of Z_{240}? Does $S = \{0, 16, 96, 160, 225\} \subseteq Z_{240}$ form a semigroup?

97. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle|\right.$ $a_i \in P = \{x_1 + x_2tg_1 + x_3g_2 \mid x_j \in Q, 1 \le j \le 3, g_1 = 4$ and $g_2 = 9 \in Z_{12}\}, 1 \le i \le 10\}$ be a general vector space of special dual like numbers over the field Q.

(i) Find a basis of S over Q?
(ii) What is the dimension of S over Q?
(iii) Find Hom (S, S).
(iv) Find eigen values and eigen vectors for some T $\in$ Hom (S, S) such that $T^2 = (0)$.
(v) Write P as a direct sum of subspaces.

98. Let M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle|\right.$ $a_i \in \langle R \cup I\rangle, 1 \le i \le 4\}$ be a general vector space of neutrosophic special dual like numbers over the field R.

(i) Find dimension of M over R.
(ii) Find a basis of M over R.
(iii) Find the algebraic structure enjoyed by Hom(M, M).
(iv) If T : M $\to$ M be defined by T $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ a_2 \\ 0 \end{bmatrix}$ find eigen values and eigen vectors associated with T.

99. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \middle|\right.$ $a_i \in \langle Z_{11} \cup I\rangle, I^2 = I\}$ be the general ring of neutrosophic polynomial of special dual like numbers.

(i) Can S have zero divisors?
(ii) Can S have units?
(iii) Is S a Smarandache ring?
(iv) Can S have ideals?
(v) Can S have subrings which are not ideals?
(vi) Can S have idempotents?

100. Let P = $\left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in \langle Z_{12} \cup I\rangle, I^2 = I\}$ be the general ring of neutrosophic polynomial of special dual like numbers.

(i) Prove P has zero divisors?
(ii) Find ideals of P.
(iii) Find subrings in P which are not ideals of P.
(iv) Can P have idempotents?
(v) Prove P is a S-ring.
(vi) Does p(x) = $x^2 - (7+3I)x + 0\ (5+3I)$ reducible in P?

101. Let S = $\left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in \langle R \cup I\rangle; I^2 = I\}$ be the general ring of neutrosophic polynomial of special dual like numbers.

(i) Does S contain polynomials which are irreducible in S?
(ii) Find the roots of the polynomial $(3+4I)x^3 + (5-3I)x^2 + 7Ix - (8I - 4)$.
(iii) Is T = $\left\{\sum_{i=0}^{\infty} a_i x^i \right| a_i \in \langle Z \cup I\rangle; I^2 = I\} \subseteq S$ an ideal of S?
(iv) Can S have zero divisors?

102. Let $W = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Q \cup I\rangle; I^2 = I \right\}$ be the general ring of neutrosophic polynomial of special dual like numbers over the field Q.
(i) Find subspaces of W.
(ii) Is W infinite dimensional?
(iii) Can linear functional from W to Q be defined?
(iv) Can eigen values of any linear operator on T be a neutrosophic special dual like numbers?

103. Let $S = \{(a_1, a_2, \ldots, a_{10}) \mid a_i = x_1 + x_2g_1 + x_3g_2$ where $1 \le i \le 10$, $x_j \in \langle Q \cup I\rangle$; $1 \le j \le 3$ $g_1 = 9$ and $g_2 = 4 \in Z_{12}\}$ be the general neutrosophic ring of special dual like elements.
(i) Find ideals of S.
(ii) Prove S has zero divisors.
(iii) Prove S has idempotents.
(iv) Does S contain subrings which are not ideals?

104. Let $V = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \,\middle|\, a_i \in \langle Q^+ \cup I\rangle; 1 \le i \le 15 \right\}$ be a general vector space of neutrosophic semivector space over the semiring $S = \langle Z^+ \cup I \cup \{0\}\rangle$.
(i) Find a basis of V over S.
(ii) What is a dimension of V over S?
(iii) Find Hom(V, V) = {T : V → V all semilinear operators on V} and the algebraic structure enjoyed by it.
(iv) Can f : V → S be defined? Find L (V,S).

105. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in \langle Q^+ \cup \{0\} \cup I \rangle; 1 \leq i \leq 9 \right\}$

be a general semivector space of neutrosophic special dual like numbers over the semifield $S = \langle Z^+ \cup \{0\} \cup \{I\} \rangle$.

(i) Find dimension of W over S.

(ii) Find the algebraic structure enjoyed by L(W,S).

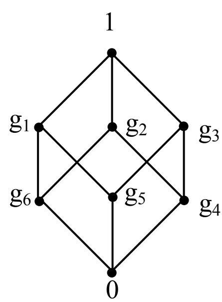
(iii) Can W be written as a direct sum of semivector subspaces?

(iv) Is W a linear semialgebra on W by define usual × product of matrices?

106. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \; a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + \right.$

$x_6g_5 + x_7g_6$ where $x_j \in Q$, $1 \leq j \leq 7$ and

$g_p \in L =$



$1 \leq p \leq 6\}$ be the general vector space of special dual like numbers over the field Q.

(i) Find dimension of S over Q.

(ii) Find $Hom_Q(S, S)$.

(iii) Can a eigen value of $T : S \to S$ be special dual like numbers?

107. Let M = $\{x_0 + x_1g_1 + x_2g_2 + x_3g_3 + x_4g_4 + x_5g_5 + x_6g_6 \mid x_i \in Q; 0 \leq i \leq 6$ and $g_1 = (I, 0,0,0,0,0)$, $g_2 = (0, I,0,0,0,0)$ $g_3 = (0, 0,I,0,0,0)$, $g_4 = (0, 0,0,I,0,0)$ $g_5 = (0, 0,0,0,I,0)$ and $g_6 = (0,0,0,0,0,I)$ with $I^2 = I\}$ be a general linear algebra of neutrosophic special dual like numbers over the field Q.
   (i) Find dimension of M over Q.
   (ii) Find a basis of M over Q.
   (iii) Write M as a pseudo direct sum of subspaces of M over Q.
   (iv) Find Hom(M, M).
   (v) Find L(M, Q).

108. Find P = $\{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 \mid x_i \in Z_{13}; 1 \leq i \leq 5$, $g_1 = (I, 0, 0, 0)$, $g_2 = (0, I, 0, 0)$, $g_3 = (0, 0, I, 0)$ and $g_4 = (0, 0, 0, I)\}$ be a general vector space of neutrosophic special dual like numbers over the field Z_{13}.
   (i) Find the number of elements in P.
   (ii) Find dimension of P over Z_{13}.
   (iii) Find a basis of P over Z_{13}.
   (iv) Can P have more than one basis over Z_{13}?
   (v) How many basis can P have over Z_{13}?

109. Let F = $\{\langle Z_{37} \cup I \rangle\}$ be the general neutrosophic ring of special dual like numbers.
   (i) Find order of F.
   (ii) Is F a S-ring?
   (iii) Find ideals in F.
   (iv) Can F have subrings which are not ideals?
   (v) Can F have zero divisors?
   (vi) Can F have idmepotents other than I?

110. Let A = $\left\{\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{bmatrix} \middle| x_j = a_1 + a_2g_1 + a_3g_2 + x_4g_3\right.$ where $a_i \in Q$, $1 \leq i \leq 4$, $1 \leq j \leq 8$, $g_1 = 6$, $g_2 = 9$ and $g_3 = 4 \in Z_{12}\}$ be the general ring of mixed dual numbers.
   (i) Can A have zero divisors?
   (ii) Find idemponents in A?
   (iii) Prove A is a commutative ring.

111. Let A = $\left\{\begin{bmatrix} a_1 & a_2 & \cdots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \\ a_{21} & a_{22} & \cdots & a_{30} \end{bmatrix} \middle| \; a_i \in \langle Z_{12} \cup I \rangle; 1 \leq i \leq 30, \right.$

$I^2 = I\}$ be the general neutrosophic matrix ring of special dual like numbers.

(i) Find zero divisors of S.
(ii) Can S have subrings which are not ideals?
(iii) Find ideals of S.
(iv) Can S have idempotents?
(v) Does S contain Smarandache zero divisors?

112. Let T = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i \in \langle Z_7 \cup I \rangle; 1 \leq i \leq 12\right\}$ be the general

neutrosophic ring of special dual like numbers.

(i) Find the numbers of elements in T.
(ii) Can T have idemponents?
(iii) Give some special features enjoyed by T.
(iv) Does T contain Smarandache ideals?

113. Let A = $\left\{\begin{bmatrix} a_1 & a_2 & \cdots & a_{15} \\ a_{16} & a_{17} & \cdots & a_{30} \end{bmatrix} \middle| \; a_i \in (c, d]\; c, d \in \{x_1 + \right.$

$x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4 + x_6g_5 + x_7g_6 + x_8g_7 \mid x_j \in [0, 1]$, $1 \leq j \leq 8$, $g_1 = 16$, $g_2 = 60$, $g_3 = 96$, $g_4 = 120$, $g_5 = 160$, $g_6 = 180$ and $g_7 = 225 \in Z_{240}\}$, $1 \leq i \leq 30\}$ be a closed open interval fuzzy semigroup of mixed dual numbers under min.

(i) Find zero divisors in M.
(ii) Can M have idempotents?
(iii) Can every elements in M be an idempotent?

(iv) Find ideals in M.
(v) Can M have subsemigroup which are not ideals?

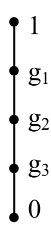
114. Find some interesting properties associated with interval fuzzy semigroup of mixed dual numbers.

115. Obtain some applications of interval fuzzy semigroups of special dual like numbers under min (or max operation).

116. Let P = $\{x_1 + x_2g_1 + x_3g_2 + \ldots + x_{18}g_{17} \mid x_j \in [0, 1], 1 \leq j \leq 18, g_p \in L$ = chain lattice of order 19$\}$ be the general fuzzy semigroup of special dual like numbers under min operation.
(i) Find fuzzy subsemigroups of P which are not fuzzy ideals.
(ii) Find ideals in P.
(iii) Under min operation can P have zero divisors?
(iv) If max operation is performed on P can P have zero divisors?

117. Obtain any interesting property / application enjoyed by general fuzzy semigroup of special dual like numbers.
Let M = $\{[a, b] \mid a, b \in S = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 + x_5g_4$ where $x_i \in Q^+ \cup \{0\}$, $1 \leq i \leq 5$, $g_p \in L$, L a chain lattice of order six, $1 \leq p \leq 4\}$ be a general closed interval semivector space over the semifield T = $Q^+ \cup \{0\}$.
(i) Find a basis of M over T.
(ii) Find Hom(M, M).
(iii) Find L(M, T).

118. Let V = $\{(a_1, a_2] \mid a_i = x_1 + x_2g_1 + x_3g_2 + x_4g_3; 1 \leq i \leq 2, x_j \in Z_{127}, 1 \leq j \leq 4,$

$g_p \in L =$ 

$1 \leq p \leq 3\}$

be a general vector space over the field Z_{127} of special dual like numbers.
(i) Find a basis of V over over Z_{127}.
(ii) Write V as a direct sum.
(iii) Find $T : V \to V$ so that T^{-1} does not exist.
(iv) How many elements does V contain?
(v) Find $L(V, Z_{127})$.

119. Is every ideal in $P = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in \langle Z_{19} \cup I \rangle\right\}$ principal? Justify.

120. Can $S = \left\{\sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_1 = x_i + x_2 g_1 + x_3 g_2 + x_4 g_3 + x_5 g_4 + x_6 g_5\right.$ with $x_p \in R$, $g_j \in L$; $1 \leq j \leq 5$, $1 \leq p \leq 6\}$ have S-ideals?

121. Let $W = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} I \,\middle|\, x_i, y_j \in Q;\ 1 \leq i, j \leq 4\right\}$ be the neutrosophic general ring of special dual like numbers.
(i) Find ideals of W.
(ii) Does W contain S-subrings which are not ideals?
(iii) Can W have S-idempotents?

122. Let $P = \left\{\begin{bmatrix} x_1 & x_2 & \ldots & x_5 \\ x_6 & x_7 & \ldots & x_{10} \\ x_{11} & x_{12} & \ldots & x_{15} \end{bmatrix} + \begin{bmatrix} y_1 & y_2 & \ldots & y_5 \\ y_6 & y_7 & \ldots & y_{10} \\ y_{11} & y_{12} & \ldots & y_{15} \end{bmatrix} I \,\middle|\, x_i,\right.$ $y_i \in R$; $1 \leq i, j \leq 15\}$ be a general neutrosophic ring of special dual like numbers.
(i) Find ideals of P.
(ii) Does P have S zero divisors?
(iii) Prove P is isomorphc to

$$S = \begin{pmatrix} x_1 + y_1 I & x_2 + y_2 I & \ldots & x_5 + y_5 I \\ x_6 + y_6 I & x_7 + y_7 I & \ldots & x_{10} + y_{10} I \\ x_{11} + y_{11} I & x_{12} + y_{12} I & \ldots & x_{15} + y_{15} I \end{pmatrix} \text{ where}$$

$x_i, y_i \in R$, $1 \leq i \leq 15\}$ as a ring of special dual like numbers.

123. Let $R = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i = \begin{bmatrix} x_1 + y_1 I \\ \vdots \\ x_9 + y_9 I \end{bmatrix}; x_i, y_i \in Q, 1 \leq i \leq 9 \right\}$

be a general neutrosophic polynomial ring of special dual like numbers.

(i) Prove R has zero divisors.
(ii) Can R have S-zero divisors?
(iii) Is R a S-ring?
(iv) Can R have S-subrings which are not ideals?

124. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \end{bmatrix} \;\middle|\; a_i \in \langle Q \cup I \rangle \; 1 \leq i \leq 21 \right\}$

be a general vector space over Q of special neutrosophic dual like number over Q.

(i) Find a basis of M over Q.
(ii) Find subspaces of M so that M is a direct sum of subspaces.
(iii) Find Hom(M,M).
(iv) Find L(M, Q).
(v) If Q is replaced $\langle Q \cup I \rangle$, M is a S-vector space find L (M, $\langle Q \cup I \rangle$).
(vi) Find S-basis of M over $\langle Q \cup I \rangle$.

125. Obtain some special properties enjoyed by general vector spaces of special dual like numbers of n-dimension $n > 2$.

126. Obtain some special features enjoyed by general semilinear algebra of special dual like numbers of t-dimension, $t \geq 3$.

127. Study problems (126) and (125) in case of mixed dual numbers of dimension > 2.

128. Let $S = Z_8 (g_1, g_2, g_3) = \{x_1 + x_2g_1 + x_3g_2 + x_4g_3 \mid x_i \in Z_8, 1 \leq i \leq 8, g_1 = 6, g_2 = 4 \text{ and } g_3 = 9 \in Z_{12}\}$, study the algebraic structure enjoyed by S.

129. Find the mixed dual number semigroup component of Z_{112}.

130. Study the mixed dual number semigroup component of Z_{352}.

131. Study the semigroup mixed dual number component of Z_{23p}, where p is a prime.

132. Study the semigroup mixed dual number of component of Z_{64m} where m is a odd and not a prime.

133. Compare problems (131) and (132) (that is the nature of the mixed semigroups).

134. Study the general ring of mixed dual numbers of dimension 9.

135. Can any other algebraic structure other than modulo integer Z_n contribute to mixed dual numbers?

136. Show we can have any desired dimensional general ring of special dual like numbers (semiring or vector space or semivector space).

137. Obtain some special properties enjoyed by fuzzy semigroup of mixed dual numbers.

138. Let $M = \{x_1 + x_2g_1 + \ldots + x_{20}g_{19} \mid x_i \in Z^+ \cup \{0\}; 1 \leq i \leq 20$ and $g_j \in L$ a chain lattice of order 21, $1 \leq j \leq 19\}$ be a semivector space over the semifield $S = Z^+ \cup \{0\}$ of special dual like numbers.
    (i) Find a basis of M over S.
    (ii) What is the dimension of M over S?
    (iii) Can M have more than one basis over S?
    (iv) Find Hom(M, M).
    (v) Find L(M, S).

139. Using the mixed dual number component semigroup of Z_{640} construct a general ring of mixed dual numbers with elements from Z_3. Study the properties of this ring.

140. Give an example of a Smarandache general ring of mixed dual numbers.

141. Study the properties of open-closed interval general ring of mixed dual numbers.

142. Characterize all Z_n which has mixed dual numbers semigroup component.

143. Characterize those Z_n which has idempotent semigroup.

144. Characterize those Z_n which has no idempotent (when n not a prime).

145. Characterize those Z_n which has no mixed dual number semigroup component (n not a prime $n \neq 2^tp$).

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ABOUT THE AUTHORS

Dr.W.B.Vasantha Kandasamy is an Associate Professor in the Department of Mathematics, Indian Institute of Technology Madras, Chennai. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 646 research papers. She has guided over 68 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 70th book.

On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

She can be contacted at vasanthakandasamy@gmail.com
Web Site: http://mat.iitm.ac.in/home/wbv/public_html/
or http://www.vasantha.in

Dr. Florentin Smarandache is a Professor of Mathematics at the University of New Mexico in USA. He published over 75 books and 200 articles and notes in mathematics, physics, philosophy, psychology, rebus, literature. In mathematics his research is in number theory, non-Euclidean geometry, synthetic geometry, algebraic structures, statistics, neutrosophic logic and set (generalizations of fuzzy logic and set respectively), neutrosophic probability (generalization of classical and imprecise probability). Also, small contributions to nuclear and particle physics, information fusion, neutrosophy (a generalization of dialectics), law of sensations and stimuli, etc. He got the 2010 Telesio-Galilei Academy of Science Gold Medal, Adjunct Professor (equivalent to Doctor Honoris Causa) of Beijing Jiaotong University in 2011, and 2011 Romanian Academy Award for Technical Science (the highest in the country). Dr. W. B. Vasantha Kandasamy and Dr. Florentin Smarandache got the 2011 New Mexico Book Award for Algebraic Structures. He can be contacted at smarand@unm.edu

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In this book we define $x = a + bg$; to be a special dual like number where a, b are reals and g is a new element such that $g^2 = g$. The new element which is an idempotent can be got from Z_n or from lattices or from linear operators. Mixed dual numbers are constructed using dual numbers and special dual like numbers. Neutrosophic numbers are a natural source of special dual like numbers.

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