Short Introduction to Standard and NonStandard Neutrosophic Set and Logic

(review paper)

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Abstract: The definitions of Single-Valued Neutrosophic Set (SVNS), Interval-Valued Neutrosophic Set (IVNS), Subset-Valued Neutrosophic Set (SVNS), and respectively NonStandard Neutrosophic Set [Single-Valued (NoS SVNS), Interval-Valued (NoS IVNS), and Subset-Valued (NoS SVNS)], are recalled together with their operators. Similarly for Standard and NonStandard Neutrosophic Logic.

1. Introduction

The neutrosophic set was introduced in 1995 by F. Smarandache as extension of Intuitionistic Fuzzy Set. The first publications were in 1998-2005 [25, 26].

2. Standard Neutrosophic Set and Logic

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{S}$ a non-empty subset of $\mathcal{U}$. Let $t(x), i(x), f(x)$ be the degrees of truth (membership), indeterminacy, and falsehood (nonmembership) respectively of the generic element x with respect to the set $\mathcal{S}$.

Let $\mathcal{S} = \left\{x_{(t(x),i(x),f(x))}, \text{for } x \in \mathcal{U}\right\}$.

2.1 Definition of Single-Valued Neutrosophic Set (SVNS)

$\mathcal{S}$ is a <u>Single-Valued Neutrosophic Set</u> if:

$$t(x), i(x), f(x): \mathcal{S} \to [0,1]$$

(or $t(x), i(x), f(x)$ are single numbers in [0, 1]) such that:

$$0 \le t(x) + i(x) + f(x) \le 3$$

2.2 Definition of Interval-Valued Neutrosophic Set (IVNS)

$\mathcal{S}$ is an Interval-Valued Neutrosophic Set if:

$$t(x), i(x), f(x): \mathcal{S} \to \mathcal{P}([0,1])$$

where $\mathcal{P}([0,1])$ is the powerset of [0, 1], and $t(x), i(x), f(x)$ are intervals included in [0, 1] such that:

$$0 \le inf\big(t(x)\big) + inf\big(i(x)\big) + inf\big(f(x)\big) \le sup\big(t(x)\big) + sup\big(i(x)\big) + sup\big(f(x)\big) \le 3$$

2.3 Definition of Subset-Valued Neutrosophic Set (SVNS)

$\mathcal{S}$ is a Subset-Valued Neutrosophic Set if:

$$t(x), i(x), f(x): \mathcal{S} \to \mathcal{P}([0,1])$$

and $t(x), i(x), f(x)$ are subsets included in [0, 1] such that:

$$0 \le inf\big(t(x)\big) + inf\big(i(x)\big) + inf\big(f(x)\big) \le sup\big(t(x)\big) + sup\big(i(x)\big) + sup\big(f(x)\big) \le 3$$

3. Necessity to Introduce the NonStandard Neutrosophic Set and Logic

In order to make distinction between Relative Truth (truth in at least one world, according to Leibniz) and Absolute Truth (truth in all possible words, again according to Leibniz), we considered the NonStandard Analysis:

$t(x) = 1$, meaning relative truth (membership),

and $t(x) = 1^{+}$, meaning absolute truth (membership).

Similarly, for Relative Indeterminacy and Absolute Indeterminacy respectively:

$i(x) = 1$, for relative indeterminacy,

and $i(x) = 1^{+}$, for absolute indeterminacy.

And for Relative Falsehood (NonMembership), and Absolute Falsehood (NonMembership):

$f(x) = 1$, for relative falsehood (nonmembership), and $f(x) = 1^{+}$, for absolute falsehood (nonmembership).

4. Introduction to NonStandard Analysis

Abraham Robinson in 1960s has developed the non-standard analysis [15, 16, 27, 28], by rigurously defining the infinitesimals and infinite numbers.

An *infinitesimal number* (ε) is a number ε such that its absolute value $|\varepsilon| < \frac{1}{n}$, for any non-null positive integer *n*. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero. Infinitesimals are used in calculus, but interpreted as tiny real numbers.

An *infinite number* (ω) is a number greater than anything:

$$1 + 1 + 1 + \cdots + 1 \text{ (for any finite number terms).}$$

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (*non-standard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

$$\frac{1}{\varepsilon} = \frac{\omega}{1}$$

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in *R* are valid in R^* as well [according to the classical NonStandard Analysis].

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to *a*.

Let's denote by R_+^* the set of positive nonzero hyperreal numbers.

Left Monad and Right Monad were defined as follows:

Left Monad {that we denote, for simplicity, by (^{-}a)}, is defined as:

$$\mu(^{-}a) = (^{-}a) = \{a - x, x \in R_+^* | x \text{ is infinitesimal}\}$$

Right Monad {that we denote, for simplicity, by (a^{+})}, is defined as:

$$\mu(a^{+}) = (a^{+}) = \{a + x, x \in R_+^* | x \text{ is infinitesimal}\}$$

5. Extensions of NonStandard Analysis

In 1998, Smarandache [25] introduced the Pierced Binad.

5.1. **Pierced Binad** {that we denote, for simplicity, by $({}^{-}a^{+})$} is defined as:

$$\mu({}^{-}a^{+}) = ({}^{-}a^{+}) =$$

$$= \{a - x, x \in R_{+}^{*} | x \text{ is a positive infinitesimal}\} \cup \{a + x, x \in R_{+}^{*} | x \text{ is a positive infinitesimal}\}$$

$$= \{a - x, x \in R^{*} | x \text{ is a positive or negative infinitesimal}\}.$$

Later on, in 2019, Smarandache [23] also introduced the Left Monad Closed to the Right, The Right Monad closed to the Left, and the Unpierced Binad (all defined as below) in order to have the Nonstandard Real Monad/Binad Set closed under arithmetic operations.

5.2. Left Monad Closed to the Right

$$\mu\binom{-0}{a} = \binom{-0}{a} = \left\{a - x | x = 0, \text{or } x \in R_{+}^{*}, \text{and } x \text{ is infinitesimal}\right\}$$

$$= \mu({}^{-}a) \cup \{a\}$$

By notation, $\mu\begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix} = \overset{0}{a} = a$.

And by $x = \overset{-0}{a}$ we understand the hyperreal $x = a - \varepsilon$, or $x = a$, where ε is a positive infinitesimal. So, x is not clearly known, $x \in \{a - \varepsilon, a\}$.

5.3. Right Monad Closed to the Left

$$\mu\begin{pmatrix} 0+ \\ a \end{pmatrix} = \begin{pmatrix} 0+ \\ a \end{pmatrix} = \{a + x | x = 0, \text{or } x \in R_{+}^{*}, \text{and } x \text{ is infinitesimal}\}$$

$$= \mu(a^{+}) \cup \{a\}$$

And by $x = \overset{0+}{a}$ we understand the *hyperreal* $x = a + \varepsilon$, or $x = a$, where ε is a positive infinitesimal. So, x is not clearly known, $x \in \{a, a + \varepsilon\}$.

5.4. Unpierced Binad

$$\mu\left(\overset{-0+}{a}\right)=\left(\overset{-0+}{a}\right)=\{a+x|x=0, \text{or } x\in R^{*}, \text{and } x \text{ is positive or negative infinitesimal}\}$$
$$=\mu(^{-}a)\cup(a^{+})\cup\{a\}=(^{-}a)\cup(a^{+})\cup\{a\}$$

And by $x=\overset{-0+}{a}$ we understand the *hyperreal* $x=a-\varepsilon$, or $x=a$, or $x=a+\varepsilon$, where ε is a positive infinitesimal. So, x is not clearly known, $x\in\{a-\varepsilon, a, a+\varepsilon\}$.

The left monad, left monad closed to the right, right monad, right monad closed to the left, the pierced binad, and the unpierced binad are subsets of R^{*}, while the above hyperreals are numbers from R^{*}.

6. NonStandard Neutrosophic Set and Logic

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{S}$ a non-empty subset of $\mathcal{U}$.

Let $\mathcal{S}=\left\{x_{(t(x),i(x),f(x))}, \text{for } x\in\mathcal{U}\right\}$.

6.1 Definition of Single-Valued NonStandard Neutrosophic Set and Logic

$\mathcal{S}$ is an Single-Valued NonStandard Neutrosophic Set and Logic, if:

$$t(x), i(x), f(x): \mathcal{S}\rightarrow\,]^{-}0, 1^{+}[$$

where $]^{-}0, 1^{+}[$ is the nonstandard unit interval, which is a set that contains, besides the ordinary real numbers between 0 and 1, also:

Left Monads, Right Monads;

As well as the newly introduced by Smarandache [23, 25] in 1998-2019:

Left Monads closed to the Right,

Right Monads closed to the Left,

Pierced Binads,

and Unpierced Binads.

Of course, $[0, 1]\subset\,]^{-}0, 1^{+}[$.

Also,

$$^{-}0 \le inf\big(t(x)\big) + inf\big(i(x)\big) + inf\big(f(x)\big) \le sup\big(t(x)\big) + sup\big(i(x)\big) + sup\big(f(x)\big) \le 3^{+}$$

6.2 Definition of Interval-Valued NonStandard Neutrosophic Set and Logic

$\mathcal{S}$ is an Interval-Valued NonStandard Neutrosophic Set and Logic, if:

$$t(x), i(x), f(x): \mathcal{S} \to \mathcal{P}(\,]^{-}0, 1^{+}[\,)$$

where $\mathcal{P}(\,]^{-}0, 1^{+}[\,)$ is the powerset of all standard and nonstandard subset of $]^{-}0, 1^{+}[$

with $t(x), i(x), f(x)$ being nonstandard intervals of the form:

$$\left]\overset{*}{a}, \overset{*}{b}\right[$$

where $0 \le a \le b \le 1$, and $\overset{*}{a} < \overset{\#}{b}$,

where $\overset{*}{a} \in \{\overset{-}{a}, \overset{0}{a}, \overset{+}{a}, \overset{-0}{a}, \overset{-+}{a}, \overset{0+}{a}, \overset{-0+}{a}\}$

similarly $\overset{\#}{b} \in \{\overset{-}{b}, \overset{0}{b}, \overset{+}{b}, \overset{-0}{b}, \overset{-+}{b}, \overset{0+}{b}, \overset{-0+}{b}\}$.

6.3 Definition of Subset-Valued NonStandard Neutrosophic Set and Logic

$\mathcal{S}$ is a Subset-Valued NonStandard Neutrosophic Set and Logic, if:

$$t(x), i(x), f(x): \mathcal{S} \to \mathcal{P}(\,]^{-}0, 1^{+}[\,)$$

with $t(x), i(x), f(x)$ being nonstandard subsets of the nonstandard interval $]^{-}0, 1^{+}[$.

7. Standard and NonStandard Neutroosphic Operators

Firstly we need to recall the t-norm and t-conorm from the fuzzy set and logic:

7.1. A *t-norm* [29] is a function t: [0, 1] × [0, 1] → [0, 1] that satisfies the following properties:

- Commutativity: t(a, b) = t(b, a)
- Monotonicity: t(a, b) ≤ t(c, d) if $a \le c$ and $b \le d$
- Associativity: t(a, t(b, c)) = t(t(a, b), c)
- The number 1 acts as identity element: t(a, 1) = a

A common notation in fuzzy set and logic is t(a, b) = a $\wedge_F$ b, meaning intersection respectively conjunction.

A *t-conorm* [29] is a function ⊥: [0, 1] × [0, 1] → [0, 1] that satisfies the following properties:

- Commutativity: ⊥(*a*, *b*) = ⊥(*b*, *a*)
- Monotonicity: ⊥(*a*, *b*) ≤ ⊥(*c*, *d*) if *a* ≤ *c* and *b* ≤ *d*
- Associativity: ⊥(*a*, ⊥(*b*, *c*)) = ⊥(⊥(*a*, *b*), *c*)
- Identity element: ⊥(*a*, 0) = *a*

A common notation in fuzzy set and logic is t(a, b) = a $\vee_F$ b, meaning union respectively disjunction.

All the above types of Neutroosphic Standard and NonStandard Operators are based on the fuzzy t-norm ($\wedge_F$) and fuzzy t-conorm ($\vee_F$).

Let x(T, I, F)

7.2. Neutroosphic Standard and NonStandard Intersection/Conjunction ($\wedge_N$)

$$x(T_1,I_1,F_1)\wedge_N x(T_2,I_2,F_2)=x(T_1\wedge_F T_2,I_1\vee_F I_2,F_1\vee_F F_2)$$

7.3. Neutroosphic Standard and NonStandard Union/Disjunction ($\vee_N$)

$$x(T_1,I_1,F_1)\vee_N x(T_2,I_2,F_2)=x(T_1\vee_F T_2,I_1\wedge_F I_2,F_1\wedge_F F_2)$$

7.4. Neutroosphic Standard and NonStandard Complement/Negation ($\neg_N$)

$$\neg_N x(T,I,F)=\bar{x}(F,1-I,T)$$

7.5. Neutrosophic Standard and NonStandard Implication ($\rightarrow_N$)

$$A_1(T_1,I_1,F_1)\rightarrow_N A_2(T_2,I_2,F_2)$$

is neutrosophically equivalent with

$$\neg_N A_1(T_1,I_1,F_1)\vee_N A_2(T_2,I_2,F_2)$$

7.6. Neutrosophic Standard and NonStandard Equivalence ($\leftrightarrow_N$)

$$A_1(T_1,I_1,F_1)\leftrightarrow_N A_2(T_2,I_2,F_2)$$

is neutrosophically equivalent with

$A_1(T_1, I_1, F_1) \rightarrow_N A_2(T_2, I_2, F_2)$ and $A_2(T_2, I_2, F_2) \rightarrow_N A_1(T_1, I_1, F_1)$.

Conclusion

In this paper we reviewed the various types of standard and nonstandard neutrosophic sets and logics, used for about three decades of scientific research and applications.

With the help of the fuzzy t-norm and fuzzy t-conorm we recalled the definitions of the neutrosophic standard and nonstandard operators.

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Received: Aug 15, 2024. Accepted: Nov 9, 2024