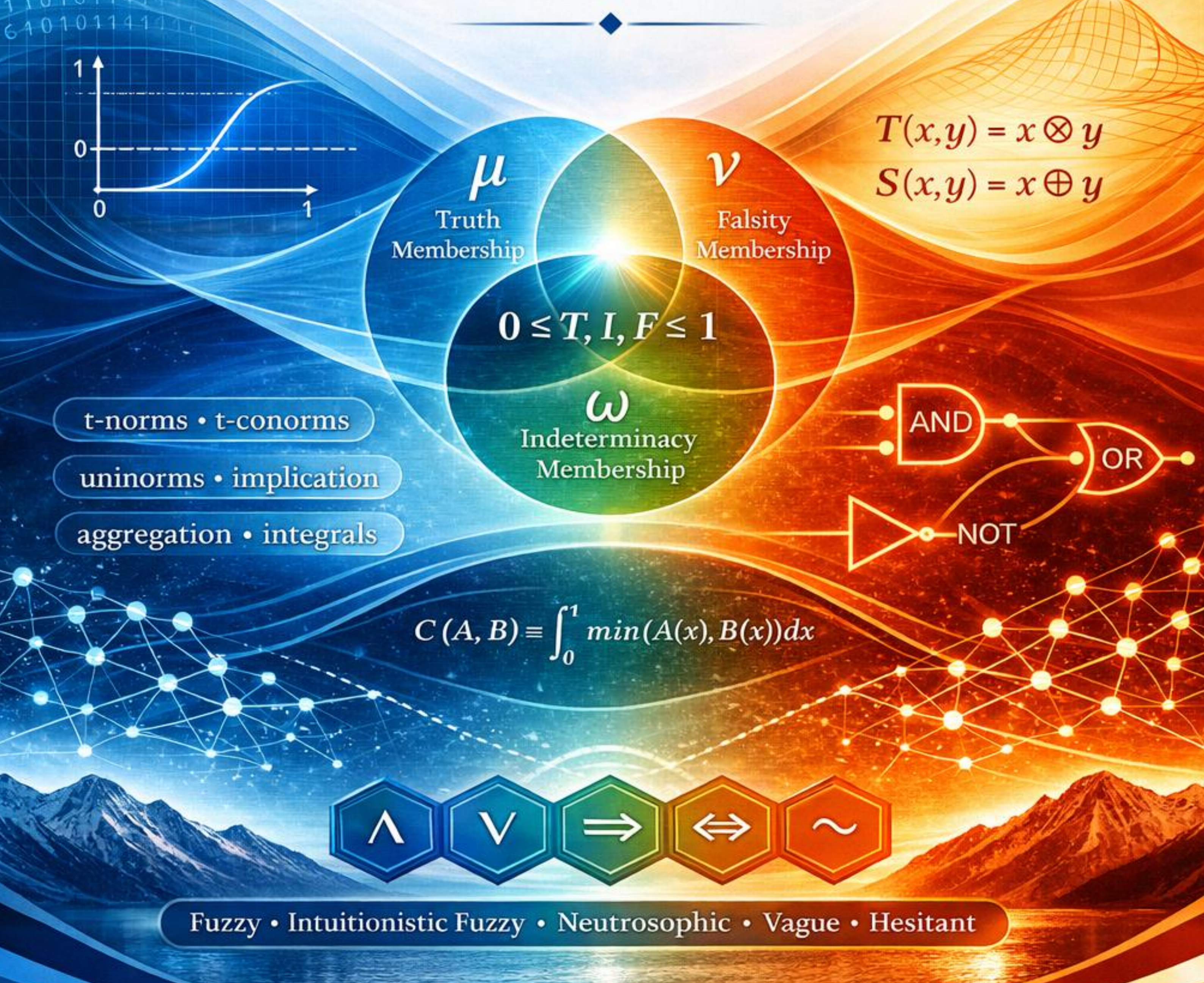
SURVEY OF FUZZY AND UNCERTAIN LOGICAL OPERATORS

A Comprehensive Study of Operators for Fuzzy, Neutrosophic, and Uncertain Set Theories



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Chapter 1

Introduction

1.1 Uncertain Set

Real-world information is often imprecise, incomplete, inconsistent, or only partially reliable. To model such situations, many generalized set-theoretic frameworks have been developed, including Fuzzy Sets [1], Intuitionistic Fuzzy Sets [2], Neutrosophic Sets [3–7], Hesitant Neutrosophic Sets [8, 9], Vague Sets [10], Hesitant Fuzzy Sets [11], Picture Fuzzy Sets [12], Quadripartitioned Neutrosophic Sets [13], and Plithogenic Sets [14]. These frameworks have been widely applied in areas such as decision science, chemistry, control, and machine learning [15]. In practice, the choice of an appropriate model depends on the nature of the phenomenon under consideration and on the number and type of uncertainty components that must be represented.

As a further unifying perspective, Uncertain Sets have been introduced [16]. They generalize classical sets by assigning to each element a graded or multi-component membership description, thereby providing a common framework capable of encompassing many uncertainty-oriented models in a unified manner.

1.2 Uncertain Logical Operators

Logical operators play a fundamental role in formal reasoning by combining propositions to produce new statements. Classical examples include conjunction, disjunction, negation, implication, and equivalence. When information is uncertain, however, classical two-valued operators are often insufficient for capturing vagueness, incompleteness, indeterminacy, or multi-component truth structures.

Uncertain logical operators extend classical logical connectives to uncertain environments. They are designed to aggregate or transform uncertain truth information while preserving mathematically meaningful reasoning behavior. Such operators are essential in the development of uncertainty-based logic, approximate reasoning, decision-making models, and related intelligent systems.

1.3 Our Contributions

A very broad variety of concepts are known under the general theme of uncertain logical operators. However, relatively few studies have examined these operators from a unified and systematic viewpoint. Motivated by this gap, in this book we present a broad survey of uncertain logical operators.

More specifically, we organize representative operators within a common uncertain-set-based framework, clarify their relationships with existing uncertainty models, and provide a structured overview of their mathematical formulations. Through this unified treatment, the book aims to offer a clearer conceptual foundation for future theoretical developments and applications involving uncertain logical operators. For reference, a high-level taxonomy of uncertain logical operators is presented in Table 1.1.

Table 1.1: High-level taxonomy of the uncertain logical operators.

Category	**Typical operators**	**Main viewpoint**
Basic set-theoretic operators	Union, Intersection, Symmetric Difference, Cartesian Product, Negation	Generalized set construction and basic logical/set manipulation
Conjunctive–disjunctive logical families	t-norm, t-conorm, uninorm, overlap function, conjunctor, disjunctor, grouping function, Water Logic	Truth-functional combination of uncertain information
Relational and inferential operators	Implication, Equivalence Operator	Conditionality, similarity, and logical relation between uncertain statements
Aggregation families	Aggregation Operator, Dombi Aggregation, OWA, Power Average, Mean Operator	Fusion of multiple uncertain inputs into a representative output
Integral / dependence-based operators	Copula, Choquet Integral, Sugeno Integral	Dependence modeling and non-additive / interaction-aware aggregation
Analytic and transformational operators	Differentiation, Upside-down Logic	Dynamic change, transformation, or reinterpretation of uncertain information
Structural algebraic operators	HyperOperation, SuperHyperOperation	Multivalued and higher-order algebraic structure generation

Chapter 2

Preliminaries

This chapter introduces the notation and fundamental concepts used in the sequel.

2.1 Fuzzy Set

Fuzzy set theory generalizes the ordinary notion of a subset by allowing each element to belong to a set with a degree in the unit interval $[0,1]$ [1,17,18]. We first recall the standard definition.

Definition 2.1.1 (Fuzzy set)**.** [1] Let X be a nonempty set. A *fuzzy set* A on X is determined by a function

$$\mu_A : X \to [0,1],$$

called the *membership function* of A. Equivalently, one may represent A as

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ expresses the degree to which x belongs to A.

2.2 Intuitionistic Fuzzy Set

Intuitionistic fuzzy sets refine fuzzy sets by assigning to each element both a membership degree and a non-membership degree, thereby leaving room for an explicit hesitation part [2,19]. The usual definition is given below.

Definition 2.2.1 (Intuitionistic fuzzy set)**.** [20] Let E be a nonempty set. An *intuitionistic fuzzy set* (IFS) A on E is of the form

$$A = \big\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in E\big\},$$

where

$$\mu_A, \nu_A : E \to [0,1]$$

denote the membership and non-membership functions, respectively, and satisfy

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1 \qquad \text{for all } x \in E.$$

The quantity

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x)$$

is called the *hesitation degree* of x.

The classical fuzzy-set case is recovered when

$$\nu_A(x) = 1 - \mu_A(x) \qquad \text{for all } x \in E,$$

or equivalently when $\pi_A(x) = 0$ for every $x \in E$.

2.3 Neutrosophic Set

Neutrosophic sets describe uncertainty by assigning to each element three quantities: truth, indeterminacy, and falsity, usually taken in the interval $[0,1]$ [4, 21–23]. Because the indeterminacy component is handled explicitly, this framework extends both fuzzy sets and intuitionistic fuzzy sets in a flexible way [24].

Definition 2.3.1 (Neutrosophic set)**.** [25, 26] Let X be a nonempty set. A *neutrosophic set* (NS) A on X is specified by three mappings

$$T_A : X \to [0,1], \qquad I_A : X \to [0,1], \qquad F_A : X \to [0,1],$$

where, for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively, of the statement "$x \in A$". These values satisfy

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \qquad \text{for all } x \in X.$$

2.4 Plithogenic Set

Plithogenic set theory extends uncertainty modeling by incorporating attribute-based appurtenance together with contradiction degrees between attribute values [27–30]. A standard formulation is as follows.

Definition 2.4.1 (Plithogenic Set)**.** [27, 28] Let P be a nonempty universe of discourse, and let v be a fixed attribute whose possible values form a nonempty set Pv. Let $s, t \in \mathbb{N}$.

A *plithogenic set* on (P, v, Pv) is a quintuple

$$PS = (P,\, v,\, Pv,\, pdf,\, pCF),$$

where

- $$pdf : P \times Pv \to [0,1]^s$$

  is the *degree of appurtenance function* (DAF); for $x \in P$ and $a \in Pv$, the value $pdf(x,a)$ gives the possibly vector-valued degree to which x belongs relative to the attribute value a;

- $$pCF : Pv \times Pv \to [0,1]^t$$

  is the *degree of contradiction function* (DCF), satisfying

  $$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a,b \in Pv.$$

In plithogenic theory, one usually selects a *dominant attribute value* $a^* \in Pv$. Set-theoretic operations such as union and intersection are then constructed by combining appurtenance degrees with contradiction degrees relative to a^*, thereby capturing interaction and opposition among different attribute values.

2.5 Rough Set

Rough set theory treats imprecision by replacing a target set with two approximations: a lower approximation, representing certainty, and an upper approximation, representing possibility. These are derived from an indiscernibility relation [31–34]. The classical Pawlak construction is recalled below.

Definition 2.5.1 (Rough set approximations)**.** [35] Let X be a nonempty universe, and let $R \subseteq X \times X$ be an equivalence relation. For each $x \in X$, define the equivalence class of x by

$$[x]_R := \{\, y \in X \mid (x,y) \in R \,\}.$$

For any subset $U \subseteq X$, define:

1. *Lower approximation*:
   $$\underline{U} := \{\, x \in X \mid [x]_R \subseteq U \,\}.$$
   Thus, $\underline{U}$ consists of those elements whose entire equivalence classes lie inside U.

2. *Upper approximation*:
   $$\overline{U} := \{\, x \in X \mid [x]_R \cap U \neq \varnothing \,\}.$$
   Hence, $\overline{U}$ consists of those elements whose equivalence classes intersect U.

The pair $(\underline{U}, \overline{U})$ is called the *rough approximation* of U, and one always has

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

2.6 Soft Set

Soft sets model uncertainty by means of parameters: each parameter determines a subset of the universe, and the entire family of such subsets forms the soft description. This framework was introduced by Molodtsov and has since been used widely in uncertainty analysis and decision-making [36, 37].

Definition 2.6.1 (Soft set)**.** [37] Let U be a universe, let E be a set of parameters, and let $A \subseteq E$. Denote by $\mathcal{P}(U)$ the power set of U. A pair (F, A) is called a *soft set* over U if

$$F : A \to \mathcal{P}(U).$$

For each parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ is called the ϵ-*approximation* of (F, A). Thus, a soft set is simply a parameterized family of subsets of the universe U.

Related concepts of the Soft Set include the HyperSoft Set [38], IndetermSoft Set [39, 40], and SuperHyperSoft Set [41–43].

2.7 Uncertain set

An *uncertain set* associates with each element a degree taken from a chosen uncertainty model, thereby providing a unifying umbrella for fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related frameworks [16, 44].

Definition 2.7.1 (Uncertain model)**.** [44] Let U denote the class of all *uncertain models*. Each $M \in U$ is determined by:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0, 1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$; and
- model-specific algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical instances include:

- **Fuzzy model:** $\mathrm{Dom}(M) = [0, 1]$;
- **Intuitionistic fuzzy model:** $\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 : \mu + \nu \leq 1\}$;
- **Neutrosophic model:** $\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 : 0 \leq T + I + F \leq 3\}$;
- **Plithogenic model,** and many further extensions.

Definition 2.7.2 (Uncertain set (U-set))**.** [44] Let X be a nonempty universe, and fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. An *uncertain set of type M* (briefly, a *U-set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is the *uncertainty-degree function* (membership map) of $\mathcal{U}$. For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to $\mathcal{U}$, as prescribed by the model M.

As noted in the remark, various generalizations are possible. For reference, Table 2.1 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ (cf. [16]).

Table 2.1: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [16].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [1,45]; N-Fuzzy Set [46–48] Shadowed Set [49–51]
2		Intuitionistic Fuzzy Set [2,19]; Vague Set [10,52]; Bipolar Fuzzy Set (two-component description) [53,54]; Pythagorean Fuzzy Set [55,56]; Fermatean fuzzy Set [57,58]; Variable Fuzzy Set [59–61]; Paraconsistent Fuzzy Set [62, 63]; Bifuzzy Set [64,65]
3		Single-Valued Neutrosophic Set [23,26]; Picture Fuzzy Set [12,66]; Ternary Fuzzy Set [67]; Hesitant Fuzzy Set [11,68]; Spherical Fuzzy Set [69,70]; Tripolar Fuzzy Set (three-component formalisms) [71–73]; Neutrosophic Vague Set [74,75]
4		Quadripartitioned Neutrosophic Set [13,76]; Double-Valued Neutrosophic Set [77, 78]; Dual Hesitant Fuzzy Set [79, 80]; Ambiguous Set [81–83]; Turiyam Neutrosophic Set [84–87]
5		Pentapartitioned Neutrosophic Set [88–90]; Triple-Valued Neutrosophic Set [91–94]
6		Hexapartitioned Neutrosophic Set [95]; Bipolar Neutrosophic Set [96, 97]; Bipolar Picture Fuzzy Sets [98,99]; Quadruple-Valued Neutrosophic Set [93, 100]
7		Heptapartitioned Neutrosophic Set [101–103]; Quintuple-Valued Neutrosophic Set [93,104,105]
8		Octapartitioned Neutrosophic Set [95]; Bipolar Quadripartitioned Neutrosophic Set [106,107]; Bipolar Double-valued Neutrosophic Set
9		Nonapartitioned Neutrosophic Set [95]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [108]; MultiFuzzy Set [109]; n-Refined Fuzzy Set [110,111]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [111]; Multi-Intuitionistic Fuzzy Set [109]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [111,112]; Multi-Neutrosophic Set [109,113,114]

Reading guide. In the U-Set scheme [44], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).

(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [24].

(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [13,76,83].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [115].

Remark 2.7.3. The notion of a U-set provides a common formal framework for various uncertainty-based set models through the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$. However, this unified representation does not replace the additional structures specific to individual theories; such model-dependent features should be retained separately when they are essential.

Example 2.7.4 (Fuzzy set as a U-set)**.** Let X be a nonempty set, and let F be a fuzzy set on X with membership function

$$\mu_F : X \to [0,1].$$

Define the uncertain model M_{FS} by

$$\mathrm{Dom}(M_{\mathrm{FS}}) := [0,1].$$

Then F can be written as the U-set

$$\mathcal{U}_F = (X, \mu_{M_{\mathrm{FS}}}),$$

where

$$\mu_{M_{\mathrm{FS}}} := \mu_F.$$

Indeed, for each $x \in X$,

$$\mu_{M_{\mathrm{FS}}}(x) = \mu_F(x) \in [0,1] = \mathrm{Dom}(M_{\mathrm{FS}}).$$

Hence every fuzzy set is a U-set of type M_{FS}.

Example 2.7.5 (Intuitionistic fuzzy set as a U-set)**.** Let X be a nonempty set, and let

$$A = (\mu_A, \nu_A)$$

be an intuitionistic fuzzy set on X, where

$$\mu_A, \nu_A : X \to [0,1]$$

satisfy

$$\mu_A(x) + \nu_A(x) \le 1 \qquad \text{for all } x \in X.$$

Define the uncertain model M_{IFS} by

$$\mathrm{Dom}(M_{\mathrm{IFS}}) := \big\{(a,b) \in [0,1]^2 \mid a + b \le 1\big\}.$$

Now define

$$\mu_{M_{\mathrm{IFS}}} : X \to \mathrm{Dom}(M_{\mathrm{IFS}})$$

by

$$\mu_{M_{\mathrm{IFS}}}(x) := \big(\mu_A(x), \nu_A(x)\big) \qquad (x \in X).$$

Then

$$\mathcal{U}_A = (X, \mu_{M_{\mathrm{IFS}}})$$

is a U-set of type M_{IFS}. Indeed, for every $x \in X$,

$$\mu_{M_{\mathrm{IFS}}}(x) = \big(\mu_A(x), \nu_A(x)\big) \in \mathrm{Dom}(M_{\mathrm{IFS}})$$

because

$$\mu_A(x), \nu_A(x) \in [0,1] \quad \text{and} \quad \mu_A(x) + \nu_A(x) \le 1.$$

Thus every intuitionistic fuzzy set can be viewed as a U-set with a two-dimensional degree-domain.

Chapter 3

Uncertain Basic Set-Theoretic Operators

In this chapter, we examine basic set-theoretic operators in frameworks such as Fuzzy Sets and Neutrosophic Sets.

3.1 Uncertain Union

An uncertain union combines uncertain sets or truth values by modeling inclusive combination under ambiguity, preserving membership possibility while accommodating incomplete, imprecise, or hesitant information.

Definition 3.1.1 (Fuzzy Union)**.** Let A and B be fuzzy sets on a common universe X, with membership functions

$$\mu_A, \mu_B : X \to [0,1].$$

The *fuzzy union* $A \cup B$ is the fuzzy set on X defined by

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\} \qquad (x \in X).$$

Definition 3.1.2 (Neutrosophic Union)**.** Let A and B be single-valued neutrosophic sets on X. The *neutrosophic union* of A and B, denoted by

$$A \cup_N B,$$

is the single-valued neutrosophic set defined by

$$T_{A\cup_N B}(x) = T_A(x) \vee_F T_B(x),$$

$$I_{A\cup_N B}(x) = I_A(x) \wedge_F I_B(x),$$

$$F_{A\cup_N B}(x) = F_A(x) \wedge_F F_B(x) \qquad \text{for all } x \in X.$$

Equivalently,

$$A \cup_N B = \{\langle x, T_A(x) \vee_F T_B(x), I_A(x) \wedge_F I_B(x), F_A(x) \wedge_F F_B(x)\rangle : x \in X\}.$$

In the uncertain-set framework, a union operation must be defined relative to a fixed uncertain model, because admissible uncertainty values are constrained by the model-specific domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Accordingly, the correct abstract definition is obtained by assuming a binary operation on $\mathrm{Dom}(M)$ that preserves admissibility.

Definition 3.1.3 (M-union operator on an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A binary mapping

$$\sqcup_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-union operator* if it is closed on $\mathrm{Dom}(M)$; that is, for all $a, b \in \mathrm{Dom}(M)$,

$$a \sqcup_M b \in \mathrm{Dom}(M).$$

If, in addition, $\sqcup_M$ satisfies one or more of the usual union-type properties such as commutativity, associativity, idempotency, and monotonicity (with respect to a chosen order on $\mathrm{Dom}(M)$), then it is regarded as a *standard uncertain union operator* on the model M.

Definition 3.1.4 (Uncertain union of U-sets)**.** Let X be a nonempty universe, let M be an uncertain model, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be two uncertain sets of the same type M on X, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Assume that $\sqcup_M$ is an M-union operator.

The *uncertain union* of $\mathcal{U}$ and $\mathcal{V}$ with respect to M, denoted by

$$\mathcal{U} \cup_M \mathcal{V},$$

is defined by

$$\mathcal{U} \cup_M \mathcal{V} := \big(X, \mu_{\mathcal{U} \cup_M \mathcal{V}}\big),$$

where the uncertainty-degree function

$$\mu_{\mathcal{U} \cup_M \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \cup_M \mathcal{V}}(x) = \mu_{\mathcal{U}}(x) \sqcup_M \mu_{\mathcal{V}}(x) \qquad (x \in X).$$

Theorem 3.1.5 (Well-definedness of uncertain union)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be two uncertain sets of type M on X. If $\sqcup_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$ is an M-union operator, then the uncertain union

$$\mathcal{U} \cup_M \mathcal{V}$$

defined above is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, their degree maps satisfy

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Because $\sqcup_M$ is an M-union operator, it is closed on $\mathrm{Dom}(M)$. Hence, for every $x \in X$,

$$\mu_{\mathcal{U}}(x) \sqcup_M \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto \mu_{\mathcal{U}}(x) \sqcup_M \mu_{\mathcal{V}}(x)$$

defines a map

$$\mu_{\mathcal{U} \cup_M \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \cup_M \mathcal{V} = \big(X, \mu_{\mathcal{U} \cup_M \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain union is well-defined. □

Proposition 3.1.6 (Basic inherited properties)**.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uncertain sets of the same type M on X.*

(i) If $\sqcup_M$ is commutative, then

$$\mathcal{U} \cup_M \mathcal{V} = \mathcal{V} \cup_M \mathcal{U}.$$

(ii) If $\sqcup_M$ is associative, then

$$(\mathcal{U} \cup_M \mathcal{V}) \cup_M \mathcal{W} = \mathcal{U} \cup_M (\mathcal{V} \cup_M \mathcal{W}).$$

(iii) If $\sqcup_M$ is idempotent, then

$$\mathcal{U} \cup_M \mathcal{U} = \mathcal{U}.$$

Proof. Each statement follows immediately from the corresponding pointwise property of $\sqcup_M$. For example, if $\sqcup_M$ is commutative, then for every $x \in X$,

$$\mu_{\mathcal{U} \cup_M \mathcal{V}}(x) = \mu_{\mathcal{U}}(x) \sqcup_M \mu_{\mathcal{V}}(x) = \mu_{\mathcal{V}}(x) \sqcup_M \mu_{\mathcal{U}}(x) = \mu_{\mathcal{V} \cup_M \mathcal{U}}(x).$$

Hence $\mathcal{U} \cup_M \mathcal{V} = \mathcal{V} \cup_M \mathcal{U}$. The other assertions are proved similarly. □

Table 3.1: A catalogue of representative union operators by the dimension k of the degree-domain.

k	note	Representative union operator(s)
1		Fuzzy Union: $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$
2		Intuitionistic Fuzzy Union: $(\mu, \nu) \mapsto (\max, \min)$.
3		Neutrosophic Union: $(T, I, F) \mapsto (\vee_F, \wedge_F, \wedge_F)$.
n	$(n \geq 1)$	Plithogenic Union (vector-valued form): contradiction-aware aggregation on $[0,1]^n$.

Reading guide. The table groups representative union operators by the dimension k of their degree values.

3.2 Uncertain Intersection

An uncertain intersection captures commonality between uncertain sets or truth values, preserving shared support under ambiguity while restricting outcomes to jointly compatible information only overall.

Definition 3.2.1 (Fuzzy Intersection)**.** Let A and B be fuzzy sets on a common universe X. The *fuzzy intersection* $A \cap B$ is the fuzzy set on X defined by

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \qquad (x \in X).$$

Definition 3.2.2 (Neutrosophic Intersection)**.** Let A and B be single-valued neutrosophic sets on X, and let $\wedge_F$ and $\vee_F$ denote a fuzzy t-norm and fuzzy t-conorm, respectively. The *neutrosophic intersection* of A and B, denoted by

$$A \cap_N B,$$

is the single-valued neutrosophic set defined by

$$T_{A\cap_N B}(x) = T_A(x) \wedge_F T_B(x),$$

$$I_{A\cap_N B}(x) = I_A(x) \vee_F I_B(x),$$

$$F_{A\cap_N B}(x) = F_A(x) \vee_F F_B(x) \qquad \text{for all } x \in X.$$

Equivalently,

$$A \cap_N B = \{\langle x,\, T_A(x) \wedge_F T_B(x),\, I_A(x) \vee_F I_B(x),\, F_A(x) \vee_F F_B(x)\rangle : x \in X\}.$$

In the uncertain-set framework, an intersection operation must be defined relative to a fixed uncertain model, because admissible uncertainty values are constrained by the model-specific domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Therefore, the appropriate abstract definition is obtained by assuming a binary operation on $\mathrm{Dom}(M)$ that preserves admissibility.

Definition 3.2.3 (M-intersection operator on an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A binary mapping

$$\sqcap_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-intersection operator* if it is closed on $\mathrm{Dom}(M)$; that is, for all $a, b \in \mathrm{Dom}(M)$,

$$a \sqcap_M b \in \mathrm{Dom}(M).$$

If, in addition, $\sqcap_M$ satisfies one or more of the usual intersection-type properties such as commutativity, associativity, idempotency, and monotonicity (with respect to a chosen order on $\mathrm{Dom}(M)$), then it is regarded as a *standard uncertain intersection operator* on the model M.

Definition 3.2.4 (Uncertain intersection of U-sets)**.** Let X be a nonempty universe, let M be an uncertain model, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be two uncertain sets of the same type M on X, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Assume that $\sqcap_M$ is an M-intersection operator.

The *uncertain intersection* of $\mathcal{U}$ and $\mathcal{V}$ with respect to M, denoted by

$$\mathcal{U} \cap_M \mathcal{V},$$

is defined by

$$\mathcal{U} \cap_M \mathcal{V} := \big(X, \mu_{\mathcal{U} \cap_M \mathcal{V}}\big),$$

where the uncertainty-degree function

$$\mu_{\mathcal{U} \cap_M \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \cap_M \mathcal{V}}(x) = \mu_{\mathcal{U}}(x) \sqcap_M \mu_{\mathcal{V}}(x) \qquad (x \in X).$$

Theorem 3.2.5 (Well-definedness of uncertain intersection)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be two uncertain sets of type M on X. If $\sqcap_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$ is an M-intersection operator, then the uncertain intersection

$$\mathcal{U} \cap_M \mathcal{V}$$

defined above is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Because $\sqcap_M$ is an M-intersection operator, it is closed on $\mathrm{Dom}(M)$. Hence, for every $x \in X$,

$$\mu_{\mathcal{U}}(x) \sqcap_M \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto \mu_{\mathcal{U}}(x) \sqcap_M \mu_{\mathcal{V}}(x)$$

defines a map

$$\mu_{\mathcal{U} \cap_M \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \cap_M \mathcal{V} = \big(X, \mu_{\mathcal{U} \cap_M \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain intersection is well-defined. □

Proposition 3.2.6 (Basic inherited properties)**.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uncertain sets of the same type M on X.*

(i) If $\sqcap_M$ is commutative, then
$$\mathcal{U} \cap_M \mathcal{V} = \mathcal{V} \cap_M \mathcal{U}.$$

(ii) If $\sqcap_M$ is associative, then
$$(\mathcal{U} \cap_M \mathcal{V}) \cap_M \mathcal{W} = \mathcal{U} \cap_M (\mathcal{V} \cap_M \mathcal{W}).$$

(iii) If $\sqcap_M$ is idempotent, then
$$\mathcal{U} \cap_M \mathcal{U} = \mathcal{U}.$$

Proof. Each assertion follows pointwise from the corresponding property of $\sqcap_M$. For example, if $\sqcap_M$ is commutative, then for every $x \in X$,
$$\mu_{\mathcal{U} \cap_M \mathcal{V}}(x) = \mu_{\mathcal{U}}(x) \sqcap_M \mu_{\mathcal{V}}(x) = \mu_{\mathcal{V}}(x) \sqcap_M \mu_{\mathcal{U}}(x) = \mu_{\mathcal{V} \cap_M \mathcal{U}}(x).$$
Hence $\mathcal{U} \cap_M \mathcal{V} = \mathcal{V} \cap_M \mathcal{U}$. The remaining statements are proved similarly. □

We list a catalogue of representative intersection operators classified by the dimension k of the degree domain in Table 3.2.

Table 3.2: A catalogue of representative intersection operators by the dimension k of the degree-domain.

k	note	Representative intersection operator(s)
1		Fuzzy Intersection [1]: $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$.
2		Intuitionistic Fuzzy Intersection: $(\mu, \nu) \mapsto (\min, \max)$.
3		Neutrosophic Intersection (cf. [23, 116]): $(T, I, F) \mapsto (\wedge_F, \vee_F, \vee_F)$.
n	$(n \geq 1)$	Plithogenic Intersection: contradiction-aware aggregation on $[0, 1]^n$.

Reading guide. The table groups representative intersection operators by the dimension k of their degree values.

3.3 Uncertain Symmetric difference

Symmetric difference is the set of elements belonging to exactly one of two sets, capturing disagreement by excluding common elements and preserving nonshared membership only [117]. Fuzzy symmetric difference is a fuzzy set operator measuring exclusive membership between two fuzzy sets, typically using membership differences while diminishing common overlapping degrees quantitatively (cf. [118]). The symmetric difference is not a primitive operator, but a derived operator. Even in the uncertain setting, it is constructed as a composition of admissible operators.

Definition 3.3.1 (Fuzzy Symmetric Difference)**.** Let A and B be fuzzy sets on a common universe X. The *fuzzy symmetric difference* of A and B, denoted by
$$A \triangle B,$$

is the fuzzy set on X defined by

$$\mu_{A\triangle B}(x) = \max\Big\{\min\big(\mu_A(x), 1-\mu_B(x)\big), \min\big(\mu_B(x), 1-\mu_A(x)\big)\Big\} \qquad (x \in X).$$

Under the standard fuzzy operations above, this is equivalently written as

$$\mu_{A\triangle B}(x) = \big|\mu_A(x) - \mu_B(x)\big| \qquad (x \in X).$$

In the uncertain-set framework, a symmetric difference should be defined relative to a fixed uncertain model, because the admissible uncertainty values are restricted by the model-specific domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

A natural abstract construction is obtained from an M-intersection operator, an M-union operator, and an M-complement operator, exactly as in the classical identity

$$A\triangle B = (A \cap B^c) \cup (B \cap A^c).$$

Definition 3.3.2 (M-complement operator on an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A mapping

$$c_M : \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-complement operator* if it preserves admissibility; that is,

$$c_M(a) \in \mathrm{Dom}(M) \qquad \text{for all } a \in \mathrm{Dom}(M).$$

Definition 3.3.3 (Induced M-symmetric difference operator)**.** Let M be an uncertain model. Assume that

$$\sqcup_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M), \qquad \sqcap_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

are an M-union operator and an M-intersection operator, respectively, and let

$$c_M : \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-complement operator.

The *induced M-symmetric difference operator*

$$\triangle_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is defined by

$$a\triangle_M b := \big(a \sqcap_M c_M(b)\big) \sqcup_M \big(b \sqcap_M c_M(a)\big) \qquad (a, b \in \mathrm{Dom}(M)).$$

Definition 3.3.4 (Uncertain symmetric difference of U-sets)**.** Let X be a nonempty universe, let M be an uncertain model, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be two uncertain sets of the same type M on X, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Assume that $\sqcup_M$, $\sqcap_M$, and c_M are as above.

The *uncertain symmetric difference* of $\mathcal{U}$ and $\mathcal{V}$ with respect to M, denoted by

$$\mathcal{U} \triangle_M \mathcal{V},$$

is the uncertain set

$$\mathcal{U} \triangle_M \mathcal{V} := \big(X, \mu_{\mathcal{U} \triangle_M \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \triangle_M \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is defined pointwise by

$$\mu_{\mathcal{U} \triangle_M \mathcal{V}}(x) = \mu_{\mathcal{U}}(x) \triangle_M \mu_{\mathcal{V}}(x) \qquad (x \in X).$$

Equivalently,

$$\mu_{\mathcal{U} \triangle_M \mathcal{V}}(x) = \Big(\mu_{\mathcal{U}}(x) \sqcap_M c_M\big(\mu_{\mathcal{V}}(x)\big)\Big) \sqcup_M \Big(\mu_{\mathcal{V}}(x) \sqcap_M c_M\big(\mu_{\mathcal{U}}(x)\big)\Big) \qquad (x \in X).$$

Theorem 3.3.5 (Well-definedness of uncertain symmetric difference)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M on X. Assume that

$$\sqcup_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M),$$

$$\sqcap_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M),$$

$$c_M : \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

are an M-union operator, an M-intersection operator, and an M-complement operator, respectively. Then the uncertain symmetric difference

$$\mathcal{U} \triangle_M \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix $x \in X$. Because c_M is an M-complement operator,

$$c_M\big(\mu_{\mathcal{U}}(x)\big) \in \mathrm{Dom}(M), \qquad c_M\big(\mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Since $\sqcap_M$ is closed on $\mathrm{Dom}(M)$, it follows that

$$\mu_{\mathcal{U}}(x) \sqcap_M c_M\big(\mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M),$$

and

$$\mu_{\mathcal{V}}(x) \sqcap_M c_M\big(\mu_{\mathcal{U}}(x)\big) \in \mathrm{Dom}(M).$$

Finally, because $\sqcup_M$ is closed on $\mathrm{Dom}(M)$, we obtain

$$\Big(\mu_{\mathcal{U}}(x) \sqcap_M c_M\big(\mu_{\mathcal{V}}(x)\big)\Big) \sqcup_M \Big(\mu_{\mathcal{V}}(x) \sqcap_M c_M\big(\mu_{\mathcal{U}}(x)\big)\Big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathcal{U}\triangle_M\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Hence the pointwise assignment

$$x \longmapsto \mu_{\mathcal{U}\triangle_M\mathcal{V}}(x)$$

defines a map

$$\mu_{\mathcal{U}\triangle_M\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U}\triangle_M\mathcal{V} = \big(X, \mu_{\mathcal{U}\triangle_M\mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain symmetric difference is well-defined. □

A catalogue of representative symmetric difference operators classified by the dimension k of the degree domain is given in Table 3.3.

Table 3.3: A catalogue of representative symmetric difference operators by the dimension k of the degree-domain.

k	note	Representative symmetric difference operator(s)
1		Fuzzy Symmetric Difference [119]: $\mu_{A\triangle B}(x) = \lvert\mu_A(x) - \mu_B(x)\rvert$.
2		Intuitionistic Fuzzy Symmetric Difference [120]: $(\mu, \nu) \mapsto (\lvert\mu_A - \mu_B\rvert, \lvert\nu_A - \nu_B\rvert)$.
3		Neutrosophic Symmetric Difference: $(T, I, F) \mapsto (\lvert T_A - T_B\rvert, \lvert I_A - I_B\rvert, \lvert F_A - F_B\rvert)$.
n	$(n \geq 1)$	Plithogenic Symmetric Difference: contradiction-aware symmetric difference on $[0, 1]^n$.

Reading guide. The table groups representative symmetric difference operators by the dimension k of their degree values.

3.4 Uncertain Cartesian product

The Cartesian product of sets X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$, respectively [121, 122]. An uncertain Cartesian product combines uncertain sets on X and Y by assigning each pair (x, y) an admissible uncertainty degree derived from both components pointwise.

Definition 3.4.1 (Fuzzy Cartesian Product)**.** [123] Let A be a fuzzy set on X and B a fuzzy set on Y, where X and Y are nonempty sets. The *fuzzy Cartesian product* $A \times B$ is the fuzzy set on $X \times Y$ defined by

$$\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\} \qquad \big((x, y) \in X \times Y\big).$$

In the uncertain-set framework, a Cartesian product must be defined relative to a fixed uncertain model, because admissible uncertainty values are constrained by the model-specific domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Accordingly, the appropriate abstract definition is obtained by assuming a binary operation on $\mathrm{Dom}(M)$ that preserves admissibility and plays the role of a product-type combination rule.

Definition 3.4.2 (M-product operator on an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. A binary mapping

$$\otimes_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-product operator* if it is closed on $\mathrm{Dom}(M)$; that is, for all $a, b \in \mathrm{Dom}(M)$,

$$a \otimes_M b \in \mathrm{Dom}(M).$$

If, in addition, $\otimes_M$ satisfies one or more natural product-type properties such as commutativity, associativity, or monotonicity with respect to a chosen order on $\mathrm{Dom}(M)$, then it is called a *standard uncertain Cartesian product operator* on M.

Definition 3.4.3 (Uncertain Cartesian product of U-sets)**.** Let X and Y be nonempty sets, let M be an uncertain model, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (Y, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}} : X \to \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}} : Y \to \mathrm{Dom}(M).$$

Assume that $\otimes_M$ is an M-product operator.

The *uncertain Cartesian product* of $\mathcal{U}$ and $\mathcal{V}$ with respect to M, denoted by

$$\mathcal{U} \times_M \mathcal{V},$$

is defined to be the uncertain set on $X \times Y$ given by

$$\mathcal{U} \times_M \mathcal{V} := \big(X \times Y, \mu_{\mathcal{U} \times_M \mathcal{V}}\big),$$

where the uncertainty-degree function

$$\mu_{\mathcal{U} \times_M \mathcal{V}} : X \times Y \to \mathrm{Dom}(M)$$

is defined by

$$\mu_{\mathcal{U} \times_M \mathcal{V}}(x, y) = \mu_{\mathcal{U}}(x) \otimes_M \mu_{\mathcal{V}}(y) \qquad \big((x, y) \in X \times Y\big).$$

Theorem 3.4.4 (Well-definedness of uncertain Cartesian product)**.** *Let X and Y be nonempty sets, let M be an uncertain model, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (Y, \mu_{\mathcal{V}})$$

be uncertain sets of type M. If

$$\otimes_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

is an M-product operator, then the uncertain Cartesian product

$$\mathcal{U} \times_M \mathcal{V}$$

is a well-defined uncertain set of type M on $X \times Y$.

Proof. Since $\mathcal{U}$ is an uncertain set of type M on X, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Similarly, since $\mathcal{V}$ is an uncertain set of type M on Y, one has

$$\mu_{\mathcal{V}}(y) \in \mathrm{Dom}(M) \qquad \text{for all } y \in Y.$$

Let $(x, y) \in X \times Y$ be arbitrary. Then

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(y) \in \mathrm{Dom}(M).$$

Because $\otimes_M$ is closed on $\mathrm{Dom}(M)$, it follows that

$$\mu_{\mathcal{U}}(x) \otimes_M \mu_{\mathcal{V}}(y) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathcal{U} \times_M \mathcal{V}}(x, y) \in \mathrm{Dom}(M) \qquad \text{for all } (x, y) \in X \times Y.$$

Therefore the pointwise assignment

$$(x, y) \longmapsto \mu_{\mathcal{U}}(x) \otimes_M \mu_{\mathcal{V}}(y)$$

defines a mapping

$$\mu_{\mathcal{U} \times_M \mathcal{V}} : X \times Y \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \times_M \mathcal{V} = \big(X \times Y, \mu_{\mathcal{U} \times_M \mathcal{V}}\big)$$

is an uncertain set of type M on $X \times Y$. Thus the uncertain Cartesian product is well-defined. □

Proposition 3.4.5 (Basic inherited properties)**.** *Let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \quad \mathcal{U}' = (X', \mu_{\mathcal{U}'}), \quad \mathcal{V} = (Y, \mu_{\mathcal{V}}), \quad \mathcal{V}' = (Y', \mu_{\mathcal{V}'})$$

be uncertain sets of type M.

(i) If $\otimes_M$ is commutative, then for all $x \in X$ and $y \in Y$,

$$\mu_{\mathcal{U} \times_M \mathcal{V}}(x, y) = \mu_{\mathcal{V} \times_M \mathcal{U}}(y, x).$$

(ii) If $\otimes_M$ is associative, then for uncertain sets

$$\mathcal{W} = (Z, \mu_{\mathcal{W}})$$

of type M, one has the canonical identification

$$(\mathcal{U} \times_M \mathcal{V}) \times_M \mathcal{W} \cong \mathcal{U} \times_M (\mathcal{V} \times_M \mathcal{W}).$$

Proof. Both statements follow directly from the corresponding pointwise properties of $\otimes_M$.

For instance, if $\otimes_M$ is commutative, then for all $x \in X$ and $y \in Y$,

$$\mu_{\mathcal{U}\times_M \mathcal{V}}(x, y) = \mu_{\mathcal{U}}(x) \otimes_M \mu_{\mathcal{V}}(y) = \mu_{\mathcal{V}}(y) \otimes_M \mu_{\mathcal{U}}(x) = \mu_{\mathcal{V}\times_M \mathcal{U}}(y, x).$$

The associative case is proved similarly. □

As a reference, a catalogue of representative Cartesian product operators classified by the dimension k of the degree-domain is presented in Table 3.4.

Table 3.4: A catalogue of representative Cartesian product operators by the dimension k of the degree-domain.

k	note	Representative Cartesian product operator(s)
1		Fuzzy Cartesian Product: $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$.
2		Intuitionistic Fuzzy Cartesian Product [124]: $(\mu, \nu) \mapsto (\min, \max)$.
3		Neutrosophic Cartesian Product [125, 126]: $(T, I, F) \mapsto (\wedge_F, \vee_F, \vee_F)$.
n	$(n \geq 1)$	Plithogenic Cartesian Product: contradiction-aware product on $[0, 1]^n$.

Reading guide. The table groups representative Cartesian product operators by the dimension k of their degree values.

Remark 3.4.6. If the emphasis is on a relational product interpretation, then a t-norm-type rule is a natural choice for the product operator. On the other hand, if one only requires admissibility on the degree-domain, then any binary rule

$$\otimes_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

that is closed on $\mathrm{Dom}(M)$ may also be used.

3.5 Uncertain negation

An uncertain negation reverses uncertain truth or membership assessments, modeling opposition, complementarity, or refusal when information is vague, incomplete, interval based, or hesitant in practice.

Definition 3.5.1 (Fuzzy Negation)**.** [127] Let

$$N : [0, 1] \to [0, 1]$$

be a mapping. Then N is called a *fuzzy negation* if it satisfies the following conditions:

1. $N(0) = 1$ and $N(1) = 0$;

2. for all $x, y \in [0,1]$, if $x \leq y$, then $N(x) \geq N(y)$.

Definition 3.5.2 (Neutrosophic Negation)**.** Let A be a single-valued neutrosophic set on X. The *neutrosophic negation* (or *neutrosophic complement*) of A, denoted by $\neg_N A$, is the single-valued neutrosophic set defined by

$$T_{\neg_N A}(x) = F_A(x), \qquad I_{\neg_N A}(x) = I_A(x), \qquad F_{\neg_N A}(x) = T_A(x) \qquad \text{for all } x \in X.$$

Equivalently,

$$\neg_N A = \{\langle x,\, F_A(x),\, I_A(x),\, T_A(x)\rangle : x \in X\}.$$

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

An uncertain negation on M is given by a unary operation on $\mathrm{Dom}(M)$ that preserves admissibility.

Definition 3.5.3 (M-negation operator)**.** A mapping

$$N_M : \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-negation operator* if it is closed on $\mathrm{Dom}(M)$; that is,

$$N_M(a) \in \mathrm{Dom}(M) \qquad \text{for all } a \in \mathrm{Dom}(M).$$

If, in addition, $\mathrm{Dom}(M)$ is equipped with a partial order $\preceq_M$ and distinguished elements $0_M, 1_M \in \mathrm{Dom}(M)$, and if N_M satisfies suitable further axioms such as boundary conditions

$$N_M(0_M) = 1_M, \qquad N_M(1_M) = 0_M,$$

and antitonicity

$$a \preceq_M b \Longrightarrow N_M(b) \preceq_M N_M(a),$$

then N_M is called a *standard uncertain negation* on M.

Definition 3.5.4 (Uncertain negation of a U-set)**.** Let X be a nonempty universe, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}})$$

be an uncertain set of type M, where

$$\mu_{\mathcal{U}} : X \to \mathrm{Dom}(M).$$

Assume that $N_M : \mathrm{Dom}(M) \to \mathrm{Dom}(M)$ is an M-negation operator.

The *uncertain negation* (or *uncertain complement*) of $\mathcal{U}$ with respect to M, denoted by

$$\neg_M \mathcal{U},$$

is defined by

$$\neg_M \mathcal{U} := \big(X, \mu_{\neg_M \mathcal{U}}\big),$$

where

$$\mu_{\neg_M \mathcal{U}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\neg_M \mathcal{U}}(x) = N_M\big(\mu_{\mathcal{U}}(x)\big) \qquad (x \in X).$$

Theorem 3.5.5 (Well-definedness of uncertain negation)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}})$$

be an uncertain set of type M, and let

$$N_M : \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-negation operator. Then

$$\neg_M \mathcal{U}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ is an uncertain set of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Because N_M is an M-negation operator, it preserves admissibility. Hence, for every $x \in X$,

$$N_M\big(\mu_{\mathcal{U}}(x)\big) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto N_M\big(\mu_{\mathcal{U}}(x)\big)$$

defines a mapping

$$\mu_{\neg_M \mathcal{U}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\neg_M \mathcal{U} = \big(X, \mu_{\neg_M \mathcal{U}}\big)$$

is an uncertain set of type M on X. Thus the uncertain negation is well-defined. □

Proposition 3.5.6 (Inherited properties)**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be uncertain sets of type M on X.*

(i) If N_M is involutive, i.e.,

$$N_M(N_M(a)) = a \qquad \textit{for all } a \in \mathrm{Dom}(M),$$

then

$$\neg_M(\neg_M \mathcal{U}) = \mathcal{U}.$$

(ii) If $\mathrm{Dom}(M)$ is equipped with a partial order $\preceq_M$, and if N_M is antitone, then

$$\mu_{\mathcal{U}}(x) \preceq_M \mu_{\mathcal{V}}(x) \textit{ for all } x \in X \quad \Longrightarrow \quad \mu_{\neg_M \mathcal{V}}(x) \preceq_M \mu_{\neg_M \mathcal{U}}(x) \textit{ for all } x \in X.$$

Proof. (i) For every $x \in X$,

$$\mu_{\neg_M(\neg_M \mathcal{U})}(x) = N_M\big(\mu_{\neg_M \mathcal{U}}(x)\big) = N_M\big(N_M(\mu_{\mathcal{U}}(x))\big) = \mu_{\mathcal{U}}(x).$$

Hence

$$\neg_M(\neg_M \mathcal{U}) = \mathcal{U}.$$

(ii) Assume

$$\mu_{\mathcal{U}}(x) \preceq_M \mu_{\mathcal{V}}(x) \qquad \text{for all } x \in X.$$

Since N_M is antitone, for every $x \in X$,

$$N_M(\mu_{\mathcal{V}}(x)) \preceq_M N_M(\mu_{\mathcal{U}}(x)).$$

That is,

$$\mu_{\neg_M \mathcal{V}}(x) \preceq_M \mu_{\neg_M \mathcal{U}}(x) \qquad \text{for all } x \in X.$$

□

As a reference, a catalogue of representative negation operators classified by the dimension k of the degree-domain is presented in Table 3.5.

Table 3.5: A catalogue of representative negation operators by the dimension k of the degree-domain.

k	note	Representative negation operator(s)
1		Fuzzy negation [128, 129]: $N : [0,1] \to [0,1]$.
2		Intuitionistic Fuzzy negation [130–132]: $(\mu, \nu) \mapsto (\nu, \mu)$.
3		Picture Fuzzy negation [133]: $(\mu, \eta, \nu) \mapsto (\nu, \eta, \mu)$; Hesitant Fuzzy negation [134, 135]: elementwise negation on hesitant degrees; Neutrosophic negation [136]: $(T, I, F) \mapsto (F, I, T)$.
n	$(n \geq 1)$	Plithogenic negation: attribute-dependent negation on $[0,1]^n$.

Reading guide. The table groups representative negation operators by the dimension k of their degree values.

Chapter 4

Uncertain Conjunctive–Disjunctive Logical Families

In this chapter, we present operators related to uncertain conjunctive–disjunctive logical families. For reference, a comparison table is provided in Table 4.1.

4.1 Uncertain t-norm

A t-norm is an associative, commutative, monotone binary operation on $[0, 1]$ with neutral element 1, commonly modeling fuzzy conjunction and generalized intersection mathematically [137, 138]. An uncertain t-norm generalizes conjunction under uncertainty, combining uncertain truth degrees through an associative, monotone, boundary respecting operation suitable for uncertain logical reasoning and tasks.

Definition 4.1.1 (Fuzzy t-norm)**.** [139] A mapping

$$T : [0, 1] \times [0, 1] \to [0, 1]$$

is called a *fuzzy t-norm* (or *triangular norm*) if, for all $x, y, z, x_1, x_2, y_1, y_2 \in [0, 1]$, it satisfies:

1. **Commutativity:**
$$T(x, y) = T(y, x).$$
2. **Associativity:**
$$T(x, T(y, z)) = T(T(x, y), z).$$
3. **Monotonicity:** if $x_1 \le x_2$ and $y_1 \le y_2$, then
$$T(x_1, y_1) \le T(x_2, y_2).$$
4. **Neutral element** 1**:**
$$T(x, 1) = x.$$

Table 4.1: A concise comparison map of major conjunctive–disjunctive operator families

Family	**Associa-tive**	**Commu-tative**	**Neutral element**	**Zero/unit characteriza-tion**	**Type**
t-norm	Yes	Yes	Yes (1)	Usually $T(x, 0) = 0$, $T(x, 1) = x$	Conjunction-like
t-conorm	Yes	Yes	Yes (0)	Usually $S(x, 1) = 1$, $S(x, 0) = x$	Disjunction-like
Uninorm	Yes	Yes	Yes ($e \in [0, 1]$)	Depends on the neutral element and the definition	Mixed / hybrid
Overlap function	Not required	Yes	Not required	Typically $O(x, y) = 0$ iff $xy = 0$, and $O(x, y) = 1$ iff $x = y = 1$	Conjunction-like
Conjunctor	Not required	Not required	Not required	Usually satisfies conjunctive boundary behavior	Conjunction-like
Disjunctor	Not required	Not required	Not required	Usually satisfies disjunctive boundary behavior	Disjunction-like
Grouping function	Not required	Usually yes	Not required	Commonly dual to overlap-type behavior	Disjunction-like
Water Logic	Depends on the chosen operators	Depends on the chosen operators	Depends on the chosen operators	Depends on the chosen operators	Logic-dependent

This table is intended only as a quick navigation map. The entries summarize common defining tendencies of each family; specific variants may satisfy stronger or weaker properties.

Definition 4.1.2 (Standard Neutrosophic Domain)**.** Define

$$D^* = \left\{ x = (x_1, x_2, x_3) \in [0, 1]^3 \ : \ x_1 + x_2 + x_3 \le 1 \right\}.$$

For

$$x = (x_1, x_2, x_3), \qquad y = (y_1, y_2, y_3) \in D^*,$$

define the order relation $\le_{D^*}$ by

$$x \le_{D^*} y$$

if and only if

$$\big(x_1 < y_1 \text{ and } x_3 \ge y_3\big) \ \vee \ \big(x_1 = y_1 \text{ and } x_3 > y_3\big) \ \vee \ \big(x_1 = y_1, \ x_3 = y_3, \ x_2 \le y_2\big).$$

Also define

$$0_{D^*} := (0, 0, 1), \qquad 1_{D^*} := (1, 0, 0).$$

For each $x = (x_1, x_2, x_3) \in D^*$, let

$$\Gamma(x) = \{ y \in D^* \ : \ y = (x_1, y_2, x_3), \ 0 \le y_2 \le x_2 \}.$$

Definition 4.1.3 (Neutrosophic t-norm)**.** A mapping

$$\mathcal{T} : (D^*)^2 \to D^*$$

is called a *standard neutrosophic t-norm* if, for all $x, y, z \in D^*$, it satisfies:

1. **Commutativity:**
$$\mathcal{T}(x, y) = \mathcal{T}(y, x).$$
2. **Associativity:**
$$\mathcal{T}\big(x, \mathcal{T}(y, z)\big) = \mathcal{T}\big(\mathcal{T}(x, y), z\big).$$
3. **Monotonicity:** if $y \leq_{D^*} z$, then
$$\mathcal{T}(x, y) \leq_{D^*} \mathcal{T}(x, z).$$
4. **Neutrality condition at 1_{D^*}:**
$$\mathcal{T}(1_{D^*}, x) \in \Gamma(x).$$

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

equipped with a partial order $\preceq_M$ and a distinguished element

$$1_M \in \mathrm{Dom}(M),$$

which plays the role of the truth-unit (or neutral element).

Definition 4.1.4 (M-t-norm)**.** A mapping

$$T_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-t-norm* if, for all $a, b, c, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M)$, the following conditions hold:

(i) **Closure:**
$$T_M(a, b) \in \mathrm{Dom}(M).$$
(ii) **Commutativity:**
$$T_M(a, b) = T_M(b, a).$$
(iii) **Associativity:**
$$T_M\big(a, T_M(b, c)\big) = T_M\big(T_M(a, b), c\big).$$
(iv) **Monotonicity:** if
$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$
then
$$T_M(a_1, b_1) \preceq_M T_M(a_2, b_2).$$

(v) **Neutral element:**

$$T_M(a, 1_M) = a.$$

Definition 4.1.5 (Uncertain t-norm of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let T_M be an M-t-norm.

The *uncertain t-norm* of $\mathcal{U}$ and $\mathcal{V}$ induced by T_M, denoted by

$$\mathcal{U} \otimes_{T_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \otimes_{T_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}}(x) = T_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.1.6 (Well-definedness of the uncertain t-norm of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$T_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-t-norm. Then

$$\mathcal{U} \otimes_{T_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of the M-t-norm,

$$T_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence, for every $x \in X$,

$$\mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto T_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \otimes_{T_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain t-norm of U-sets is well-defined. □

Proposition 4.1.7 (Inherited algebraic properties)**.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uncertain sets of type M on X.*

(i)

$$\mathcal{U} \otimes_{T_M} \mathcal{V} = \mathcal{V} \otimes_{T_M} \mathcal{U}.$$

(ii)

$$\big(\mathcal{U} \otimes_{T_M} \mathcal{V}\big) \otimes_{T_M} \mathcal{W} = \mathcal{U} \otimes_{T_M} \big(\mathcal{V} \otimes_{T_M} \mathcal{W}\big).$$

(iii) If $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ is the constant U-set defined by

$$\mu_{\mathbf{1}_M}(x) = 1_M \qquad \text{for all } x \in X,$$

then

$$\mathcal{U} \otimes_{T_M} \mathbf{1}_M = \mathcal{U}.$$

Proof. For each $x \in X$, commutativity of T_M yields

$$\mu_{\mathcal{U} \otimes_{T_M} \mathcal{V}}(x) = T_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = T_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{V} \otimes_{T_M} \mathcal{U}}(x),$$

proving (i).

For (ii), associativity of T_M gives

$$\mu_{(\mathcal{U} \otimes_{T_M} \mathcal{V}) \otimes_{T_M} \mathcal{W}}(x) = T_M\Big(T_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big), \mu_{\mathcal{W}}(x)\Big)$$

$$= T_M\Big(\mu_{\mathcal{U}}(x), T_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{W}}(x)\big)\Big) = \mu_{\mathcal{U} \otimes_{T_M} (\mathcal{V} \otimes_{T_M} \mathcal{W})}(x),$$

for all $x \in X$.

For (iii), the neutral-element property of T_M implies

$$\mu_{\mathcal{U} \otimes_{T_M} \mathbf{1}_M}(x) = T_M\big(\mu_{\mathcal{U}}(x), 1_M\big) = \mu_{\mathcal{U}}(x) \qquad \text{for all } x \in X.$$

Hence

$$\mathcal{U} \otimes_{T_M} \mathbf{1}_M = \mathcal{U}.$$

□

As a reference, a catalogue of representative t-norm operators classified by the dimension k of the degree-domain is presented in Table 4.2.

Uncertain t-norms are related to several other concepts, including continuous t-norms [150], monoidal t-norms [151, 152], pseudo-t-norms [153, 154], interval t-norms [155, 156], vector t-norms [157], and s-norms [158, 159].

Table 4.2: A catalogue of representative t-norm operators by the dimension k of the degree-domain.

k	note	Representative t-norm operator(s)
1		Fuzzy t-norm [140,141]: $T : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy t-norm [142,143]: $(\mu,\nu) \mapsto (T,S)$.
3		Picture Fuzzy t-norm [140,144]: $(\mu,\eta,\nu) \mapsto (T,S,S)$; Hesitant Fuzzy t-norm [145,146]: componentwise / induced t-norm; Spherical Fuzzy t-norm [147]: $(\mu,\eta,\nu) \mapsto (T,S,S)$; Neutrosophic t-norm [148,149]: $(T,I,F) \mapsto (T,S,S)$.
n	$(n \geq 1)$	Plithogenic t-norm: contradiction-aware aggregation on $[0,1]^n$.

Reading guide. The table groups representative t-norm operators by the dimension k of their degree values.

4.2 Uncertain t-conorm

A t-conorm is an associative, commutative, monotone binary operation on $[0,1]$ with neutral element 0, commonly modeling fuzzy disjunction and generalized union in logic mathematically [160]. An uncertain t-conorm generalizes disjunction under uncertainty, combining uncertain truth degrees through an associative, monotone, boundary respecting operation suitable for uncertain logical aggregation and inference.

Definition 4.2.1 (Fuzzy t-conorm)**.** [161] A mapping

$$S : [0,1] \times [0,1] \to [0,1]$$

is called a *fuzzy t-conorm* (or *triangular conorm*, also called an *s-norm*) if, for all $x, y, z, x_1, x_2, y_1, y_2 \in [0,1]$, it satisfies:

1. **Commutativity:**
$$S(x,y) = S(y,x).$$
2. **Associativity:**
$$S(x, S(y,z)) = S(S(x,y), z).$$
3. **Monotonicity:** if $x_1 \leq x_2$ and $y_1 \leq y_2$, then
$$S(x_1, y_1) \leq S(x_2, y_2).$$
4. **Neutral element** 0**:**
$$S(x,0) = x.$$

Definition 4.2.2 (Neutrosophic t-conorm)**.** A mapping

$$\mathcal{S} : (D^*)^2 \to D^*$$

is called a *standard neutrosophic t-conorm* if, for all $x, y, z \in D^*$, it satisfies:

1. **Commutativity:**
$$\mathcal{S}(x,y) = \mathcal{S}(y,x).$$
2. **Associativity:**
$$\mathcal{S}\big(x, \mathcal{S}(y,z)\big) = \mathcal{S}\big(\mathcal{S}(x,y), z\big).$$

3. **Monotonicity:** if $y \leq_{D^*} z$, then

$$\mathcal{S}(x, y) \leq_{D^*} \mathcal{S}(x, z).$$

4. **Neutrality condition at 0_{D^*}:**

$$\mathcal{S}(0_{D^*}, x) \in \Gamma(x).$$

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

equipped with a partial order $\preceq_M$ and a distinguished element

$$0_M \in \mathrm{Dom}(M),$$

which plays the role of the falsity-unit (or neutral element for disjunction-type aggregation).

Definition 4.2.3 (M-t-conorm)**.** A mapping

$$S_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-t-conorm* if, for all $a, b, c, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M)$, the following conditions hold:

(i) **Closure:**

$$S_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Commutativity:**

$$S_M(a, b) = S_M(b, a).$$

(iii) **Associativity:**

$$S_M\big(a, S_M(b, c)\big) = S_M\big(S_M(a, b), c\big).$$

(iv) **Monotonicity:** if

$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$

then

$$S_M(a_1, b_1) \preceq_M S_M(a_2, b_2).$$

(v) **Neutral element:**

$$S_M(a, 0_M) = a.$$

Definition 4.2.4 (Uncertain t-conorm of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let S_M be an M-t-conorm.

The *uncertain t-conorm* of $\mathcal{U}$ and $\mathcal{V}$ induced by S_M, denoted by

$$\mathcal{U} \oplus_{S_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \oplus_{S_M} \mathcal{V} := \big(X, \mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}}(x) = S_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.2.5 (Well-definedness of the uncertain t-conorm of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$S_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-t-conorm. Then

$$\mathcal{U} \oplus_{S_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of the M-t-conorm,

$$S_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence, for every $x \in X$,

$$\mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto S_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \oplus_{S_M} \mathcal{V} = \big(X, \mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain t-conorm of U-sets is well-defined. □

Proposition 4.2.6 (Inherited algebraic properties)**.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uncertain sets of type M on X.*

(i)

$$\mathcal{U} \oplus_{S_M} \mathcal{V} = \mathcal{V} \oplus_{S_M} \mathcal{U}.$$

(ii)

$$(\mathcal{U} \oplus_{S_M} \mathcal{V}) \oplus_{S_M} \mathcal{W} = \mathcal{U} \oplus_{S_M} (\mathcal{V} \oplus_{S_M} \mathcal{W}).$$

(iii) If $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ *is the constant U-set defined by*

$$\mu_{\mathbf{0}_M}(x) = 0_M \qquad \textit{for all } x \in X,$$

then

$$\mathcal{U} \oplus_{S_M} \mathbf{0}_M = \mathcal{U}.$$

Proof. For each $x \in X$, commutativity of S_M yields

$$\mu_{\mathcal{U}\oplus_{S_M}\mathcal{V}}(x) = S_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = S_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{V}\oplus_{S_M}\mathcal{U}}(x),$$

proving (i).

For (ii), associativity of S_M gives

$$\mu_{(\mathcal{U}\oplus_{S_M}\mathcal{V})\oplus_{S_M}\mathcal{W}}(x) = S_M\Big(S_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big), \mu_{\mathcal{W}}(x)\Big)$$

$$= S_M\Big(\mu_{\mathcal{U}}(x), S_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{W}}(x)\big)\Big) = \mu_{\mathcal{U}\oplus_{S_M}(\mathcal{V}\oplus_{S_M}\mathcal{W})}(x),$$

for all $x \in X$.

For (iii), the neutral-element property of S_M implies

$$\mu_{\mathcal{U}\oplus_{S_M}\mathbf{0}_M}(x) = S_M\big(\mu_{\mathcal{U}}(x), 0_M\big) = \mu_{\mathcal{U}}(x) \qquad \text{for all } x \in X.$$

Hence

$$\mathcal{U} \oplus_{S_M} \mathbf{0}_M = \mathcal{U}.$$

□

As a reference, a catalogue of representative t-conorm operators classified by the dimension k of the degree-domain is presented in Table 4.3.

Table 4.3: A catalogue of representative t-conorm operators by the dimension k of the degree-domain.

k	note	Representative t-conorm operator(s)
1		Fuzzy t-conorm [139,162]: $S : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy t-conorm [163,164]: $(\mu,\nu) \mapsto (S,T)$; pythagorean fuzzy t-conorm [165,166]
3		Neutrosophic t-conorm [167–169]: $(T,I,F) \mapsto (S,T,T)$; spherical fuzzy t-conorm; hesitant fuzzy t-conorm [170]
n	$(n \geq 1)$	Plithogenic t-conorm: contradiction-aware aggregation on $[0,1]^n$.

Reading guide. The table groups representative t-conorm operators by the dimension k of their degree values.

Related concepts include continuous t-conorms [171] and Archimedean t-conorms [172,173].

4.3 Uncertain uninorm

An uncertain uninorm unifies conjunction and disjunction under uncertainty using a variable neutral element, allowing flexible combination behavior across low, medium, and high assessments regimes.

Definition 4.3.1 (Fuzzy Uninorm)**.** Let

$$U : [0, 1] \times [0, 1] \to [0, 1]$$

be a binary operation. Then U is called a *fuzzy uninorm* if there exists an element $e \in [0, 1]$ such that, for all $x, y, z, x_1, x_2, y_1, y_2 \in [0, 1]$, the following conditions hold:

1. **Commutativity:**
$$U(x, y) = U(y, x).$$
2. **Associativity:**
$$U(x, U(y, z)) = U(U(x, y), z).$$
3. **Monotonicity:** if $x_1 \le x_2$ and $y_1 \le y_2$, then
$$U(x_1, y_1) \le U(x_2, y_2).$$
4. **Neutral element:**
$$U(e, x) = x \qquad \text{for all } x \in [0, 1].$$

The element e is called the *neutral element* of U.

Remark 4.3.2. If $e = 1$, then a fuzzy uninorm reduces to a fuzzy t-norm. If $e = 0$, then a fuzzy uninorm reduces to a fuzzy t-conorm.

Definition 4.3.3 (Neutrosophic Uninorm)**.** A mapping

$$U_N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \qquad U_N(x, y) = \big(U_N^T(x, y),\, U_N^I(x, y),\, U_N^F(x, y)\big),$$

is called a *neutrosophic uninorm* if there exists an element

$$e \in \mathbb{N}$$

such that, for all $x, y, z \in \mathbb{N}$, the following conditions hold:

1. **Commutativity:**
$$U_N(x, y) = U_N(y, x).$$
2. **Associativity:**
$$U_N\big(U_N(x, y), z\big) = U_N\big(x, U_N(y, z)\big).$$
3. **Monotonicity:** if $y \le_N z$, then
$$U_N(x, y) \le_N U_N(x, z).$$

4. **Neutral element:**
$$U_N(e, x) = x.$$

The element e is called the *neutral element* of U_N.

Remark 4.3.4. If the neutral element is

$$e = 1_N = (1, 0, 0),$$

then the neutrosophic uninorm behaves as a neutrosophic t-norm type operator. If

$$e = 0_N = (0, 1, 1),$$

then it behaves as a neutrosophic t-conorm type operator.

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

equipped with a partial order $\preceq_M$ and a distinguished element

$$e_M \in \mathrm{Dom}(M),$$

which plays the role of the neutral element.

Definition 4.3.5 (M-uninorm)**.** A mapping

$$U_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-uninorm* if, for all

$$a, b, c, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M),$$

the following conditions hold:

(i) **Closure:**
$$U_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Commutativity:**
$$U_M(a, b) = U_M(b, a).$$

(iii) **Associativity:**
$$U_M\big(a, U_M(b, c)\big) = U_M\big(U_M(a, b), c\big).$$

(iv) **Monotonicity:** if
$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$
then
$$U_M(a_1, b_1) \preceq_M U_M(a_2, b_2).$$

(v) **Neutral element:**
$$U_M(a, e_M) = a.$$

Definition 4.3.6 (Uncertain uninorm of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let U_M be an M-uninorm.

The *uncertain uninorm* of $\mathcal{U}$ and $\mathcal{V}$ induced by U_M, denoted by

$$\mathcal{U} \odot_{U_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \odot_{U_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \odot_{U_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \odot_{U_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \odot_{U_M} \mathcal{V}}(x) = U_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.3.7 (Well-definedness of the uncertain uninorm of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$U_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-uninorm. Then

$$\mathcal{U} \odot_{U_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of U_M,

$$U_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence, for every $x \in X$,

$$\mu_{\mathcal{U} \odot_{U_M} \mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto U_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \odot_{U_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \odot_{U_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \odot_{U_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain uninorm of U-sets is well-defined. □

Proposition 4.3.8 (Inherited algebraic properties)**.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be uncertain sets of type M on X.*

(i)

$$\mathcal{U} \odot_{U_M} \mathcal{V} = \mathcal{V} \odot_{U_M} \mathcal{U}.$$

(ii)

$$\big(\mathcal{U} \odot_{U_M} \mathcal{V}\big) \odot_{U_M} \mathcal{W} = \mathcal{U} \odot_{U_M} \big(\mathcal{V} \odot_{U_M} \mathcal{W}\big).$$

(iii) If $\mathbf{e}_M = (X, \mu_{\mathbf{e}_M})$ is the constant U-set defined by

$$\mu_{\mathbf{e}_M}(x) = e_M \qquad \text{for all } x \in X,$$

then

$$\mathcal{U} \odot_{U_M} \mathbf{e}_M = \mathcal{U}.$$

Proof. For each $x \in X$, commutativity of U_M yields

$$\mu_{\mathcal{U}\odot_{U_M}\mathcal{V}}(x) = U_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = U_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{V}\odot_{U_M}\mathcal{U}}(x),$$

which proves (i).

For (ii), associativity of U_M gives

$$\mu_{(\mathcal{U}\odot_{U_M}\mathcal{V})\odot_{U_M}\mathcal{W}}(x) = U_M\Big(U_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big), \mu_{\mathcal{W}}(x)\Big)$$

$$= U_M\Big(\mu_{\mathcal{U}}(x), U_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{W}}(x)\big)\Big) = \mu_{\mathcal{U}\odot_{U_M}(\mathcal{V}\odot_{U_M}\mathcal{W})}(x),$$

for all $x \in X$.

For (iii), the neutral-element property implies

$$\mu_{\mathcal{U}\odot_{U_M}\mathbf{e}_M}(x) = U_M\big(\mu_{\mathcal{U}}(x), e_M\big) = \mu_{\mathcal{U}}(x) \qquad \text{for all } x \in X.$$

Hence

$$\mathcal{U} \odot_{U_M} \mathbf{e}_M = \mathcal{U}.$$

□

As a reference, a catalogue of representative uninorm operators classified by the dimension k of the degree-domain is presented in Table 4.4.

As related concepts other than the above, off-uninorms [178] and nullnorms [179,180], among others, are also known.

Table 4.4: A catalogue of representative uninorm operators by the dimension k of the degree-domain.

k	note	Representative uninorm operator(s)
1		Fuzzy uninorm [174]: $U : [0,1]^2 \to [0,1]$ with neutral element $e \in [0,1]$.
2		Intuitionistic Fuzzy uninorm (cf. [175]): componentwise uninorm on (μ, ν).
3		Neutrosophic uninorm [176,177]: componentwise uninorm on (T, I, F).
n	$(n \geq 1)$	Plithogenic uninorm: contradiction-aware uninorm on $[0,1]^n$.

Reading guide. The table groups representative uninorm operators by the dimension k of their degree values.

4.4 Uncertain overlap function

An uncertain overlap function measures uncertain jointness without requiring full associativity, quantifying simultaneous support between uncertain inputs and emphasizing overlap sensitive conjunction behavior in settings.

Definition 4.4.1 (Fuzzy Overlap Function)**.** [181] A function

$$O : [0,1]^2 \to [0,1]$$

is called a *fuzzy overlap function* if it is symmetric, continuous, nondecreasing in each argument, and satisfies

$$O(x,y) = 0 \iff x = 0 \text{ or } y = 0,$$

and

$$O(x,y) = 1 \iff x = y = 1.$$

Definition 4.4.2 (Neutrosophic Overlap Function)**.** Let

$$D^* = [0,1]^3$$

be equipped with the order

$$(s_1, s_2, s_3) \leq_1 (t_1, t_2, t_3) \iff s_1 \leq t_1,\ s_2 \geq t_2,\ s_3 \geq t_3.$$

Set

$$0_{D^*} = (0,1,1), \qquad 1_{D^*} = (1,0,0).$$

A function

$$O : D^* \times D^* \to D^*$$

is called a *neutrosophic overlap function* if it is commutative, nondecreasing with respect to $\leq_1$ in each argument, continuous, and satisfies

$$O(0_{D^*}, t) = O(t, 0_{D^*}) = 0_{D^*} \qquad \text{for all } t \in D^*,$$

together with

$$O(1_{D^*}, 1_{D^*}) = 1_{D^*}.$$

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

equipped with a partial order $\preceq_M$ and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which play the roles of the bottom and top degrees, respectively. We endow $\mathrm{Dom}(M)$ with the subspace topology inherited from $[0,1]^k$.

Definition 4.4.3 (M-overlap function)**.** A mapping

$$O_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-overlap function* if, for all $a, b, c, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M)$, the following conditions hold:

(i) **Closure:**

$$O_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Commutativity:**

$$O_M(a, b) = O_M(b, a).$$

(iii) **Monotonicity:** if

$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$

then

$$O_M(a_1, b_1) \preceq_M O_M(a_2, b_2).$$

(iv) **Continuity:** O_M is continuous with respect to the subspace topology on $\mathrm{Dom}(M)$.

(v) **Zero condition:**

$$O_M(a, b) = 0_M \iff a = 0_M \text{ or } b = 0_M.$$

(vi) **Unit condition:**

$$O_M(a, b) = 1_M \iff a = b = 1_M.$$

Definition 4.4.4 (Uncertain overlap of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let O_M be an M-overlap function.

The *uncertain overlap* of $\mathcal{U}$ and $\mathcal{V}$ induced by O_M, denoted by

$$\mathcal{U} \diamond_{O_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \diamond_{O_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}}(x) = O_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.4.5 (Well-definedness of the uncertain overlap of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$O_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-overlap function. Then

$$\mathcal{U} \diamond_{O_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of O_M,

$$O_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto O_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a map

$$\mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \diamond_{O_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain overlap of U-sets is well-defined. □

Proposition 4.4.6 (Inherited properties)**.** *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}, \mathcal{Z}$ be uncertain sets of type M on X.*

(i) ***Commutativity:***

$$\mathcal{U} \diamond_{O_M} \mathcal{V} = \mathcal{V} \diamond_{O_M} \mathcal{U}.$$

(ii) ***Monotonicity:*** *if*

$$\mu_{\mathcal{U}}(x) \preceq_M \mu_{\mathcal{W}}(x) \quad \textit{and} \quad \mu_{\mathcal{V}}(x) \preceq_M \mu_{\mathcal{Z}}(x) \qquad \textit{for all } x \in X,$$

then

$$\mu_{\mathcal{U} \diamond_{O_M} \mathcal{V}}(x) \preceq_M \mu_{\mathcal{W} \diamond_{O_M} \mathcal{Z}}(x) \qquad \textit{for all } x \in X.$$

(iii) ***Zero characterization:*** *if $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ is the constant U-set defined by*

$$\mu_{\mathbf{0}_M}(x) = 0_M \qquad \textit{for all } x \in X,$$

then

$$\mathcal{U} \diamond_{O_M} \mathbf{0}_M = \mathbf{0}_M.$$

(iv) ***Unit characterization:*** *if* $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ *is the constant U-set defined by*

$$\mu_{\mathbf{1}_M}(x) = 1_M \qquad \textit{for all } x \in X,$$

then

$$\mathcal{U} \diamond_{O_M} \mathbf{1}_M = \mathbf{1}_M \iff \mathcal{U} = \mathbf{1}_M.$$

Proof. (i) For every $x \in X$, by commutativity of O_M,

$$\mu_{\mathcal{U}\diamond_{O_M}\mathcal{V}}(x) = O_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = O_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{V}\diamond_{O_M}\mathcal{U}}(x).$$

Hence

$$\mathcal{U} \diamond_{O_M} \mathcal{V} = \mathcal{V} \diamond_{O_M} \mathcal{U}.$$

(ii) For each $x \in X$, the assumed pointwise inequalities and monotonicity of O_M imply

$$O_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \preceq_M O_M\big(\mu_{\mathcal{W}}(x), \mu_{\mathcal{Z}}(x)\big).$$

Thus

$$\mu_{\mathcal{U}\diamond_{O_M}\mathcal{V}}(x) \preceq_M \mu_{\mathcal{W}\diamond_{O_M}\mathcal{Z}}(x) \qquad \text{for all } x \in X.$$

(iii) For every $x \in X$,

$$\mu_{\mathcal{U}\diamond_{O_M}\mathbf{0}_M}(x) = O_M\big(\mu_{\mathcal{U}}(x), 0_M\big) = 0_M$$

by the zero condition of O_M. Hence

$$\mathcal{U} \diamond_{O_M} \mathbf{0}_M = \mathbf{0}_M.$$

(iv) Assume first that

$$\mathcal{U} \diamond_{O_M} \mathbf{1}_M = \mathbf{1}_M.$$

Then for every $x \in X$,

$$O_M\big(\mu_{\mathcal{U}}(x), 1_M\big) = 1_M.$$

By the unit condition of O_M, this implies

$$\mu_{\mathcal{U}}(x) = 1_M \qquad \text{for all } x \in X.$$

Hence $\mathcal{U} = \mathbf{1}_M$.

Conversely, if $\mathcal{U} = \mathbf{1}_M$, then for every $x \in X$,

$$\mu_{\mathcal{U}\diamond_{O_M}\mathbf{1}_M}(x) = O_M(1_M, 1_M) = 1_M,$$

so

$$\mathcal{U} \diamond_{O_M} \mathbf{1}_M = \mathbf{1}_M.$$

□

As a reference, a catalogue of representative overlap functions classified by the dimension k of the degree-domain is presented in Table 4.5.

Table 4.5: A catalogue of representative overlap functions by the dimension k of the degree-domain.

k	note	Representative overlap function(s)
1		Fuzzy overlap function: $O : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy overlap function [182]: componentwise overlap on (μ, ν).
3		Neutrosophic overlap function [183]: componentwise overlap on (T, I, F).
n	$(n \geq 1)$	Plithogenic overlap function: contradiction-aware overlap on $[0,1]^n$.

Reading guide. The table groups representative overlap functions by the dimension k of their degree values.

4.5 Uncertain conjunctor

An uncertain conjunctor is a broad conjunction type operator under uncertainty, extending beyond uncertain t-norms while preserving essential monotonic and boundary oriented combination behavior practically.

Definition 4.5.1 (Fuzzy Conjunctor)**.** [184–187] A function

$$C : [0,1]^2 \to [0,1]$$

is called a *fuzzy conjunctor* if it is nondecreasing in each argument and satisfies

$$C(0,0) = C(0,1) = C(1,0) = 0, \qquad C(1,1) = 1.$$

Remark 4.5.2. A fuzzy conjunctor C is called a *border conjunctor* if

$$C(1,x) = x \qquad \text{for all } x \in [0,1].$$

A commutative and associative border conjunctor is a fuzzy t-norm.

Definition 4.5.3 (Neutrosophic Conjunctor)**.** A mapping

$$C_N : \mathbb{N}_{SV} \times \mathbb{N}_{SV} \to \mathbb{N}_{SV}$$

is called a *neutrosophic conjunctor* if it satisfies the following conditions:

1. **Monotonicity:** for all $x_1, x_2, y_1, y_2 \in \mathbb{N}_{SV}$,

$$x_1 \leq_N x_2 \text{ and } y_1 \leq_N y_2 \quad \Longrightarrow \quad C_N(x_1, y_1) \leq_N C_N(x_2, y_2).$$

2. **Boundary conditions:**

$$C_N(0_N, 0_N) = 0_N, \qquad C_N(0_N, 1_N) = 0_N, \qquad C_N(1_N, 0_N) = 0_N, \qquad C_N(1_N, 1_N) = 1_N.$$

Equivalently, C_N is a monotone extension of the crisp conjunction on $\{0_N, 1_N\}^2$.

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

equipped with a partial order $\preceq_M$ and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which represent the bottom and top uncertainty values, respectively.

Definition 4.5.4 (M-conjunctor)**.** A mapping

$$C_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-conjunctor* if, for all

$$a, b, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M),$$

the following conditions hold:

(i) **Closure:**

$$C_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Monotonicity:** if

$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$

then

$$C_M(a_1, b_1) \preceq_M C_M(a_2, b_2).$$

(iii) **Boundary conditions:**

$$C_M(0_M, 0_M) = 0_M, \qquad C_M(0_M, 1_M) = 0_M, \qquad C_M(1_M, 0_M) = 0_M, \qquad C_M(1_M, 1_M) = 1_M.$$

Definition 4.5.5 (Uncertain conjunctor of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let C_M be an M-conjunctor.

The *uncertain conjunctor* of $\mathcal{U}$ and $\mathcal{V}$ induced by C_M, denoted by

$$\mathcal{U} \curlywedge_{C_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \curlywedge_{C_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \curlywedge_{C_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \curlywedge_{C_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \curlywedge_{C_M} \mathcal{V}}(x) = C_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.5.6 (Well-definedness of the uncertain conjunctor of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$C_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-conjunctor. Then

$$\mathcal{U} \curlywedge_{C_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of C_M,

$$C_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathcal{U} \curlywedge_{C_M} \mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto C_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \curlywedge_{C_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \curlywedge_{C_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \curlywedge_{C_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain conjunctor of U-sets is well-defined. □

Proposition 4.5.7 (Inherited properties)**.** *Let*

$$\mathcal{U}_1 = (X, \mu_{\mathcal{U}_1}), \quad \mathcal{U}_2 = (X, \mu_{\mathcal{U}_2}), \quad \mathcal{V}_1 = (X, \mu_{\mathcal{V}_1}), \quad \mathcal{V}_2 = (X, \mu_{\mathcal{V}_2})$$

be uncertain sets of type M on X. Then the following statements hold.

(i) ***Monotonicity.*** *If*

$$\mu_{\mathcal{U}_1}(x) \preceq_M \mu_{\mathcal{U}_2}(x) \quad \text{and} \quad \mu_{\mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{V}_2}(x) \qquad \text{for all } x \in X,$$

then

$$\mu_{\mathcal{U}_1 \curlywedge_{C_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_2 \curlywedge_{C_M} \mathcal{V}_2}(x) \qquad \text{for all } x \in X.$$

(ii) ***Boundary behavior.*** *Let $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ and $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ be the constant bottom and top U-sets defined by*

$$\mu_{\mathbf{0}_M}(x) = 0_M, \qquad \mu_{\mathbf{1}_M}(x) = 1_M \qquad (x \in X).$$

Then

$$\mathbf{0}_M \curlywedge_{C_M} \mathbf{0}_M = \mathbf{0}_M, \qquad \mathbf{0}_M \curlywedge_{C_M} \mathbf{1}_M = \mathbf{0}_M, \qquad \mathbf{1}_M \curlywedge_{C_M} \mathbf{0}_M = \mathbf{0}_M, \qquad \mathbf{1}_M \curlywedge_{C_M} \mathbf{1}_M = \mathbf{1}_M.$$

Proof. (i) Fix $x \in X$. By assumption,

$$\mu_{\mathcal{U}_1}(x) \preceq_M \mu_{\mathcal{U}_2}(x) \quad \text{and} \quad \mu_{\mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{V}_2}(x).$$

Hence, by monotonicity of C_M,

$$C_M\big(\mu_{\mathcal{U}_1}(x), \mu_{\mathcal{V}_1}(x)\big) \preceq_M C_M\big(\mu_{\mathcal{U}_2}(x), \mu_{\mathcal{V}_2}(x)\big).$$

That is,

$$\mu_{\mathcal{U}_1 \curlywedge_{C_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_2 \curlywedge_{C_M} \mathcal{V}_2}(x).$$

(ii) For every $x \in X$, one has

$$\mu_{\mathbf{0}_M \curlywedge_{C_M} \mathbf{0}_M}(x) = C_M(0_M, 0_M) = 0_M,$$

$$\mu_{\mathbf{0}_M \curlywedge_{C_M} \mathbf{1}_M}(x) = C_M(0_M, 1_M) = 0_M,$$

$$\mu_{\mathbf{1}_M \curlywedge_{C_M} \mathbf{0}_M}(x) = C_M(1_M, 0_M) = 0_M,$$

and

$$\mu_{\mathbf{1}_M \curlywedge_{C_M} \mathbf{1}_M}(x) = C_M(1_M, 1_M) = 1_M.$$

Hence the stated identities follow.

□

As a reference, a catalogue of representative conjunctors classified by the dimension k of the degree-domain is presented in Table 4.6.

Table 4.6: A catalogue of representative conjunctors by the dimension k of the degree-domain.

k	note	Representative conjunctor(s)
1		Fuzzy conjunctor [188, 189]: $C : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy conjunctor [190, 191]: componentwise conjunctor on (μ, ν).
3		Neutrosophic conjunctor (cf. [192, 193]): componentwise conjunctor on (T, I, F).
n	$(n \geq 1)$	Hesitant Fuzzy conjunctor: conjunctor on hesitant degrees; Plithogenic conjunctor: contradiction-aware conjunctor on $[0,1]^n$.

Reading guide. The table groups representative conjunctors by the dimension k of their degree values.

4.6 Uncertain disjunctor

An uncertain disjunctor is a broad disjunction type operator under uncertainty, extending beyond uncertain t-conorms while preserving essential monotonic and boundary oriented combination behavior practically (cf. [188, 194]).

Definition 4.6.1 (Fuzzy Disjunctor)**.** A mapping

$$D : [0,1] \times [0,1] \to [0,1]$$

is called a *fuzzy disjunctor* if, for all $x_1, x_2, y_1, y_2 \in [0,1]$, it satisfies:

1. **Monotonicity:** if $x_1 \leq x_2$ and $y_1 \leq y_2$, then

$$D(x_1, y_1) \leq D(x_2, y_2).$$

2. **Boundary conditions:**

$$D(0,0) = 0, \qquad D(1,0) = 1, \qquad D(0,1) = 1, \qquad D(1,1) = 1.$$

Definition 4.6.2 (Neutrosophic Disjunctor)**.** A mapping

$$D_N : \mathbb{N}_{SV} \times \mathbb{N}_{SV} \to \mathbb{N}_{SV}$$

is called a *neutrosophic disjunctor* if it satisfies the following conditions:

1. **Monotonicity:** for all $x_1, x_2, y_1, y_2 \in \mathbb{N}_{SV}$,

$$x_1 \leq_N x_2 \text{ and } y_1 \leq_N y_2 \quad \Longrightarrow \quad D_N(x_1, y_1) \leq_N D_N(x_2, y_2).$$

2. **Boundary conditions:**

$$D_N(0_N, 0_N) = 0_N, \qquad D_N(1_N, 0_N) = 1_N, \qquad D_N(0_N, 1_N) = 1_N, \qquad D_N(1_N, 1_N) = 1_N.$$

Equivalently, D_N is a monotone extension of the crisp disjunction on $\{0_N, 1_N\}^2$.

Remark 4.6.3. A neutrosophic disjunctor D_N is called a *border neutrosophic disjunctor* if

$$D_N(0_N, x) = x \qquad \text{for all } x \in \mathbb{N}_{SV}.$$

If a border neutrosophic disjunctor is also commutative and associative, then it is a neutrosophic n-conorm.

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

equipped with a partial order $\preceq_M$ and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which represent the bottom and top uncertainty values, respectively.

Definition 4.6.4 (M-disjunctor)**.** A mapping

$$D_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-disjunctor* if, for all

$$a, b, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M),$$

the following conditions hold:

(i) **Closure:**

$$D_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Monotonicity:** if

$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$

then

$$D_M(a_1, b_1) \preceq_M D_M(a_2, b_2).$$

(iii) **Boundary conditions:**

$$D_M(0_M, 0_M) = 0_M, \qquad D_M(1_M, 0_M) = 1_M, \qquad D_M(0_M, 1_M) = 1_M, \qquad D_M(1_M, 1_M) = 1_M.$$

Definition 4.6.5 (Uncertain disjunctor of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let D_M be an M-disjunctor.

The *uncertain disjunctor* of $\mathcal{U}$ and $\mathcal{V}$ induced by D_M, denoted by

$$\mathcal{U} \curlyvee_{D_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \curlyvee_{D_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \curlyvee_{D_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \curlyvee_{D_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \curlyvee_{D_M} \mathcal{V}}(x) = D_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.6.6 (Well-definedness of the uncertain disjunctor of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$D_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-disjunctor. Then

$$\mathcal{U} \curlyvee_{D_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of D_M,

$$D_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathcal{U} \curlyvee_{D_M} \mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto D_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \curlyvee_{D_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \curlyvee_{D_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \curlyvee_{D_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain disjunctor of U-sets is well-defined. □

Proposition 4.6.7 (Inherited properties)**.** *Let*

$$\mathcal{U}_1 = (X, \mu_{\mathcal{U}_1}), \quad \mathcal{U}_2 = (X, \mu_{\mathcal{U}_2}), \quad \mathcal{V}_1 = (X, \mu_{\mathcal{V}_1}), \quad \mathcal{V}_2 = (X, \mu_{\mathcal{V}_2})$$

be uncertain sets of type M on X. Then the following statements hold.

(i) ***Monotonicity.*** *If*

$$\mu_{\mathcal{U}_1}(x) \preceq_M \mu_{\mathcal{U}_2}(x) \quad \text{and} \quad \mu_{\mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{V}_2}(x) \qquad \text{for all } x \in X,$$

then

$$\mu_{\mathcal{U}_1 \curlyvee_{D_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_2 \curlyvee_{D_M} \mathcal{V}_2}(x) \qquad \text{for all } x \in X.$$

(ii) ***Boundary behavior.*** *Let $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ and $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ be the constant bottom and top U-sets defined by*

$$\mu_{\mathbf{0}_M}(x) = 0_M, \qquad \mu_{\mathbf{1}_M}(x) = 1_M \qquad (x \in X).$$

Then

$$\mathbf{0}_M \curlyvee_{D_M} \mathbf{0}_M = \mathbf{0}_M, \qquad \mathbf{1}_M \curlyvee_{D_M} \mathbf{0}_M = \mathbf{1}_M, \qquad \mathbf{0}_M \curlyvee_{D_M} \mathbf{1}_M = \mathbf{1}_M, \qquad \mathbf{1}_M \curlyvee_{D_M} \mathbf{1}_M = \mathbf{1}_M.$$

Proof. (i) Fix $x \in X$. By assumption,

$$\mu_{\mathcal{U}_1}(x) \preceq_M \mu_{\mathcal{U}_2}(x) \quad \text{and} \quad \mu_{\mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{V}_2}(x).$$

Hence, by monotonicity of D_M,

$$D_M\big(\mu_{\mathcal{U}_1}(x), \mu_{\mathcal{V}_1}(x)\big) \preceq_M D_M\big(\mu_{\mathcal{U}_2}(x), \mu_{\mathcal{V}_2}(x)\big).$$

That is,

$$\mu_{\mathcal{U}_1 \curlyvee_{D_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_2 \curlyvee_{D_M} \mathcal{V}_2}(x).$$

(ii) For every $x \in X$, one has

$$\mu_{\mathbf{0}_M \curlyvee_{D_M} \mathbf{0}_M}(x) = D_M(0_M, 0_M) = 0_M,$$

$$\mu_{\mathbf{1}_M \curlyvee_{D_M} \mathbf{0}_M}(x) = D_M(1_M, 0_M) = 1_M,$$

$$\mu_{\mathbf{0}_M \curlyvee_{D_M} \mathbf{1}_M}(x) = D_M(0_M, 1_M) = 1_M,$$

and

$$\mu_{\mathbf{1}_M \curlyvee_{D_M} \mathbf{1}_M}(x) = D_M(1_M, 1_M) = 1_M.$$

Hence the stated identities follow.

□

As a reference, a catalogue of representative disjunctors classified by the dimension k of the degree-domain is presented in Table 4.7.

Table 4.7: A catalogue of representative disjunctors by the dimension k of the degree-domain.

k	note	Representative disjunctor(s)
1		Fuzzy disjunctor [188, 194]: $D : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy disjunctor [190, 195]: componentwise disjunctor on (μ, ν).
3		Neutrosophic disjunctor (cf. [196]): componentwise disjunctor on (T, I, F).
n	$(n \geq 1)$	Plithogenic disjunctor: contradiction-aware disjunctor on $[0,1]^n$.

Reading guide. The table groups representative disjunctors by the dimension k of their degree values.

4.7 Uncertain Grouping Function

A grouping function combines input degrees into a single value, usually increasing with each argument, modeling collective reinforcement, aggregation strength, and generalized disjunctive behavior mathematically (cf. [197]). A fuzzy grouping function maps membership degrees in $[0, 1]$ to an aggregated degree, preserving monotonicity and boundary conditions for cooperative uncertain information fusion processes mathematically(cf. [197]).

Definition 4.7.1 (Fuzzy Grouping Function)**.** (cf. [197]) A function

$$G : [0,1]^2 \to [0,1]$$

is called a *fuzzy grouping function* if it is symmetric, continuous, nondecreasing in each argument, and satisfies

$$G(x, y) = 0 \iff x = y = 0,$$

and

$$G(x, y) = 1 \iff x = 1 \text{ or } y = 1.$$

A grouping function combines input degrees into a single value, usually increasing with each argument, modeling collective reinforcement, aggregation strength, and generalized disjunctive behavior mathematically. In the uncertain-set framework, the appropriate extension must be defined on the admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

of a fixed uncertain model M, and then lifted pointwise to uncertain sets.

Definition 4.7.2 (M-grouping function)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$. Assume that $\mathrm{Dom}(M)$ is equipped with

- a partial order $\preceq_M$,
- distinguished elements $0_M, 1_M \in \mathrm{Dom}(M)$,
- the subspace topology inherited from $[0,1]^k$.

A mapping
$$G_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$
is called an *M-grouping function* if, for all
$$a, b, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M),$$
the following conditions hold:

(i) **Closure:**
$$G_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Symmetry:**
$$G_M(a, b) = G_M(b, a).$$

(iii) **Monotonicity:** if
$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$
then
$$G_M(a_1, b_1) \preceq_M G_M(a_2, b_2).$$

(iv) **Continuity:** G_M is continuous with respect to the subspace topology on $\mathrm{Dom}(M)$.

(v) **Zero condition:**
$$G_M(a, b) = 0_M \iff a = b = 0_M.$$

(vi) **Unit condition:**
$$G_M(a, b) = 1_M \iff a = 1_M \text{ or } b = 1_M.$$

Remark 4.7.3. When M is the fuzzy model, that is,
$$\mathrm{Dom}(M) = [0, 1], \qquad 0_M = 0, \qquad 1_M = 1,$$
the above notion reduces exactly to the ordinary fuzzy grouping function.

Definition 4.7.4 (Uncertain grouping of U-sets)**.** Let X be a nonempty set, let M be an uncertain model, and let
$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$
be uncertain sets of type M on X, where
$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$
Assume that G_M is an M-grouping function.

The *uncertain grouping* of $\mathcal{U}$ and $\mathcal{V}$ induced by G_M, denoted by
$$\mathcal{U} \star_{G_M} \mathcal{V},$$
is defined by
$$\mathcal{U} \star_{G_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \star_{G_M} \mathcal{V}}\big),$$
where the uncertainty-degree function
$$\mu_{\mathcal{U} \star_{G_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$
is given pointwise by
$$\mu_{\mathcal{U} \star_{G_M} \mathcal{V}}(x) = G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 4.7.5 (Well-definedness of uncertain grouping)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M on X. If

$$G_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

is an M-grouping function, then

$$\mathcal{U} \star_{G_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of G_M,

$$G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathcal{U} \star_{G_M} \mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \star_{G_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \star_{G_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \star_{G_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain grouping is well-defined. □

Proposition 4.7.6 (Inherited properties)**.** *Let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}}), \qquad \mathcal{W} = (X, \mu_{\mathcal{W}}), \qquad \mathcal{Z} = (X, \mu_{\mathcal{Z}})$$

be uncertain sets of type M on X. Then the following statements hold.

(i) ***Symmetry:***

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathcal{V} \star_{G_M} \mathcal{U}.$$

(ii) ***Monotonicity:*** *if*

$$\mu_{\mathcal{U}}(x) \preceq_M \mu_{\mathcal{W}}(x) \quad \textit{and} \quad \mu_{\mathcal{V}}(x) \preceq_M \mu_{\mathcal{Z}}(x) \qquad \textit{for all } x \in X,$$

then

$$\mu_{\mathcal{U} \star_{G_M} \mathcal{V}}(x) \preceq_M \mu_{\mathcal{W} \star_{G_M} \mathcal{Z}}(x) \qquad \textit{for all } x \in X.$$

(iii) ***Zero characterization:*** *if* $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ *is the constant U-set defined by*

$$\mu_{\mathbf{0}_M}(x) = 0_M \qquad \text{for all } x \in X,$$

then

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathbf{0}_M \iff \mathcal{U} = \mathbf{0}_M \text{ and } \mathcal{V} = \mathbf{0}_M.$$

(iv) ***Unit characterization:*** *if* $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ *is the constant U-set defined by*

$$\mu_{\mathbf{1}_M}(x) = 1_M \qquad \text{for all } x \in X,$$

then

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathbf{1}_M \iff \mathcal{U} = \mathbf{1}_M \text{ or } \mathcal{V} = \mathbf{1}_M.$$

Proof. (i) For every $x \in X$, by symmetry of G_M,

$$\mu_{\mathcal{U}\star_{G_M}\mathcal{V}}(x) = G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = G_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{V}\star_{G_M}\mathcal{U}}(x).$$

Hence

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathcal{V} \star_{G_M} \mathcal{U}.$$

(ii) Fix $x \in X$. By assumption,

$$\mu_{\mathcal{U}}(x) \preceq_M \mu_{\mathcal{W}}(x) \quad \text{and} \quad \mu_{\mathcal{V}}(x) \preceq_M \mu_{\mathcal{Z}}(x).$$

Therefore, by monotonicity of G_M,

$$G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \preceq_M G_M\big(\mu_{\mathcal{W}}(x), \mu_{\mathcal{Z}}(x)\big).$$

That is,

$$\mu_{\mathcal{U}\star_{G_M}\mathcal{V}}(x) \preceq_M \mu_{\mathcal{W}\star_{G_M}\mathcal{Z}}(x).$$

(iii) Assume first that

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathbf{0}_M.$$

Then for every $x \in X$,

$$G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = 0_M.$$

By the zero condition of G_M, it follows that

$$\mu_{\mathcal{U}}(x) = 0_M \qquad \text{and} \qquad \mu_{\mathcal{V}}(x) = 0_M$$

for all $x \in X$. Hence

$$\mathcal{U} = \mathbf{0}_M \qquad \text{and} \qquad \mathcal{V} = \mathbf{0}_M.$$

Conversely, if

$$\mathcal{U} = \mathbf{0}_M \qquad \text{and} \qquad \mathcal{V} = \mathbf{0}_M,$$

then for every $x \in X$,

$$\mu_{\mathcal{U}\star_{G_M}\mathcal{V}}(x) = G_M(0_M, 0_M) = 0_M,$$

so

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathbf{0}_M.$$

(iv) Assume first that

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathbf{1}_M.$$

Then for every $x \in X$,

$$G_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = 1_M.$$

By the unit condition of G_M, for every $x \in X$,

$$\mu_{\mathcal{U}}(x) = 1_M \quad \text{or} \quad \mu_{\mathcal{V}}(x) = 1_M.$$

In particular, if $\mathcal{U} = \mathbf{1}_M$, then the conclusion holds; likewise if $\mathcal{V} = \mathbf{1}_M$, the conclusion also holds.

Conversely, if $\mathcal{U} = \mathbf{1}_M$, then for every $x \in X$,

$$\mu_{\mathcal{U} \star_{G_M} \mathcal{V}}(x) = G_M\big(1_M, \mu_{\mathcal{V}}(x)\big) = 1_M$$

by the unit condition of G_M. Hence

$$\mathcal{U} \star_{G_M} \mathcal{V} = \mathbf{1}_M.$$

The case $\mathcal{V} = \mathbf{1}_M$ is analogous.

□

4.8 Uncertain Water logic

Fuzzy Water Logic is a many-valued logical system on $[0, 1]$, using water-style conjunction and disjunction to model gradual truth, saturation, flow, and uncertainty consistently mathematically [198].

Definition 4.8.1 (Fuzzy Water Logic)**.** [198] Let Var be a nonempty set of propositional variables, and let $\mathcal{L}_{\mathrm{FW}}$ be the set of formulas generated by

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi), \qquad p \in \mathsf{Var}.$$

A *Fuzzy Water Logic* is the many-valued propositional logic whose set of truth values is

$$L = [0, 1],$$

equipped with the following operations:

$$\begin{aligned} a \vee_W b &:= \min\{1, a + b\}, \\ a \wedge_W b &:= \max\{0, a + b - 1\}, \\ \neg_W a &:= 1 - a, \end{aligned}$$

for all $a, b \in [0, 1]$.

A *valuation* is a mapping

$$v : \mathsf{Var} \to [0, 1],$$

which is extended recursively to all formulas by

$$\begin{aligned} v(\bot) &:= 0, \\ v(\top) &:= 1, \\ v(\neg\varphi) &:= 1 - v(\varphi), \\ v(\varphi \vee \psi) &:= v(\varphi) \vee_W v(\psi) = \min\{1,\, v(\varphi) + v(\psi)\}, \\ v(\varphi \wedge \psi) &:= v(\varphi) \wedge_W v(\psi) = \max\{0,\, v(\varphi) + v(\psi) - 1\}. \end{aligned}$$

The structure

$$\mathbf{FW} = \big([0,1], \vee_W, \wedge_W, \neg_W, 0, 1\big)$$

is called the *truth-functional algebra* of Fuzzy Water Logic.

Remark 4.8.2. The operation $\vee_W$ is the *flow* operation and $\wedge_W$ is the *bypass* operation. Thus Fuzzy Water Logic formalizes Water Logic as a fuzzy-valued logical system.

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$, and assume that $\mathrm{Dom}(M)$ is equipped with a partial order

$$\preceq_M$$

and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which play the roles of the bottom and top uncertainty values, respectively.

Definition 4.8.3 (M-Water operator system)**.** An *M-Water operator system* is a triple

$$\mathbf{W}_M = \big(\oplus_{W,M}, \otimes_{W,M}, N_{W,M}\big),$$

where

$$\begin{aligned} \oplus_{W,M} &: \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M), \\ \otimes_{W,M} &: \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M), \end{aligned}$$

and

$$N_{W,M} : \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

satisfy the following conditions:

(i) **Closure:**

$$a \oplus_{W,M} b \in \mathrm{Dom}(M), \qquad a \otimes_{W,M} b \in \mathrm{Dom}(M), \qquad N_{W,M}(a) \in \mathrm{Dom}(M)$$

for all $a, b \in \mathrm{Dom}(M)$.

(ii) **Commutativity of the water connectives:**

$$a \oplus_{W,M} b = b \oplus_{W,M} a, \qquad a \otimes_{W,M} b = b \otimes_{W,M} a$$

for all $a, b \in \mathrm{Dom}(M)$.

(iii) **Monotonicity:** if

$$a_1 \preceq_M a_2 \qquad \text{and} \qquad b_1 \preceq_M b_2,$$

then

$$a_1 \oplus_{W,M} b_1 \preceq_M a_2 \oplus_{W,M} b_2, \qquad a_1 \otimes_{W,M} b_1 \preceq_M a_2 \otimes_{W,M} b_2.$$

(iv) **Boundary conditions:**

$$a \oplus_{W,M} 0_M = a, \qquad a \oplus_{W,M} 1_M = 1_M,$$

$$a \otimes_{W,M} 1_M = a, \qquad a \otimes_{W,M} 0_M = 0_M$$

for all $a \in \mathrm{Dom}(M)$.

(v) **Water negation boundary conditions:**

$$N_{W,M}(0_M) = 1_M, \qquad N_{W,M}(1_M) = 0_M.$$

(vi) **Antitonicity of negation:** if

$$a \preceq_M b,$$

then

$$N_{W,M}(b) \preceq_M N_{W,M}(a).$$

Remark 4.8.4. When M is the fuzzy model, that is,

$$\mathrm{Dom}(M) = [0,1], \qquad 0_M = 0, \qquad 1_M = 1,$$

the standard choice

$$a \oplus_{W,M} b = \min\{1, a+b\}, \qquad a \otimes_{W,M} b = \max\{0, a+b-1\}, \qquad N_{W,M}(a) = 1-a$$

recovers the ordinary fuzzy Water Logic.

Definition 4.8.5 (Pointwise uncertain Water operations on U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M on X, that is,

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

The *uncertain Water disjunction* of $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$\mathcal{U} \vee_{W,M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \vee_{W,M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \vee_{W,M} \mathcal{V}}(x) := \mu_{\mathcal{U}}(x) \oplus_{W,M} \mu_{\mathcal{V}}(x) \qquad (x \in X).$$

The *uncertain Water conjunction* of $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$\mathcal{U} \wedge_{W,M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \wedge_{W,M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \wedge_{W,M} \mathcal{V}}(x) := \mu_{\mathcal{U}}(x) \otimes_{W,M} \mu_{\mathcal{V}}(x) \qquad (x \in X).$$

The *uncertain Water negation* of $\mathcal{U}$ is defined by

$$\neg_{W,M}\mathcal{U} := \big(X, \mu_{\neg_{W,M}\mathcal{U}}\big),$$

where

$$\mu_{\neg_{W,M}\mathcal{U}}(x) := N_{W,M}\big(\mu_{\mathcal{U}}(x)\big) \qquad (x \in X).$$

Theorem 4.8.6 (Well-definedness of the induced uncertain Water operations)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M on X. If $\mathbf{W}_M = \big(\oplus_{W,M}, \otimes_{W,M}, N_{W,M}\big)$ is an M-Water operator system, then

$$\mathcal{U} \vee_{W,M} \mathcal{V}, \qquad \mathcal{U} \wedge_{W,M} \mathcal{V}, \qquad \neg_{W,M}\mathcal{U}$$

are all well-defined uncertain sets of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix $x \in X$. By closure of $\oplus_{W,M}$,

$$\mu_{\mathcal{U}}(x) \oplus_{W,M} \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Hence the assignment

$$x \longmapsto \mu_{\mathcal{U}}(x) \oplus_{W,M} \mu_{\mathcal{V}}(x)$$

defines a map

$$\mu_{\mathcal{U} \vee_{W,M} \mathcal{V}} : X \to \mathrm{Dom}(M),$$

so

$$\mathcal{U} \vee_{W,M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \vee_{W,M} \mathcal{V}}\big)$$

is a well-defined uncertain set of type M.

Similarly, by closure of $\otimes_{W,M}$,

$$\mu_{\mathcal{U}}(x) \otimes_{W,M} \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X,$$

so

$$\mathcal{U} \wedge_{W,M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \wedge_{W,M} \mathcal{V}}\big)$$

is a well-defined uncertain set of type M.

Finally, by closure of $N_{W,M}$,

$$N_{W,M}\big(\mu_{\mathcal{U}}(x)\big) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X,$$

hence

$$\neg_{W,M}\mathcal{U} = \big(X, \mu_{\neg_{W,M}\mathcal{U}}\big)$$

is also a well-defined uncertain set of type M.

Therefore all three induced uncertain Water operations are well-defined. □

Definition 4.8.7 (Uncertain Water model)**.** Let Var be a nonempty set of propositional variables, and let $\mathcal{L}_{UW}$ be the set of formulas generated by

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi), \qquad p \in \mathsf{Var}.$$

Let X be a nonempty set. An *uncertain Water model of type M on X* is a pair

$$\mathfrak{M} = (X, V),$$

where

$$V : \mathsf{Var} \to \mathsf{U}_M(X)$$

assigns to each propositional variable p an uncertain set

$$V(p) = (X, \mu_p)$$

of type M on X, and where an M-Water operator system

$$\mathbf{W}_M = \big(\oplus_{W,M}, \otimes_{W,M}, N_{W,M}\big)$$

has been fixed.

Definition 4.8.8 (Semantics of Uncertain Water Logic)**.** Let

$$\mathfrak{M} = (X, V)$$

be an uncertain Water model of type M on X. The interpretation

$$[\![\cdot]\!]_{\mathfrak{M}} : \mathcal{L}_{UW} \to \mathsf{U}_M(X)$$

is defined recursively as follows:

(i) for each propositional variable $p \in \mathsf{Var}$,

$$[\![p]\!]_{\mathfrak{M}} := V(p);$$

(ii)

$$[\![\bot]\!]_{\mathfrak{M}} := \big(X, \mu_\bot\big), \qquad \mu_\bot(x) := 0_M \quad (x \in X);$$

(iii)

$$[\![\top]\!]_{\mathfrak{M}} := \big(X, \mu_\top\big), \qquad \mu_\top(x) := 1_M \quad (x \in X);$$

(iv)

$$[\![\neg\varphi]\!]_{\mathfrak{M}} := \neg_{W,M}[\![\varphi]\!]_{\mathfrak{M}};$$

(v)

$$[\![\varphi \vee \psi]\!]_{\mathfrak{M}} := [\![\varphi]\!]_{\mathfrak{M}} \vee_{W,M} [\![\psi]\!]_{\mathfrak{M}};$$

(vi)

$$[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} := [\![\varphi]\!]_{\mathfrak{M}} \wedge_{W,M} [\![\psi]\!]_{\mathfrak{M}}.$$

Definition 4.8.9 (Uncertain Water Logic)**.** The propositional system

$$\mathbf{UW}_M = \big(\mathcal{L}_{UW}, \mathrm{Dom}(M), \oplus_{W,M}, \otimes_{W,M}, N_{W,M}, 0_M, 1_M\big)$$

together with the above U-set semantics is called the *Uncertain Water Logic* associated with the uncertain model M.

Theorem 4.8.10 (Well-definedness of Uncertain Water Logic)**.** *Let*

$$\mathfrak{M} = (X, V)$$

be an uncertain Water model of type M on X. Then, for every formula

$$\varphi \in \mathcal{L}_{UW},$$

the interpretation

$$[\![\varphi]\!]_{\mathfrak{M}}$$

is a well-defined uncertain set of type M on X. Moreover, the recursive interpretation is unique.

Proof. We prove by structural induction on the formation of formulas that

$$[\![\varphi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X) \qquad \text{for all } \varphi \in \mathcal{L}_{UW}.$$

Base cases.

(a) If $\varphi = p \in \mathsf{Var}$, then by definition of V,

$$V(p) \in \mathsf{U}_M(X).$$

Hence

$$[\![p]\!]_{\mathfrak{M}} = V(p)$$

is a well-defined uncertain set of type M.

(b) For $\varphi = \bot$, the map

$$\mu_\bot : X \to \mathrm{Dom}(M), \qquad \mu_\bot(x) = 0_M,$$

is well-defined because $0_M \in \mathrm{Dom}(M)$. Therefore

$$[\![\bot]\!]_{\mathfrak{M}} = \big(X, \mu_\bot\big) \in \mathsf{U}_M(X).$$

(c) Similarly, for $\varphi = \top$, the map

$$\mu_\top : X \to \mathrm{Dom}(M), \qquad \mu_\top(x) = 1_M,$$

is well-defined because $1_M \in \mathrm{Dom}(M)$. Hence

$$[\![\top]\!]_{\mathfrak{M}} = \big(X, \mu_\top\big) \in \mathsf{U}_M(X).$$

Inductive steps.

Assume that

$$[\![\varphi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X) \qquad \text{and} \qquad [\![\psi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X).$$

(a) For negation, by the previous theorem on induced uncertain Water operations,

$$\neg_{W,M}[\![\varphi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X).$$

Hence

$$[\![\neg\varphi]\!]_{\mathfrak{M}} = \neg_{W,M}[\![\varphi]\!]_{\mathfrak{M}}$$

is well-defined.

(b) For disjunction, again by the induced-operation theorem,

$$[\![\varphi]\!]_{\mathfrak{M}} \vee_{W,M} [\![\psi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X).$$

Therefore

$$[\![\varphi \vee \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \vee_{W,M} [\![\psi]\!]_{\mathfrak{M}}$$

is well-defined.

(c) For conjunction, similarly,

$$[\![\varphi]\!]_{\mathfrak{M}} \wedge_{W,M} [\![\psi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X),$$

and thus

$$[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \wedge_{W,M} [\![\psi]\!]_{\mathfrak{M}}$$

is well-defined.

Hence, by structural induction,

$$[\![\varphi]\!]_{\mathfrak{M}} \in \mathsf{U}_M(X) \qquad \text{for every } \varphi \in \mathcal{L}_{UW}.$$

Therefore Uncertain Water Logic is well-defined.

For uniqueness, observe that the interpretation clauses specify uniquely the value of each atomic formula, the constants $\bot, \top$, and the value of every compound formula from the values of its immediate subformulas. A standard induction on formula complexity therefore yields uniqueness of the recursive interpretation. □

Chapter 5

Uncertain Relational and Inferential Operators

In this chapter, we examine several uncertain relational and inferential operators.

5.1 Uncertain implication

An uncertain implication models conditional reasoning under uncertainty, expressing how uncertain antecedent information supports uncertain consequent information within vague, incomplete, interval valued, or hesitant settings (cf. [199–202]).

Definition 5.1.1 (Fuzzy Implication)**.** (cf. [203–205]) A function

$$I : [0,1]^2 \to [0,1]$$

is called a *fuzzy implication* if it is nonincreasing in its first argument, nondecreasing in its second argument, and satisfies

$$I(0,0) = 1, \qquad I(1,1) = 1, \qquad I(1,0) = 0.$$

Remark 5.1.2. From the axioms above, it follows that

$$I(0,y) = 1 \quad \text{and} \quad I(x,1) = 1 \qquad \text{for all } x,y \in [0,1].$$

Definition 5.1.3 (Neutrosophic Implication)**.** [206, 207] Let

$$\mathbb{N} = [0,1]^3, \qquad 0_N = (0,1,1), \qquad 1_N = (1,0,0),$$

and define

$$(T_x, I_x, F_x) \leq_N (T_y, I_y, F_y) \iff T_x \leq T_y, \;\; I_x \geq I_y, \;\; F_x \geq F_y.$$

A function

$$I_N : \mathbb{N}^2 \to \mathbb{N}$$

is called a *neutrosophic implication* if it is decreasing in the first argument, increasing in the second argument with respect to $\leq_N$, and satisfies

$$I_N(0_N, 0_N) = I_N(0_N, 1_N) = I_N(1_N, 1_N) = 1_N, \qquad I_N(1_N, 0_N) = 0_N.$$

Example 5.1.4 (A concrete example of a neutrosophic implication)**.** Define

$$I_N : \mathbb{N}^2 \to \mathbb{N}$$

by

$$I_N\big((T_x, I_x, F_x), (T_y, I_y, F_y)\big) := (\max\{1 - T_x,\ T_y\},\ \min\{T_x,\ 1 - T_y\},\ \min\{T_x,\ 1 - T_y\})\,.$$

Then I_N is a neutrosophic implication.

Indeed, let

$$x = (T_x, I_x, F_x), \quad x' = (T_{x'}, I_{x'}, F_{x'}), \quad y = (T_y, I_y, F_y), \quad y' = (T_{y'}, I_{y'}, F_{y'}).$$

First, suppose that

$$x \le_N x'.$$

Then

$$T_x \le T_{x'}.$$

Hence

$$1 - T_{x'} \le 1 - T_x,$$

and therefore

$$\max\{1 - T_{x'}, T_y\} \le \max\{1 - T_x, T_y\}.$$

Also,

$$\min\{T_{x'}, 1 - T_y\} \ge \min\{T_x, 1 - T_y\}.$$

Thus

$$I_N(x', y) \le_N I_N(x, y),$$

so I_N is decreasing in the first argument.

Next, suppose that

$$y \le_N y'.$$

Then

$$T_y \le T_{y'}.$$

Therefore

$$\max\{1 - T_x, T_y\} \le \max\{1 - T_x, T_{y'}\},$$

and

$$1 - T_{y'} \le 1 - T_y,$$

so

$$\min\{T_x, 1 - T_{y'}\} \le \min\{T_x, 1 - T_y\}.$$

Hence

$$I_N(x, y) \le_N I_N(x, y'),$$

showing that I_N is increasing in the second argument.

Finally, the boundary conditions are satisfied:

$$I_N(0_N, 0_N) = I_N\big((0, 1, 1), (0, 1, 1)\big) = \big(\max\{1, 0\}, \min\{0, 1\}, \min\{0, 1\}\big) = (1, 0, 0) = 1_N,$$

$$I_N(0_N, 1_N) = I_N\big((0,1,1),(1,0,0)\big) = \big(\max\{1,1\}, \min\{0,0\}, \min\{0,0\}\big) = (1,0,0) = 1_N,$$

$$I_N(1_N, 1_N) = I_N\big((1,0,0),(1,0,0)\big) = \big(\max\{0,1\}, \min\{1,0\}, \min\{1,0\}\big) = (1,0,0) = 1_N,$$

and

$$I_N(1_N, 0_N) = I_N\big((1,0,0),(0,1,1)\big) = \big(\max\{0,0\}, \min\{1,1\}, \min\{1,1\}\big) = (0,1,1) = 0_N.$$

Therefore, I_N is a neutrosophic implication.

Next, we explain the uncertain implication. Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

equipped with a partial order $\preceq_M$ and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which play the roles of the bottom and top degrees, respectively.

Definition 5.1.5 (M-implication)**.** A mapping

$$I_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an M*-implication* if, for all

$$a, b, a_1, a_2, b_1, b_2 \in \mathrm{Dom}(M),$$

the following conditions hold:

(i) **Closure:**

$$I_M(a, b) \in \mathrm{Dom}(M).$$

(ii) **Antitonicity in the first argument:** if

$$a_1 \preceq_M a_2,$$

then

$$I_M(a_2, b) \preceq_M I_M(a_1, b).$$

(iii) **Monotonicity in the second argument:** if

$$b_1 \preceq_M b_2,$$

then

$$I_M(a, b_1) \preceq_M I_M(a, b_2).$$

(iv) **Boundary conditions:**

$$I_M(0_M, 0_M) = 1_M, \qquad I_M(1_M, 1_M) = 1_M, \qquad I_M(1_M, 0_M) = 0_M.$$

Remark 5.1.6. Under the preceding axioms, one also obtains the standard consequence-type inequalities

$$I_M(0_M, b) = 1_M \qquad \text{and} \qquad I_M(a, 1_M) = 1_M$$

for all $a, b \in \mathrm{Dom}(M)$. Indeed, since $0_M \preceq_M b$, monotonicity in the second argument yields

$$I_M(0_M, 0_M) \preceq_M I_M(0_M, b),$$

hence

$$1_M \preceq_M I_M(0_M, b),$$

so $I_M(0_M, b) = 1_M$. Similarly, since $a \preceq_M 1_M$, antitonicity in the first argument gives

$$I_M(1_M, 1_M) \preceq_M I_M(a, 1_M),$$

hence

$$1_M \preceq_M I_M(a, 1_M),$$

so $I_M(a, 1_M) = 1_M$.

Definition 5.1.7 (Uncertain implication of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

Let I_M be an M-implication.

The *uncertain implication* from $\mathcal{U}$ to $\mathcal{V}$ induced by I_M, denoted by

$$\mathcal{U} \Rightarrow_{I_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \Rightarrow_{I_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \Rightarrow_{I_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \Rightarrow_{I_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \Rightarrow_{I_M} \mathcal{V}}(x) = I_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 5.1.8 (Well-definedness of the uncertain implication of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M, and let

$$I_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-implication. Then

$$\mathcal{U} \Rightarrow_{I_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix any $x \in X$. By the closure property of I_M,

$$I_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence, for every $x \in X$,

$$\mu_{\mathcal{U} \Rightarrow_{I_M} \mathcal{V}}(x) \in \mathrm{Dom}(M).$$

Therefore the pointwise assignment

$$x \longmapsto I_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \Rightarrow_{I_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \Rightarrow_{I_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \Rightarrow_{I_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain implication of U-sets is well-defined. □

Proposition 5.1.9 (Inherited order properties)**.** *Let*

$$\mathcal{U}_1 = (X, \mu_{\mathcal{U}_1}), \quad \mathcal{U}_2 = (X, \mu_{\mathcal{U}_2}), \quad \mathcal{V}_1 = (X, \mu_{\mathcal{V}_1}), \quad \mathcal{V}_2 = (X, \mu_{\mathcal{V}_2})$$

be uncertain sets of type M on X.

(i) If

$$\mu_{\mathcal{U}_1}(x) \preceq_M \mu_{\mathcal{U}_2}(x) \qquad \textit{for all } x \in X,$$

then

$$\mu_{\mathcal{U}_2 \Rightarrow_{I_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_1 \Rightarrow_{I_M} \mathcal{V}_1}(x) \qquad \textit{for all } x \in X.$$

(ii) If

$$\mu_{\mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{V}_2}(x) \qquad \textit{for all } x \in X,$$

then

$$\mu_{\mathcal{U}_1 \Rightarrow_{I_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_1 \Rightarrow_{I_M} \mathcal{V}_2}(x) \qquad \textit{for all } x \in X.$$

(iii) If

$$\mathbf{0}_M = (X, \mu_{\mathbf{0}_M}), \qquad \mu_{\mathbf{0}_M}(x) = 0_M \quad (x \in X),$$

and

$$\mathbf{1}_M = (X, \mu_{\mathbf{1}_M}), \qquad \mu_{\mathbf{1}_M}(x) = 1_M \quad (x \in X),$$

then

$$\mathbf{0}_M \Rightarrow_{I_M} \mathcal{V} = \mathbf{1}_M, \qquad \mathcal{U} \Rightarrow_{I_M} \mathbf{1}_M = \mathbf{1}_M, \qquad \mathbf{1}_M \Rightarrow_{I_M} \mathbf{0}_M = \mathbf{0}_M.$$

Proof. (i) For every $x \in X$, the assumption

$$\mu_{\mathcal{U}_1}(x) \preceq_M \mu_{\mathcal{U}_2}(x)$$

and antitonicity in the first argument imply

$$I_M\big(\mu_{\mathcal{U}_2}(x), \mu_{\mathcal{V}_1}(x)\big) \preceq_M I_M\big(\mu_{\mathcal{U}_1}(x), \mu_{\mathcal{V}_1}(x)\big).$$

Hence

$$\mu_{\mathcal{U}_2 \Rightarrow_{I_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_1 \Rightarrow_{I_M} \mathcal{V}_1}(x) \qquad \text{for all } x \in X.$$

(ii) For every $x \in X$, the assumption

$$\mu_{\mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{V}_2}(x)$$

and monotonicity in the second argument imply

$$I_M\big(\mu_{\mathcal{U}_1}(x), \mu_{\mathcal{V}_1}(x)\big) \preceq_M I_M\big(\mu_{\mathcal{U}_1}(x), \mu_{\mathcal{V}_2}(x)\big).$$

Hence

$$\mu_{\mathcal{U}_1 \Rightarrow_{I_M} \mathcal{V}_1}(x) \preceq_M \mu_{\mathcal{U}_1 \Rightarrow_{I_M} \mathcal{V}_2}(x) \qquad \text{for all } x \in X.$$

(iii) For every $x \in X$, by the remark above,

$$\mu_{\mathbf{0}_M \Rightarrow_{I_M} \mathcal{V}}(x) = I_M\big(0_M, \mu_{\mathcal{V}}(x)\big) = 1_M = \mu_{\mathbf{1}_M}(x),$$

so

$$\mathbf{0}_M \Rightarrow_{I_M} \mathcal{V} = \mathbf{1}_M.$$

Likewise,

$$\mu_{\mathcal{U} \Rightarrow_{I_M} \mathbf{1}_M}(x) = I_M\big(\mu_{\mathcal{U}}(x), 1_M\big) = 1_M = \mu_{\mathbf{1}_M}(x),$$

hence

$$\mathcal{U} \Rightarrow_{I_M} \mathbf{1}_M = \mathbf{1}_M.$$

Finally,

$$\mu_{\mathbf{1}_M \Rightarrow_{I_M} \mathbf{0}_M}(x) = I_M(1_M, 0_M) = 0_M = \mu_{\mathbf{0}_M}(x),$$

so

$$\mathbf{1}_M \Rightarrow_{I_M} \mathbf{0}_M = \mathbf{0}_M.$$

□

A catalogue of representative implications by the dimension k of the degree-domain is presented in Table 5.1.

Table 5.1: A catalogue of representative implications by the dimension k of the degree-domain.

k	note	Representative implication(s)
1		Fuzzy implication [208, 209]: $I : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy implication [210, 211]: componentwise implication on (μ, ν); Bipolar Fuzzy implication [212, 213]; Pythagorean fuzzy implication [214, 215]
3		Neutrosophic implication [216,217]: componentwise implication on (T, I, F); Hesitant Fuzzy implication [218, 219]; Spherical Fuzzy implication; picture Fuzzy implication [144, 220];
n	$(n \geq 1)$	Hesitant Fuzzy implication [218, 221]: implication on hesitant degrees; Plithogenic implication: contradiction-aware implication on $[0,1]^n$.

Reading guide. The table groups representative implications by the dimension k of their degree values.

In addition, as related concepts to the above, residuation operators [222] and fuzzy co-implications [223, 224], among others, are also known.

5.2 Uncertain equivalence operator

An uncertain equivalence operator evaluates uncertain similarity or logical biconditionality, measuring how closely uncertain inputs agree, coincide, or mutually support each other under uncertainty contexts.

Definition 5.2.1 (Fuzzy Equivalence Operator)**.** A mapping

$$E : [0,1] \times [0,1] \to [0,1]$$

is called a *fuzzy equivalence operator* if, for all $x, y, z \in [0,1]$, it satisfies:

1. **Symmetry:**
$$E(x,y) = E(y,x).$$
2. **Compatibility with classical equivalence:**
$$E(0,1) = E(1,0) = 0, \qquad E(0,0) = E(1,1) = 1.$$
3. **Reflexivity:**
$$E(x,x) = 1.$$
4. **Associativity:**
$$E\big(x, E(y,z)\big) = E\big(E(x,y), z\big).$$
5. **Neutrality principle:**
$$E(1,x) = x.$$

Example 5.2.2 (An algebraic example of a fuzzy equivalence operator)**.** Since

$$|[0,1]| = \big|\{0,1\}^{\mathbb{N}}\big|,$$

there exists a bijection

$$\varphi : [0,1] \longrightarrow \{0,1\}^{\mathbb{N}}$$

such that

$$\varphi(1) = \mathbf{0} \qquad \text{and} \qquad \varphi(0) = \mathbf{e},$$

where $\mathbf{0} = (0,0,0,\dots)$ and $\mathbf{e} = (1,0,0,\dots)$.

Define

$$E : [0,1] \times [0,1] \to [0,1]$$

by

$$E(x,y) := \varphi^{-1}\big(\varphi(x) \oplus \varphi(y)\big),$$

where $\oplus$ denotes coordinatewise addition modulo 2.

Then E is a fuzzy equivalence operator in the sense of Definition **??**. Indeed:

1. *Symmetry:*
$$E(x,y) = \varphi^{-1}\big(\varphi(x) \oplus \varphi(y)\big) = \varphi^{-1}\big(\varphi(y) \oplus \varphi(x)\big) = E(y,x).$$

2. *Compatibility with classical equivalence:*

$$E(0,1) = \varphi^{-1}(\mathbf{e} \oplus \mathbf{0}) = \varphi^{-1}(\mathbf{e}) = 0,$$

$$E(1,0) = \varphi^{-1}(\mathbf{0} \oplus \mathbf{e}) = 0,$$

$$E(0,0) = \varphi^{-1}(\mathbf{e} \oplus \mathbf{e}) = \varphi^{-1}(\mathbf{0}) = 1,$$

$$E(1,1) = \varphi^{-1}(\mathbf{0} \oplus \mathbf{0}) = 1.$$

3. *Reflexivity:* for every $x \in [0,1]$,

$$E(x,x) = \varphi^{-1}\big(\varphi(x) \oplus \varphi(x)\big) = \varphi^{-1}(\mathbf{0}) = 1.$$

4. *Associativity:* for all $x, y, z \in [0,1]$,

$$E\big(x, E(y,z)\big) = \varphi^{-1}\Big(\varphi(x) \oplus \big(\varphi(y) \oplus \varphi(z)\big)\Big),$$

$$E\big(E(x,y), z\big) = \varphi^{-1}\Big(\big(\varphi(x) \oplus \varphi(y)\big) \oplus \varphi(z)\Big).$$

Since $\oplus$ is associative on $\{0,1\}^{\mathbb{N}}$, these two values are equal.

5. *Neutrality principle:* for every $x \in [0,1]$,

$$E(1,x) = \varphi^{-1}\big(\mathbf{0} \oplus \varphi(x)\big) = \varphi^{-1}\big(\varphi(x)\big) = x.$$

Therefore, E satisfies all the required axioms.

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

equipped with a partial order $\preceq_M$ and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which play the roles of the bottom and top uncertainty values, respectively.

Definition 5.2.3 (M-equivalence operator)**.** A mapping

$$E_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \longrightarrow \mathrm{Dom}(M)$$

is called an *M-equivalence operator* if, for all $a, b \in \mathrm{Dom}(M)$, the following conditions hold:

(i) **Closure:**

$$E_M(a,b) \in \mathrm{Dom}(M).$$

(ii) **Symmetry:**

$$E_M(a,b) = E_M(b,a).$$

(iii) **Reflexivity:**

$$E_M(a,a) = 1_M.$$

(iv) **Compatibility with the extreme values:**

$$E_M(0_M, 1_M) = E_M(1_M, 0_M) = 0_M, \qquad E_M(0_M, 0_M) = E_M(1_M, 1_M) = 1_M.$$

Definition 5.2.4 (Uncertain equivalence of U-sets)**.** Let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}) \qquad \text{and} \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M on a common nonempty universe X, and let

$$E_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-equivalence operator.

The *uncertain equivalence* of $\mathcal{U}$ and $\mathcal{V}$ induced by E_M, denoted by

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V},$$

is defined by

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V} := \big(X, \mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}}\big),$$

where

$$\mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}}(x) = E_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 5.2.5 (Well-definedness of the uncertain equivalence of U-sets)**.** *Let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}) \qquad \textit{and} \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of type M on a common nonempty set X, and let

$$E_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

be an M-equivalence operator. Then

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}$$

is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has

$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Fix $x \in X$. By the closure property of the M-equivalence operator,

$$E_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the assignment

$$x \longmapsto E_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$

defines a mapping

$$\mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V} = \big(X, \mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}}\big)$$

is an uncertain set of type M on X. Thus the uncertain equivalence of U-sets is well-defined. □

Proposition 5.2.6 (Inherited properties)**.** *Let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}}), \qquad \mathcal{W} = (X, \mu_{\mathcal{W}})$$

be uncertain sets of type M on X. Then the following statements hold.

(i) ***Symmetry:***

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V} = \mathcal{V} \Leftrightarrow_{E_M} \mathcal{U}.$$

(ii) ***Reflexivity:*** *if $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ denotes the constant top U-set given by*

$$\mu_{\mathbf{1}_M}(x) = 1_M \qquad (x \in X),$$

then

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{U} = \mathbf{1}_M.$$

(iii) ***Extreme-value compatibility:*** *if $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ and $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ are the constant bottom and top U-sets on X, then*

$$\mathbf{0}_M \Leftrightarrow_{E_M} \mathbf{1}_M = \mathbf{1}_M \Leftrightarrow_{E_M} \mathbf{0}_M = \mathbf{0}_M.$$

Proof. (i) For every $x \in X$, by symmetry of E_M,

$$\mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V}}(x) = E_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) = E_M\big(\mu_{\mathcal{V}}(x), \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{V} \Leftrightarrow_{E_M} \mathcal{U}}(x).$$

Hence

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{V} = \mathcal{V} \Leftrightarrow_{E_M} \mathcal{U}.$$

(ii) For every $x \in X$, reflexivity of E_M gives

$$\mu_{\mathcal{U} \Leftrightarrow_{E_M} \mathcal{U}}(x) = E_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{U}}(x)\big) = 1_M = \mu_{\mathbf{1}_M}(x).$$

Therefore

$$\mathcal{U} \Leftrightarrow_{E_M} \mathcal{U} = \mathbf{1}_M.$$

(iii) For every $x \in X$, compatibility of E_M with the extreme values yields

$$\mu_{\mathbf{0}_M \Leftrightarrow_{E_M} \mathbf{1}_M}(x) = E_M(0_M, 1_M) = 0_M = \mu_{\mathbf{0}_M}(x),$$

and similarly,

$$\mu_{\mathbf{1}_M \Leftrightarrow_{E_M} \mathbf{0}_M}(x) = E_M(1_M, 0_M) = 0_M = \mu_{\mathbf{0}_M}(x).$$

Hence

$$\mathbf{0}_M \Leftrightarrow_{E_M} \mathbf{1}_M = \mathbf{1}_M \Leftrightarrow_{E_M} \mathbf{0}_M = \mathbf{0}_M.$$

□

As a reference, a catalogue of representative equivalence operators classified by the dimension k of the degree-domain is presented in Table 5.2.

For reference, a concise comparison between implication and equivalence operators is given in Table 5.3.

Table 5.2: A catalogue of representative equivalence operators by the dimension k of the degree-domain.

k	note	Representative equivalence operator(s)
1		Fuzzy equivalence operator [225, 226]: $E : [0,1]^2 \to [0,1]$.
2		Intuitionistic Fuzzy equivalence operator: componentwise equivalence on (μ, ν).
3		Neutrosophic equivalence operator: componentwise equivalence on (T, I, F).
n	$(n \geq 1)$	Plithogenic equivalence operator: contradiction-aware equivalence on $[0,1]^n$.

Reading guide. The table groups representative equivalence operators by the dimension k of their degree values.

Table 5.3: A concise comparison between implication and equivalence operators

Aspect	**Implication operator**	**Equivalence operator**
Basic role	Models conditional reasoning of the form “if a, then b”	Models mutual agreement or logical similarity between a and b
Arity	Binary	Binary
Symmetry	Generally not symmetric	Usually symmetric
Monotonicity	Typically antitone in the first argument and monotone in the second argument	Typically monotone with respect to both arguments under the underlying order
Reflexive behavior	Not necessarily satisfies $I(a,a) = 1$	Usually satisfies $E(a,a) = 1$
Typical boundary conditions	Commonly $$I(0,b) = 1, \quad I(a,1) = 1, \quad I(1,0) = 0$$	Commonly $$E(0,0) = 1, \quad E(1,1) = 1, \quad E(0,1) = E(1,0) = 0$$
Logical interpretation	Expresses directed entailment from the first input to the second	Expresses bidirectional compatibility or sameness of truth status
Relation between them	Serves as a primitive inferential connective in many logical systems	Is often derived from implication-type connectives, for example by combining two opposite implications

Chapter 6

Uncertain Aggregation Families

In this chapter, we discuss uncertain aggregation families. For convenience, the aggregation families considered in this chapter are grouped into three broad categories, as summarized in Table 6.1.

Table 6.1: A concise grouping of the aggregation families covered in this chapter

Group	**Representative families**
General foundations	Aggregation operator, Mean
Interaction-aware means	Bonferroni mean, Heronian mean, Maclaurin symmetric mean
Order/weight/parametric families	Dombi operators, Power Average, OWA, Einstein operators, Geometric operators

In addition, a concise comparison of representative aggregation families is provided in Table 6.2.

6.1 Uncertain aggregation operator

Aggregation operators are functions that combine multiple input values into one representative output, preserving chosen structural properties such as monotonicity, boundedness, idempotency, or compensatory behavior [227–229]. An uncertain aggregation operator combines multiple uncertain values into one summary assessment, preserving chosen structural principles such as monotonicity, idempotency, compensation, or weighting schemes consistently.

Definition 6.1.1 (Fuzzy Aggregation Operator)**.** [230, 231] Let $n \geq 2$ be an integer. A mapping

$$A : [0,1]^n \to [0,1]$$

is called a *fuzzy aggregation operator* (or simply an *aggregation operator* on $[0,1]$) if it satisfies the following conditions:

Table 6.2: A concise comparison of representative aggregation families

Family	**Order-dependent**	**Uses weights**	**Interaction**	**Penalty for low values**	**Typical application**
Mean	No	Optional	No	Moderate	Basic averaging and baseline aggregation
Bonferroni mean	No	Optional	Yes	Moderate to strong	Modeling pairwise interaction among inputs
Heronian mean	No	Optional	Yes	Moderate	Interaction-aware averaging with balanced compensation
Maclaurin symmetric mean	No	Optional	Yes	Moderate	Capturing multi-input interaction symmetrically
Dombi operators	No	Usually no	Limited	Adjustable by parameter	Flexible parametric aggregation
Power Average	No	Yes	Yes	Data-dependent	Support-based adaptive aggregation
OWA	Yes	Yes	No	Adjustable by ordered weights	Ordered aggregation under attitudinal preference
Einstein operators	No	Optional	Limited	Moderate	Smooth nonlinear aggregation
Geometric operators	No	Yes	No	Strong	Multiplicative aggregation sensitive to small values

This table is intended as a quick comparison map. The entries indicate typical tendencies of the corresponding families, and specific variants may exhibit different properties.

1. **Boundary conditions:**

$$A(0,\dots,0)=0, \qquad A(1,\dots,1)=1.$$

2. **Monotonicity:** for all

$$(x_1,\dots,x_n),(y_1,\dots,y_n)\in[0,1]^n,$$

if

$$x_i\le y_i \qquad \text{for all } i=1,\dots,n,$$

then

$$A(x_1,\dots,x_n)\le A(y_1,\dots,y_n).$$

Definition 6.1.2 (Neutrosophic Aggregation Operator)**.** Let $n\ge 2$. A mapping

$$\mathcal{A}:\mathbb{N}_{SV}^n\to\mathbb{N}_{SV}$$

is called a *neutrosophic aggregation operator* if it satisfies the following conditions:

1. **Boundary conditions:**

$$\mathcal{A}(0_N,\dots,0_N)=0_N, \qquad \mathcal{A}(1_N,\dots,1_N)=1_N.$$

2. **Monotonicity:** for any

$$a_j, b_j \in \mathbb{N}_{SV} \qquad (j = 1, \dots, n),$$

if

$$a_j \preceq b_j \qquad \text{for all } j = 1, \dots, n,$$

then

$$\mathcal{A}(a_1, \dots, a_n) \preceq \mathcal{A}(b_1, \dots, b_n).$$

Remark 6.1.3. If, in addition,

$$\mathcal{A}(a, \dots, a) = a \qquad \text{for all } a \in \mathbb{N}_{SV},$$

then $\mathcal{A}$ is called an *idempotent neutrosophic aggregation operator*.

Example 6.1.4 (Neutrosophic arithmetic aggregation operator)**.** Assume that

$$\mathbb{N}_{SV} = \left\{(T, I, F) \in [0,1]^3\right\},$$

with

$$0_N = (0, 1, 1), \qquad 1_N = (1, 0, 0),$$

and let the order $\preceq$ on $\mathbb{N}_{SV}$ be defined by

$$(T_1, I_1, F_1) \preceq (T_2, I_2, F_2) \quad \Longleftrightarrow \quad T_1 \le T_2, \quad I_1 \ge I_2, \quad F_1 \ge F_2.$$

For $n \ge 2$, define

$$\mathcal{A}_{\text{avg}} : \mathbb{N}_{SV}^n \to \mathbb{N}_{SV}$$

by

$$\mathcal{A}_{\text{avg}}\big((T_1, I_1, F_1), \dots, (T_n, I_n, F_n)\big) := \left(\frac{1}{n}\sum_{j=1}^{n} T_j,\ \frac{1}{n}\sum_{j=1}^{n} I_j,\ \frac{1}{n}\sum_{j=1}^{n} F_j\right).$$

Then $\mathcal{A}_{\text{avg}}$ is a neutrosophic aggregation operator.

Indeed, the boundary conditions hold:

$$\mathcal{A}_{\text{avg}}(0_N, \dots, 0_N) = \left(\frac{1}{n}\sum_{j=1}^{n} 0,\ \frac{1}{n}\sum_{j=1}^{n} 1,\ \frac{1}{n}\sum_{j=1}^{n} 1\right) = (0, 1, 1) = 0_N,$$

and

$$\mathcal{A}_{\text{avg}}(1_N, \dots, 1_N) = \left(\frac{1}{n}\sum_{j=1}^{n} 1,\ \frac{1}{n}\sum_{j=1}^{n} 0,\ \frac{1}{n}\sum_{j=1}^{n} 0\right) = (1, 0, 0) = 1_N.$$

Moreover, let

$$a_j = (T_j, I_j, F_j), \qquad b_j = (T'_j, I'_j, F'_j) \in \mathbb{N}_{SV} \quad (j = 1, \dots, n),$$

and suppose that

$$a_j \preceq b_j \qquad \text{for all } j = 1, \dots, n.$$

Then

$$T_j \le T'_j, \qquad I_j \ge I'_j, \qquad F_j \ge F'_j \qquad (j = 1, \dots, n).$$

Hence,

$$\frac{1}{n}\sum_{j=1}^{n} T_j \le \frac{1}{n}\sum_{j=1}^{n} T'_j, \qquad \frac{1}{n}\sum_{j=1}^{n} I_j \ge \frac{1}{n}\sum_{j=1}^{n} I'_j, \qquad \frac{1}{n}\sum_{j=1}^{n} F_j \ge \frac{1}{n}\sum_{j=1}^{n} F'_j.$$

Therefore,

$$\mathcal{A}_{\text{avg}}(a_1, \dots, a_n) \preceq \mathcal{A}_{\text{avg}}(b_1, \dots, b_n).$$

So $\mathcal{A}_{\text{avg}}$ is monotone and thus is a neutrosophic aggregation operator.

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

equipped with a partial order $\preceq_M$ and distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M),$$

which play the roles of the bottom and top uncertainty values, respectively.

Definition 6.1.5 (M-aggregation operator)**.** Let $n \ge 2$. A mapping

$$A_M : \mathrm{Dom}(M)^n \longrightarrow \mathrm{Dom}(M)$$

is called an *n-ary M-aggregation operator* if, for all

$$a_1, \dots, a_n,\ b_1, \dots, b_n \in \mathrm{Dom}(M),$$

the following conditions hold:

(i) **Closure:**

$$A_M(a_1, \dots, a_n) \in \mathrm{Dom}(M).$$

(ii) **Boundary conditions:**

$$A_M(0_M, \dots, 0_M) = 0_M, \qquad A_M(1_M, \dots, 1_M) = 1_M.$$

(iii) **Monotonicity:** if

$$a_i \preceq_M b_i \qquad \text{for all } i = 1, \dots, n,$$

then

$$A_M(a_1, \dots, a_n) \preceq_M A_M(b_1, \dots, b_n).$$

If, in addition,

$$A_M(a, \dots, a) = a \qquad \text{for all } a \in \mathrm{Dom}(M),$$

then A_M is called *idempotent*.

Definition 6.1.6 (Uncertain aggregation of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of the same type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Let

$$A_M : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

be an n-ary M-aggregation operator.

The *uncertain aggregation* of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ induced by A_M, denoted by

$$A_M(\mathcal{U}_1, \ldots, \mathcal{U}_n),$$

is defined by

$$A_M(\mathcal{U}_1, \ldots, \mathcal{U}_n) := \big(X, \mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)}\big),$$

where

$$\mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) = A_M\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.1.7 (Well-definedness of uncertain aggregation)**.** *Let X be a nonempty set, let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M, and let

$$A_M : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

be an n-ary M-aggregation operator. Then

$$A_M(\mathcal{U}_1, \ldots, \mathcal{U}_n)$$

is a well-defined uncertain set of type M on X.

Proof. Since each $\mathcal{U}_i$ is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X \text{ and all } i = 1, \ldots, n.$$

Fix any $x \in X$. Then

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the closure property of A_M,

$$A_M\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto A_M\big(\mu_1(x), \ldots, \mu_n(x)\big)$$

defines a mapping

$$\mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M).$$

Consequently,

$$A_M(\mathcal{U}_1, \ldots, \mathcal{U}_n) = \big(X, \mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)}\big)$$

is an uncertain set of type M on X. Thus the uncertain aggregation is well-defined. □

Proposition 6.1.8 (Inherited properties)**.** *Let*

$$\mathcal{U}_i = (X, \mu_i), \qquad \mathcal{V}_i = (X, \nu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M on X. Then the following statements hold.

(i) ***Monotonicity.*** *If*

$$\mu_i(x) \preceq_M \nu_i(x) \qquad \text{for all } x \in X \text{ and all } i = 1, \ldots, n,$$

then

$$\mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) \preceq_M \mu_{A_M(\mathcal{V}_1,\ldots,\mathcal{V}_n)}(x) \qquad \text{for all } x \in X.$$

(ii) ***Boundary behavior.*** *If $\mathbf{0}_M = (X, \mu_{\mathbf{0}_M})$ and $\mathbf{1}_M = (X, \mu_{\mathbf{1}_M})$ are the constant bottom and top U-sets defined by*

$$\mu_{\mathbf{0}_M}(x) = 0_M, \qquad \mu_{\mathbf{1}_M}(x) = 1_M \qquad (x \in X),$$

then

$$A_M(\mathbf{0}_M, \ldots, \mathbf{0}_M) = \mathbf{0}_M, \qquad A_M(\mathbf{1}_M, \ldots, \mathbf{1}_M) = \mathbf{1}_M.$$

(iii) ***Idempotency.*** *If A_M is idempotent, then for every uncertain set $\mathcal{U} = (X, \mu_{\mathcal{U}})$ of type M,*

$$A_M(\mathcal{U}, \ldots, \mathcal{U}) = \mathcal{U}.$$

Proof. (i) Fix $x \in X$. By assumption,

$$\mu_i(x) \preceq_M \nu_i(x) \qquad (i = 1, \ldots, n).$$

Hence, by monotonicity of A_M,

$$A_M\big(\mu_1(x), \ldots, \mu_n(x)\big) \preceq_M A_M\big(\nu_1(x), \ldots, \nu_n(x)\big).$$

That is,

$$\mu_{A_M(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) \preceq_M \mu_{A_M(\mathcal{V}_1,\ldots,\mathcal{V}_n)}(x).$$

(ii) For every $x \in X$,

$$\mu_{A_M(\mathbf{0}_M,\ldots,\mathbf{0}_M)}(x) = A_M(0_M, \ldots, 0_M) = 0_M = \mu_{\mathbf{0}_M}(x),$$

and similarly,

$$\mu_{A_M(\mathbf{1}_M,\ldots,\mathbf{1}_M)}(x) = A_M(1_M, \ldots, 1_M) = 1_M = \mu_{\mathbf{1}_M}(x).$$

Hence

$$A_M(\mathbf{0}_M, \ldots, \mathbf{0}_M) = \mathbf{0}_M, \qquad A_M(\mathbf{1}_M, \ldots, \mathbf{1}_M) = \mathbf{1}_M.$$

(iii) Assume that A_M is idempotent. Then for every $x \in X$,

$$\mu_{A_M(\mathcal{U},\ldots,\mathcal{U})}(x) = A_M\big(\mu_{\mathcal{U}}(x), \ldots, \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{U}}(x).$$

Therefore

$$A_M(\mathcal{U}, \ldots, \mathcal{U}) = \mathcal{U}.$$

□

As a reference, a catalogue of representative aggregation operators classified by the dimension k of the degree-domain is presented in Table 6.3.

As related concepts other than the above, prioritized aggregation operators [247, 248] and hybrid aggregation operators [249, 250], among others, are also known.

Table 6.3: A catalogue of representative aggregation operators by the dimension k of the degree-domain.

k	note	Representative aggregation operator(s)
1		Fuzzy aggregation operator [232]: $A : [0,1]^n \to [0,1]$.
2		Intuitionistic Fuzzy aggregation operator [233]; Pythagorean Fuzzy aggregation operator [234]; Fermatean fuzzy aggregation operator [235]; Bipolar Fuzzy aggregation operator [236–238]
3		Picture Fuzzy aggregation operator [239, 240]; Hesitant Fuzzy aggregation operator [241, 242]; Spherical Fuzzy aggregation operator [243, 244]; Neutrosophic aggregation operator [245, 246].
n	$(n \geq 1)$	Plithogenic aggregation operator [27].

Reading guide. The table groups representative aggregation operators by the dimension k of their degree values.

6.2 Uncertain Dombi aggregation operator

A Dombi aggregation operator combines multiple membership values using Dombi's parametric t-norm or t-conorm, providing flexible, smooth, adjustable fusion controlled by a tunable interaction parameter [251, 252]. An uncertain Dombi aggregation operator fuses uncertain-valued inputs through Dombi-type parametric aggregation, preserving uncertainty information while allowing flexible, parameter-driven control of conjunctive or disjunctive behavior.

Definition 6.2.1 (Fuzzy Dombi Aggregation Operator)**.** [253–255] Let $n \geq 2$, let $w = (w_1, \ldots, w_n)$ be a weight vector with

$$w_i \in [0,1], \qquad \sum_{i=1}^{n} w_i = 1,$$

and let $k \geq 1$. The *fuzzy Dombi aggregation operator* is the mapping

$$\mathrm{FDWA}_{k,w} : [0,1]^n \to [0,1]$$

defined by

$$\mathrm{FDWA}_{k,w}(x_1, \ldots, x_n) = 1 - \frac{1}{1 + \left(\sum_{i=1}^{n} w_i \left(\frac{x_i}{1-x_i} \right)^k \right)^{1/k}},$$

for $(x_1, \ldots, x_n) \in (0,1)^n$, and extended to $[0,1]^n$ by continuity.

Definition 6.2.2 (Neutrosophic Dombi Aggregation Operators)**.** [256–258] Let

$$s_j = \langle t_j, u_j, v_j \rangle \in \mathbb{SVN} \qquad (j = 1, 2, \ldots, n),$$

let

$$w = (w_1, w_2, \ldots, w_n)$$

be a weight vector such that

$$w_j \in [0,1], \qquad \sum_{j=1}^{n} w_j = 1,$$

and let $\rho \geq 1$ be a fixed Dombi parameter.

Assume first that

$$t_j, u_j, v_j \in (0,1) \qquad (j = 1, 2, \ldots, n),$$

and extend the formulas below to boundary values by continuity.

1. The *single-valued neutrosophic Dombi weighted arithmetic average* (briefly, *SVNDWAA*) operator is defined by

$$\mathrm{SVNDWAA}_{\rho,w}(s_1,\ldots,s_n)=\bigoplus_{j=1}^{n} w_j s_j$$

with the explicit form

$$\mathrm{SVNDWAA}_{\rho,w}(s_1,\ldots,s_n)$$
$$=\left\langle 1-\frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{t_j}{1-t_j}\right)^{\rho}\right)^{1/\rho}},\ \frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{1-u_j}{u_j}\right)^{\rho}\right)^{1/\rho}},\ \frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{1-v_j}{v_j}\right)^{\rho}\right)^{1/\rho}}\right\rangle.$$

2. The *single-valued neutrosophic Dombi weighted geometric average* (briefly, *SVNDWGA*) operator is defined by

$$\mathrm{SVNDWGA}_{\rho,w}(s_1,\ldots,s_n)$$
$$=\bigotimes_{j=1}^{n} s_j^{w_j}$$

with the explicit form

$$\mathrm{SVNDWGA}_{\rho,w}(s_1,\ldots,s_n)=\left\langle \frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{1-t_j}{t_j}\right)^{\rho}\right)^{1/\rho}},\ 1-\frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{u_j}{1-u_j}\right)^{\rho}\right)^{1/\rho}},\ 1-\frac{}{1+\left(\sum\right.}\right.$$

Each of the operators

$$\mathrm{SVNDWAA}_{\rho,w},\ \mathrm{SVNDWGA}_{\rho,w}:\mathbb{SVN}^n\to\mathbb{SVN}$$

is called a *neutrosophic Dombi aggregation operator*.

Example 6.2.3 (A concrete example of the SVNDWAA operator)**.** Let

$$s_1=\left\langle\frac{1}{2},\frac{1}{3},\frac{1}{4}\right\rangle,\qquad s_2=\left\langle\frac{3}{4},\frac{2}{3},\frac{1}{2}\right\rangle$$

be two single-valued neutrosophic values, and let

$$w=\left(\frac{1}{2},\frac{1}{2}\right),\qquad \rho=1.$$

Then the single-valued neutrosophic Dombi weighted arithmetic average is

$$\mathrm{SVNDWAA}_{1,w}(s_1,s_2)=\langle T,U,V\rangle,$$

where

$$T=1-\frac{1}{1+\left(\frac{1}{2}\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)+\frac{1}{2}\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right)\right)}=1-\frac{1}{1+\left(\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 3\right)}=1-\frac{1}{3}=\frac{2}{3},$$

$$U=\frac{1}{1+\left(\frac{1}{2}\left(\frac{1-\frac{1}{3}}{\frac{1}{3}}\right)+\frac{1}{2}\left(\frac{1-\frac{2}{3}}{\frac{2}{3}}\right)\right)}=\frac{1}{1+\left(\frac{1}{2}\cdot 2+\frac{1}{2}\cdot\frac{1}{2}\right)}=\frac{1}{1+\frac{5}{4}}=\frac{4}{9},$$

and

$$V=\frac{1}{1+\left(\frac{1}{2}\left(\frac{1-\frac{1}{4}}{\frac{1}{4}}\right)+\frac{1}{2}\left(\frac{1-\frac{1}{2}}{\frac{1}{2}}\right)\right)}=\frac{1}{1+\left(\frac{1}{2}\cdot 3+\frac{1}{2}\cdot 1\right)}=\frac{1}{3}.$$

Hence

$$\mathrm{SVNDWAA}_{1,w}(s_1,s_2)=\left\langle\frac{2}{3},\frac{4}{9},\frac{1}{3}\right\rangle.$$

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

To define a Dombi-type aggregation on $\mathrm{Dom}(M)$, one must specify how elements of $\mathrm{Dom}(M)$ are represented by scalar coordinates on which Dombi aggregation is performed, and how the aggregated coordinates are converted back into admissible uncertain values.

Definition 6.2.4 (Dombi-admissible representation of an uncertain model)**.** Let $n \geq 2$, let

$$w = (w_1, \dots, w_n) \in [0,1]^n, \qquad \sum_{i=1}^{n} w_i = 1,$$

and let $\rho \geq 1$.

A *Dombi-admissible representation* of the uncertain model M consists of:

(i) a nonempty set
$$C_M \subseteq [0,1]^q$$
for some integer $q \geq 1$;

(ii) a representation map
$$\Phi_M : \mathrm{Dom}(M) \to C_M;$$

(iii) a reconstruction map
$$\Psi_M : C_M \to \mathrm{Dom}(M)$$
such that
$$\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)};$$

(iv) scalar Dombi aggregation functions
$$d_{j,\rho,w} : [0,1]^n \to [0,1] \qquad (j = 1, \dots, q),$$
such that the induced coordinatewise map
$$\mathbf{D}_{M,\rho,w} : C_M^n \to [0,1]^q$$
given by
$$\mathbf{D}_{M,\rho,w}\big(u^{(1)}, \dots, u^{(n)}\big) := \Big(d_{1,\rho,w}(u_1^{(1)}, \dots, u_1^{(n)}), \dots, d_{q,\rho,w}(u_q^{(1)}, \dots, u_q^{(n)})\Big)$$
satisfies the closure condition
$$\mathbf{D}_{M,\rho,w}\big(C_M^n\big) \subseteq C_M.$$

Definition 6.2.5 (M-Dombi aggregation operator)**.** Assume that a Dombi-admissible representation of M has been fixed. The induced *M-Dombi aggregation operator*

$$\mathfrak{D}_{M,\rho,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is defined by

$$\mathfrak{D}_{M,\rho,w}(a_1, \dots, a_n) := \Psi_M\Big(\mathbf{D}_{M,\rho,w}\big(\Phi_M(a_1), \dots, \Phi_M(a_n)\big)\Big)$$

for all

$$a_1, \dots, a_n \in \mathrm{Dom}(M).$$

Definition 6.2.6 (Uncertain Dombi aggregation of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$

be uncertain sets of the same type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$

The *uncertain Dombi aggregation* of

$$\mathcal{U}_1, \dots, \mathcal{U}_n$$

induced by $\mathfrak{D}_{M,\rho,w}$ is the uncertain set

$$\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1, \dots, \mathcal{U}_n) := \big(X, \mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1,\dots,\mathcal{U}_n)}\big),$$

where

$$\mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1,\dots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M)$$

is defined pointwise by

$$\mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1,\dots,\mathcal{U}_n)}(x) = \mathfrak{D}_{M,\rho,w}\big(\mu_1(x), \dots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.2.7 (Well-definedness of the M-Dombi aggregation operator)**.** *Under the assumptions of the previous definitions, the mapping*

$$\mathfrak{D}_{M,\rho,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Then, for each $i = 1, \dots, n$,

$$\Phi_M(a_i) \in C_M.$$

Hence

$$\big(\Phi_M(a_1), \dots, \Phi_M(a_n)\big) \in C_M^n.$$

By the closure assumption on $\mathbf{D}_{M,\rho,w}$,

$$\mathbf{D}_{M,\rho,w}\big(\Phi_M(a_1), \dots, \Phi_M(a_n)\big) \in C_M.$$

Since

$$\Psi_M : C_M \to \mathrm{Dom}(M),$$

it follows that

$$\Psi_M\Big(\mathbf{D}_{M,\rho,w}\big(\Phi_M(a_1), \dots, \Phi_M(a_n)\big)\Big) \in \mathrm{Dom}(M).$$

Therefore

$$\mathfrak{D}_{M,\rho,w}(a_1, \dots, a_n) \in \mathrm{Dom}(M)$$

for all

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Thus $\mathfrak{D}_{M,\rho,w}$ is well-defined. □

Theorem 6.2.8 (Well-definedness of uncertain Dombi aggregation of U-sets)**.** *Let X be a nonempty set, and let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M. Then

$$\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1, \ldots, \mathcal{U}_n)$$

is a well-defined uncertain set of type M on X.

Proof. Since each $\mathcal{U}_i$ is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X \text{ and all } i = 1, \ldots, n.$$

Fix $x \in X$. Then

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the preceding theorem,

$$\mathfrak{D}_{M,\rho,w}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto \mathfrak{D}_{M,\rho,w}\big(\mu_1(x), \ldots, \mu_n(x)\big)$$

defines a mapping

$$\mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = \big(X, \mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}\big)$$

is an uncertain set of type M on X. Thus the uncertain Dombi aggregation of U-sets is well-defined. □

Proposition 6.2.9 (Idempotency under coordinatewise idempotency)**.** *Assume, in addition, that for each $j = 1, \ldots, q$,*

$$d_{j,\rho,w}(t, \ldots, t) = t \qquad \textit{for all } t \in [0, 1].$$

Then

$$\mathfrak{D}_{M,\rho,w}(a, \ldots, a) = a \qquad \textit{for all } a \in \mathrm{Dom}(M).$$

Consequently, for every uncertain set $\mathcal{U} = (X, \mu_{\mathcal{U}})$ of type M,

$$\mathrm{UDA}_{M,\rho,w}(\mathcal{U}, \ldots, \mathcal{U}) = \mathcal{U}.$$

Proof. Let $a \in \mathrm{Dom}(M)$, and write

$$\Phi_M(a) = (c_1, \ldots, c_q) \in C_M.$$

By the coordinatewise idempotency assumption,

$$\mathbf{D}_{M,\rho,w}\big(\Phi_M(a), \ldots, \Phi_M(a)\big) = (c_1, \ldots, c_q) = \Phi_M(a).$$

Therefore

$$\mathfrak{D}_{M,\rho,w}(a,\dots,a) = \Psi_M\Big(\mathbf{D}_{M,\rho,w}\big(\Phi_M(a),\dots,\Phi_M(a)\big)\Big) = \Psi_M(\Phi_M(a)) = a,$$

because $\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)}$.

The U-set statement follows pointwise: for every $x \in X$,

$$\mu_{\mathrm{UDA}_{M,\rho,w}(\mathcal{U},\dots,\mathcal{U})}(x) = \mathfrak{D}_{M,\rho,w}\big(\mu_{\mathcal{U}}(x),\dots,\mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{U}}(x).$$

Hence

$$\mathrm{UDA}_{M,\rho,w}(\mathcal{U},\dots,\mathcal{U}) = \mathcal{U}.$$

□

As a reference, a catalogue of representative Dombi aggregation operators classified by the dimension k of the degree-domain is presented in Table 6.4.

Table 6.4: A catalogue of representative Dombi aggregation operators by the dimension k of the degree-domain.

k	note	Representative Dombi aggregation operator(s)
1		Fuzzy Dombi aggregation operator.
2		Intuitionistic Fuzzy Dombi aggregation operator [259, 260]; Bipolar Fuzzy Dombi aggregation operator [261]; Pythagorean Fuzzy Dombi aggregation operator [262, 263].
3		Picture Fuzzy Dombi aggregation operator [264,265]; Spherical Fuzzy Dombi aggregation operator [266,267]; Hesitant Fuzzy Dombi aggregation operator [268]; Neutrosophic Dombi aggregation operator [269–271].
n	$(n \geq 1)$	Plithogenic Dombi aggregation operator.

Reading guide. The table groups representative Dombi aggregation operators by the dimension k of their degree values.

As another related concept besides the above, Dombi prioritized aggregation operators [272] are also known.

6.3 Uncertain Power Average Operator

A power average operator aggregates inputs using support-based adaptive weights, emphasizing mutually reinforcing values while reducing the influence of isolated or weakly supported information effectively [273, 274]. An uncertain power average operator extends power averaging to uncertain information, assigning support-dependent weights under uncertainty and aggregating values while preserving adaptive collective behavior mathematically.

Definition 6.3.1 (Fuzzy Power Average Operator)**.** [275, 276] Let $\mathcal{F}_c(\mathbb{R})$ denote the class of all normal, convex fuzzy numbers on $\mathbb{R}$ with compact support. For each $A \in \mathcal{F}_c(\mathbb{R})$ and $\alpha \in [0, 1]$, write its α-cut as

$$[A]_\alpha = [A^-_\alpha, A^+_\alpha], \qquad A^-_\alpha \leq A^+_\alpha.$$

Let $n \geq 2$, and let

$$A_1, A_2, \ldots, A_n \in \mathcal{F}_c(\mathbb{R}).$$

Assume that a support function

$$\mathrm{Sup} : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0,1]$$

is given, satisfying, for all $A, B, C, D \in \mathcal{F}_c(\mathbb{R})$,

1. $\mathrm{Sup}(A, B) \in [0, 1]$;
2. $\mathrm{Sup}(A, B) = \mathrm{Sup}(B, A)$;
3. if $d(A, B) \leq d(C, D)$, then
$$\mathrm{Sup}(A, B) \geq \mathrm{Sup}(C, D),$$
where d is a prescribed metric on $\mathcal{F}_c(\mathbb{R})$.

For each $i \in \{1, \ldots, n\}$, define

$$T_i := \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Sup}(A_i, A_j), \qquad \lambda_i := \frac{1 + T_i}{\sum_{k=1}^{n}(1 + T_k)}.$$

Then $\lambda_i \geq 0$ for all i and

$$\sum_{i=1}^{n} \lambda_i = 1.$$

The *fuzzy power average operator* of $(A_1, \ldots, A_n)$ is defined by

$$\mathrm{FPA}(A_1, \ldots, A_n) := \bigoplus_{i=1}^{n} \lambda_i A_i,$$

where $\lambda_i A_i$ and $\bigoplus$ denote the usual scalar multiplication and addition of fuzzy numbers, equivalently described by α-cuts as

$$\big[\mathrm{FPA}(A_1, \ldots, A_n)\big]_\alpha = \sum_{i=1}^{n} \lambda_i [A_i]_\alpha = \left[\sum_{i=1}^{n} \lambda_i A_{i,\alpha}^{-}, \sum_{i=1}^{n} \lambda_i A_{i,\alpha}^{+}\right] \qquad (\alpha \in [0, 1]).$$

In the uncertain setting, the same idea can be formulated on the degree-domain of an uncertain model. Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

where $k \geq 1$. We assume that $\mathrm{Dom}(M)$ is nonempty and convex.

Definition 6.3.2 (Uncertain power average operator)**.** Let $n \geq 2$, and let

$$a_1, a_2, \dots, a_n \in \mathrm{Dom}(M).$$

Assume that a support function

$$\mathrm{Sup}_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to [0,1]$$

is given, satisfying, for all $a, b, c, d \in \mathrm{Dom}(M)$,

1. $\mathrm{Sup}_M(a,b) \in [0,1]$;
2. $\mathrm{Sup}_M(a,b) = \mathrm{Sup}_M(b,a)$;
3. if $\delta_M(a,b) \leq \delta_M(c,d)$, then

$$\mathrm{Sup}_M(a,b) \geq \mathrm{Sup}_M(c,d),$$

   where δ_M is a prescribed metric on $\mathrm{Dom}(M)$.

For each $i \in \{1, \dots, n\}$, define

$$T_i := \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Sup}_M(a_i, a_j), \qquad \lambda_i := \frac{1 + T_i}{\sum_{m=1}^{n}(1 + T_m)}.$$

The *uncertain power average operator* associated with M is defined by

$$\mathrm{UPA}_M(a_1, \dots, a_n) := \sum_{i=1}^{n} \lambda_i a_i.$$

Theorem 6.3.3 (Basic properties of the adaptive weights)**.** *For the coefficients* $\lambda_1, \dots, \lambda_n$ *defined above, the following hold:*

1. $\lambda_i \geq 0$ *for all* $i = 1, \dots, n$;
2. $\sum_{i=1}^{n} \lambda_i = 1$.

Proof. Since $\mathrm{Sup}_M(a_i, a_j) \in [0,1]$ for all i, j, one has

$$T_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Sup}_M(a_i, a_j) \geq 0 \qquad (i = 1, \dots, n).$$

Hence

$$1 + T_i > 0 \qquad (i = 1, \dots, n).$$

Moreover,

$$\sum_{m=1}^{n} (1 + T_m) > 0.$$

Therefore

$$\lambda_i = \frac{1 + T_i}{\sum_{m=1}^{n}(1 + T_m)} \geq 0 \qquad (i = 1, \ldots, n).$$

Also,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \frac{1 + T_i}{\sum_{m=1}^{n}(1 + T_m)} = \frac{\sum_{i=1}^{n}(1 + T_i)}{\sum_{m=1}^{n}(1 + T_m)} = 1.$$

Thus the assertions follow. □

Theorem 6.3.4 (Well-definedness of the uncertain power average operator)**.** *The mapping*

$$\mathrm{UPA}_M : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

By the previous theorem, the coefficients $\lambda_1, \ldots, \lambda_n$ satisfy

$$\lambda_i \geq 0 \qquad (i = 1, \ldots, n), \qquad \sum_{i=1}^{n} \lambda_i = 1.$$

Hence

$$\mathrm{UPA}_M(a_1, \ldots, a_n) = \sum_{i=1}^{n} \lambda_i a_i$$

is a convex combination of the points $a_1, \ldots, a_n \in \mathrm{Dom}(M)$. Since $\mathrm{Dom}(M)$ is convex, every convex combination of its elements again belongs to $\mathrm{Dom}(M)$. Therefore,

$$\mathrm{UPA}_M(a_1, \ldots, a_n) \in \mathrm{Dom}(M).$$

Thus the formula indeed defines a mapping

$$\mathrm{UPA}_M : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M),$$

so the uncertain power average operator is well-defined. □

Definition 6.3.5 (Induced uncertain power average on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

The *induced uncertain power average* of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is defined by

$$\mathrm{UPA}_M(\mathcal{U}_1, \ldots, \mathcal{U}_n) := (X, \mu_{\mathrm{UPA}}),$$

where

$$\mu_{\mathrm{UPA}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UPA}}(x) := \mathrm{UPA}_M\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.3.6 (Well-definedness of the induced uncertain power average)**.** *Let X be a nonempty set, and let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$

be uncertain sets of type M on X. Then

$$\mathrm{UPA}_M(\mathcal{U}_1, \dots, \mathcal{U}_n) = (X, \mu_{\mathrm{UPA}})$$

is well-defined.

Proof. Fix $x \in X$. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$

Hence

$$\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the well-definedness of the uncertain power average operator,

$$\mathrm{UPA}_M\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathrm{UPA}}(x) \in \mathrm{Dom}(M) \qquad (x \in X).$$

It follows that

$$\mu_{\mathrm{UPA}} : X \to \mathrm{Dom}(M)$$

is a well-defined function. Consequently,

$$\mathrm{UPA}_M(\mathcal{U}_1, \dots, \mathcal{U}_n) = (X, \mu_{\mathrm{UPA}})$$

is a well-defined uncertain set of type M. □

Remark 6.3.7. If $k = 1$ and $\mathrm{Dom}(M) = [0, 1]$, then UPA_M reduces to the ordinary scalar-valued power average on the unit interval. More generally, when $\mathrm{Dom}(M)$ is a convex degree-domain of a concrete uncertainty model (such as an interval-valued, intuitionistic, or neutrosophic degree-domain), the above construction gives the corresponding uncertain power average operator in a unified form.

6.4 Uncertain Mean Operator

A mean operator aggregates multiple inputs into a representative average, preserving central tendency, balancing contributions, and providing a basic numerical summary of collective magnitude overall. An uncertain mean operator averages uncertain values or structures into a representative result, preserving central tendency while incorporating ambiguity, variability, and incomplete information mathematically consistently.

Definition 6.4.1 (Fuzzy Mean Operator)**.** [277, 278] Let $\mathcal{F}_c(\mathbb{R})$ denote the class of all normal, convex fuzzy numbers on $\mathbb{R}$ with compact support. For each $A \in \mathcal{F}_c(\mathbb{R})$ and $\alpha \in [0, 1]$, write

$$[A]_\alpha = [A_\alpha^-, A_\alpha^+], \qquad A_\alpha^- \le A_\alpha^+,$$

for the α-cut of A.

For any integer $n \geq 1$, the *fuzzy mean operator*

$$\mathrm{FM}_n : \mathcal{F}_c(\mathbb{R})^n \to \mathcal{F}_c(\mathbb{R})$$

is defined by

$$\mathrm{FM}_n(A_1, \ldots, A_n) := \frac{1}{n} \odot (A_1 \oplus A_2 \oplus \cdots \oplus A_n),$$

where $\oplus$ and $\odot$ denote the usual addition and nonnegative scalar multiplication of fuzzy numbers induced by Zadeh's extension principle.

Equivalently, for every $\alpha \in [0, 1]$,

$$\big[\mathrm{FM}_n(A_1, \ldots, A_n)\big]_\alpha = \left[\frac{1}{n}\sum_{i=1}^{n} A^-_{i,\alpha}, \ \frac{1}{n}\sum_{i=1}^{n} A^+_{i,\alpha}\right],$$

where

$$[A_i]_\alpha = [A^-_{i,\alpha}, A^+_{i,\alpha}] \qquad (i = 1, \ldots, n).$$

Remark 6.4.2. The fuzzy mean operator is the natural arithmetic averaging operator on fuzzy numbers. If each input A_i degenerates to a crisp real number a_i, then

$$\mathrm{FM}_n(a_1, \ldots, a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i,$$

so it reduces to the ordinary arithmetic mean.

Remark 6.4.3. More generally, for a weight vector

$$w = (w_1, \ldots, w_n), \qquad w_i \geq 0, \qquad \sum_{i=1}^{n} w_i = 1,$$

the weighted fuzzy mean operator is defined by

$$\mathrm{WFM}_w(A_1, \ldots, A_n) := \bigoplus_{i=1}^{n} (w_i \odot A_i),$$

or equivalently,

$$\big[\mathrm{WFM}_w(A_1, \ldots, A_n)\big]_\alpha = \left[\sum_{i=1}^{n} w_i A^-_{i,\alpha}, \ \sum_{i=1}^{n} w_i A^+_{i,\alpha}\right].$$

The unweighted fuzzy mean operator is the special case $w_i = \frac{1}{n}$ for all i.

A mean operator aggregates multiple inputs into a representative average, preserving central tendency, balancing contributions, and providing a basic numerical summary of collective magnitude overall. In the uncertain-set framework, however, a direct arithmetic mean on

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

need not remain inside $\mathrm{Dom}(M)$. Therefore, a mathematically correct model-independent definition must be based on a representation of uncertain values in a convex coordinate domain.

Definition 6.4.4 (Mean-admissible representation of an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

A *mean-admissible representation* of M consists of the following data:

(i) a nonempty convex set
$$C_M \subseteq [0,1]^q$$
for some integer $q \geq 1$;

(ii) a representation map
$$\Phi_M : \mathrm{Dom}(M) \to C_M;$$

(iii) a reconstruction map
$$\Psi_M : C_M \to \mathrm{Dom}(M)$$
such that
$$\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)} .$$

Remark 6.4.5. The convexity of C_M ensures that coordinatewise weighted averages of represented uncertain values remain inside C_M, which is the key condition needed for mean-type aggregation.

Definition 6.4.6 (Weighted M-mean operator)**.** Let

$$w = (w_1, \ldots, w_n) \in [0,1]^n, \qquad \sum_{i=1}^{n} w_i = 1,$$

with $n \geq 1$. Assume that a mean-admissible representation

$$(C_M, \Phi_M, \Psi_M)$$

of M has been fixed.

For
$$a_1, \ldots, a_n \in \mathrm{Dom}(M),$$
write
$$\Phi_M(a_i) = u^{(i)} = (u_1^{(i)}, \ldots, u_q^{(i)}) \in C_M \qquad (i = 1, \ldots, n).$$
The *weighted M-mean operator*

$$\mathrm{WMean}_{M,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is defined by
$$\mathrm{WMean}_{M,w}(a_1, \ldots, a_n) := \Psi_M\left(\sum_{i=1}^{n} w_i \, \Phi_M(a_i)\right),$$
that is,
$$\mathrm{WMean}_{M,w}(a_1, \ldots, a_n) = \Psi_M\left(\left(\sum_{i=1}^{n} w_i u_1^{(i)}, \ldots, \sum_{i=1}^{n} w_i u_q^{(i)}\right)\right).$$

In the special case

$$w_i = \frac{1}{n} \qquad (i = 1, \ldots, n),$$

the resulting operator is called the *unweighted M-mean operator* and is denoted by

$$\mathrm{Mean}_{M,n}(a_1, \ldots, a_n).$$

Explicitly,

$$\mathrm{Mean}_{M,n}(a_1, \ldots, a_n) = \Psi_M\left(\frac{1}{n}\sum_{i=1}^{n} \Phi_M(a_i)\right).$$

Theorem 6.4.7 (Well-definedness of the weighted M-mean operator)**.** *Under the assumptions above, the mapping*

$$\mathrm{WMean}_{M,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined. Consequently,

$$\mathrm{Mean}_{M,n} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is also well-defined.

Proof. Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

Since

$$\Phi_M : \mathrm{Dom}(M) \to C_M,$$

one has

$$\Phi_M(a_i) \in C_M \qquad (i = 1, \ldots, n).$$

Because C_M is convex and

$$w_i \geq 0, \qquad \sum_{i=1}^{n} w_i = 1,$$

the convex combination

$$\sum_{i=1}^{n} w_i \, \Phi_M(a_i)$$

belongs to C_M. Since

$$\Psi_M : C_M \to \mathrm{Dom}(M),$$

it follows that

$$\Psi_M\left(\sum_{i=1}^{n} w_i \, \Phi_M(a_i)\right) \in \mathrm{Dom}(M).$$

Therefore

$$\mathrm{WMean}_{M,w}(a_1, \ldots, a_n) \in \mathrm{Dom}(M) \qquad \text{for all } (a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

Hence $\mathrm{WMean}_{M,w}$ is well-defined.

The unweighted case is obtained by taking

$$w_i = \frac{1}{n} \qquad (i = 1, \ldots, n),$$

which is again a valid weight vector. Thus $\mathrm{Mean}_{M,n}$ is well-defined as well. □

Definition 6.4.8 (Uncertain mean of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

Let

$$w = (w_1, \ldots, w_n) \in [0, 1]^n, \qquad \sum_{i=1}^{n} w_i = 1.$$

The *weighted uncertain mean* of

$$\mathcal{U}_1, \ldots, \mathcal{U}_n$$

is the uncertain set

$$\mathrm{UWMean}_{M,w}(\mathcal{U}_1, \ldots, \mathcal{U}_n) := \big(X, \mu_{\mathrm{UWMean}_{M,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}\big),$$

where

$$\mu_{\mathrm{UWMean}_{M,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M)$$

is defined pointwise by

$$\mu_{\mathrm{UWMean}_{M,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) = \mathrm{WMean}_{M,w}\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

In the special case

$$w_i = \frac{1}{n} \qquad (i = 1, \ldots, n),$$

the resulting operator is called the *uncertain mean operator* and is denoted by

$$\mathrm{UMean}_{M,n}(\mathcal{U}_1, \ldots, \mathcal{U}_n).$$

Theorem 6.4.9 (Well-definedness of the uncertain mean of U-sets)**.** *Let X be a nonempty set, let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M, and let

$$\mathrm{WMean}_{M,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

be the weighted M-mean operator associated with a mean-admissible representation of M. Then

$$\mathrm{UWMean}_{M,w}(\mathcal{U}_1, \ldots, \mathcal{U}_n)$$

is a well-defined uncertain set of type M on X. In particular,

$$\mathrm{UMean}_{M,n}(\mathcal{U}_1, \ldots, \mathcal{U}_n)$$

is well-defined.

Proof. Since each

$$\mathcal{U}_i = (X, \mu_i)$$

is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X \text{ and all } i = 1, \dots, n.$$

Fix $x \in X$. Then

$$\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the previous theorem,

$$\mathrm{WMean}_{M,w}\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathrm{UWMean}_{M,w}(\mathcal{U}_1,\dots,\mathcal{U}_n)}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto \mathrm{WMean}_{M,w}\big(\mu_1(x), \dots, \mu_n(x)\big)$$

defines a mapping

$$\mu_{\mathrm{UWMean}_{M,w}(\mathcal{U}_1,\dots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathrm{UWMean}_{M,w}(\mathcal{U}_1, \dots, \mathcal{U}_n) = \big(X, \mu_{\mathrm{UWMean}_{M,w}(\mathcal{U}_1,\dots,\mathcal{U}_n)}\big)$$

is an uncertain set of type M on X. Thus the weighted uncertain mean is well-defined.

The unweighted uncertain mean is the special case

$$w_i = \frac{1}{n} \qquad (i = 1, \dots, n),$$

so it is also well-defined. □

Proposition 6.4.10 (Basic properties)**.** *Let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$

be uncertain sets of type M on X, and let

$$w = (w_1, \dots, w_n)$$

be a weight vector.

(i) ***Idempotency:*** *for every uncertain set*

$$\mathcal{U} = (X, \mu_{\mathcal{U}}),$$

one has

$$\mathrm{UWMean}_{M,w}(\mathcal{U}, \dots, \mathcal{U}) = \mathcal{U}.$$

(ii) ***Symmetry in the unweighted case:*** *for every permutation σ of $\{1, \dots, n\}$,*

$$\mathrm{UMean}_{M,n}(\mathcal{U}_1, \dots, \mathcal{U}_n) = \mathrm{UMean}_{M,n}(\mathcal{U}_{\sigma(1)}, \dots, \mathcal{U}_{\sigma(n)}).$$

Proof. (i) For every $a \in \mathrm{Dom}(M)$,

$$\mathrm{WMean}_{M,w}(a, \ldots, a) = \Psi_M\left(\sum_{i=1}^{n} w_i\, \Phi_M(a)\right) = \Psi_M(\Phi_M(a)) = a,$$

because

$$\sum_{i=1}^{n} w_i = 1$$

and

$$\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)}\,.$$

Hence, for every $x \in X$,

$$\mu_{\mathrm{UWMean}_{M,w}(\mathcal{U},\ldots,\mathcal{U})}(x) = \mathrm{WMean}_{M,w}\big(\mu_{\mathcal{U}}(x), \ldots, \mu_{\mathcal{U}}(x)\big) = \mu_{\mathcal{U}}(x).$$

Therefore

$$\mathrm{UWMean}_{M,w}(\mathcal{U}, \ldots, \mathcal{U}) = \mathcal{U}.$$

(ii) In the unweighted case,

$$\mathrm{UMean}_{M,n}(a_1, \ldots, a_n) = \Psi_M\left(\frac{1}{n}\sum_{i=1}^{n} \Phi_M(a_i)\right).$$

Since ordinary vector addition is commutative, the value of

$$\frac{1}{n}\sum_{i=1}^{n} \Phi_M(a_i)$$

is unchanged by permuting the inputs. Hence $\mathrm{Mean}_{M,n}$, and therefore also $\mathrm{UMean}_{M,n}$, is symmetric.

□

6.5 Uncertain Bonferroni Mean Operator

A Bonferroni mean operator aggregates inputs through pairwise interaction terms, capturing interdependence among arguments and producing a representative value reflecting collective relationships quantitatively in aggregation [279, 280]. A fuzzy Bonferroni mean operator extends the Bonferroni mean to fuzzy membership degrees in $[0, 1]$, aggregating pairwise interacting values while preserving boundedness mathematically and structure [281, 282].

Definition 6.5.1 (Fuzzy Bonferroni Mean Operator)**.** [281, 282] Let

$$n \geq 2, \qquad p, q \geq 0, \qquad p + q > 0.$$

The *fuzzy Bonferroni mean operator* of parameters (p, q) is the mapping

$$\mathrm{FBM}_{p,q} : [0, 1]^n \to [0, 1]$$

defined by

$$\mathrm{FBM}_{p,q}(x_1, x_2, \ldots, x_n) := \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} x_i^p x_j^q\right)^{1/(p+q)},$$

for all

$$(x_1, \ldots, x_n) \in [0, 1]^n.$$

Remark 6.5.2. Since each $x_i \in [0,1]$, one has

$$0 \le x_i^p x_j^q \le 1 \qquad (i \neq j).$$

Hence

$$0 \le \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} x_i^p x_j^q \le 1,$$

and therefore

$$\mathrm{FBM}_{p,q}(x_1,\ldots,x_n) \in [0,1].$$

Thus $\mathrm{FBM}_{p,q}$ is well-defined on $[0,1]^n$.

Remark 6.5.3. The fuzzy Bonferroni mean captures pairwise interrelationships among the input arguments, unlike the ordinary arithmetic mean, which treats them only through direct averaging.

Definition 6.5.4 (Fuzzy Weighted Bonferroni Mean Operator)**.** Let

$$w = (w_1,\ldots,w_n) \in [0,1]^n$$

be a weight vector such that

$$\sum_{i=1}^{n} w_i = 1.$$

The *fuzzy weighted Bonferroni mean operator* is defined by

$$\mathrm{FWBM}_{p,q,w} : [0,1]^n \to [0,1],$$

$$\mathrm{FWBM}_{p,q,w}(x_1,\ldots,x_n) := \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{w_i w_j}{1-w_i}\, x_i^p x_j^q \right)^{1/(p+q)},$$

provided $w_i \neq 1$ for all $i = 1,\ldots,n$.

In the uncertain setting, this idea can be extended to multi-dimensional degree values in a componentwise manner.

Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) = [0,1]^k, \qquad k \ge 1.$$

For each

$$a = (a^{(1)},\ldots,a^{(k)}) \in [0,1]^k$$

and each real number $r \ge 0$, define

$$a^{\odot r} := \big((a^{(1)})^r,\ldots,(a^{(k)})^r\big).$$

For

$$a = (a^{(1)},\ldots,a^{(k)}), \quad b = (b^{(1)},\ldots,b^{(k)}) \in [0,1]^k,$$

define their componentwise product by

$$a \odot b := \big(a^{(1)}b^{(1)},\ldots,a^{(k)}b^{(k)}\big).$$

Also, for $s > 0$ and $c = (c^{(1)},\ldots,c^{(k)}) \in [0,1]^k$, define

$$c^{\odot 1/s} := \big((c^{(1)})^{1/s},\ldots,(c^{(k)})^{1/s}\big).$$

Definition 6.5.5 (Uncertain Bonferroni mean operator)**.** Let

$$n \geq 2, \qquad p, q \geq 0, \qquad p + q > 0.$$

The *uncertain Bonferroni mean operator* of parameters (p, q) on M is the mapping

$$\mathrm{UBM}_{M;p,q} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UBM}_{M;p,q}(a_1, \dots, a_n) := \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^{\odot p} \odot a_j^{\odot q} \right)^{\odot 1/(p+q)}$$

for all

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Theorem 6.5.6 (Well-definedness of the uncertain Bonferroni mean operator)**.** *The operator*

$$\mathrm{UBM}_{M;p,q} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Take arbitrary

$$a_1, \dots, a_n \in \mathrm{Dom}(M) = [0,1]^k.$$

Write

$$a_i = \left(a_i^{(1)}, \dots, a_i^{(k)}\right) \qquad (i = 1, \dots, n),$$

where

$$0 \leq a_i^{(\ell)} \leq 1 \qquad (\ell = 1, \dots, k).$$

Fix $\ell \in \{1, \dots, k\}$. Since $p, q \geq 0$, one has

$$0 \leq \left(a_i^{(\ell)}\right)^p \left(a_j^{(\ell)}\right)^q \leq 1 \qquad (i \neq j).$$

Hence each ℓ-th component of

$$a_i^{\odot p} \odot a_j^{\odot q}$$

belongs to $[0, 1]$, and therefore

$$a_i^{\odot p} \odot a_j^{\odot q} \in [0,1]^k \qquad (i \neq j).$$

Consequently, for each ℓ,

$$0 \leq \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(a_i^{(\ell)}\right)^p \left(a_j^{(\ell)}\right)^q \leq 1.$$

Thus

$$\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^{\odot p} \odot a_j^{\odot q} \in [0,1]^k.$$

Since $p + q > 0$, the exponent $1/(p + q)$ is well-defined and nonnegative, so taking the componentwise $1/(p + q)$-power again yields a vector in $[0, 1]^k$. Therefore,

$$\mathrm{UBM}_{M;p,q}(a_1, \dots, a_n) \in [0,1]^k = \mathrm{Dom}(M).$$

Hence $\mathrm{UBM}_{M;p,q}$ is well-defined. □

Remark 6.5.7. When $k = 1$, the above definition reduces to the ordinary scalar-valued fuzzy Bonferroni mean operator on $[0, 1]$. Thus the uncertain Bonferroni mean operator is a natural multi-dimensional extension of the classical fuzzy Bonferroni mean.

Definition 6.5.8 (Uncertain weighted Bonferroni mean operator)**.** Let

$$w = (w_1, \dots, w_n) \in [0, 1]^n$$

be a weight vector such that

$$\sum_{i=1}^{n} w_i = 1$$

and

$$w_i \neq 1 \qquad (i = 1, \dots, n).$$

The *uncertain weighted Bonferroni mean operator* of parameters (p, q) with respect to w is the mapping

$$\mathrm{UWBM}_{M;p,q,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UWBM}_{M;p,q,w}(a_1, \dots, a_n) := \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{w_i w_j}{1 - w_i} \, a_i^{\odot p} \odot a_j^{\odot q} \right)^{\odot 1/(p+q)}$$

for all

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Lemma 6.5.9 (Basic properties of the weighted coefficients)**.** *For*

$$\eta_{ij} := \frac{w_i w_j}{1 - w_i} \qquad (i \neq j),$$

the following hold:

1. $\eta_{ij} \geq 0$ *for all* $i \neq j$*;*
2. $$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \eta_{ij} = 1.$$

Proof. Since $w_i, w_j \in [0, 1]$ and $w_i \neq 1$, one has $1 - w_i > 0$. Hence

$$\eta_{ij} = \frac{w_i w_j}{1 - w_i} \geq 0 \qquad (i \neq j).$$

Moreover,

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \eta_{ij} = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{w_i w_j}{1 - w_i} = \sum_{i=1}^{n} \frac{w_i}{1 - w_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j.$$

Since
$$\sum_{\substack{j=1\\j\neq i}}^{n} w_j = 1 - w_i,$$
it follows that
$$\sum_{\substack{i,j=1\\i\neq j}}^{n} \eta_{ij} = \sum_{i=1}^{n} \frac{w_i}{1-w_i}(1-w_i) = \sum_{i=1}^{n} w_i = 1.$$
This proves the result. □

Theorem 6.5.10 (Well-definedness of the uncertain weighted Bonferroni mean operator)**.** *The operator*
$$\mathrm{UWBM}_{M;p,q,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$
is well-defined.

Proof. Take arbitrary
$$a_1, \dots, a_n \in \mathrm{Dom}(M) = [0,1]^k.$$
As in the proof of the previous theorem,
$$a_i^{\odot p} \odot a_j^{\odot q} \in [0,1]^k \qquad (i \neq j).$$
For each $\ell \in \{1, \dots, k\}$, the ℓ-th component of
$$a_i^{\odot p} \odot a_j^{\odot q}$$
lies in $[0,1]$. By the lemma, the coefficients
$$\eta_{ij} = \frac{w_i w_j}{1-w_i} \qquad (i \neq j)$$
are nonnegative and sum to 1. Therefore, for each ℓ,
$$0 \leq \sum_{\substack{i,j=1\\i\neq j}}^{n} \eta_{ij} \big(a_i^{(\ell)}\big)^p \big(a_j^{(\ell)}\big)^q \leq 1.$$
Hence
$$\sum_{\substack{i,j=1\\i\neq j}}^{n} \frac{w_i w_j}{1-w_i}\, a_i^{\odot p} \odot a_j^{\odot q} \in [0,1]^k.$$
Since $p+q>0$, taking the componentwise $1/(p+q)$-power is well-defined and again produces an element of $[0,1]^k$. Therefore,
$$\mathrm{UWBM}_{M;p,q,w}(a_1, \dots, a_n) \in [0,1]^k = \mathrm{Dom}(M).$$
Thus $\mathrm{UWBM}_{M;p,q,w}$ is well-defined. □

Definition 6.5.11 (Induced uncertain Bonferroni mean on uncertain sets)**.** Let X be a nonempty universe, and let
$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$

be uncertain sets of type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

The *induced uncertain Bonferroni mean* of $\mathcal{U}_1, \dots, \mathcal{U}_n$ is defined by

$$\mathrm{UBM}_{M;p,q}(\mathcal{U}_1, \dots, \mathcal{U}_n) := (X, \mu_{\mathrm{UBM}}),$$

where

$$\mu_{\mathrm{UBM}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UBM}}(x) := \mathrm{UBM}_{M;p,q}\big(\mu_1(x), \dots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.5.12 (Well-definedness of the induced uncertain Bonferroni mean)**.** *The pair*

$$\mathrm{UBM}_{M;p,q}(\mathcal{U}_1, \dots, \mathcal{U}_n) = (X, \mu_{\mathrm{UBM}})$$

is a well-defined uncertain set of type M.

Proof. Fix $x \in X$. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M,

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$

Hence

$$\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the well-definedness of $\mathrm{UBM}_{M;p,q}$,

$$\mathrm{UBM}_{M;p,q}\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathrm{UBM}}(x) \in \mathrm{Dom}(M) \qquad (x \in X),$$

so

$$\mu_{\mathrm{UBM}} : X \to \mathrm{Dom}(M)$$

is a well-defined function. Thus

$$(X, \mu_{\mathrm{UBM}})$$

is a well-defined uncertain set of type M. □

Definition 6.5.13 (Induced uncertain weighted Bonferroni mean on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$

be uncertain sets of type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

The *induced uncertain weighted Bonferroni mean* of $\mathcal{U}_1, \dots, \mathcal{U}_n$ is defined by

$$\mathrm{UWBM}_{M;p,q,w}(\mathcal{U}_1, \dots, \mathcal{U}_n) := (X, \mu_{\mathrm{UWBM}}),$$

where

$$\mu_{\mathrm{UWBM}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UWBM}}(x) := \mathrm{UWBM}_{M;p,q,w}\big(\mu_1(x), \dots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.5.14 (Well-definedness of the induced uncertain weighted Bonferroni mean)**.** *The pair*

$$\mathrm{UWBM}_{M;p,q,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n) = (X, \mu_{\mathrm{UWBM}})$$

is a well-defined uncertain set of type M.

Proof. Fix $x \in X$. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M,

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Hence

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the well-definedness of $\mathrm{UWBM}_{M;p,q,w}$,

$$\mathrm{UWBM}_{M;p,q,w}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathrm{UWBM}}(x) \in \mathrm{Dom}(M) \qquad (x \in X),$$

and so

$$\mu_{\mathrm{UWBM}} : X \to \mathrm{Dom}(M)$$

is well-defined. Consequently,

$$(X, \mu_{\mathrm{UWBM}})$$

is a well-defined uncertain set of type M. □

Table 6.5 presents a catalogue of representative Bonferroni mean operators classified by the dimension k of the degree-domain.

Table 6.5: A catalogue of representative Bonferroni mean operators by the dimension k of the degree-domain.

k	note	Representative Bonferroni mean operator(s)
1		Fuzzy Bonferroni mean operator.
2		Intuitionistic Fuzzy Bonferroni mean operators [260, 283]; Pythagorean Fuzzy Bonferroni mean operators [284, 285]
3		Picture Fuzzy Bonferroni mean operators [286, 287]; Spherical Fuzzy Bonferroni mean operators [288, 289]; Hesitant Fuzzy Bonferroni mean operators [290, 291]; Neutrosophic Bonferroni mean operators [292, 293].
n	$(n \geq 1)$	Plithogenic Bonferroni mean operators.

Reading guide. The table groups representative Bonferroni mean operators by the dimension k of their degree values.

6.6 Uncertain Heronian Mean Operator

A Heronian mean operator aggregates inputs using pairwise interaction terms, reflecting mutual relationships among arguments and providing a mean between arithmetic and geometric behavior [294, 295]. A fuzzy Heronian mean operator extends the Heronian mean to membership degrees in $[0, 1]$, aggregating pairwise interacting fuzzy values while preserving boundedness mathematically [296, 297].

Definition 6.6.1 (Fuzzy Heronian Mean Operator)**.** (cf. [296, 297]) Let

$$n \geq 2, \qquad p, q \geq 0, \qquad p + q > 0.$$

The *fuzzy Heronian mean operator* of parameters (p, q) is the mapping

$$\mathrm{FHM}_{p,q} : [0,1]^n \to [0,1]$$

defined by

$$\mathrm{FHM}_{p,q}(x_1, x_2, \ldots, x_n) := \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i^p x_j^q \right)^{1/(p+q)},$$

for all

$$(x_1, \ldots, x_n) \in [0,1]^n.$$

Remark 6.6.2. If

$$p = q = 1,$$

then $\mathrm{FHM}_{p,q}$ reduces to the classical Heronian mean on $[0,1]^n$:

$$\mathrm{FHM}_{1,1}(x_1, \ldots, x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i x_j \right)^{1/2}.$$

Remark 6.6.3. Since

$$0 \leq x_i^p x_j^q \leq 1 \qquad (i = 1, \ldots, n,\ j = i, \ldots, n),$$

one has

$$0 \leq \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i^p x_j^q \leq 1.$$

Therefore

$$\mathrm{FHM}_{p,q}(x_1, \ldots, x_n) \in [0,1],$$

so the operator is well-defined.

Definition 6.6.4 (Fuzzy Weighted Heronian Mean Operator)**.** Let

$$w = (w_1, \ldots, w_n) \in [0,1]^n$$

be a weight vector such that

$$\sum_{i=1}^{n} w_i = 1.$$

For parameters

$$p, q \geq 0, \qquad p + q > 0,$$

the *fuzzy weighted Heronian mean operator* is defined by

$$\mathrm{FWHM}_{p,q,w} : [0,1]^n \to [0,1],$$

$$\mathrm{FWHM}_{p,q,w}(x_1, \ldots, x_n) := \left(\sum_{i=1}^{n} \sum_{j=i}^{n} \omega_{ij}\, x_i^p x_j^q \right)^{1/(p+q)},$$

where the coefficients $\omega_{ij} \geq 0$ satisfy

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_{ij} = 1.$$

A common symmetric choice is

$$\omega_{ij} = \begin{cases} w_i w_j, & i < j, \\ w_i^2, & i = j. \end{cases}$$

In the uncertain-set setting, this idea can be extended to multi-dimensional degree values in a componentwise manner.

Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) = [0,1]^k, \qquad k \geq 1.$$

For each

$$a = (a^{(1)}, \dots, a^{(k)}) \in [0,1]^k$$

and each real number $r \geq 0$, define

$$a^{\odot r} := \big((a^{(1)})^r, \dots, (a^{(k)})^r\big).$$

For

$$a = (a^{(1)}, \dots, a^{(k)}), \quad b = (b^{(1)}, \dots, b^{(k)}) \in [0,1]^k,$$

define their componentwise product by

$$a \odot b := \big(a^{(1)}b^{(1)}, \dots, a^{(k)}b^{(k)}\big).$$

Also, for $s > 0$ and $c = (c^{(1)}, \dots, c^{(k)}) \in [0,1]^k$, define

$$c^{\odot 1/s} := \big((c^{(1)})^{1/s}, \dots, (c^{(k)})^{1/s}\big).$$

Definition 6.6.5 (Uncertain Heronian mean operator)**.** Let

$$n \geq 2, \qquad p, q \geq 0, \qquad p + q > 0.$$

The *uncertain Heronian mean operator* of parameters (p, q) on M is the mapping

$$\mathrm{UHM}_{M;p,q} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UHM}_{M;p,q}(a_1, \dots, a_n) := \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n} a_i^{\odot p} \odot a_j^{\odot q}\right)^{\odot 1/(p+q)}$$

for all

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Remark 6.6.6. If
$$p = q = 1,$$
then $\mathrm{UHM}_{M;p,q}$ reduces componentwise to the classical Heronian-type mean formula:
$$\mathrm{UHM}_{M;1,1}(a_1, \ldots, a_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} a_i \odot a_j \right)^{\odot 1/2}.$$
When $k = 1$, this coincides with the ordinary scalar-valued Heronian mean on $[0, 1]$.

Theorem 6.6.7 (Well-definedness of the uncertain Heronian mean operator)**.** *The operator*
$$\mathrm{UHM}_{M;p,q} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$
is well-defined.

Proof. Take arbitrary
$$a_1, \ldots, a_n \in \mathrm{Dom}(M) = [0, 1]^k.$$
Write
$$a_i = \big(a_i^{(1)}, \ldots, a_i^{(k)}\big) \qquad (i = 1, \ldots, n),$$
where
$$0 \le a_i^{(\ell)} \le 1 \qquad (\ell = 1, \ldots, k).$$

Fix $\ell \in \{1, \ldots, k\}$. Since $p, q \ge 0$, one has
$$0 \le \big(a_i^{(\ell)}\big)^p \big(a_j^{(\ell)}\big)^q \le 1 \qquad (1 \le i \le j \le n).$$
Hence each ℓ-th component of
$$a_i^{\odot p} \odot a_j^{\odot q}$$
belongs to $[0, 1]$, so
$$a_i^{\odot p} \odot a_j^{\odot q} \in [0, 1]^k \qquad (1 \le i \le j \le n).$$
Therefore, for each ℓ,
$$0 \le \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \big(a_i^{(\ell)}\big)^p \big(a_j^{(\ell)}\big)^q \le 1,$$
because there are exactly $n(n+1)/2$ terms in the double sum. Thus
$$\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} a_i^{\odot p} \odot a_j^{\odot q} \in [0, 1]^k.$$

Since $p + q > 0$, the exponent $1/(p + q)$ is well-defined and nonnegative. Hence the componentwise $1/(p + q)$-power again belongs to $[0, 1]^k$. Consequently,
$$\mathrm{UHM}_{M;p,q}(a_1, \ldots, a_n) \in [0, 1]^k = \mathrm{Dom}(M).$$
Thus $\mathrm{UHM}_{M;p,q}$ is well-defined. □

Definition 6.6.8 (Uncertain weighted Heronian mean operator)**.** Let

$$w = (w_1, \dots, w_n) \in [0,1]^n$$

be a weight vector such that

$$\sum_{i=1}^{n} w_i = 1.$$

Define

$$\Gamma_w := \sum_{1 \le r \le s \le n} w_r w_s.$$

Since $w_i \ge 0$ and $\sum_{i=1}^{n} w_i = 1$, one has $\Gamma_w > 0$. For

$$1 \le i \le j \le n,$$

define

$$\omega_{ij} := \frac{w_i w_j}{\Gamma_w}.$$

The *uncertain weighted Heronian mean operator* of parameters (p, q) with respect to w is the mapping

$$\mathrm{UWHM}_{M;p,q,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UWHM}_{M;p,q,w}(a_1, \dots, a_n) := \left(\sum_{i=1}^{n} \sum_{j=i}^{n} \omega_{ij}\, a_i^{\odot p} \odot a_j^{\odot q} \right)^{\odot 1/(p+q)}$$

for all

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Lemma 6.6.9 (Basic properties of the coefficients)**.** *The coefficients ω_{ij} satisfy:*

1. $\omega_{ij} \ge 0$ *for all* $1 \le i \le j \le n$*;*
2. $$\sum_{i=1}^{n} \sum_{j=i}^{n} \omega_{ij} = 1.$$

Proof. Since $w_i, w_j \ge 0$ and $\Gamma_w > 0$, one has

$$\omega_{ij} = \frac{w_i w_j}{\Gamma_w} \ge 0 \qquad (1 \le i \le j \le n).$$

Moreover,

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \omega_{ij} = \frac{1}{\Gamma_w} \sum_{i=1}^{n} \sum_{j=i}^{n} w_i w_j = \frac{\Gamma_w}{\Gamma_w} = 1.$$

Thus the result follows. □

Remark 6.6.10. If

$$w_1 = \cdots = w_n = \frac{1}{n},$$

then

$$\Gamma_w = \sum_{1 \le r \le s \le n} \frac{1}{n^2} = \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = \frac{n+1}{2n},$$

and hence

$$\omega_{ij} = \frac{(1/n)(1/n)}{(n+1)/(2n)} = \frac{2}{n(n+1)} \qquad (1 \le i \le j \le n).$$

Therefore $\mathrm{UWHM}_{M;p,q,w}$ reduces to $\mathrm{UHM}_{M;p,q}$ in the uniform-weight case.

Theorem 6.6.11 (Well-definedness of the uncertain weighted Heronian mean operator)**.** *The operator*

$$\mathrm{UWHM}_{M;p,q,w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Take arbitrary

$$a_1, \ldots, a_n \in \mathrm{Dom}(M) = [0,1]^k.$$

As in the proof of the previous theorem,

$$a_i^{\odot p} \odot a_j^{\odot q} \in [0,1]^k \qquad (1 \le i \le j \le n).$$

Fix $\ell \in \{1, \ldots, k\}$. Then

$$0 \le \big(a_i^{(\ell)}\big)^p \big(a_j^{(\ell)}\big)^q \le 1 \qquad (1 \le i \le j \le n).$$

By the lemma, the coefficients ω_{ij} are nonnegative and sum to 1. Hence

$$0 \le \sum_{i=1}^{n} \sum_{j=i}^{n} \omega_{ij} \big(a_i^{(\ell)}\big)^p \big(a_j^{(\ell)}\big)^q \le 1.$$

Therefore,

$$\sum_{i=1}^{n} \sum_{j=i}^{n} \omega_{ij}\, a_i^{\odot p} \odot a_j^{\odot q} \in [0,1]^k.$$

Since $p + q > 0$, taking the componentwise $1/(p+q)$-power is well-defined and again yields an element of $[0,1]^k$. Thus,

$$\mathrm{UWHM}_{M;p,q,w}(a_1, \ldots, a_n) \in [0,1]^k = \mathrm{Dom}(M).$$

Hence $\mathrm{UWHM}_{M;p,q,w}$ is well-defined. □

Definition 6.6.12 (Induced uncertain Heronian mean on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

The *induced uncertain Heronian mean* of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is defined by

$$\mathrm{UHM}_{M;p,q}(\mathcal{U}_1, \ldots, \mathcal{U}_n) := (X, \mu_{\mathrm{UHM}}),$$

where

$$\mu_{\mathrm{UHM}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UHM}}(x) := \mathrm{UHM}_{M;p,q}\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.6.13 (Well-definedness of the induced uncertain Heronian mean)**.** *The pair*

$$\mathrm{UHM}_{M;p,q}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = (X, \mu_{\mathrm{UHM}})$$

is a well-defined uncertain set of type M.

Proof. Fix $x \in X$. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M,

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Hence

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the well-definedness of $\mathrm{UHM}_{M;p,q}$,

$$\mathrm{UHM}_{M;p,q}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathrm{UHM}}(x) \in \mathrm{Dom}(M) \qquad (x \in X),$$

so

$$\mu_{\mathrm{UHM}} : X \to \mathrm{Dom}(M)$$

is well-defined. Consequently,

$$(X, \mu_{\mathrm{UHM}})$$

is a well-defined uncertain set of type M. □

Definition 6.6.14 (Induced uncertain weighted Heronian mean on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

The *induced uncertain weighted Heronian mean* of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is defined by

$$\mathrm{UWHM}_{M;p,q,w}(\mathcal{U}_1, \ldots, \mathcal{U}_n) := (X, \mu_{\mathrm{UWHM}}),$$

where

$$\mu_{\mathrm{UWHM}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UWHM}}(x) := \mathrm{UWHM}_{M;p,q,w}\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.6.15 (Well-definedness of the induced uncertain weighted Heronian mean)**.** *The pair*

$$\mathrm{UWHM}_{M;p,q,w}(\mathcal{U}_1,\ldots,\mathcal{U}_n) = (X, \mu_{\mathrm{UWHM}})$$

is a well-defined uncertain set of type M.

Proof. Fix $x \in X$. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M,

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Hence

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the well-definedness of $\mathrm{UWHM}_{M;p,q,w}$,

$$\mathrm{UWHM}_{M;p,q,w}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathrm{UWHM}}(x) \in \mathrm{Dom}(M) \qquad (x \in X),$$

and thus

$$\mu_{\mathrm{UWHM}} : X \to \mathrm{Dom}(M)$$

is well-defined. Consequently,

$$(X, \mu_{\mathrm{UWHM}})$$

is a well-defined uncertain set of type M. □

As a reference, a catalogue of representative Heronian mean operators classified by the dimension k of the degree-domain is presented in Table 6.6.

Table 6.6: A catalogue of representative Heronian mean operators by the dimension k of the degree-domain.

k	note	Representative Heronian mean operator(s)
1		Fuzzy Heronian mean operator.
2		Intuitionistic Fuzzy Heronian mean operators [298, 299];
3		Picture Fuzzy Heronian mean operators [239,300]; Spherical Fuzzy Heronian mean operators [301,302]; Neutrosophic Heronian mean operators [303,304]; Hesitant Fuzzy Heronian mean operators [305, 306].

Reading guide. The table groups representative Heronian mean operators by the dimension k of their degree values.

Related concepts other than those mentioned above include Archimedean Her than those mentioned above include Archimedean Heronian mean operators [307, 308], power Heronian mean operators [309, 310], and geometric Heronian mean operators [311–313].

6.7 Uncertain Maclaurin Symmetric Mean Operator

A Maclaurin symmetric mean operator aggregates inputs through averaged elementary symmetric products, capturing collective interaction among several arguments and generalizing arithmetic and geometric means [314, 315]. Fuzzy Maclaurin symmetric mean operators extend Maclaurin symmetric means to fuzzy degrees in $[0, 1]$, aggregating interacting membership values while preserving symmetry and boundedness.

Definition 6.7.1 (Fuzzy Maclaurin Symmetric Mean Operators)**.** [316, 317] Let

$$n \geq 1, \qquad k \in \{1, 2, \ldots, n\}.$$

The *k-th fuzzy Maclaurin symmetric mean operator* is the mapping

$$\mathrm{FMSM}^{(k)} : [0,1]^n \to [0,1]$$

defined by

$$\mathrm{FMSM}^{(k)}(x_1, x_2, \ldots, x_n) := \left(\frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{j=1}^{k} x_{i_j} \right)^{1/k},$$

for all

$$(x_1, \ldots, x_n) \in [0,1]^n.$$

The family

$$\left\{ \mathrm{FMSM}^{(k)} \right\}_{k=1}^{n}$$

is called the family of *fuzzy Maclaurin symmetric mean operators.*

In the uncertain-set setting, this idea can be extended naturally to multi-dimensional degree values in a componentwise manner.

Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) = [0,1]^d, \qquad d \geq 1.$$

For vectors

$$a_i = \left(a_i^{(1)}, \ldots, a_i^{(d)}\right) \in [0,1]^d \qquad (i = 1, \ldots, n),$$

and for indices

$$1 \leq i_1 < i_2 < \cdots < i_r \leq n,$$

define their componentwise product by

$$\bigodot_{t=1}^{r} a_{i_t} := \left(\prod_{t=1}^{r} a_{i_t}^{(1)}, \ \prod_{t=1}^{r} a_{i_t}^{(2)}, \ \ldots, \ \prod_{t=1}^{r} a_{i_t}^{(d)} \right).$$

Also, for

$$c = (c^{(1)}, \ldots, c^{(d)}) \in [0,1]^d$$

and $r \geq 1$, define the componentwise r-th root by

$$c^{\odot 1/r} := \left((c^{(1)})^{1/r}, \ldots, (c^{(d)})^{1/r} \right).$$

Definition 6.7.2 (Uncertain Maclaurin symmetric mean operators)**.** Let

$$n \geq 1, \qquad r \in \{1, 2, \ldots, n\}.$$

The *r-th uncertain Maclaurin symmetric mean operator* on M is the mapping

$$\mathrm{UMSM}_M^{(r)} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UMSM}_M^{(r)}(a_1, \ldots, a_n) := \left(\frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \bigodot_{t=1}^{r} a_{i_t} \right)^{\odot 1/r},$$

for all

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

The family

$$\left\{ \mathrm{UMSM}_M^{(r)} \right\}_{r=1}^{n}$$

is called the family of *uncertain Maclaurin symmetric mean operators.*

Remark 6.7.3. When $d = 1$, the above definition reduces to the usual scalar-valued Maclaurin symmetric mean operator on $[0, 1]^n$. Thus the uncertain Maclaurin symmetric mean is a natural multi-dimensional extension of the classical fuzzy Maclaurin symmetric mean.

Remark 6.7.4. The family $\{\mathrm{UMSM}_M^{(r)}\}_{r=1}^{n}$ interpolates between important special cases:

$$\mathrm{UMSM}_M^{(1)}(a_1, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i,$$

which is the componentwise arithmetic mean, and

$$\mathrm{UMSM}_M^{(n)}(a_1, \ldots, a_n) = \left(\bigodot_{i=1}^{n} a_i \right)^{\odot 1/n},$$

which is the componentwise geometric mean.

Theorem 6.7.5 (Well-definedness of the uncertain Maclaurin symmetric mean)**.** *For each*

$$r \in \{1, 2, \ldots, n\},$$

the operator

$$\mathrm{UMSM}_M^{(r)} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Fix

$$r \in \{1, 2, \ldots, n\}$$

and take arbitrary

$$a_1, \ldots, a_n \in \mathrm{Dom}(M) = [0, 1]^d.$$

Write

$$a_i = \left(a_i^{(1)}, \ldots, a_i^{(d)}\right) \qquad (i = 1, \ldots, n),$$

where

$$0 \le a_i^{(\ell)} \le 1 \qquad (\ell = 1, \dots, d).$$

Let

$$1 \le i_1 < i_2 < \cdots < i_r \le n$$

be arbitrary, and fix $\ell \in \{1, \dots, d\}$. Then

$$0 \le \prod_{t=1}^{r} a_{i_t}^{(\ell)} \le 1.$$

Hence each ℓ-th component of

$$\bigodot_{t=1}^{r} a_{i_t}$$

belongs to $[0,1]$, and therefore

$$\bigodot_{t=1}^{r} a_{i_t} \in [0,1]^d.$$

Now consider

$$\frac{1}{\binom{n}{r}} \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \bigodot_{t=1}^{r} a_{i_t}.$$

For each fixed component ℓ, this is the arithmetic mean of $\binom{n}{r}$ numbers from $[0,1]$. Therefore its ℓ-th component also belongs to $[0,1]$. Hence

$$\frac{1}{\binom{n}{r}} \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \bigodot_{t=1}^{r} a_{i_t} \in [0,1]^d.$$

Finally, if

$$c = (c^{(1)}, \dots, c^{(d)}) \in [0,1]^d,$$

then for every ℓ,

$$0 \le c^{(\ell)} \le 1 \quad \Longrightarrow \quad 0 \le \left(c^{(\ell)}\right)^{1/r} \le 1.$$

Thus

$$c^{\odot 1/r} \in [0,1]^d.$$

Applying this to the preceding average yields

$$\mathrm{UMSM}_M^{(r)}(a_1, \dots, a_n) \in [0,1]^d = \mathrm{Dom}(M).$$

Therefore the mapping

$$\mathrm{UMSM}_M^{(r)} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined. □

Definition 6.7.6 (Induced uncertain Maclaurin symmetric mean on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

For each

$$r \in \{1, 2, \ldots, n\},$$

the *r-th induced uncertain Maclaurin symmetric mean* of $\mathcal{U}_1, \ldots, \mathcal{U}_n$ is defined by

$$\mathrm{UMSM}_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_n) := (X, \mu_{\mathrm{UMSM}}^{(r)}),$$

where

$$\mu_{\mathrm{UMSM}}^{(r)} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UMSM}}^{(r)}(x) := \mathrm{UMSM}_M^{(r)}\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.7.7 (Well-definedness of the induced uncertain Maclaurin symmetric mean)**.** *For each*

$$r \in \{1, 2, \ldots, n\},$$

the pair

$$\mathrm{UMSM}_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = (X, \mu_{\mathrm{UMSM}}^{(r)})$$

is a well-defined uncertain set of type M.

Proof. Fix

$$r \in \{1, 2, \ldots, n\}$$

and $x \in X$. Since each

$$\mathcal{U}_i = (X, \mu_i)$$

is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Hence

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the well-definedness of $\mathrm{UMSM}_M^{(r)}$,

$$\mathrm{UMSM}_M^{(r)}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore

$$\mu_{\mathrm{UMSM}}^{(r)}(x) \in \mathrm{Dom}(M) \qquad (x \in X).$$

Thus

$$\mu_{\mathrm{UMSM}}^{(r)} : X \to \mathrm{Dom}(M)$$

is well-defined, and consequently

$$(X, \mu_{\mathrm{UMSM}}^{(r)})$$

is a well-defined uncertain set of type M. □

A catalogue of representative Maclaurin symmetric mean operators classified by the dimension k of the degree-domain is presented in Table 6.7.

Table 6.7: A catalogue of representative Maclaurin symmetric mean operators by the dimension k of the degree-domain.

k	note	Representative Maclaurin symmetric mean operator(s)
1		Fuzzy Maclaurin symmetric mean operators.
2		Intuitionistic Fuzzy Maclaurin symmetric mean operators [318, 319].
3		Picture Fuzzy Maclaurin symmetric mean operators [320, 321]; Spherical Fuzzy Maclaurin symmetric mean operators [322, 323]; Neutrosophic Maclaurin symmetric mean operators [324–326].
n	$(n \geq 1)$	Hesitant Fuzzy Maclaurin symmetric mean operators [327, 328].

Reading guide. The table groups representative Maclaurin symmetric mean operators by the dimension k of their degree values.

6.8 Uncertain OWA operator

An uncertain OWA operator aggregates uncertain inputs after ordering them, enabling attitudinal decision control between optimistic and pessimistic combination without fixed source weights or priorities.

Definition 6.8.1 (Fuzzy OWA Operator)**.** [329, 330] Let $n \geq 1$ be an integer, and let

$$W = (w_1, w_2, \ldots, w_n)$$

be a weight vector such that

$$w_i \in [0,1] \qquad (i = 1, 2, \ldots, n), \qquad \sum_{i=1}^{n} w_i = 1.$$

The *fuzzy ordered weighted averaging operator* (briefly, *fuzzy OWA operator*) associated with W is the mapping

$$\mathrm{OWA}_W : [0,1]^n \to [0,1]$$

defined by

$$\mathrm{OWA}_W(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} w_i\, y_i,$$

where

$$(y_1, y_2, \ldots, y_n)$$

is a permutation of

$$(x_1, x_2, \ldots, x_n)$$

such that

$$y_1 \geq y_2 \geq \cdots \geq y_n.$$

Definition 6.8.2 (Neutrosophic OWA Operator)**.** [331] Let

$$\tilde{a}_j = \langle T_j, I_j, F_j \rangle \in \mathbb{SVN} \qquad (j = 1, \ldots, n),$$

let

$$W = (w_1, \ldots, w_n), \qquad w_j \in [0,1], \qquad \sum_{j=1}^{n} w_j = 1,$$

and let $\rho : \mathbb{SVN} \to \mathbb{R}$ be a fixed ranking function. If

$$\rho(\tilde{a}_{\sigma(1)}) \geq \rho(\tilde{a}_{\sigma(2)}) \geq \cdots \geq \rho(\tilde{a}_{\sigma(n)}),$$

then the *neutrosophic OWA operator* is defined by

$$\mathrm{NOWA}_{W,\rho}(\tilde{a}_1, \ldots, \tilde{a}_n) = \left\langle \sum_{j=1}^{n} w_j T_{\sigma(j)},\ \sum_{j=1}^{n} w_j I_{\sigma(j)},\ \sum_{j=1}^{n} w_j F_{\sigma(j)} \right\rangle.$$

In the uncertain-set setting, such an operator can be defined on the degree-domain of an uncertain model by combining a ranking map with a convex weighted average.

Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

where $k \geq 1$. Assume that $\mathrm{Dom}(M)$ is nonempty and convex.

Let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map. For each $n \geq 1$, let

$$W = (w_1, w_2, \ldots, w_n) \in [0,1]^n$$

be a weight vector satisfying

$$\sum_{i=1}^{n} w_i = 1.$$

Definition 6.8.3 (Canonical ρ_M-ordering)**.** Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

A permutation σ of $\{1, \ldots, n\}$ is called the *canonical ρ_M-ordering permutation* of $(a_1, \ldots, a_n)$ if the following conditions hold:

1. $$\rho_M(a_{\sigma(1)}) \geq \rho_M(a_{\sigma(2)}) \geq \cdots \geq \rho_M(a_{\sigma(n)});$$

2. whenever
$$\rho_M(a_{\sigma(i)}) = \rho_M(a_{\sigma(j)}) \quad \text{and} \quad i < j,$$
one has
$$\sigma(i) < \sigma(j).$$

In other words, the inputs are ordered in decreasing order of their ranking values, and ties are broken by preserving the smaller original index first.

Lemma 6.8.4 (Existence and uniqueness of the canonical ordering)**.** *For every*

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$

there exists a unique canonical ρ_M-ordering permutation σ.

Proof. Define a binary relation $\preccurlyeq$ on the index set $\{1, \dots, n\}$ by

$$u \preccurlyeq v \iff \big(\rho_M(a_u) > \rho_M(a_v)\big) \text{ or } \big(\rho_M(a_u) = \rho_M(a_v) \text{ and } u \leq v\big).$$

This relation totally orders the finite set $\{1, \dots, n\}$: any two indices are comparable, and ties of the ranking values are resolved by the natural order of indices. Hence there exists a unique listing

$$\sigma(1), \sigma(2), \dots, \sigma(n)$$

of the indices in decreasing order with respect to this rule. By construction, σ satisfies conditions (1) and (2) of the definition. Therefore the canonical ρ_M-ordering permutation exists and is unique. □

Definition 6.8.5 (Uncertain OWA operator)**.** Let $n \geq 1$, let

$$W = (w_1, \dots, w_n) \in [0, 1]^n, \qquad \sum_{i=1}^{n} w_i = 1,$$

and let $\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$ be a fixed ranking map. For

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n,$$

let σ be the unique canonical ρ_M-ordering permutation of $(a_1, \dots, a_n)$, and write

$$b_i := a_{\sigma(i)} \qquad (i = 1, \dots, n).$$

The *uncertain OWA operator* associated with M, W, and ρ_M is the mapping

$$\mathrm{UOWA}_{M;W,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UOWA}_{M;W,\rho_M}(a_1, \dots, a_n) := \sum_{i=1}^{n} w_i \, b_i.$$

Here the sum is the usual vector sum in $\mathbb{R}^k$.

Remark 6.8.6. If $k = 1$, $\mathrm{Dom}(M) = [0, 1]$, and

$$\rho_M(a) = a \qquad (a \in [0, 1]),$$

then $\mathrm{UOWA}_{M;W,\rho_M}$ reduces to the ordinary fuzzy OWA operator. Thus the present definition is a natural uncertain-set-based extension of the classical scalar OWA operator.

Remark 6.8.7. For specific uncertainty models, different choices of the ranking map ρ_M lead to different ordering policies. For example, in neutrosophic-type models one may choose a score or ranking function tailored to the intended semantics of truth, indeterminacy, and falsity.

Theorem 6.8.8 (Well-definedness of the uncertain OWA operator)**.** *The mapping*

$$\mathrm{UOWA}_{M;W,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n$$

be arbitrary. By the previous lemma, there exists a unique canonical ρ_M-ordering permutation σ, so the reordered tuple

$$(b_1, \ldots, b_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

is uniquely determined.

Since each $b_i \in \mathrm{Dom}(M) \subseteq [0,1]^k$, and since

$$w_i \geq 0 \qquad (i = 1, \ldots, n), \qquad \sum_{i=1}^{n} w_i = 1,$$

the vector

$$\sum_{i=1}^{n} w_i b_i$$

is a convex combination of points of $\mathrm{Dom}(M)$. Because $\mathrm{Dom}(M)$ is convex, every convex combination of its elements again belongs to $\mathrm{Dom}(M)$. Therefore,

$$\mathrm{UOWA}_{M;W,\rho_M}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i b_i \in \mathrm{Dom}(M).$$

Hence the formula defines a mapping

$$\mathrm{UOWA}_{M;W,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M),$$

so the uncertain OWA operator is well-defined. □

Definition 6.8.9 (Induced uncertain OWA operator on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

The *induced uncertain OWA operator* on $(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ is defined by

$$\mathrm{UOWA}_{M;W,\rho_M}(\mathcal{U}_1, \ldots, \mathcal{U}_n) := (X, \mu_{\mathrm{UOWA}}),$$

where

$$\mu_{\mathrm{UOWA}} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{UOWA}}(x) := \mathrm{UOWA}_{M;W,\rho_M}\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 6.8.10 (Well-definedness of the induced uncertain OWA operator)**.** *The pair*

$$\mathrm{UOWA}_{M;W,\rho_M}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = (X, \mu_{\mathrm{UOWA}})$$

is a well-defined uncertain set of type M.

Proof. Fix $x \in X$. Since each
$$\mathcal{U}_i = (X, \mu_i)$$
is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Hence
$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$
By the well-definedness of the uncertain OWA operator,

$$\mathrm{UOWA}_{M;W,\rho_M}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore,
$$\mu_{\mathrm{UOWA}}(x) \in \mathrm{Dom}(M) \qquad (x \in X).$$
Thus
$$\mu_{\mathrm{UOWA}} : X \to \mathrm{Dom}(M)$$
is a well-defined function, and consequently

$$(X, \mu_{\mathrm{UOWA}})$$

is a well-defined uncertain set of type M. □

A catalogue of representative OWA operators by the dimension k of the degree-domain is presented in Table 6.8.

Table 6.8: A catalogue of representative OWA operators by the dimension k of the degree-domain.

k	note	Representative OWA operator(s)
1		Fuzzy OWA operator [330].
2		Intuitionistic Fuzzy OWA operator [332, 333].
3		Neutrosophic OWA operator [334, 335].
n	$(n \geq 1)$	Plithogenic OWA operator.

Reading guide. The table groups representative OWA operators by the dimension k of their degree values.

As related concepts to the above, induced OWA operators [336, 337], linguistic OWA operators [338, 339], heavy OWA operators [340], weighted owa operators [341, 342] and generalized OWA operators [329, 330], among others, are also known.

6.9 Uncertain Einstein Aggregation Operators

Einstein aggregation operators combine multiple values using Einstein sum and product laws, providing smooth, parameter-free fusion that balances interaction, boundedness, and generalized averaging behavior effectively [343–345]. Uncertain Einstein aggregation operators extend Einstein-based fusion to uncertain-valued information, aggregating vague or multi-component inputs while preserving boundedness, interaction structure, and representative collective assessment consistently.

Definition 6.9.1 (Fuzzy Einstein Aggregation Operators)**.** [346] Let

$$I = [0,1], \qquad n \geq 1,$$

and let

$$w = (w_1, \dots, w_n) \in I^n$$

be a weight vector satisfying

$$\sum_{i=1}^{n} w_i = 1.$$

First, define the *Einstein sum* and *Einstein product* on I by

$$x \oplus_\varepsilon y := \frac{x+y}{1+xy}, \qquad x \otimes_\varepsilon y := \frac{xy}{1+(1-x)(1-y)}, \qquad x, y \in I.$$

For $\lambda > 0$ and $x \in I$, define the *Einstein scalar multiplication* of x by λ as

$$\lambda \odot_\varepsilon x := \frac{(1+x)^\lambda - (1-x)^\lambda}{(1+x)^\lambda + (1-x)^\lambda},$$

and define the *Einstein power* of x with exponent λ as

$$x^{\lambda_\varepsilon} := \frac{2x^\lambda}{(2-x)^\lambda + x^\lambda}.$$

Then the following operators are called *fuzzy Einstein aggregation operators.*

1. The *fuzzy Einstein weighted averaging operator* (FEWA) is the mapping

   $$\mathrm{FEWA}_w : I^n \to I$$

   defined by

   $$\mathrm{FEWA}_w(x_1, \dots, x_n) := (w_1 \odot_\varepsilon x_1) \oplus_\varepsilon \cdots \oplus_\varepsilon (w_n \odot_\varepsilon x_n).$$

   Equivalently,

   $$\mathrm{FEWA}_w(x_1, \dots, x_n) = \frac{\prod_{i=1}^{n}(1+x_i)^{w_i} - \prod_{i=1}^{n}(1-x_i)^{w_i}}{\prod_{i=1}^{n}(1+x_i)^{w_i} + \prod_{i=1}^{n}(1-x_i)^{w_i}}.$$

2. The *fuzzy Einstein weighted geometric operator* (FEWG) is the mapping

   $$\mathrm{FEWG}_w : I^n \to I$$

   defined by

   $$\mathrm{FEWG}_w(x_1, \dots, x_n) := x_1^{w_{1\varepsilon}} \otimes_\varepsilon \cdots \otimes_\varepsilon x_n^{w_{n\varepsilon}}.$$

   Equivalently,

   $$\mathrm{FEWG}_w(x_1, \dots, x_n) = \frac{2\prod_{i=1}^{n} x_i^{w_i}}{\prod_{i=1}^{n}(2-x_i)^{w_i} + \prod_{i=1}^{n} x_i^{w_i}}.$$

3. Let $(y_1, \dots, y_n)$ be a permutation of $(x_1, \dots, x_n)$ such that
$$y_1 \geq y_2 \geq \cdots \geq y_n.$$
Then the *fuzzy Einstein ordered weighted averaging operator* (FEOWA) is defined by
$$\mathrm{FEOWA}_w(x_1, \dots, x_n) := (w_1 \odot_\varepsilon y_1) \oplus_\varepsilon \cdots \oplus_\varepsilon (w_n \odot_\varepsilon y_n).$$
Equivalently,
$$\mathrm{FEOWA}_w(x_1, \dots, x_n) = \frac{\prod_{i=1}^{n}(1+y_i)^{w_i} - \prod_{i=1}^{n}(1-y_i)^{w_i}}{\prod_{i=1}^{n}(1+y_i)^{w_i} + \prod_{i=1}^{n}(1-y_i)^{w_i}}.$$

4. With the same ordering $y_1 \geq \cdots \geq y_n$, the *fuzzy Einstein ordered weighted geometric operator* (FEOWG) is defined by
$$\mathrm{FEOWG}_w(x_1, \dots, x_n) := y_1^{w_1 \varepsilon} \otimes_\varepsilon \cdots \otimes_\varepsilon y_n^{w_n \varepsilon}.$$
Equivalently,
$$\mathrm{FEOWG}_w(x_1, \dots, x_n) = \frac{2\prod_{i=1}^{n} y_i^{w_i}}{\prod_{i=1}^{n}(2-y_i)^{w_i} + \prod_{i=1}^{n} y_i^{w_i}}.$$

In the uncertain-set setting, these operators can be extended naturally to multi-dimensional degree values by applying the corresponding scalar formulas componentwise.

Throughout this subsection, let M be a fixed uncertain model with degree-domain
$$\mathrm{Dom}(M) = [0,1]^d, \qquad d \geq 1.$$
Let
$$n \geq 1$$
be an integer, and let
$$w = (w_1, \dots, w_n) \in [0,1]^n$$
be a weight vector satisfying
$$\sum_{i=1}^{n} w_i = 1.$$
Define
$$I_w^+ := \{\, i \in \{1, \dots, n\} \mid w_i > 0 \,\}.$$
Since $\sum_{i=1}^{n} w_i = 1$, the set I_w^+ is nonempty.

Remark 6.9.2. For scalar values $x, y \in [0,1]$, the Einstein sum and Einstein product are given by
$$x \oplus_\varepsilon y := \frac{x+y}{1+xy}, \qquad x \otimes_\varepsilon y := \frac{xy}{1+(1-x)(1-y)}.$$
The uncertain Einstein aggregation operators introduced below are multi-dimensional extensions of the corresponding scalar Einstein aggregation formulas.

Definition 6.9.3 (Uncertain Einstein weighted averaging operator)**.** For

$$a_i = \big(a_i^{(1)}, \dots, a_i^{(d)}\big) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n),$$

define

$$\mathrm{UEWA}_{M;w}(a_1, \dots, a_n) := \big(\eta^{(1)}, \dots, \eta^{(d)}\big),$$

where, for each $\ell \in \{1, \dots, d\}$,

$$\eta^{(\ell)} := \frac{\prod_{i \in I_w^+} (1 + a_i^{(\ell)})^{w_i} - \prod_{i \in I_w^+} (1 - a_i^{(\ell)})^{w_i}}{\prod_{i \in I_w^+} (1 + a_i^{(\ell)})^{w_i} + \prod_{i \in I_w^+} (1 - a_i^{(\ell)})^{w_i}}.$$

This mapping is called the *uncertain Einstein weighted averaging operator* associated with M and w.

Definition 6.9.4 (Uncertain Einstein weighted geometric operator)**.** For

$$a_i = \big(a_i^{(1)}, \dots, a_i^{(d)}\big) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n),$$

define

$$\mathrm{UEWG}_{M;w}(a_1, \dots, a_n) := \big(\gamma^{(1)}, \dots, \gamma^{(d)}\big),$$

where, for each $\ell \in \{1, \dots, d\}$,

$$\gamma^{(\ell)} := \frac{2 \prod_{i \in I_w^+} (a_i^{(\ell)})^{w_i}}{\prod_{i \in I_w^+} (2 - a_i^{(\ell)})^{w_i} + \prod_{i \in I_w^+} (a_i^{(\ell)})^{w_i}}.$$

This mapping is called the *uncertain Einstein weighted geometric operator* associated with M and w.

Definition 6.9.5 (Canonical ρ_M-ordering)**.** Let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map. For

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n,$$

a permutation σ of $\{1, \dots, n\}$ is called the *canonical ρ_M-ordering permutation* if:

1. $$\rho_M(a_{\sigma(1)}) \ge \rho_M(a_{\sigma(2)}) \ge \cdots \ge \rho_M(a_{\sigma(n)});$$

2. whenever
$$\rho_M(a_{\sigma(i)}) = \rho_M(a_{\sigma(j)}) \quad \text{and} \quad i < j,$$
one has
$$\sigma(i) < \sigma(j).$$

Thus the inputs are ordered in decreasing order of the ranking values, and ties are broken by preserving the smaller original index first.

Lemma 6.9.6 (Existence and uniqueness of the canonical ordering)**.** *For every*

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$

there exists a unique canonical ρ_M-ordering permutation.

Proof. Define a binary relation $\preccurlyeq$ on $\{1, \ldots, n\}$ by

$$u \preccurlyeq v \iff \big(\rho_M(a_u) > \rho_M(a_v)\big) \text{ or } \big(\rho_M(a_u) = \rho_M(a_v) \text{ and } u \leq v\big).$$

This relation totally orders the finite index set $\{1, \ldots, n\}$, because every two indices are comparable and ties are resolved by the natural order of indices. Hence there exists a unique permutation

$$\sigma(1), \ldots, \sigma(n)$$

listing the indices in decreasing order with respect to this rule. By construction, σ is exactly the unique canonical ρ_M-ordering permutation. □

Definition 6.9.7 (Uncertain Einstein ordered weighted averaging operator)**.** Let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map, and let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

Let σ be the unique canonical ρ_M-ordering permutation of $(a_1, \ldots, a_n)$, and write

$$b_i := a_{\sigma(i)} \qquad (i = 1, \ldots, n).$$

The *uncertain Einstein ordered weighted averaging operator* associated with M, w, and ρ_M is defined by

$$\mathrm{UEOWA}_{M;w,\rho_M}(a_1, \ldots, a_n) := \mathrm{UEWA}_{M;w}(b_1, \ldots, b_n).$$

Definition 6.9.8 (Uncertain Einstein ordered weighted geometric operator)**.** Let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map, and let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

Let σ be the unique canonical ρ_M-ordering permutation of $(a_1, \ldots, a_n)$, and write

$$b_i := a_{\sigma(i)} \qquad (i = 1, \ldots, n).$$

The *uncertain Einstein ordered weighted geometric operator* associated with M, w, and ρ_M is defined by

$$\mathrm{UEOWG}_{M;w,\rho_M}(a_1, \ldots, a_n) := \mathrm{UEWG}_{M;w}(b_1, \ldots, b_n).$$

Remark 6.9.9. If $d = 1$, $\mathrm{Dom}(M) = [0, 1]$, and

$$\rho_M(a) = a \qquad (a \in [0, 1]),$$

then $\mathrm{UEWA}_{M;w}$, $\mathrm{UEWG}_{M;w}$, $\mathrm{UEOWA}_{M;w,\rho_M}$, and $\mathrm{UEOWG}_{M;w,\rho_M}$ reduce to the usual scalar fuzzy Einstein weighted averaging, fuzzy Einstein weighted geometric, fuzzy Einstein ordered weighted averaging, and fuzzy Einstein ordered weighted geometric operators, respectively.

Theorem 6.9.10 (Well-definedness of the uncertain Einstein weighted averaging operator)**.** *The mapping*

$$\mathrm{UEWA}_{M;w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Take arbitrary

$$a_1, \ldots, a_n \in \mathrm{Dom}(M) = [0,1]^d.$$

Fix $\ell \in \{1, \ldots, d\}$, and define

$$A_\ell := \prod_{i \in I_w^+} (1 + a_i^{(\ell)})^{w_i}, \qquad B_\ell := \prod_{i \in I_w^+} (1 - a_i^{(\ell)})^{w_i}.$$

Since

$$0 \le a_i^{(\ell)} \le 1 \qquad (i = 1, \ldots, n),$$

one has

$$1 + a_i^{(\ell)} \in [1,2], \qquad 1 - a_i^{(\ell)} \in [0,1].$$

Hence

$$A_\ell > 0, \qquad B_\ell \ge 0.$$

Moreover, for every $i \in I_w^+$,

$$1 + a_i^{(\ell)} \ge 1 - a_i^{(\ell)},$$

and since all exponents w_i are positive on I_w^+, it follows that

$$A_\ell \ge B_\ell.$$

Therefore,

$$0 \le A_\ell - B_\ell \le A_\ell + B_\ell, \qquad A_\ell + B_\ell > 0.$$

Thus

$$0 \le \frac{A_\ell - B_\ell}{A_\ell + B_\ell} \le 1.$$

Hence

$$\eta^{(\ell)} \in [0,1] \qquad (\ell = 1, \ldots, d).$$

Consequently,

$$\mathrm{UEWA}_{M;w}(a_1, \ldots, a_n) = \big(\eta^{(1)}, \ldots, \eta^{(d)}\big) \in [0,1]^d = \mathrm{Dom}(M).$$

Therefore $\mathrm{UEWA}_{M;w}$ is well-defined. □

Theorem 6.9.11 (Well-definedness of the uncertain Einstein weighted geometric operator)**.** *The mapping*

$$\mathrm{UEWG}_{M;w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Take arbitrary

$$a_1, \ldots, a_n \in \mathrm{Dom}(M) = [0,1]^d.$$

Fix $\ell \in \{1, \dots, d\}$, and define

$$C_\ell := \prod_{i \in I_w^+} (a_i^{(\ell)})^{w_i}, \qquad D_\ell := \prod_{i \in I_w^+} (2 - a_i^{(\ell)})^{w_i}.$$

Since

$$0 \le a_i^{(\ell)} \le 1,$$

one has

$$C_\ell \ge 0, \qquad 2 - a_i^{(\ell)} \in [1, 2],$$

and therefore

$$D_\ell > 0.$$

Also, for every $i \in I_w^+$,

$$a_i^{(\ell)} \le 2 - a_i^{(\ell)}.$$

Hence

$$C_\ell \le D_\ell.$$

Therefore,

$$0 \le 2C_\ell \le C_\ell + D_\ell, \qquad C_\ell + D_\ell > 0,$$

which implies

$$0 \le \frac{2C_\ell}{C_\ell + D_\ell} \le 1.$$

Thus

$$\gamma^{(\ell)} \in [0, 1] \qquad (\ell = 1, \dots, d).$$

Consequently,

$$\mathrm{UEWG}_{M;w}(a_1, \dots, a_n) = \left(\gamma^{(1)}, \dots, \gamma^{(d)}\right) \in [0, 1]^d = \mathrm{Dom}(M).$$

Hence $\mathrm{UEWG}_{M;w}$ is well-defined. □

Theorem 6.9.12 (Well-definedness of the ordered uncertain Einstein aggregation operators)**.** *The mappings*

$$\mathrm{UEOWA}_{M;w,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

and

$$\mathrm{UEOWG}_{M;w,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

are well-defined.

Proof. Let

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n$$

be arbitrary. By the lemma on canonical ordering, there exists a unique canonical ρ_M-ordering permutation σ. Hence the reordered tuple

$$(b_1, \dots, b_n) = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

is uniquely determined and satisfies

$$b_i \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$

By the previous two theorems,

$$\mathrm{UEWA}_{M;w}(b_1,\ldots,b_n) \in \mathrm{Dom}(M), \qquad \mathrm{UEWG}_{M;w}(b_1,\ldots,b_n) \in \mathrm{Dom}(M).$$

Therefore,

$$\mathrm{UEOWA}_{M;w,\rho_M}(a_1,\ldots,a_n) = \mathrm{UEWA}_{M;w}(b_1,\ldots,b_n) \in \mathrm{Dom}(M),$$

and

$$\mathrm{UEOWG}_{M;w,\rho_M}(a_1,\ldots,a_n) = \mathrm{UEWG}_{M;w}(b_1,\ldots,b_n) \in \mathrm{Dom}(M).$$

Thus both ordered operators are well-defined. □

Definition 6.9.13 (Induced uncertain Einstein aggregation operators on uncertain sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X,\mu_i) \qquad (i=1,\ldots,n)$$

be uncertain sets of type M, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

For

$$\Phi \in \{\mathrm{UEWA}_{M;w}, \mathrm{UEWG}_{M;w}, \mathrm{UEOWA}_{M;w,\rho_M}, \mathrm{UEOWG}_{M;w,\rho_M}\},$$

define

$$\Phi(\mathcal{U}_1,\ldots,\mathcal{U}_n) := (X,\mu_\Phi),$$

where

$$\mu_\Phi : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_\Phi(x) := \Phi\big(\mu_1(x),\ldots,\mu_n(x)\big) \qquad (x\in X).$$

These are called the *induced uncertain Einstein aggregation operators* on uncertain sets.

Theorem 6.9.14 (Well-definedness of the induced uncertain Einstein aggregation operators)**.** *For each*

$$\Phi \in \{\mathrm{UEWA}_{M;w}, \mathrm{UEWG}_{M;w}, \mathrm{UEOWA}_{M;w,\rho_M}, \mathrm{UEOWG}_{M;w,\rho_M}\},$$

the pair

$$\Phi(\mathcal{U}_1,\ldots,\mathcal{U}_n) = (X,\mu_\Phi)$$

is a well-defined uncertain set of type M.

Proof. Fix

$$\Phi \in \{\mathrm{UEWA}_{M;w}, \mathrm{UEWG}_{M;w}, \mathrm{UEOWA}_{M;w,\rho_M}, \mathrm{UEOWG}_{M;w,\rho_M}\}$$

and $x\in X$. Since each

$$\mathcal{U}_i = (X,\mu_i)$$

is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i=1,\ldots,n).$$

Hence

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the preceding well-definedness theorems,

$$\Phi\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Therefore,

$$\mu_\Phi(x) \in \mathrm{Dom}(M) \qquad (x \in X).$$

Thus

$$\mu_\Phi : X \to \mathrm{Dom}(M)$$

is a well-defined function, and consequently

$$(X, \mu_\Phi)$$

is a well-defined uncertain set of type M. □

A catalogue of representative Einstein aggregation operators classified by the dimension k of the degree-domain is presented in Table 6.9.

Table 6.9: A catalogue of representative Einstein aggregation operators by the dimension k of the degree-domain.

k	note	Representative Einstein aggregation operator(s)
1		Fuzzy Einstein Aggregation Operators [346, 347].
2		Intuitionistic Fuzzy Einstein Aggregation Operators [348,349]; Pythagorean Fuzzy Einstein Aggregation Operators [350, 351].
3		Hesitant Fuzzy Einstein Aggregation Operators [352–354]; Spherical fuzzy Einstein Aggregation Operators [355–357]; Picture fuzzy Einstein Aggregation Operators [358–360]; Neutrosophic Einstein Aggregation Operators [361, 362].

Reading guide. The table groups representative Einstein aggregation operators by the dimension k of their degree values.

6.10 Uncertain Geometric Aggregation Operators

Geometric aggregation operators combine multiple inputs multiplicatively, usually through weighted products, capturing proportional interaction, penalizing low values, and producing a representative fused assessment geometrically overall [363,364]. Uncertain geometric aggregation operators multiplicatively combine uncertain inputs, often via weighted products on uncertainty-valued data, preserving proportional structure while producing a representative aggregated uncertain assessment.

Definition 6.10.1 (Fuzzy Geometric Aggregation Operators)**.** (cf. [365, 366]) Let

$$I := [0, 1], \qquad n \geq 1,$$

and adopt the standard convention

$$0^0 := 1.$$

1. The *fuzzy geometric mean* is the mapping

$$\mathrm{FGM}_n : I^n \to I$$

defined by

$$\mathrm{FGM}_n(x_1, \ldots, x_n) := \prod_{i=1}^{n} x_i^{1/n}.$$

2. Let

$$w = (w_1, \ldots, w_n) \in I^n, \qquad \sum_{i=1}^{n} w_i = 1.$$

The *fuzzy weighted geometric operator* is the mapping

$$\mathrm{FWG}_w : I^n \to I$$

defined by

$$\mathrm{FWG}_w(x_1, \ldots, x_n) := \prod_{i=1}^{n} x_i^{w_i}.$$

3. Let

$$w = (w_1, \ldots, w_n) \in I^n, \qquad \sum_{i=1}^{n} w_i = 1.$$

For

$$(x_1, \ldots, x_n) \in I^n,$$

let

$$(y_1, \ldots, y_n)$$

be a permutation of

$$(x_1, \ldots, x_n)$$

such that

$$y_1 \geq y_2 \geq \cdots \geq y_n.$$

The *fuzzy ordered weighted geometric operator* is the mapping

$$\mathrm{FOWG}_w : I^n \to I$$

defined by

$$\mathrm{FOWG}_w(x_1, \ldots, x_n) := \prod_{i=1}^{n} y_i^{w_i}.$$

4. Let

$$\omega = (\omega_1, \ldots, \omega_n) \in I^n, \qquad \sum_{i=1}^{n} \omega_i = 1,$$

and let

$$w = (w_1, \ldots, w_n) \in I^n, \qquad \sum_{i=1}^{n} w_i = 1.$$

For

$$(x_1, \ldots, x_n) \in I^n,$$

define the preweighted values

$$u_i := x_i^{\omega_i} \qquad (i = 1, \ldots, n).$$

Let

$$(z_1, \ldots, z_n)$$

be a permutation of

$$(u_1, \ldots, u_n)$$

such that

$$z_1 \geq z_2 \geq \cdots \geq z_n.$$

The *fuzzy hybrid geometric operator* is the mapping

$$\mathrm{FHG}_{\omega,w} : I^n \to I$$

defined by

$$\mathrm{FHG}_{\omega,w}(x_1, \ldots, x_n) := \prod_{i=1}^{n} z_i^{w_i}.$$

In the uncertain-set setting, such operators can be extended naturally to multi-dimensional degree values by using componentwise powers and products.

Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) = [0,1]^d, \qquad d \geq 1.$$

Let

$$\mathbf{1}_d := (1, \ldots, 1) \in [0,1]^d.$$

For

$$a = (a^{(1)}, \ldots, a^{(d)}) \in [0,1]^d$$

and $\lambda \in [0,1]$, define the *componentwise weighted power* by

$$a^{\odot\lambda} := \begin{cases} \big((a^{(1)})^{\lambda}, \ldots, (a^{(d)})^{\lambda}\big), & \lambda > 0, \\ \mathbf{1}_d, & \lambda = 0. \end{cases}$$

For

$$a_1, \ldots, a_n \in [0,1]^d,$$

define their *componentwise product* by

$$\bigodot_{i=1}^{n} a_i := \left(\prod_{i=1}^{n} a_i^{(1)},\ \prod_{i=1}^{n} a_i^{(2)},\ \ldots,\ \prod_{i=1}^{n} a_i^{(d)} \right).$$

Lemma 6.10.2 (Closure under componentwise weighted powers and products)**.** *Let*

$$a_1, \ldots, a_n \in [0,1]^d$$

and

$$\lambda_1, \ldots, \lambda_n \in [0,1].$$

Then

$$a_i^{\odot\lambda_i} \in [0,1]^d \qquad (i = 1, \ldots, n),$$

and

$$\bigodot_{i=1}^{n} a_i^{\odot\lambda_i} \in [0,1]^d.$$

Proof. Fix $i \in \{1, \ldots, n\}$ and $\ell \in \{1, \ldots, d\}$. If $\lambda_i > 0$, then

$$0 \leq a_i^{(\ell)} \leq 1 \quad \Longrightarrow \quad 0 \leq \big(a_i^{(\ell)}\big)^{\lambda_i} \leq 1.$$

If $\lambda_i = 0$, then by definition

$$a_i^{\odot 0} = \mathbf{1}_d \in [0,1]^d.$$

Hence

$$a_i^{\odot \lambda_i} \in [0,1]^d \qquad (i = 1, \ldots, n).$$

Now fix $\ell \in \{1, \ldots, d\}$. Since each ℓ-th component of $a_i^{\odot \lambda_i}$ lies in $[0,1]$, their product also lies in $[0,1]$. Therefore each component of

$$\bigodot_{i=1}^{n} a_i^{\odot \lambda_i}$$

belongs to $[0,1]$, and so

$$\bigodot_{i=1}^{n} a_i^{\odot \lambda_i} \in [0,1]^d.$$

□

Definition 6.10.3 (Uncertain geometric mean)**.** Let $n \geq 1$. The *uncertain geometric mean* on M is the mapping

$$\mathrm{UGM}_{M,n} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UGM}_{M,n}(a_1, \ldots, a_n) := \bigodot_{i=1}^{n} a_i^{\odot 1/n}.$$

Definition 6.10.4 (Uncertain weighted geometric operator)**.** Let

$$w = (w_1, \ldots, w_n) \in [0,1]^n, \qquad \sum_{i=1}^{n} w_i = 1.$$

The *uncertain weighted geometric operator* on M associated with w is the mapping

$$\mathrm{UWG}_{M;w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UWG}_{M;w}(a_1, \ldots, a_n) := \bigodot_{i=1}^{n} a_i^{\odot w_i}.$$

Definition 6.10.5 (Canonical ρ_M-ordering)**.** Let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map. For

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$

a permutation σ of $\{1, \ldots, n\}$ is called the *canonical ρ_M-ordering permutation* if:

1. $$\rho_M(a_{\sigma(1)}) \geq \rho_M(a_{\sigma(2)}) \geq \cdots \geq \rho_M(a_{\sigma(n)});$$

2. whenever
$$\rho_M(a_{\sigma(i)}) = \rho_M(a_{\sigma(j)}) \quad \text{and} \quad i < j,$$
one has
$$\sigma(i) < \sigma(j).$$

Thus the values are ordered in decreasing order of their ranking values, and ties are broken by preserving the smaller original index first.

Lemma 6.10.6 (Existence and uniqueness of the canonical ordering)**.** *For every*

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n,$$

there exists a unique canonical ρ_M-ordering permutation.

Proof. Define a binary relation $\preccurlyeq$ on $\{1, \dots, n\}$ by

$$u \preccurlyeq v \iff \big(\rho_M(a_u) > \rho_M(a_v)\big) \text{ or } \big(\rho_M(a_u) = \rho_M(a_v) \text{ and } u \le v\big).$$

This relation totally orders the finite set $\{1, \dots, n\}$, since any two indices are comparable and ties are resolved by the natural order of indices. Hence there exists a unique permutation

$$\sigma(1), \dots, \sigma(n)$$

listing the indices in decreasing order with respect to this rule. By construction, σ is exactly the unique canonical ρ_M-ordering permutation. ☐

Definition 6.10.7 (Uncertain ordered weighted geometric operator)**.** Let

$$w = (w_1, \dots, w_n) \in [0,1]^n, \qquad \sum_{i=1}^{n} w_i = 1,$$

and let
$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$
be a fixed ranking map. For
$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n,$$
let σ be the unique canonical ρ_M-ordering permutation of $(a_1, \dots, a_n)$, and write

$$b_i := a_{\sigma(i)} \qquad (i = 1, \dots, n).$$

The *uncertain ordered weighted geometric operator* on M associated with w and ρ_M is the mapping
$$\mathrm{UOWG}_{M;w,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$
defined by
$$\mathrm{UOWG}_{M;w,\rho_M}(a_1, \dots, a_n) := \bigodot_{i=1}^{n} b_i^{\odot w_i}.$$

Definition 6.10.8 (Uncertain hybrid geometric operator)**.** Let

$$\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n, \qquad \sum_{i=1}^{n} \omega_i = 1,$$

and let

$$w = (w_1, \ldots, w_n) \in [0,1]^n, \qquad \sum_{i=1}^{n} w_i = 1.$$

Let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map. For

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$

define the preweighted values

$$u_i := a_i^{\odot \omega_i} \qquad (i = 1, \ldots, n).$$

Let σ be the unique canonical ρ_M-ordering permutation of

$$(u_1, \ldots, u_n),$$

and write

$$z_i := u_{\sigma(i)} \qquad (i = 1, \ldots, n).$$

The *uncertain hybrid geometric operator* on M associated with ω, w, and ρ_M is the mapping

$$\mathrm{UHG}_{M;\omega,w,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

defined by

$$\mathrm{UHG}_{M;\omega,w,\rho_M}(a_1, \ldots, a_n) := \bigodot_{i=1}^{n} z_i^{\odot w_i}.$$

Remark 6.10.9. If $d = 1$, $\mathrm{Dom}(M) = [0,1]$, and

$$\rho_M(a) = a \qquad (a \in [0,1]),$$

then $\mathrm{UGM}_{M,n}$, $\mathrm{UWG}_{M;w}$, $\mathrm{UOWG}_{M;w,\rho_M}$, and $\mathrm{UHG}_{M;\omega,w,\rho_M}$ reduce to the ordinary scalar fuzzy geometric mean, fuzzy weighted geometric operator, fuzzy ordered weighted geometric operator, and fuzzy hybrid geometric operator, respectively.

Theorem 6.10.10 (Well-definedness of the uncertain geometric mean)**.** *The mapping*

$$\mathrm{UGM}_{M,n} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

Since

$$\frac{1}{n} \in [0,1],$$

the preceding closure lemma yields

$$\bigodot_{i=1}^{n} a_i^{\odot 1/n} \in [0,1]^d = \mathrm{Dom}(M).$$

Therefore

$$\mathrm{UGM}_{M,n}(a_1, \ldots, a_n) \in \mathrm{Dom}(M),$$

so $\mathrm{UGM}_{M,n}$ is well-defined. □

Theorem 6.10.11 (Well-definedness of the uncertain weighted geometric operator)**.** *The mapping*

$$\mathrm{UWG}_{M;w} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

Since

$$w_i \in [0, 1] \qquad (i = 1, \ldots, n),$$

the closure lemma gives

$$\bigodot_{i=1}^{n} a_i^{\odot w_i} \in [0, 1]^d = \mathrm{Dom}(M).$$

Hence

$$\mathrm{UWG}_{M;w}(a_1, \ldots, a_n) \in \mathrm{Dom}(M),$$

and thus $\mathrm{UWG}_{M;w}$ is well-defined. □

Theorem 6.10.12 (Well-definedness of the uncertain ordered weighted geometric operator)**.** *The mapping*

$$\mathrm{UOWG}_{M;w,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$

By the lemma on canonical ordering, there exists a unique canonical ρ_M-ordering permutation σ. Hence the reordered tuple

$$(b_1, \ldots, b_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

is uniquely determined and satisfies

$$b_i \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Since each $w_i \in [0, 1]$, the closure lemma implies

$$\bigodot_{i=1}^{n} b_i^{\odot w_i} \in [0, 1]^d = \mathrm{Dom}(M).$$

Therefore

$$\mathrm{UOWG}_{M;w,\rho_M}(a_1, \ldots, a_n) \in \mathrm{Dom}(M),$$

so $\mathrm{UOWG}_{M;w,\rho_M}$ is well-defined. □

Theorem 6.10.13 (Well-definedness of the uncertain hybrid geometric operator)**.** *The mapping*

$$\mathrm{UHG}_{M;\omega,w,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let
$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$
For each $i \in \{1, \dots, n\}$, since $\omega_i \in [0, 1]$, the closure lemma gives
$$u_i = a_i^{\odot \omega_i} \in [0, 1]^d = \mathrm{Dom}(M).$$
Hence
$$(u_1, \dots, u_n) \in \mathrm{Dom}(M)^n.$$
By the lemma on canonical ordering, there exists a unique canonical ρ_M-ordering permutation σ of $(u_1, \dots, u_n)$. Therefore the reordered tuple
$$(z_1, \dots, z_n) = (u_{\sigma(1)}, \dots, u_{\sigma(n)})$$
is uniquely determined and satisfies
$$z_i \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$
Since each $w_i \in [0, 1]$, the closure lemma yields
$$\bigodot_{i=1}^{n} z_i^{\odot w_i} \in [0, 1]^d = \mathrm{Dom}(M).$$
Thus
$$\mathrm{UHG}_{M;\omega,w,\rho_M}(a_1, \dots, a_n) \in \mathrm{Dom}(M),$$
and therefore $\mathrm{UHG}_{M;\omega,w,\rho_M}$ is well-defined. □

Definition 6.10.14 (Induced uncertain geometric aggregation operators on uncertain sets)**.** Let X be a nonempty universe, and let
$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$
be uncertain sets of type M, where
$$\mu_i : X \to \mathrm{Dom}(M).$$
For
$$\Phi \in \{\mathrm{UGM}_{M,n}, \mathrm{UWG}_{M;w}, \mathrm{UOWG}_{M;w,\rho_M}, \mathrm{UHG}_{M;\omega,w,\rho_M}\},$$
define
$$\Phi(\mathcal{U}_1, \dots, \mathcal{U}_n) := (X, \mu_\Phi),$$
where
$$\mu_\Phi : X \to \mathrm{Dom}(M)$$
is given pointwise by
$$\mu_\Phi(x) := \Phi\big(\mu_1(x), \dots, \mu_n(x)\big) \qquad (x \in X).$$
These are called the *induced uncertain geometric aggregation operators* on uncertain sets.

Theorem 6.10.15 (Well-definedness of the induced uncertain geometric aggregation operators)**.** *For each*
$$\Phi \in \{\mathrm{UGM}_{M,n}, \mathrm{UWG}_{M;w}, \mathrm{UOWG}_{M;w,\rho_M}, \mathrm{UHG}_{M;\omega,w,\rho_M}\},$$
the pair
$$\Phi(\mathcal{U}_1, \dots, \mathcal{U}_n) = (X, \mu_\Phi)$$
is a well-defined uncertain set of type M.

Proof. Fix
$$\Phi \in \{\mathrm{UGM}_{M,n}, \mathrm{UWG}_{M;w}, \mathrm{UOWG}_{M;w,\rho_M}, \mathrm{UHG}_{M;\omega,w,\rho_M}\}$$
and $x \in X$. Since each
$$\mathcal{U}_i = (X, \mu_i)$$
is an uncertain set of type M, one has
$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$
Hence
$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$
By the preceding well-definedness theorems,
$$\Phi\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$
Therefore
$$\mu_\Phi(x) \in \mathrm{Dom}(M) \qquad (x \in X).$$
Thus
$$\mu_\Phi : X \to \mathrm{Dom}(M)$$
is a well-defined function, and consequently
$$(X, \mu_\Phi)$$
is a well-defined uncertain set of type M. □

A catalogue of representative geometric aggregation operators classified by the dimension k of the degree-domain is presented in Table 6.10.

Table 6.10: Catalogue of representative geometric aggregation operators classified by the dimension k of the degree-domain.

k	**Framework**	**Representative geometric aggregation operators**
1	Fuzzy	Fuzzy Geometric Aggregation Operators [367]
2	Intuitionistic Fuzzy	Intuitionistic Fuzzy Geometric Aggregation Operators [368–370]
2	Pythagorean Fuzzy	Pythagorean Fuzzy Geometric Aggregation Operators [371, 372]
3	Picture Fuzzy	Picture Fuzzy Geometric Aggregation Operators [373]
3	Neutrosophic	Neutrosophic Geometric Aggregation Operators [374, 375]

Note. Hesitant fuzzy geometric aggregation operators [376] are not placed in the table above, because the degree information of a hesitant fuzzy element is typically a finite subset of $[0, 1]$, rather than an element of a fixed finite-dimensional domain $[0, 1]^k$.

Chapter 7

Uncertain Integral / Dependence-Based Operators

In this chapter, we discuss uncertain integral / dependence-based operators. For reference, a concise comparison of copula, Choquet integral, and Sugeno integral is given in Table 7.1.

Table 7.1: A concise comparison of copula, Choquet integral, and Sugeno integral

Aspect	**Copula**	**Choquet integral**	**Sugeno integral**
Basic role	Models dependence among components	Aggregates inputs under a capacity with interaction effects	Aggregates inputs under a capacity in an ordinal max–min style
Main structure	Joint dependence function on $[0,1]^n$	Capacity and weighted summation of ordered increments	Capacity and max–min combination of ordered inputs
Additivity	Not an additive aggregator	Generalizes weighted averaging beyond additivity	Non-additive and non-linear
Order dependence	Not primarily order-based	Yes, via ranking or sorting of inputs	Yes, via ranking or sorting of inputs
Interaction handling	Captures dependence structure directly	Captures redundancy and synergy through the capacity	Captures ordinal importance and qualitative interaction
Output style	Describes joint behavior or dependence	Numerical aggregated value	Ordinal or lattice-style aggregated value
Typical use	Dependence modeling and multivariate uncertainty	Interaction-aware decision making and information fusion	Qualitative decision making and ordinal aggregation

7.1 Uncertain copula

An uncertain copula models dependence between uncertain variables or memberships, separating marginal uncertainty from joint interaction while preserving grounded boundary and monotonicity conditions under uncertainty.

Definition 7.1.1 (Fuzzy Copula)**.** [377, 378] A function

$$C : [0,1]^2 \to [0,1]$$

is called a *fuzzy copula* if

$$C(x,0) = C(0,x) = 0, \qquad C(x,1) = C(1,x) = x \qquad (x \in [0,1]),$$

and

$$C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geq 0$$

for all $x_1 \leq x_2$ and $y_1 \leq y_2$ in $[0,1]$.

Definition 7.1.2 (Neutrosophic Copula)**.** Let

$$\mathbb{SVN} = \left\{(T,I,F) \in [0,1]^3 : \ 0 \leq T+I+F \leq 3\right\},$$

let

$$C : [0,1]^2 \to [0,1]$$

be a copula, and let

$$C^*(x,y) = 1 - C(1-x, 1-y).$$

A *neutrosophic copula* is the mapping

$$\mathcal{C}_N : \mathbb{SVN}^2 \to \mathbb{SVN}$$

given by

$$\mathcal{C}_N\big((T_1,I_1,F_1),(T_2,I_2,F_2)\big) = \big(C(T_1,T_2),\, C^*(I_1,I_2),\, C^*(F_1,F_2)\big).$$

Let M be a fixed uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

In order to define a copula-type operation on $\mathrm{Dom}(M)$ in a model-independent and mathematically consistent way, we use a coordinate representation of uncertain values and apply classical scalar copulas componentwise.

Definition 7.1.3 (Copula-admissible representation of an uncertain model)**.** A *copula-admissible representation* of the uncertain model M consists of the following data:

(i) a nonempty set

$$C_M \subseteq [0,1]^q$$

for some integer $q \geq 1$;

(ii) a representation map

$$\Phi_M : \mathrm{Dom}(M) \to C_M;$$

(iii) a reconstruction map

$$\Psi_M : C_M \to \mathrm{Dom}(M)$$

such that

$$\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)};$$

(iv) distinguished elements

$$0_M, 1_M \in \mathrm{Dom}(M)$$

whose coordinate representatives are

$$\Phi_M(0_M) = \mathbf{0} := (0, \dots, 0), \qquad \Phi_M(1_M) = \mathbf{1} := (1, \dots, 1);$$

(v) classical scalar copulas

$$C_1, \dots, C_q : [0,1]^2 \to [0,1]$$

such that the induced coordinatewise map

$$\mathbf{C}_M : C_M \times C_M \to [0,1]^q$$

defined by

$$\mathbf{C}_M(u,v) := \big(C_1(u_1,v_1), \dots, C_q(u_q,v_q)\big), \qquad u = (u_1, \dots, u_q),\ v = (v_1, \dots, v_q),$$

satisfies the closure condition

$$\mathbf{C}_M(C_M \times C_M) \subseteq C_M.$$

Definition 7.1.4 (M-copula)**.** Assume that a copula-admissible representation of M has been fixed. The induced *M-copula* is the mapping

$$\mathcal{C}_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

defined by

$$\mathcal{C}_M(a,b) := \Psi_M\Big(\mathbf{C}_M\big(\Phi_M(a), \Phi_M(b)\big)\Big) \qquad (a, b \in \mathrm{Dom}(M)).$$

Definition 7.1.5 (Uncertain copula of U-sets)**.** Let X be a nonempty set, and let

$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$

be uncertain sets of the same type M, where

$$\mu_{\mathcal{U}}, \mu_{\mathcal{V}} : X \to \mathrm{Dom}(M).$$

The *uncertain copula* of $\mathcal{U}$ and $\mathcal{V}$ induced by $\mathcal{C}_M$, denoted by

$$\mathrm{Cop}_M(\mathcal{U}, \mathcal{V}),$$

is defined by

$$\mathrm{Cop}_M(\mathcal{U}, \mathcal{V}) := \big(X, \mu_{\mathrm{Cop}_M(\mathcal{U},\mathcal{V})}\big),$$

where

$$\mu_{\mathrm{Cop}_M(\mathcal{U},\mathcal{V})} : X \to \mathrm{Dom}(M)$$

is given pointwise by

$$\mu_{\mathrm{Cop}_M(\mathcal{U},\mathcal{V})}(x) = \mathcal{C}_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \qquad (x \in X).$$

Theorem 7.1.6 (Well-definedness of the M-copula)**.** *Under the assumptions above, the mapping*

$$\mathcal{C}_M : \mathrm{Dom}(M) \times \mathrm{Dom}(M) \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let
$$a, b \in \mathrm{Dom}(M).$$
Then
$$\Phi_M(a) \in C_M, \qquad \Phi_M(b) \in C_M.$$
Hence
$$\big(\Phi_M(a), \Phi_M(b)\big) \in C_M \times C_M.$$
By the closure assumption on $\mathbf{C}_M$,
$$\mathbf{C}_M\big(\Phi_M(a), \Phi_M(b)\big) \in C_M.$$
Since
$$\Psi_M : C_M \to \mathrm{Dom}(M),$$
it follows that
$$\Psi_M\Big(\mathbf{C}_M\big(\Phi_M(a), \Phi_M(b)\big)\Big) \in \mathrm{Dom}(M).$$
Therefore
$$\mathcal{C}_M(a, b) \in \mathrm{Dom}(M) \qquad \text{for all } a, b \in \mathrm{Dom}(M).$$
Thus $\mathcal{C}_M$ is well-defined. □

Theorem 7.1.7 (Well-definedness of uncertain copula of U-sets)**.** *Let X be a nonempty set, and let*
$$\mathcal{U} = (X, \mu_{\mathcal{U}}), \qquad \mathcal{V} = (X, \mu_{\mathcal{V}})$$
be uncertain sets of type M. Then
$$\mathrm{Cop}_M(\mathcal{U}, \mathcal{V})$$
is a well-defined uncertain set of type M on X.

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ are uncertain sets of type M, one has
$$\mu_{\mathcal{U}}(x) \in \mathrm{Dom}(M), \qquad \mu_{\mathcal{V}}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$
Fix $x \in X$. By the preceding theorem,
$$\mathcal{C}_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big) \in \mathrm{Dom}(M).$$
Hence
$$\mu_{\mathrm{Cop}_M(\mathcal{U},\mathcal{V})}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$
Therefore the pointwise assignment
$$x \longmapsto \mathcal{C}_M\big(\mu_{\mathcal{U}}(x), \mu_{\mathcal{V}}(x)\big)$$
defines a mapping
$$\mu_{\mathrm{Cop}_M(\mathcal{U},\mathcal{V})} : X \to \mathrm{Dom}(M).$$
Consequently,
$$\mathrm{Cop}_M(\mathcal{U}, \mathcal{V}) = \big(X, \mu_{\mathrm{Cop}_M(\mathcal{U},\mathcal{V})}\big)$$
is an uncertain set of type M on X. Thus the uncertain copula of U-sets is well-defined. □

Proposition 7.1.8 (Basic inherited properties)**.** *Assume that each scalar copula C_j is a classical copula on $[0, 1]$ for $j = 1, \ldots, q$. Then the induced M-copula $\mathcal{C}_M$ satisfies the following properties:*

(i) ***Groundedness:***

$$\mathcal{C}_M(a, 0_M) = \mathcal{C}_M(0_M, a) = 0_M \qquad \textit{for all } a \in \mathrm{Dom}(M).$$

(ii) ***Identity at the top element:***

$$\mathcal{C}_M(a, 1_M) = \mathcal{C}_M(1_M, a) = a \qquad \textit{for all } a \in \mathrm{Dom}(M).$$

(iii) ***Commutativity:***

$$\mathcal{C}_M(a, b) = \mathcal{C}_M(b, a) \qquad \textit{for all } a, b \in \mathrm{Dom}(M).$$

Proof. Let

$$a \in \mathrm{Dom}(M), \qquad \Phi_M(a) = (u_1, \dots, u_q).$$

(i) Since $\Phi_M(0_M) = \mathbf{0} = (0, \dots, 0)$, one has

$$\mathbf{C}_M\big(\Phi_M(a), \Phi_M(0_M)\big) = \big(C_1(u_1, 0), \dots, C_q(u_q, 0)\big).$$

Because each C_j is a classical copula,

$$C_j(u_j, 0) = 0 \qquad (j = 1, \dots, q).$$

Hence

$$\mathbf{C}_M\big(\Phi_M(a), \Phi_M(0_M)\big) = \mathbf{0}.$$

Therefore

$$\mathcal{C}_M(a, 0_M) = \Psi_M(\mathbf{0}) = \Psi_M(\Phi_M(0_M)) = 0_M.$$

Similarly,

$$\mathcal{C}_M(0_M, a) = 0_M.$$

(ii) Since $\Phi_M(1_M) = \mathbf{1} = (1, \dots, 1)$, one has

$$\mathbf{C}_M\big(\Phi_M(a), \Phi_M(1_M)\big) = \big(C_1(u_1, 1), \dots, C_q(u_q, 1)\big).$$

Because each C_j is a classical copula,

$$C_j(u_j, 1) = u_j \qquad (j = 1, \dots, q).$$

Thus

$$\mathbf{C}_M\big(\Phi_M(a), \Phi_M(1_M)\big) = \Phi_M(a).$$

Hence

$$\mathcal{C}_M(a, 1_M) = \Psi_M(\Phi_M(a)) = a.$$

Similarly,

$$\mathcal{C}_M(1_M, a) = a.$$

(iii) For $a, b \in \mathrm{Dom}(M)$, write

$$\Phi_M(a) = (u_1, \ldots, u_q), \qquad \Phi_M(b) = (v_1, \ldots, v_q).$$

Since each scalar copula C_j is commutative,

$$C_j(u_j, v_j) = C_j(v_j, u_j) \qquad (j = 1, \ldots, q).$$

Therefore

$$\mathbf{C}_M\big(\Phi_M(a), \Phi_M(b)\big) = \mathbf{C}_M\big(\Phi_M(b), \Phi_M(a)\big),$$

and hence

$$\mathcal{C}_M(a, b) = \mathcal{C}_M(b, a).$$

□

A catalogue of representative copulas classified by the dimension k of the degree-domain is presented in Table 7.2.

Table 7.2: A catalogue of representative copulas by the dimension k of the degree-domain.

k	note	Representative copula(s)
1		Fuzzy copula.
2		Intuitionistic Fuzzy copula [379]; Pythagorean fuzzy copula [380]; Fermatean Fuzzy copula [381].
3		Spherical Fuzzy copula [378]; Neutrosophic copula [382].
n	$(n \geq 1)$	Plithogenic copula.

Reading guide. The table groups representative copulas by the dimension k of their degree values.

As related concepts other than the above, quasi-copulas [383, 384], co-copulas [385, 386], and semi-copulas [387, 388], among others, are also known.

7.2 Uncertain Choquet integral

An uncertain Choquet integral aggregates uncertain inputs with respect to an uncertain capacity, capturing interaction, redundancy, and importance beyond additive weighting assumptions in decision contexts.

Definition 7.2.1 (Fuzzy Measure)**.** Let

$$N = \{1, 2, \ldots, n\}.$$

A set function

$$\mu : 2^N \to [0, 1]$$

is called a *fuzzy measure* if it satisfies

$$\mu(\varnothing) = 0, \qquad \mu(N) = 1,$$

and

$$A \subseteq B \implies \mu(A) \leq \mu(B) \qquad \text{for all } A, B \subseteq N.$$

Definition 7.2.2 (Fuzzy Choquet Integral)**.** [389, 390] Let

$$N = \{1, 2, \ldots, n\},$$

and let

$$\mu : 2^N \to [0, 1]$$

be a fuzzy measure on N. For

$$x = (x_1, x_2, \ldots, x_n) \in [0, 1]^n,$$

choose a permutation σ of N such that

$$x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)}.$$

Set

$$x_{\sigma(0)} := 0,$$

and for each $i = 1, 2, \ldots, n$, define

$$A_i := \{\sigma(i), \sigma(i+1), \ldots, \sigma(n)\}.$$

Then the *fuzzy Choquet integral* of x with respect to μ is defined by

$$C_\mu(x) = \sum_{i=1}^{n} \big(x_{\sigma(i)} - x_{\sigma(i-1)}\big)\mu(A_i).$$

Definition 7.2.3 (Neutrosophic Choquet Integral)**.** (cf. [391, 392]) Let

$$\tilde{f} : X \to L$$

be an interval neutrosophic number function on a finite set

$$X = \{x_1, x_2, \ldots, x_n\},$$

and let

$$\mu : 2^X \to [0, 1]$$

be a fuzzy measure. Let $\preceq$ be a fixed total order on L, and choose a permutation σ such that

$$\tilde{f}(x_{\sigma(1)}) \preceq \tilde{f}(x_{\sigma(2)}) \preceq \cdots \preceq \tilde{f}(x_{\sigma(n)}).$$

Set

$$A_i := \{x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\}, \qquad A_{n+1} := \varnothing.$$

Then the *neutrosophic Choquet integral* of $\tilde{f}$ with respect to μ is

$$(C)\int \tilde{f}\, d\mu = \bigoplus_{i=1}^{n} \big(\mu(A_i) - \mu(A_{i+1})\big)\tilde{f}(x_{\sigma(i)}).$$

An uncertain Choquet integral aggregates finitely many uncertain values with respect to a monotone capacity, thereby capturing importance and interaction effects beyond additive weighting.

Definition 7.2.4 (Uncertain capacity)**.** Let

$$N = \{1, 2, \ldots, n\}.$$

A set function

$$\nu : 2^N \to [0, 1]$$

is called an *uncertain capacity* on N if

$$\nu(\varnothing) = 0, \qquad \nu(N) = 1,$$

and

$$A \subseteq B \implies \nu(A) \leq \nu(B) \qquad \text{for all } A, B \subseteq N.$$

Definition 7.2.5 (Choquet-admissible representation)**.** Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

A triple

$$(C_M, \Phi_M, \Psi_M)$$

is called a *Choquet-admissible representation* for M if there exists an integer $m \geq 1$ such that:

1. $C_M \subseteq \mathbb{R}^m$ is a nonempty convex set;
2. $$\Phi_M : \mathrm{Dom}(M) \to C_M$$
   is an encoding map;
3. $$\Psi_M : C_M \to \mathrm{Dom}(M)$$
   is a decoding map satisfying
   $$\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)}.$$

Definition 7.2.6 (Ranking map)**.** Let M be an uncertain model. A mapping

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

is called a *ranking map* for M.

Definition 7.2.7 (Uncertain Choquet integral)**.** Let

$$N = \{1, 2, \ldots, n\},$$

let M be an uncertain model with degree-domain $\mathrm{Dom}(M)$, let

$$(C_M, \Phi_M, \Psi_M)$$

be a Choquet-admissible representation for M, let

$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map, and let

$$\nu : 2^N \to [0, 1]$$

be an uncertain capacity.

For

$$a = (a_1, a_2, \dots, a_n) \in \mathrm{Dom}(M)^n,$$

let σ be the unique permutation of N such that the sequence

$$\big(\rho_M(a_{\sigma(1)}), \sigma(1)\big), \big(\rho_M(a_{\sigma(2)}), \sigma(2)\big), \dots, \big(\rho_M(a_{\sigma(n)}), \sigma(n)\big)$$

is lexicographically nondecreasing. Thus the values are ordered increasingly by ρ_M, with ties broken by the indices.

For $i = 1, 2, \dots, n$, define

$$A_i := \{\sigma(i), \sigma(i+1), \dots, \sigma(n)\}, \qquad A_{n+1} := \varnothing.$$

The *uncertain Choquet integral* of a with respect to ν is defined by

$$\mathcal{C}_\nu^M(a_1, \dots, a_n) := \Psi_M\left(\sum_{i=1}^{n} \big(\nu(A_i) - \nu(A_{i+1})\big)\, \Phi_M(a_{\sigma(i)})\right).$$

Theorem 7.2.8 (Well-definedness of the uncertain Choquet integral)**.** *With the notation above, the mapping*

$$\mathcal{C}_\nu^M : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Fix

$$a = (a_1, \dots, a_n) \in \mathrm{Dom}(M)^n.$$

Since $\rho_M(a_j) \in \mathbb{R}$ for each $j \in N$, the lexicographic rule determines a unique permutation σ of N. Hence the sets

$$A_i = \{\sigma(i), \sigma(i+1), \dots, \sigma(n)\} \qquad (i = 1, \dots, n+1)$$

are uniquely determined.

Now set

$$\alpha_i := \nu(A_i) - \nu(A_{i+1}) \qquad (i = 1, \dots, n).$$

Because

$$A_{i+1} \subseteq A_i$$

for each i, and ν is monotone, we have

$$\alpha_i \geq 0 \qquad (i = 1, \dots, n).$$

Moreover, since $A_1 = N$ and $A_{n+1} = \varnothing$,

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \big(\nu(A_i) - \nu(A_{i+1})\big) = \nu(A_1) - \nu(A_{n+1}) = \nu(N) - \nu(\varnothing) = 1.$$

Thus $(\alpha_1, \dots, \alpha_n)$ is a system of nonnegative coefficients summing to 1.

For each i,

$$a_{\sigma(i)} \in \mathrm{Dom}(M),$$

so

$$\Phi_M(a_{\sigma(i)}) \in C_M.$$

Since C_M is convex and the coefficients α_i are nonnegative with total sum 1, it follows that

$$z := \sum_{i=1}^{n} \alpha_i \, \Phi_M(a_{\sigma(i)}) \in C_M.$$

Therefore $\Psi_M(z)$ is defined and belongs to $\mathrm{Dom}(M)$, because

$$\Psi_M : C_M \to \mathrm{Dom}(M).$$

Hence

$$\mathcal{C}_\nu^M(a_1, \ldots, a_n) = \Psi_M(z) \in \mathrm{Dom}(M).$$

Since every step in the construction is uniquely determined, the value

$$\mathcal{C}_\nu^M(a_1, \ldots, a_n)$$

exists uniquely in $\mathrm{Dom}(M)$. Therefore

$$\mathcal{C}_\nu^M : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined. □

A catalogue of representative Choquet integrals classified by the dimension k of the degree-domain is presented in Table 7.3.

Table 7.3: A catalogue of representative Choquet integrals by the dimension k of the degree-domain.

k	note	Representative Choquet integral(s)
1		Fuzzy Choquet integral [393].
2		Intuitionistic Fuzzy Choquet integral [394,395]; Pythagorean Fuzzy Choquet integral [396–398]
3		Neutrosophic Choquet integral [399, 400]; Hesitant Fuzzy Choquet integral [401, 402]; Picture Fuzzy Choquet integral [403, 404]; Spherical Fuzzy Choquet integral [405, 406].
n	$(n \geq 1)$	Plithogenic Choquet integral.

Reading guide. The table groups representative Choquet integrals by the dimension k of their degree values.

As related concepts, the generalized Choquet integral [407–409], the Balancing Choquet integral [410, 411], the Symmetric Choquet integral [412, 413], and the Level-dependent Choquet integral [414–416] are also known.

7.3 Uncertain Sugeno integral

A Sugeno integral is a nonlinear aggregation operator based on a fuzzy measure, combining values through max–min calculus to model ordinal, interactive decision information [417–420]. An uncertain Sugeno integral aggregates uncertain inputs using max min style calculus with an uncertain capacity, emphasizing ordinal structure and qualitative interaction in decision making.

Definition 7.3.1 (Fuzzy Sugeno Integral)**.** (cf. [421]) Let

$$N = \{1, 2, \ldots, n\},$$

let

$$\mu : 2^N \to [0, 1]$$

be a fuzzy measure on N, and let

$$x = (x_1, x_2, \ldots, x_n) \in [0, 1]^n.$$

Choose a permutation σ of N such that

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}.$$

For each $i = 1, 2, \ldots, n$, define

$$A_i := \{\sigma(i), \sigma(i+1), \ldots, \sigma(n)\}.$$

Then the *fuzzy Sugeno integral* of x with respect to μ is defined by

$$S_\mu(x) = \bigvee_{i=1}^{n} \Big(x_{\sigma(i)} \wedge \mu(A_i) \Big),$$

where $\vee$ and $\wedge$ denote the maximum and minimum operators, respectively.

Definition 7.3.2 (Neutrosophic Sugeno Integral)**.** Let

$$\tilde{f} : X \to [0, 1]^3, \qquad \tilde{f}(x) = \big(T_{\tilde{f}}(x),\, I_{\tilde{f}}(x),\, F_{\tilde{f}}(x)\big),$$

be a single-valued neutrosophic-valued measurable function, where

$$T_{\tilde{f}},\ I_{\tilde{f}},\ F_{\tilde{f}} : X \to [0, 1].$$

Let μ be a fuzzy measure on $(X, \mathcal{A})$.

The *neutrosophic Sugeno integral* of $\tilde{f}$ with respect to μ is defined componentwise by

$$\mathrm{NSug}_\mu(\tilde{f}) := \Big\langle \mathrm{Sug}_\mu(T_{\tilde{f}}), \mathrm{Sug}_\mu(I_{\tilde{f}}), \mathrm{Sug}_\mu(F_{\tilde{f}}) \Big\rangle .$$

Equivalently,

$$\mathrm{NSug}_\mu(\tilde{f}) = \Bigg\langle \sup_{\alpha \in [0,1]} \Big(\alpha \wedge \mu(\{x \in X : T_{\tilde{f}}(x) \geq \alpha\}) \Big),$$

$$\sup_{\alpha \in [0,1]} \Big(\alpha \wedge \mu(\{x \in X : I_{\tilde{f}}(x) \geq \alpha\}) \Big),\ \sup_{\alpha \in [0,1]} \Big(\alpha \wedge \mu(\{x \in X : F_{\tilde{f}}(x) \geq \alpha\}) \Big) \Bigg\rangle .$$

Remark 7.3.3. Since each component Sugeno integral belongs to $[0,1]$, one has

$$\mathrm{NSug}_\mu(\tilde{f}) \in [0,1]^3.$$

Hence $\mathrm{NSug}_\mu(\tilde{f})$ is a well-defined single-valued neutrosophic value.

In the uncertain-set setting, it is natural to extend this construction to multi-component degree values by using an uncertain capacity together with componentwise meet and join operations.

Throughout this subsection, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^d, \qquad d \geq 1.$$

Assume that $\mathrm{Dom}(M)$ is a nonempty bounded sublattice of $[0,1]^d$ with respect to the componentwise order; that is:

1. $$\mathbf{0}_d := (0,\ldots,0) \in \mathrm{Dom}(M), \qquad \mathbf{1}_d := (1,\ldots,1) \in \mathrm{Dom}(M);$$
2. for all $a, b \in \mathrm{Dom}(M)$, both the componentwise meet $a \wedge b$ and the componentwise join $a \vee b$ belong to $\mathrm{Dom}(M)$.

For

$$a = (a^{(1)},\ldots,a^{(d)}), \qquad b = (b^{(1)},\ldots,b^{(d)}) \in \mathrm{Dom}(M),$$

write

$$a \preceq b \iff a^{(\ell)} \leq b^{(\ell)} \quad (\ell = 1,\ldots,d).$$

Moreover, define

$$a \wedge b := \big(\min\{a^{(1)}, b^{(1)}\},\ldots,\min\{a^{(d)}, b^{(d)}\}\big),$$

and

$$a \vee b := \big(\max\{a^{(1)}, b^{(1)}\},\ldots,\max\{a^{(d)}, b^{(d)}\}\big).$$

Definition 7.3.4 (Uncertain capacity)**.** Let

$$N := \{1,2,\ldots,n\}, \qquad n \geq 1.$$

A mapping

$$\nu_M : 2^N \to \mathrm{Dom}(M)$$

is called an *uncertain capacity on* N if the following conditions hold:

1. $$\nu_M(\varnothing) = \mathbf{0}_d;$$
2. $$\nu_M(N) = \mathbf{1}_d;$$

3. whenever $A \subseteq B \subseteq N$, one has
$$\nu_M(A) \preceq \nu_M(B).$$

Definition 7.3.5 (Canonical ascending ρ_M-ordering)**.** Let
$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$
be a fixed ranking map. For
$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$
a permutation σ of N is called the *canonical ascending ρ_M-ordering permutation* if:

1. $$\rho_M(a_{\sigma(1)}) \leq \rho_M(a_{\sigma(2)}) \leq \cdots \leq \rho_M(a_{\sigma(n)});$$
2. whenever
$$\rho_M(a_{\sigma(i)}) = \rho_M(a_{\sigma(j)}) \quad \text{and} \quad i < j,$$
one has
$$\sigma(i) < \sigma(j).$$

Thus the arguments are ordered increasingly according to their ranking values, and ties are broken by preserving the smaller original index first.

Lemma 7.3.6 (Existence and uniqueness of the canonical ascending ordering)**.** *For every*
$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$
there exists a unique canonical ascending ρ_M-ordering permutation.

Proof. Define a binary relation $\preccurlyeq$ on $N = \{1, \ldots, n\}$ by
$$u \preccurlyeq v \iff \big(\rho_M(a_u) < \rho_M(a_v)\big) \text{ or } \big(\rho_M(a_u) = \rho_M(a_v) \text{ and } u \leq v\big).$$
This relation totally orders the finite set N, because every two indices are comparable and ties of ranking values are resolved by the natural order of indices. Hence there exists a unique permutation
$$\sigma(1), \ldots, \sigma(n)$$
listing the indices in increasing order with respect to this rule. By construction, σ satisfies the required two conditions. Therefore the canonical ascending ρ_M-ordering permutation exists and is unique. □

Definition 7.3.7 (Uncertain Sugeno integral)**.** Let
$$N = \{1, 2, \ldots, n\}, \qquad (a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n,$$
let
$$\nu_M : 2^N \to \mathrm{Dom}(M)$$
be an uncertain capacity, and let
$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$

be a fixed ranking map. Let σ be the unique canonical ascending ρ_M-ordering permutation of $(a_1, \dots, a_n)$, and for each $i = 1, \dots, n$, define

$$A_i := \{\sigma(i), \sigma(i+1), \dots, \sigma(n)\}.$$

The *uncertain Sugeno integral* of $(a_1, \dots, a_n)$ with respect to ν_M and ρ_M is defined by

$$\mathrm{USug}_{M;\nu_M,\rho_M}(a_1, \dots, a_n) := \bigvee_{i=1}^{n} \Big(a_{\sigma(i)} \wedge \nu_M(A_i) \Big).$$

Remark 7.3.8. When $d = 1$, $\mathrm{Dom}(M) = [0, 1]$, $\rho_M(a) = a$, and

$$\nu_M : 2^N \to [0, 1]$$

is an ordinary fuzzy measure, the above definition reduces to the classical Sugeno integral

$$S_{\nu_M}(x) = \bigvee_{i=1}^{n} \big(x_{\sigma(i)} \wedge \nu_M(A_i) \big).$$

Hence the uncertain Sugeno integral is a natural uncertain-set-based generalization of the ordinary Sugeno integral.

Theorem 7.3.9 (Well-definedness of the uncertain Sugeno integral)**.** *The mapping*

$$\mathrm{USug}_{M;\nu_M,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let

$$(a_1, \dots, a_n) \in \mathrm{Dom}(M)^n$$

be arbitrary. By the preceding lemma, there exists a unique canonical ascending ρ_M-ordering permutation σ. Hence the sets

$$A_i = \{\sigma(i), \sigma(i+1), \dots, \sigma(n)\} \qquad (i = 1, \dots, n)$$

are uniquely determined.

Since

$$a_{\sigma(i)} \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n),$$

and

$$\nu_M(A_i) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n),$$

the assumption that $\mathrm{Dom}(M)$ is closed under the componentwise meet implies

$$a_{\sigma(i)} \wedge \nu_M(A_i) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$

Again, since $\mathrm{Dom}(M)$ is closed under the componentwise join, the finite join

$$\bigvee_{i=1}^{n} \Big(a_{\sigma(i)} \wedge \nu_M(A_i) \Big)$$

also belongs to $\mathrm{Dom}(M)$. Therefore,

$$\mathrm{USug}_{M;\nu_M,\rho_M}(a_1, \dots, a_n) \in \mathrm{Dom}(M).$$

Thus the formula defines a mapping

$$\mathrm{USug}_{M;\nu_M,\rho_M} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M),$$

and the uncertain Sugeno integral is well-defined. □

Remark 7.3.10. If
$$\nu_M(A) = \big(\nu_1(A), \dots, \nu_d(A)\big) \qquad (A \subseteq N),$$
then the uncertain Sugeno integral may be viewed as a multi-component Sugeno-type aggregation sharing a common ordering determined by ρ_M, while the meet and join are performed componentwise.

Definition 7.3.11 (Induced uncertain Sugeno integral on uncertain sets)**.** Let X be a nonempty universe, and let
$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \dots, n)$$
be uncertain sets of type M, where
$$\mu_i : X \to \mathrm{Dom}(M).$$
Let
$$\nu_M : 2^N \to \mathrm{Dom}(M)$$
be an uncertain capacity on
$$N = \{1, \dots, n\},$$
and let
$$\rho_M : \mathrm{Dom}(M) \to \mathbb{R}$$
be a fixed ranking map. The *induced uncertain Sugeno integral* of $\mathcal{U}_1, \dots, \mathcal{U}_n$ is defined by
$$\mathrm{USug}_{M;\nu_M,\rho_M}(\mathcal{U}_1, \dots, \mathcal{U}_n) := (X, \mu_{\mathrm{USug}}),$$
where
$$\mu_{\mathrm{USug}} : X \to \mathrm{Dom}(M)$$
is given pointwise by
$$\mu_{\mathrm{USug}}(x) := \mathrm{USug}_{M;\nu_M,\rho_M}\big(\mu_1(x), \dots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 7.3.12 (Well-definedness of the induced uncertain Sugeno integral)**.** *The pair*
$$\mathrm{USug}_{M;\nu_M,\rho_M}(\mathcal{U}_1, \dots, \mathcal{U}_n) = (X, \mu_{\mathrm{USug}})$$
is a well-defined uncertain set of type M.

Proof. Fix $x \in X$. Since each
$$\mathcal{U}_i = (X, \mu_i)$$
is an uncertain set of type M, one has
$$\mu_i(x) \in \mathrm{Dom}(M) \qquad (i = 1, \dots, n).$$
Hence
$$\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$
By the well-definedness of the uncertain Sugeno integral,
$$\mathrm{USug}_{M;\nu_M,\rho_M}\big(\mu_1(x), \dots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$
Therefore
$$\mu_{\mathrm{USug}}(x) \in \mathrm{Dom}(M) \qquad (x \in X).$$

Thus

$$\mu_{\mathrm{USug}} : X \to \mathrm{Dom}(M)$$

is a well-defined function, and consequently

$$(X, \mu_{\mathrm{USug}})$$

is a well-defined uncertain set of type M. □

A catalogue of representative Sugeno integrals classified by the dimension k of the degree-domain is presented in Table 7.4.

Table 7.4: A catalogue of representative Sugeno integrals by the dimension k of the degree-domain.

k	note	Representative Sugeno integral(s)
1		Fuzzy Sugeno integral [422, 423].
2		Intuitionistic Fuzzy Sugeno integral.
3		Neutrosophic Sugeno integral.
n	$(n \geq 1)$	Plithogenic Sugeno integral.

Reading guide. The table groups representative Sugeno integrals by the dimension k of their degree values.

Chapter 8

Uncertain Analytic and Transformational Operators

In this chapter, we examine uncertain analytic and transformational operators.

8.1 Uncertain Differentiation

Uncertain differentiation extends differentiation to uncertain-valued functions, measuring change rates while incorporating ambiguity, indeterminacy, or incomplete information, and preserving mathematically consistent derivative-like behavior under uncertainty.

Definition 8.1.1 (Fuzzy Differentiation)**.** [424, 425] Let $\mathbb{R}_{\mathcal{F}}$ denote the set of all fuzzy numbers on $\mathbb{R}$. For each $u \in \mathbb{R}_{\mathcal{F}}$ and $\alpha \in [0,1]$, write

$$[u]_\alpha = [u_\alpha^-, u_\alpha^+]$$

for the α-cut of u. Define the metric D on $\mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{\alpha\in[0,1]} \max\bigl\{|u_\alpha^- - v_\alpha^-|,\ |u_\alpha^+ - v_\alpha^+|\bigr\}, \qquad u,v \in \mathbb{R}_{\mathcal{F}}.$$

Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that

$$x = y \oplus z,$$

then z is called the *Hukuhara difference* of x and y, and is denoted by

$$x \ominus_H y.$$

Let $I \subseteq \mathbb{R}$ be an interval, let

$$f : I \to \mathbb{R}_{\mathcal{F}},$$

and let $x_0 \in \mathrm{int}(I)$. We say that f is *fuzzy differentiable at* x_0 (or *H-differentiable at* x_0) if, for all sufficiently small $h > 0$, the Hukuhara differences

$$f(x_0 + h) \ominus_H f(x_0) \qquad \text{and} \qquad f(x_0) \ominus_H f(x_0 - h)$$

exist, and there is a fuzzy number $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\lim_{h \to 0^+} D\left(\frac{f(x_0 + h) \ominus_H f(x_0)}{h},\ f'(x_0)\right) = 0$$

and

$$\lim_{h \to 0^+} D\left(\frac{f(x_0) \ominus_H f(x_0 - h)}{h},\ f'(x_0)\right) = 0.$$

The fuzzy number $f'(x_0)$ is called the *fuzzy derivative* of f at x_0. If f is fuzzy differentiable at every point of $\mathrm{int}(I)$, then f is called a *fuzzy differentiable function* on I.

Unlike union, intersection, aggregation, and similar operators, differentiation is in general not closed on the admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Indeed, even when a curve takes values in $\mathrm{Dom}(M)$, its difference quotients need not belong to $\mathrm{Dom}(M)$. Therefore, the mathematically correct codomain of an uncertain derivative is not $\mathrm{Dom}(M)$ itself, but the tangent object associated with $\mathrm{Dom}(M)$.

Definition 8.1.2 (Tangent cone of an uncertain degree-domain)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k$$

for some integer $k \geq 1$, regarded as a subset of $\mathbb{R}^k$.

For $a \in \mathrm{Dom}(M)$, the *Bouligand tangent cone* (or *contingent cone*) of $\mathrm{Dom}(M)$ at a is defined by

$$T_a \,\mathrm{Dom}(M) := \left\{ v \in \mathbb{R}^k \ :\ \exists\, h_n \downarrow 0,\ \exists\, a_n \in \mathrm{Dom}(M) \text{ such that } a_n \to a \text{ and } \frac{a_n - a}{h_n} \to v \right\}.$$

The *tangent uncertain domain* of M is

$$T\,\mathrm{Dom}(M) := \left\{ (a, v) \in \mathrm{Dom}(M) \times \mathbb{R}^k \ :\ v \in T_a \,\mathrm{Dom}(M) \right\}.$$

Definition 8.1.3 (Uncertain derivative of an M-degree curve)**.** Let $I \subseteq \mathbb{R}$ be an interval, let

$$\gamma : I \to \mathrm{Dom}(M),$$

and let $t_0 \in \mathrm{int}(I)$.

We say that γ is *M-differentiable at t_0* if the limit

$$\dot{\gamma}(t_0) := \lim_{h \to 0} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h}$$

exists in $\mathbb{R}^k$.

In this case, the *uncertain derivative* of γ at t_0 is defined by

$$D_M\gamma(t_0) := \big(\gamma(t_0), \dot{\gamma}(t_0)\big).$$

Theorem 8.1.4 (Well-definedness of the uncertain derivative of an M-degree curve)**.** *Let $I \subseteq \mathbb{R}$ be an interval, let*

$$\gamma : I \to \mathrm{Dom}(M),$$

and let $t_0 \in \mathrm{int}(I)$. If γ is M-differentiable at t_0, then

$$D_M\gamma(t_0) \in T\,\mathrm{Dom}(M).$$

Hence the uncertain derivative of γ is well-defined as an element of the tangent uncertain domain. Moreover, this derivative is unique.

Proof. Assume that γ is M-differentiable at t_0, and write

$$v := \dot{\gamma}(t_0) = \lim_{h \to 0} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h} \in \mathbb{R}^k.$$

We must show that

$$v \in T_{\gamma(t_0)}\,\mathrm{Dom}(M).$$

Choose any sequence $h_n > 0$ such that $h_n \downarrow 0$, and define

$$a_n := \gamma(t_0 + h_n) \qquad (n \in \mathbb{N}).$$

Since $\gamma(t) \in \mathrm{Dom}(M)$ for every $t \in I$, one has

$$a_n \in \mathrm{Dom}(M) \qquad \text{for all } n.$$

Because γ is differentiable at t_0, it is continuous at t_0, so

$$a_n = \gamma(t_0 + h_n) \to \gamma(t_0).$$

Also,

$$\frac{a_n - \gamma(t_0)}{h_n} = \frac{\gamma(t_0 + h_n) - \gamma(t_0)}{h_n} \to v.$$

By the definition of the tangent cone, this implies

$$v \in T_{\gamma(t_0)}\,\mathrm{Dom}(M).$$

Therefore

$$D_M\gamma(t_0) = \big(\gamma(t_0), v\big) \in T\,\mathrm{Dom}(M),$$

so the uncertain derivative is well-defined.

Finally, uniqueness follows from the uniqueness of the ordinary derivative in the Euclidean space $\mathbb{R}^k$. □

Definition 8.1.5 (Tangent uncertain set)**.** Let X be a nonempty set. A *tangent uncertain set of type* M on X is a pair

$$\dot{\mathcal{U}} = (X, \nu),$$

where

$$\nu : X \to T\,\mathrm{Dom}(M).$$

We write

$$\mathsf{TU}_M(X)$$

for the class of all tangent uncertain sets of type M on X.

Definition 8.1.6 (Uncertain differentiation of a family of U-sets)**.** Let X be a nonempty set, let $I \subseteq \mathbb{R}$ be an interval, and let

$$\mathsf{U}_M(X)$$

denote the class of all uncertain sets of type M on X.

Consider a family of U-sets

$$\mathcal{U} : I \to \mathsf{U}_M(X), \qquad t \longmapsto \mathcal{U}(t) = (X, \mu_t),$$

where

$$\mu_t : X \to \mathrm{Dom}(M) \qquad (t \in I).$$

Fix $t_0 \in \mathrm{int}(I)$. For each $x \in X$, define the associated degree-curve

$$\gamma_x : I \to \mathrm{Dom}(M), \qquad \gamma_x(t) := \mu_t(x).$$

We say that $\mathcal{U}$ is *uncertain differentiable at* t_0 if, for every $x \in X$, the curve γ_x is M-differentiable at t_0.

In that case, the *uncertain derivative* of $\mathcal{U}$ at t_0 is the tangent uncertain set

$$\frac{d\mathcal{U}}{dt}(t_0) := \big(X, \dot{\mu}_{t_0}\big),$$

where

$$\dot{\mu}_{t_0} : X \to T\,\mathrm{Dom}(M)$$

is defined pointwise by

$$\dot{\mu}_{t_0}(x) := D_M\gamma_x(t_0) = \left(\mu_{t_0}(x), \lim_{h\to 0} \frac{\mu_{t_0+h}(x) - \mu_{t_0}(x)}{h}\right).$$

Theorem 8.1.7 (Well-definedness of uncertain differentiation of U-sets)**.** *Let*

$$\mathcal{U} : I \to \mathsf{U}_M(X), \qquad \mathcal{U}(t) = (X, \mu_t),$$

be a family of uncertain sets of type M, and let

$$t_0 \in \mathrm{int}(I).$$

If $\mathcal{U}$ is uncertain differentiable at t_0, then

$$\frac{d\mathcal{U}}{dt}(t_0) \in \mathsf{TU}_M(X).$$

Hence the uncertain differentiation of $\mathcal{U}$ is well-defined.

Proof. Assume that $\mathcal{U}$ is uncertain differentiable at t_0. Then, for every $x \in X$, the degree-curve

$$\gamma_x(t) = \mu_t(x)$$

is M-differentiable at t_0. By the previous theorem,

$$D_M \gamma_x(t_0) \in T\,\mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto D_M \gamma_x(t_0)$$

defines a mapping

$$\dot{\mu}_{t_0} : X \to T\,\mathrm{Dom}(M).$$

Consequently,

$$\frac{d\mathcal{U}}{dt}(t_0) = \big(X, \dot{\mu}_{t_0}\big)$$

is a tangent uncertain set of type M on X, that is,

$$\frac{d\mathcal{U}}{dt}(t_0) \in \mathsf{TU}_M(X).$$

Thus uncertain differentiation is well-defined. □

8.2 Upside-down Logic in Neutrosophic Set

Upside-Down Logic reverses truth and falsity under a defined transformation, while preserving or inverting indeterminacy, yielding a consistent neutrosophic framework for contextual opposition and reasoning [426–430].

Definition 8.2.1 (Upside-Down Logic)**.** [431, 432] An *Upside-Down Logic* is obtained from a given system $\mathcal{M}$ by introducing a transformation U acting on propositions and/or contexts, with the following property:

1. For any $A \in \mathcal{P}$ having value $v(A)$ in context $\mathcal{C}$, there exist a transformed proposition $U(A)$ and/or a transformed context $U(\mathcal{C})$ such that:
   - If $v(A) = \text{True}$ in $\mathcal{C}$, then $v\big(U(A)\big) = \text{False}$ in $U(\mathcal{C})$;
   - If $v(A) = \text{False}$ in $\mathcal{C}$, then $v\big(U(A)\big) = \text{True}$ in $U(\mathcal{C})$.
2. The operator U is well defined and yields a consistent resulting system $\mathcal{M}'$.

First, we present the definition of Neutrosophic Logic below [26, 433]. Note that Neutrosophic Logic is known to generalize Fuzzy Logic (cf. [26]).

Definition 8.2.2 (Neutrosophic Logic valuation)**.** (cf. [434]) A *Neutrosophic Logic valuation* is a mapping

$$v : \mathcal{P} \longrightarrow [0,1]^3, \qquad v(A) = (T, I, F),$$

where T is the *truth-degree*, I the *indeterminacy-degree*, and F the *falsity-degree* of a proposition $A \in \mathcal{P}$. No global normalization is required, but typically $0 \le T, I, F \le 1$ and $0 \le T + I + F \le 3$.

Definition 8.2.3 (Upside-Down Logic in Neutrosophic Logic)**.** [431] Let $v(A) = (T, I, F)$ be the neutrosophic valuation of a proposition A. Fix an *Upside-Down operator* $U : [0,1]^3 \to [0,1]^3$ defined by

$$U(T, I, F) \;:=\; (F,\; I',\; T),$$

where I' is determined according to the chosen convention:

$$I' = \begin{cases} I, & \text{if indeterminacy is preserved,} \\ 1 - I, & \text{if indeterminacy is inverted.} \end{cases}$$

Chapter 9

Structural Algebraic Operators

In this chapter, we examine structural algebraic operators.

9.1 Uncertain HyperOperation

A hyperstructure is an algebraic system whose operations may assign sets of outputs to inputs, generalizing classical algebraic structures through multivalued composition laws and relations [435–437]. A hyperoperation maps one or more inputs to a nonempty set of possible outputs, extending ordinary operations by allowing multivalued algebraic combination behavior in general [438–441]. An uncertain hyperoperation maps inputs to sets of outputs under uncertainty, representing ambiguous, indeterminate, or weighted multivalued combinations while preserving generalized hyperstructural interaction patterns mathematically.

Definition 9.1.1 (Fuzzy HyperOperation)**.** (cf. [442–444]) Let H be a nonempty set, and let

$$\mathcal{F}(H) := [0,1]^H$$

denote the family of all fuzzy subsets of H. For $\mu \in \mathcal{F}(H)$, define its support by

$$\operatorname{supp}(\mu) := \{x \in H : \mu(x) > 0\}.$$

Write

$$\mathcal{F}^*(H) := \{\mu \in \mathcal{F}(H) : \operatorname{supp}(\mu) \neq \emptyset\}$$

for the family of all nonnull fuzzy subsets of H.

Let $m \geq 1$ be an integer. An *m-ary fuzzy hyperoperation* on H is a mapping

$$\star : H^m \to \mathcal{F}^*(H).$$

Equivalently, for each $(x_1, \ldots, x_m) \in H^m$, the value

$$\star(x_1, \ldots, x_m)$$

is a fuzzy subset of H with nonempty support, and one may write

$$\mu_\star(x_1,\ldots,x_m;y) := \big(\star(x_1,\ldots,x_m)\big)(y) \in [0,1] \qquad (y \in H)$$

for the membership degree of y in the hyper-result of $(x_1,\ldots,x_m)$.

When $m = 2$, $\star$ is called a *binary fuzzy hyperoperation*. A set H endowed with a fuzzy hyperoperation is called a *fuzzy hypergroupoid*.

Within the uncertain-set framework, the natural extension is obtained by replacing ordinary subsets or fuzzy subsets with uncertain sets whose membership values lie in a fixed admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

of an uncertain model M.

Definition 9.1.2 (Uncertain support and nonnull U-set)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and assume that a distinguished bottom element

$$0_M \in \mathrm{Dom}(M)$$

has been fixed.

Let H be a nonempty set, and let

$$\mathsf{U}_M(H) := \{(H,\mu) \;:\; \mu : H \to \mathrm{Dom}(M)\}$$

denote the class of all uncertain sets of type M on H.

For

$$\mathcal{U} = (H,\mu_{\mathcal{U}}) \in \mathsf{U}_M(H),$$

define the M*-support* of $\mathcal{U}$ by

$$\mathrm{supp}_M(\mathcal{U}) := \{\, x \in H : \mu_{\mathcal{U}}(x) \neq 0_M \,\}.$$

We say that $\mathcal{U}$ is *nonnull* if

$$\mathrm{supp}_M(\mathcal{U}) \neq \emptyset.$$

The class of all nonnull uncertain sets of type M on H is denoted by

$$\mathsf{U}^*_M(H) := \{\mathcal{U} \in \mathsf{U}_M(H) : \mathrm{supp}_M(\mathcal{U}) \neq \emptyset\}.$$

Definition 9.1.3 (Kernel form of an uncertain hyperoperation)**.** Let H be a nonempty set, let $m \geq 1$, and let M be an uncertain model with bottom element $0_M \in \mathrm{Dom}(M)$. A mapping

$$\eta : H^m \times H \to \mathrm{Dom}(M)$$

is called an *m-ary uncertain hyperoperation kernel of type M* on H if, for every

$$(x_1, \ldots, x_m) \in H^m,$$

there exists at least one $y \in H$ such that

$$\eta(x_1, \ldots, x_m; y) \neq 0_M.$$

Equivalently,

$$\{\, y \in H : \eta(x_1, \ldots, x_m; y) \neq 0_M \,\} \neq \emptyset \qquad \text{for all } (x_1, \ldots, x_m) \in H^m.$$

Definition 9.1.4 (Uncertain hyperoperation)**.** Let H be a nonempty set, let $m \geq 1$, and let M be an uncertain model with bottom element $0_M \in \mathrm{Dom}(M)$.

An *m-ary uncertain hyperoperation of type M* on H is a mapping

$$\star_M : H^m \to \mathsf{U}^*_M(H).$$

Equivalently, an uncertain hyperoperation may be represented by a kernel

$$\mu_{\star_M} : H^m \times H \to \mathrm{Dom}(M)$$

such that, for every

$$(x_1, \ldots, x_m) \in H^m,$$

the assignment

$$y \longmapsto \mu_{\star_M}(x_1, \ldots, x_m; y)$$

defines a nonnull uncertain set on H.

In that case one writes

$$\star_M(x_1, \ldots, x_m) = \big(H, \mu_{(x_1,\ldots,x_m)}\big),$$

where

$$\mu_{(x_1,\ldots,x_m)}(y) := \mu_{\star_M}(x_1, \ldots, x_m; y) \qquad (y \in H).$$

When $m = 2$, $\star_M$ is called a *binary uncertain hyperoperation.* A set endowed with an uncertain hyperoperation is called an *uncertain hypergroupoid of type M.*

Theorem 9.1.5 (Well-definedness of the uncertain hyperoperation induced by a kernel)**.** *Let H be a nonempty set, let $m \geq 1$, let M be an uncertain model with bottom element $0_M \in \mathrm{Dom}(M)$, and let*

$$\eta : H^m \times H \to \mathrm{Dom}(M)$$

be an m-ary uncertain hyperoperation kernel of type M on H.

For each

$$(x_1, \ldots, x_m) \in H^m,$$

define

$$\mu_{(x_1,\ldots,x_m)} : H \to \mathrm{Dom}(M)$$

by

$$\mu_{(x_1,\ldots,x_m)}(y) := \eta(x_1,\ldots,x_m;y) \qquad (y \in H),$$

and set

$$\star_M(x_1,\ldots,x_m) := \big(H, \mu_{(x_1,\ldots,x_m)}\big).$$

Then

$$\star_M : H^m \to \mathsf{U}^*_M(H)$$

is a well-defined m-ary uncertain hyperoperation of type M on H.

Proof. Fix any

$$(x_1,\ldots,x_m) \in H^m.$$

By definition of η,

$$\eta(x_1,\ldots,x_m;y) \in \mathrm{Dom}(M) \qquad \text{for all } y \in H.$$

Hence the assignment

$$y \longmapsto \mu_{(x_1,\ldots,x_m)}(y) = \eta(x_1,\ldots,x_m;y)$$

defines a mapping

$$\mu_{(x_1,\ldots,x_m)} : H \to \mathrm{Dom}(M).$$

Therefore

$$\big(H, \mu_{(x_1,\ldots,x_m)}\big) \in \mathsf{U}_M(H).$$

It remains to prove that this uncertain set is nonnull. Since η is an uncertain hyperoperation kernel, there exists at least one element

$$y_0 \in H$$

such that

$$\eta(x_1,\ldots,x_m;y_0) \neq 0_M.$$

Thus

$$\mu_{(x_1,\ldots,x_m)}(y_0) \neq 0_M,$$

which implies

$$y_0 \in \mathrm{supp}_M\big(H, \mu_{(x_1,\ldots,x_m)}\big).$$

Hence

$$\mathrm{supp}_M\big(H, \mu_{(x_1,\ldots,x_m)}\big) \neq \emptyset.$$

Therefore

$$\big(H, \mu_{(x_1,\ldots,x_m)}\big) \in \mathsf{U}^*_M(H).$$

Since the choice of $(x_1,\ldots,x_m) \in H^m$ was arbitrary, it follows that

$$\star_M(x_1,\ldots,x_m) \in \mathsf{U}^*_M(H) \qquad \text{for all } (x_1,\ldots,x_m) \in H^m.$$

Consequently,

$$\star_M : H^m \to \mathsf{U}^*_M(H)$$

is well-defined. □

Proposition 9.1.6 (Equivalent kernel representation)**.** *Let*

$$\star_M : H^m \to \mathsf{U}^*_M(H)$$

be an uncertain hyperoperation of type M on H. For each

$$(x_1, \dots, x_m) \in H^m,$$

write

$$\star_M(x_1, \dots, x_m) = \big(H, \mu_{(x_1,\dots,x_m)}\big).$$

Define

$$\mu_{\star_M} : H^m \times H \to \mathrm{Dom}(M)$$

by

$$\mu_{\star_M}(x_1, \dots, x_m; y) := \mu_{(x_1,\dots,x_m)}(y).$$

Then $\mu_{\star_M}$ is an uncertain hyperoperation kernel of type M, and $\star_M$ is recovered from $\mu_{\star_M}$ by the construction of the preceding theorem.

Proof. Since

$$\star_M(x_1, \dots, x_m) \in \mathsf{U}^*_M(H)$$

for every

$$(x_1, \dots, x_m) \in H^m,$$

the map

$$\mu_{(x_1,\dots,x_m)} : H \to \mathrm{Dom}(M)$$

is well-defined and has nonempty M-support. Hence there exists some $y \in H$ such that

$$\mu_{(x_1,\dots,x_m)}(y) \neq 0_M.$$

Therefore

$$\mu_{\star_M}(x_1, \dots, x_m; y) \neq 0_M$$

for some $y \in H$, showing that $\mu_{\star_M}$ is an uncertain hyperoperation kernel.

By construction, the uncertain set associated with $(x_1, \dots, x_m)$ through $\mu_{\star_M}$ is precisely

$$\big(H, \mu_{(x_1,\dots,x_m)}\big) = \star_M(x_1, \dots, x_m).$$

Thus $\star_M$ is recovered exactly. □

As a reference, a catalogue of representative hyperoperations classified by the dimension k of the degree-domain is presented in Table 9.1.

Table 9.1: A catalogue of representative hyperoperations by the dimension k of the degree-domain.

k	note	Representative hyperoperation(s)
1		Fuzzy HyperOperations.
2		Intuitionistic Fuzzy HyperOperations (cf. [445, 446]).
3		Neutrosophic HyperOperations [447].
n	$(n \geq 1)$	Plithogenic HyperOperations.

Reading guide. The table groups representative hyperoperations by the dimension k of their degree values.

9.2 Uncertain SuperHyperOperation

A SuperHyperStructure generalizes hyperstructures by allowing operations and relations to act on higher-order collections, recursively producing structured families of subsets across multiple levels of abstraction [448–450]. A SuperHyperOperation maps elements or higher-order subsets to nonempty families of higher-level subsets, extending hyperoperations through recursive powerset-like constructions and multilevel combinational behavior systematically mathematically [449, 451, 452]. An Uncertain SuperHyperOperation extends superhyperoperations to uncertain settings, assigning higher-level outputs with uncertainty-aware interpretation, thereby modeling multilevel indeterminacy, ambiguity, and generalized collective interaction.

Definition 9.2.1 (Iterated nonempty powersets)**.** [450] Let H be a nonempty set. For every nonempty set X, define

$$\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}.$$

Recursively define

$$\mathcal{P}^{*(0)}(H) := H, \qquad \mathcal{P}^{*(n)}(H) := \mathcal{P}^*\big(\mathcal{P}^{*(n-1)}(H)\big) \quad (n \geq 1).$$

Definition 9.2.2 (Fuzzy SuperHyperOperation)**.** Let H be a nonempty set, and let

$$\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\emptyset\}$$

for every nonempty set X. Define recursively the iterated nonempty powersets by

$$\mathcal{P}^{*(0)}(H) := H, \qquad \mathcal{P}^{*(n)}(H) := \mathcal{P}^*\big(\mathcal{P}^{*(n-1)}(H)\big) \quad (n \geq 1).$$

For any nonempty set X, let

$$\mathcal{F}(X) := [0,1]^X, \qquad \mathcal{F}^*(X) := \{\mu \in \mathcal{F}(X) : \mathrm{supp}(\mu) \neq \emptyset\}.$$

Let $m \geq 1$ and $r, n \geq 0$ be integers. An $(m; r, n)$-*fuzzy superhyperoperation* on H is a mapping

$$\odot : \big(\mathcal{P}^{*(r)}(H)\big)^m \to \mathcal{F}^*\big(\mathcal{P}^{*(n)}(H)\big).$$

Thus, for every

$$(A_1, \ldots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m,$$

the value

$$\odot(A_1, \ldots, A_m)$$

is a fuzzy subset of $\mathcal{P}^{*(n)}(H)$ with nonempty support. Equivalently, one may represent $\odot$ by its membership function

$$\mu_{\odot} : \big(\mathcal{P}^{*(r)}(H)\big)^m \times \mathcal{P}^{*(n)}(H) \to [0,1],$$

defined by

$$\mu_{\odot}(A_1,\dots,A_m;B) := \big(\odot(A_1,\dots,A_m)\big)(B), \qquad B \in \mathcal{P}^{*(n)}(H).$$

If $r = 0$ and $n = 0$, then $\odot$ reduces to an m-ary fuzzy hyperoperation on H. If, for every input $(A_1,\dots,A_m)$, there exists a unique

$$B_{A_1,\dots,A_m} \in \mathcal{P}^{*(n)}(H)$$

such that

$$\odot(A_1,\dots,A_m) = \chi_{\{B_{A_1,\dots,A_m}\}},$$

then $\odot$ encodes the corresponding crisp $(m;r,n)$-superhyperoperation

$$\# : \big(\mathcal{P}^{*(r)}(H)\big)^m \to \mathcal{P}^{*(n)}(H), \qquad \#(A_1,\dots,A_m) := B_{A_1,\dots,A_m}.$$

A set equipped with such an operator is called a *fuzzy superhypergroupoid.*

In the uncertain-set framework, the natural extension is obtained by replacing fuzzy subsets with uncertain sets whose values lie in a fixed admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

of an uncertain model M.

Definition 9.2.3 (Nonnull uncertain sets on higher-order domains)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and assume that a distinguished bottom element

$$0_M \in \mathrm{Dom}(M)$$

has been fixed.

For any nonempty set X, let

$$\mathsf{U}_M(X) := \{(X,\mu) : \mu : X \to \mathrm{Dom}(M)\}$$

denote the class of all uncertain sets of type M on X.

For

$$\mathcal{U} = (X, \mu_{\mathcal{U}}) \in \mathsf{U}_M(X),$$

define the M-support of $\mathcal{U}$ by

$$\mathrm{supp}_M(\mathcal{U}) := \{x \in X : \mu_{\mathcal{U}}(x) \neq 0_M\}.$$

We say that $\mathcal{U}$ is *nonnull* if

$$\mathrm{supp}_M(\mathcal{U}) \neq \emptyset.$$

The class of all nonnull uncertain sets of type M on X is denoted by

$$\mathsf{U}^*_M(X) := \{\mathcal{U} \in \mathsf{U}_M(X) : \mathrm{supp}_M(\mathcal{U}) \neq \emptyset\}.$$

Definition 9.2.4 (Kernel form of an uncertain superhyperoperation)**.** Let H be a nonempty set, let $m \geq 1$, let $r, n \geq 0$, and let M be an uncertain model with bottom element $0_M \in \mathrm{Dom}(M)$.

A mapping

$$\eta : \big(\mathcal{P}^{*(r)}(H)\big)^m \times \mathcal{P}^{*(n)}(H) \longrightarrow \mathrm{Dom}(M)$$

is called an $(m; r, n)$-*uncertain superhyperoperation kernel of type* M on H if, for every

$$(A_1, \dots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m,$$

there exists at least one

$$B \in \mathcal{P}^{*(n)}(H)$$

such that

$$\eta(A_1, \dots, A_m; B) \neq 0_M.$$

Equivalently,

$$\{\, B \in \mathcal{P}^{*(n)}(H) : \eta(A_1, \dots, A_m; B) \neq 0_M \,\} \neq \emptyset$$

for every input $(A_1, \dots, A_m)$.

Definition 9.2.5 (Uncertain SuperHyperOperation)**.** Let H be a nonempty set, let $m \geq 1$, let $r, n \geq 0$, and let M be an uncertain model with bottom element $0_M \in \mathrm{Dom}(M)$.

An $(m; r, n)$-*uncertain superhyperoperation of type* M on H is a mapping

$$\odot_M : \big(\mathcal{P}^{*(r)}(H)\big)^m \longrightarrow \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big).$$

Thus, for every

$$(A_1, \dots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m,$$

the value

$$\odot_M(A_1, \dots, A_m)$$

is a nonnull uncertain set of type M on $\mathcal{P}^{*(n)}(H)$.

Equivalently, $\odot_M$ may be represented by a kernel

$$\mu_{\odot_M} : \big(\mathcal{P}^{*(r)}(H)\big)^m \times \mathcal{P}^{*(n)}(H) \longrightarrow \mathrm{Dom}(M),$$

defined by

$$\mu_{\odot_M}(A_1, \dots, A_m; B) := \big(\odot_M(A_1, \dots, A_m)\big)(B), \qquad B \in \mathcal{P}^{*(n)}(H).$$

If $r = 0$ and $n = 0$, then $\odot_M$ reduces to an m-ary uncertain hyperoperation of type M on H. A set endowed with such an operation is called an *uncertain superhypergroupoid of type* M.

Theorem 9.2.6 (Well-definedness of the uncertain superhyperoperation induced by a kernel)**.** *Let H be a nonempty set, let $m \geq 1$, let $r, n \geq 0$, let M be an uncertain model with bottom element $0_M \in \mathrm{Dom}(M)$, and let*

$$\eta : \big(\mathcal{P}^{*(r)}(H)\big)^m \times \mathcal{P}^{*(n)}(H) \to \mathrm{Dom}(M)$$

be an $(m; r, n)$-uncertain superhyperoperation kernel of type M on H.

For every

$$(A_1, \dots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m,$$

define

$$\mu_{(A_1,\dots,A_m)} : \mathcal{P}^{*(n)}(H) \to \mathrm{Dom}(M)$$

by

$$\mu_{(A_1,\dots,A_m)}(B) := \eta(A_1, \dots, A_m; B) \qquad \big(B \in \mathcal{P}^{*(n)}(H)\big),$$

and set

$$\odot_M(A_1, \dots, A_m) := \big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\dots,A_m)}\big).$$

Then

$$\odot_M : \big(\mathcal{P}^{*(r)}(H)\big)^m \to \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big)$$

is a well-defined $(m; r, n)$-uncertain superhyperoperation of type M on H.

Proof. Fix any

$$(A_1, \dots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m.$$

For each

$$B \in \mathcal{P}^{*(n)}(H),$$

the value

$$\eta(A_1, \dots, A_m; B)$$

belongs to $\mathrm{Dom}(M)$ by definition of η. Hence the assignment

$$B \longmapsto \mu_{(A_1,\dots,A_m)}(B) = \eta(A_1, \dots, A_m; B)$$

defines a mapping

$$\mu_{(A_1,\dots,A_m)} : \mathcal{P}^{*(n)}(H) \to \mathrm{Dom}(M).$$

Therefore

$$\big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\dots,A_m)}\big) \in \mathsf{U}_M\big(\mathcal{P}^{*(n)}(H)\big).$$

It remains to prove that this uncertain set is nonnull. Since η is an uncertain superhyperoperation kernel, there exists at least one

$$B_0 \in \mathcal{P}^{*(n)}(H)$$

such that

$$\eta(A_1, \dots, A_m; B_0) \neq 0_M.$$

Hence

$$\mu_{(A_1,\dots,A_m)}(B_0) \neq 0_M,$$

which implies

$$B_0 \in \operatorname{supp}_M\big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\ldots,A_m)}\big).$$

Thus

$$\operatorname{supp}_M\big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\ldots,A_m)}\big) \neq \emptyset.$$

Therefore

$$\big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\ldots,A_m)}\big) \in \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big).$$

Since the input $(A_1, \ldots, A_m)$ was arbitrary, it follows that

$$\odot_M(A_1, \ldots, A_m) \in \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big)$$

for all

$$(A_1, \ldots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m.$$

Consequently,

$$\odot_M : \big(\mathcal{P}^{*(r)}(H)\big)^m \to \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big)$$

is well-defined. □

Proposition 9.2.7 (Equivalent kernel representation)**.** *Let*

$$\odot_M : \big(\mathcal{P}^{*(r)}(H)\big)^m \to \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big)$$

be an $(m; r, n)$-uncertain superhyperoperation of type M on H. For each input

$$(A_1, \ldots, A_m) \in \big(\mathcal{P}^{*(r)}(H)\big)^m,$$

write

$$\odot_M(A_1, \ldots, A_m) = \big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\ldots,A_m)}\big).$$

Define

$$\mu_{\odot_M} : \big(\mathcal{P}^{*(r)}(H)\big)^m \times \mathcal{P}^{*(n)}(H) \to \mathrm{Dom}(M)$$

by

$$\mu_{\odot_M}(A_1, \ldots, A_m; B) := \mu_{(A_1,\ldots,A_m)}(B).$$

Then $\mu_{\odot_M}$ is an $(m; r, n)$-uncertain superhyperoperation kernel of type M, and $\odot_M$ is recovered from $\mu_{\odot_M}$ by the construction of the preceding theorem.

Proof. Since

$$\odot_M(A_1, \ldots, A_m) \in \mathsf{U}^*_M\big(\mathcal{P}^{*(n)}(H)\big)$$

for every input $(A_1, \ldots, A_m)$, the map

$$\mu_{(A_1,\ldots,A_m)} : \mathcal{P}^{*(n)}(H) \to \mathrm{Dom}(M)$$

is well-defined and has nonempty M-support. Therefore there exists some

$$B \in \mathcal{P}^{*(n)}(H)$$

such that

$$\mu_{(A_1,\ldots,A_m)}(B) \neq 0_M.$$

Hence

$$\mu_{\odot_M}(A_1, \ldots, A_m; B) \neq 0_M,$$

so $\mu_{\odot_M}$ satisfies the nonnullity condition required of a kernel.

By construction, the uncertain set induced by $\mu_{\odot_M}$ at the input

$$(A_1, \ldots, A_m)$$

is precisely

$$\big(\mathcal{P}^{*(n)}(H), \mu_{(A_1,\ldots,A_m)}\big) = \odot_M(A_1, \ldots, A_m).$$

Thus $\odot_M$ is recovered exactly from $\mu_{\odot_M}$. □

9.3 Fuzzy MultiOperations

MultiOperations are operations whose outputs are multisets rather than single elements, allowing repeated occurrences and thus representing multiplicity-aware algebraic combination in generalized structures mathematically. Fuzzy MultiOperations extend multioperations by assigning graded membership to possible multiset outputs, thereby modeling multiplicity-aware algebraic combination under uncertainty within fuzzy structural frameworks.

Definition 9.3.1 (Nonnull fuzzy subsets)**.** Let X be a nonempty set. Define

$$\mathcal{F}(X) := [0,1]^X$$

to be the family of all fuzzy subsets of X. For $\mu \in \mathcal{F}(X)$, define its support by

$$\operatorname{supp}(\mu) := \{x \in X : \mu(x) > 0\}.$$

Write

$$\mathcal{F}^*(X) := \{\mu \in \mathcal{F}(X) : \operatorname{supp}(\mu) \neq \emptyset\}$$

for the family of all nonnull fuzzy subsets of X.

Definition 9.3.2 (Fuzzy MultiOperation)**.** Let H be a nonempty set, and let $M(H)$ denote the set of all finite multisets on H. Let $m \geq 1$ be an integer.

An *m-ary fuzzy multioperation* on H is a mapping

$$\sharp_F^{(m)} : H^m \to \mathcal{F}^*(M(H)).$$

Thus, for each

$$(x_1, \ldots, x_m) \in H^m,$$

the value

$$\sharp_F^{(m)}(x_1, \ldots, x_m)$$

is a nonnull fuzzy subset of the multiset space $M(H)$.

Equivalently, one may represent $\sharp_F^{(m)}$ by its membership kernel

$$\mu_{\sharp_F^{(m)}} : H^m \times M(H) \to [0,1],$$

defined by

$$\mu_{\sharp_F^{(m)}}(x_1,\ldots,x_m;M) := \big(\sharp_F^{(m)}(x_1,\ldots,x_m)\big)(M), \qquad M \in M(H).$$

If, for every input $(x_1,\ldots,x_m) \in H^m$, there exists a unique multiset

$$M_{x_1,\ldots,x_m} \in M(H)$$

such that

$$\sharp_F^{(m)}(x_1,\ldots,x_m) = \chi_{\{M_{x_1,\ldots,x_m}\}},$$

then $\sharp_F^{(m)}$ encodes the corresponding crisp m-ary multioperation

$$\sharp^{(m)} : H^m \to M(H), \qquad \sharp^{(m)}(x_1,\ldots,x_m) := M_{x_1,\ldots,x_m}.$$

Remark 9.3.3. A family

$$\big\{\sharp_F^{(m)} : H^m \to \mathcal{F}^*(M(H))\big\}_{m\in I}$$

on a nonempty carrier H may be called a *fuzzy multistructure.*

We introduce uncertain multioperations in the framework of uncertain sets. Throughout, let M be a fixed uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \geq 1$, and let

$$\mathcal{P}^*(A) := \mathcal{P}(A) \setminus \{\varnothing\}$$

denote the family of all nonempty subsets of a set A.

Definition 9.3.4 (r-ary M-multioperation)**.** Let $r \geq 1$ be an integer. A mapping

$$\star_M^{(r)} : \mathrm{Dom}(M)^r \longrightarrow \mathcal{P}^*(\mathrm{Dom}(M))$$

is called an *r-ary uncertain multioperation on M* (or briefly, an *r-ary M-multioperation*) if, for every

$$(a_1,\ldots,a_r) \in \mathrm{Dom}(M)^r,$$

the value

$$\star_M^{(r)}(a_1,\ldots,a_r)$$

is a nonempty subset of $\mathrm{Dom}(M)$.

Thus, instead of producing a single admissible degree, an M-multioperation assigns a nonempty set of admissible output degrees.

Definition 9.3.5 (Uncertain multioperation induced on U-sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, r)$$

be uncertain sets of the same type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

Let $\star_M^{(r)}$ be an r-ary M-multioperation.

The *uncertain multioperation induced by* $\star_M^{(r)}$ *on*

$$(\mathcal{U}_1, \ldots, \mathcal{U}_r)$$

is the pair

$$\star_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_r) := \big(X, \mathcal{M}_\star\big),$$

where

$$\mathcal{M}_\star : X \longrightarrow \mathcal{P}^*(\mathrm{Dom}(M))$$

is defined pointwise by

$$\mathcal{M}_\star(x) = \star_M^{(r)}\big(\mu_1(x), \ldots, \mu_r(x)\big) \qquad (x \in X).$$

We call $\big(X, \mathcal{M}_\star\big)$ the *M-multi-valued uncertain structure* generated by $\star_M^{(r)}$.

Theorem 9.3.6 (Well-definedness of uncertain multioperations)**.** *Let X be a nonempty set, let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, r)$$

be uncertain sets of type M on X, and let

$$\star_M^{(r)} : \mathrm{Dom}(M)^r \to \mathcal{P}^*(\mathrm{Dom}(M))$$

be an r-ary M-multioperation. Then

$$\star_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_r) = \big(X, \mathcal{M}_\star\big)$$

is well-defined; that is, the mapping

$$\mathcal{M}_\star : X \to \mathcal{P}^*(\mathrm{Dom}(M))$$

is a well-defined function.

Proof. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X \text{ and } i = 1, \ldots, r.$$

Fix any $x \in X$. Then

$$\big(\mu_1(x), \ldots, \mu_r(x)\big) \in \mathrm{Dom}(M)^r.$$

Because $\star_M^{(r)}$ is an r-ary M-multioperation, it follows that

$$\star_M^{(r)}\big(\mu_1(x), \ldots, \mu_r(x)\big) \in \mathcal{P}^*(\mathrm{Dom}(M)).$$

Hence

$$\mathcal{M}_\star(x) = \star_M^{(r)}\big(\mu_1(x), \dots, \mu_r(x)\big) \in \mathcal{P}^*(\mathrm{Dom}(M)) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto \mathcal{M}_\star(x)$$

defines a mapping

$$\mathcal{M}_\star : X \to \mathcal{P}^*(\mathrm{Dom}(M)).$$

Consequently,

$$\star_M^{(r)}(\mathcal{U}_1, \dots, \mathcal{U}_r) = \big(X, \mathcal{M}_\star\big)$$

is well-defined. □

Remark 9.3.7 (Singleton lifting of ordinary uncertain operations)**.** Let

$$f_M : \mathrm{Dom}(M)^r \to \mathrm{Dom}(M)$$

be an ordinary r-ary uncertain operation on M. Then the mapping

$$\star_{f,M}^{(r)} : \mathrm{Dom}(M)^r \to \mathcal{P}^*(\mathrm{Dom}(M)), \qquad \star_{f,M}^{(r)}(a_1, \dots, a_r) := \{f_M(a_1, \dots, a_r)\},$$

is an r-ary M-multioperation. Hence every ordinary uncertain operation can be viewed as a special case of an uncertain multioperation.

9.4 Fuzzy MetaOperations

MetaOperations are operations acting on structures themselves, transforming or combining entire systems into new structures, thereby modeling higher-level construction, interaction, and organization beyond object-level operations. Fuzzy MetaOperations extend meta-level operations by assigning graded membership to possible output structures, enabling uncertainty-aware transformation, combination, and evaluation of structures within higher-order frameworks mathematically.

Definition 9.4.1 (Fuzzy MetaOperation)**.** Fix a single-sorted finitary signature

$$\Sigma = (\mathrm{Func}, \mathrm{Rel}, \mathrm{arFunc}, \mathrm{arRel}),$$

and let U be a nonempty set of finite Σ-structures. Let $k \geq 1$ be an integer.

A *k-ary fuzzy metaoperation* on U is a mapping

$$\Phi_F : U^k \to \mathcal{F}^*(U).$$

Hence, for each

$$(C_1, \dots, C_k) \in U^k,$$

the value

$$\Phi_F(C_1, \dots, C_k)$$

is a nonnull fuzzy subset of U, that is, a fuzzy family of possible output structures.

Equivalently, Φ_F may be represented by its membership kernel

$$\mu_{\Phi_F} : U^k \times U \to [0,1],$$

defined by

$$\mu_{\Phi_F}(C_1,\ldots,C_k;D) := \big(\Phi_F(C_1,\ldots,C_k)\big)(D), \qquad D \in U.$$

If, for every input $(C_1,\ldots,C_k) \in U^k$, there exists a unique structure

$$D_{C_1,\ldots,C_k} \in U$$

such that

$$\Phi_F(C_1,\ldots,C_k) = \chi_{\{D_{C_1,\ldots,C_k}\}},$$

then Φ_F encodes the corresponding crisp metaoperation

$$\Phi : U^k \to U, \qquad \Phi(C_1,\ldots,C_k) := D_{C_1,\ldots,C_k}.$$

Definition 9.4.2 (Isomorphism-invariant Fuzzy MetaOperation)**.** Let $\Phi_F : U^k \to \mathcal{F}^*(U)$ be a fuzzy metaoperation. We say that Φ_F is *isomorphism-invariant* if, whenever

$$C_i \cong D_i \qquad (i = 1,\ldots,k),$$

there exists a bijection

$$\beta : \operatorname{supp}\big(\Phi_F(C_1,\ldots,C_k)\big) \to \operatorname{supp}\big(\Phi_F(D_1,\ldots,D_k)\big)$$

such that, for every

$$E \in \operatorname{supp}\big(\Phi_F(C_1,\ldots,C_k)\big),$$

one has

$$\beta(E) \cong E$$

and

$$\mu_{\Phi_F}(C_1,\ldots,C_k;E) = \mu_{\Phi_F}(D_1,\ldots,D_k;\beta(E)).$$

Remark 9.4.3. A pair

$$\mathcal{M}_F = \big(U,(\Phi_{F,\ell})_{\ell\in\Lambda}\big),$$

where each

$$\Phi_{F,\ell} : U^{k_\ell} \to \mathcal{F}^*(U),$$

may be called a *fuzzy metastructure*.

We next define uncertain metaoperations as one level higher than uncertain multioperations. For this purpose, let

$$\mathcal{P}^{**}(A) := \mathcal{P}^*\big(\mathcal{P}^*(A)\big) = \mathcal{P}\big(\mathcal{P}^*(A)\big) \setminus \{\varnothing\}.$$

Thus, an element of $\mathcal{P}^{**}(A)$ is a nonempty family of nonempty subsets of A.

Definition 9.4.4 (r-ary M-metaoperation)**.** Let $r \geq 1$ be an integer. A mapping

$$\diamond_M^{(r)} : \mathrm{Dom}(M)^r \longrightarrow \mathcal{P}^{**}(\mathrm{Dom}(M))$$

is called an *r-ary uncertain metaoperation on M* (or briefly, an *r-ary M-metaoperation*) if, for every

$$(a_1, \ldots, a_r) \in \mathrm{Dom}(M)^r,$$

the value

$$\diamond_M^{(r)}(a_1, \ldots, a_r)$$

is a nonempty family of nonempty subsets of $\mathrm{Dom}(M)$; equivalently,

$$\diamond_M^{(r)}(a_1, \ldots, a_r) \in \mathcal{P}^{**}(\mathrm{Dom}(M)).$$

Hence an M-metaoperation does not return a single admissible degree, nor merely a single nonempty set of admissible degrees, but rather a nonempty family of such admissible output-sets.

Definition 9.4.5 (Uncertain metaoperation induced on U-sets)**.** Let X be a nonempty universe, and let

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, r)$$

be uncertain sets of the same type M on X, where

$$\mu_i : X \to \mathrm{Dom}(M).$$

Let $\diamond_M^{(r)}$ be an r-ary M-metaoperation.

The *uncertain metaoperation induced by $\diamond_M^{(r)}$ on*

$$(\mathcal{U}_1, \ldots, \mathcal{U}_r)$$

is the pair

$$\diamond_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_r) := \big(X, \mathcal{M}_\diamond\big),$$

where

$$\mathcal{M}_\diamond : X \longrightarrow \mathcal{P}^{**}(\mathrm{Dom}(M))$$

is defined pointwise by

$$\mathcal{M}_\diamond(x) = \diamond_M^{(r)}\big(\mu_1(x), \ldots, \mu_r(x)\big) \qquad (x \in X).$$

We call $\big(X, \mathcal{M}_\diamond\big)$ the *M-meta-valued uncertain structure* generated by $\diamond_M^{(r)}$.

Theorem 9.4.6 (Well-definedness of uncertain metaoperations)**.** *Let X be a nonempty set, let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, r)$$

be uncertain sets of type M on X, and let

$$\diamond_M^{(r)} : \mathrm{Dom}(M)^r \to \mathcal{P}^{**}(\mathrm{Dom}(M))$$

be an r-ary M-metaoperation. Then

$$\diamond_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_r) = \bigl(X, \mathcal{M}_\diamond\bigr)$$

is well-defined; that is, the mapping

$$\mathcal{M}_\diamond : X \to \mathcal{P}^{**}(\mathrm{Dom}(M))$$

is a well-defined function.

Proof. Since each $\mathcal{U}_i = (X, \mu_i)$ is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X \text{ and } i = 1, \ldots, r.$$

Fix any $x \in X$. Then

$$\bigl(\mu_1(x), \ldots, \mu_r(x)\bigr) \in \mathrm{Dom}(M)^r.$$

Because $\diamond_M^{(r)}$ is an r-ary M-metaoperation, it follows that

$$\diamond_M^{(r)}\bigl(\mu_1(x), \ldots, \mu_r(x)\bigr) \in \mathcal{P}^{**}(\mathrm{Dom}(M)).$$

Hence

$$\mathcal{M}_\diamond(x) = \diamond_M^{(r)}\bigl(\mu_1(x), \ldots, \mu_r(x)\bigr) \in \mathcal{P}^{**}(\mathrm{Dom}(M)) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto \mathcal{M}_\diamond(x)$$

defines a mapping

$$\mathcal{M}_\diamond : X \to \mathcal{P}^{**}(\mathrm{Dom}(M)).$$

Consequently,

$$\diamond_M^{(r)}(\mathcal{U}_1, \ldots, \mathcal{U}_r) = \bigl(X, \mathcal{M}_\diamond\bigr)$$

is well-defined. □

Remark 9.4.7 (Hierarchy)**.** The above constructions form the following hierarchy:

$$\text{ordinary operation} \implies \text{multioperation} \implies \text{metaoperation}.$$

More precisely:

1. every ordinary r-ary operation

   $$f_M : \mathrm{Dom}(M)^r \to \mathrm{Dom}(M)$$

   induces a multioperation by singleton lifting

   $$(a_1, \ldots, a_r) \longmapsto \{f_M(a_1, \ldots, a_r)\};$$

2. every r-ary multioperation

   $$\star_M^{(r)} : \mathrm{Dom}(M)^r \to \mathcal{P}^*(\mathrm{Dom}(M))$$

   induces a metaoperation by singleton-family lifting

   $$(a_1, \ldots, a_r) \longmapsto \bigl\{\star_M^{(r)}(a_1, \ldots, a_r)\bigr\}.$$

Thus uncertain metaoperations extend uncertain multioperations in a natural higher-order manner.

Chapter 10

Other Concepts

In this chapter, we present other operations and related concepts.

10.1 Uncertain expected values

Expected values are representative averages of random quantities, weighting each possible outcome by its probability and summarizing central tendency in probabilistic and decision-making models mathematically [453, 454]. Fuzzy expected values extend expectation to fuzzy events or variables, assigning monotone, boundary-preserving average-like assessments under graded membership, uncertainty, and capacity-based integration in general settings [455, 456, 456, 457].

Definition 10.1.1 (Fuzzy event)**.** Let $(X, \mathcal{A})$ be a measurable space. A *fuzzy event* on X is a fuzzy set

$$A = \{(x, \mu_A(x)) : x \in X\},$$

whose membership function

$$\mu_A : X \to [0, 1]$$

is $\mathcal{A}$-measurable. The family of all fuzzy events on $(X, \mathcal{A})$ is denoted by

$$\mathcal{A}^{\text{fuzzy}}.$$

For each $\alpha \in [0, 1]$, the α-cut of A is

$$A_\alpha := \{x \in X : \mu_A(x) \geq \alpha\}.$$

Definition 10.1.2 (Capacity)**.** Let $(X, \mathcal{A})$ be a measurable space. A mapping

$$m : \mathcal{A} \to [0, 1]$$

is called a *capacity* if it satisfies

$$m(\emptyset) = 0, \qquad m(X) = 1,$$

and

$$U \subseteq V \Longrightarrow m(U) \leq m(V) \qquad (U, V \in \mathcal{A}).$$

Definition 10.1.3 (Fuzzy expected value)**.** [458, 459] Let $(X, \mathcal{A})$ be a measurable space. A mapping
$$E : \mathcal{A}^{\text{fuzzy}} \to [0, 1]$$
is called an *expected value of fuzzy events* (or *fuzzy expected value*) if it satisfies:

1. **Boundary conditions:**
$$E(\emptyset) = 0, \qquad E(X) = 1;$$
2. **Monotonicity:** if $A, B \in \mathcal{A}^{\text{fuzzy}}$ satisfy
$$\mu_A(x) \le \mu_B(x) \qquad \text{for all } x \in X,$$
then
$$E(A) \le E(B).$$

Remark 10.1.4. Every fuzzy expected value E induces a capacity $m_E : \mathcal{A} \to [0, 1]$ on crisp events by
$$m_E(U) := E(U) \qquad (U \in \mathcal{A}),$$
where a crisp set U is identified with its characteristic function 1_U. Thus a fuzzy expected value can be viewed as a monotone extension of a capacity from crisp events to fuzzy events.

Definition 10.1.5 (Neutrosophic expected value of a neutrosophic random variable)**.** Let X be a finite neutrosophic discrete probability space as above, and let
$$Y : X \to \mathbb{R}$$
be a neutrosophic random variable. Write
$$Y(x_j) = n_j \qquad (j = 1, \dots, r),$$
and
$$Y(\text{indet}_k) = m_k \qquad (k = 1, \dots, s).$$
Then the *neutrosophic expected value* of Y is defined by
$$\mathbb{E}_N[Y] := \sum_{j=1}^{r} Y(x_j)\,\text{ch}(x_j) + \sum_{k=1}^{s} Y(\text{indet}_k)\,\text{ch}(\text{indet}_k).$$

Within the uncertain-set framework, the appropriate codomain is the model-dependent degree-domain
$$\text{Dom}(M) \subseteq [0, 1]^k,$$
so a mathematically consistent definition must ensure that the expected value of uncertain degrees remains admissible.

Definition 10.1.6 (Expected-value-admissible representation of an uncertain model)**.** Let M be an uncertain model with admissible degree-domain
$$\text{Dom}(M) \subseteq [0, 1]^k.$$
An *expected-value-admissible representation* of M consists of the following data:

(i) a nonempty convex set
$$C_M \subseteq [0,1]^q$$
for some integer $q \geq 1$;

(ii) a representation map
$$\Phi_M : \mathrm{Dom}(M) \to C_M;$$

(iii) a reconstruction map
$$\Psi_M : C_M \to \mathrm{Dom}(M)$$
such that
$$\Psi_M \circ \Phi_M = \mathrm{id}_{\mathrm{Dom}(M)}\,.$$

Remark 10.1.7. The role of C_M is to provide a convex coordinate domain in which probability-weighted averages are well-defined. The map Φ_M embeds uncertain degrees into this coordinate space, while Ψ_M converts the averaged coordinates back into admissible uncertain values.

Definition 10.1.8 (Finite probability space)**.** A *finite probability space* is a pair
$$(X,p),$$
where
$$X = \{x_1, x_2, \ldots, x_n\}$$
is a nonempty finite set and
$$p = (p_1, \ldots, p_n) \in [0,1]^n$$
satisfies
$$\sum_{i=1}^{n} p_i = 1.$$
The number p_i is interpreted as the probability of the outcome x_i.

Definition 10.1.9 (Uncertain expected value)**.** Let M be an uncertain model, let (X,p) be a finite probability space with
$$X = \{x_1, \ldots, x_n\}, \qquad p = (p_1, \ldots, p_n),$$
and let
$$\mathcal{U} = (X, \mu_{\mathcal{U}})$$
be an uncertain set of type M on X, where
$$\mu_{\mathcal{U}} : X \to \mathrm{Dom}(M).$$
Assume that an expected-value-admissible representation
$$(C_M, \Phi_M, \Psi_M)$$
of M has been fixed.

The *uncertain expected value* of $\mathcal{U}$ with respect to p is defined by
$$\mathbb{E}_{M,p}[\mathcal{U}] := \Psi_M\left(\sum_{i=1}^{n} p_i\, \Phi_M\big(\mu_{\mathcal{U}}(x_i)\big)\right).$$

Theorem 10.1.10 (Well-definedness of uncertain expected values). *Let M be an uncertain model, let (X, p) be a finite probability space, and let*

$$\mathcal{U} = (X, \mu_{\mathcal{U}})$$

be an uncertain set of type M on X. If

$$(C_M, \Phi_M, \Psi_M)$$

is an expected-value-admissible representation of M, then

$$\mathbb{E}_{M,p}[\mathcal{U}] \in \mathrm{Dom}(M).$$

Hence the uncertain expected value is well-defined.

Proof. Write

$$X = \{x_1, \ldots, x_n\}, \qquad p = (p_1, \ldots, p_n).$$

Since $\mathcal{U}$ is an uncertain set of type M, one has

$$\mu_{\mathcal{U}}(x_i) \in \mathrm{Dom}(M) \qquad (i = 1, \ldots, n).$$

Because

$$\Phi_M : \mathrm{Dom}(M) \to C_M,$$

it follows that

$$\Phi_M\big(\mu_{\mathcal{U}}(x_i)\big) \in C_M \qquad (i = 1, \ldots, n).$$

Now

$$p_i \geq 0 \qquad (i = 1, \ldots, n), \qquad \sum_{i=1}^{n} p_i = 1.$$

Therefore

$$\sum_{i=1}^{n} p_i \, \Phi_M\big(\mu_{\mathcal{U}}(x_i)\big)$$

is a convex combination of points of C_M. Since C_M is convex, one obtains

$$\sum_{i=1}^{n} p_i \, \Phi_M\big(\mu_{\mathcal{U}}(x_i)\big) \in C_M.$$

Finally, because

$$\Psi_M : C_M \to \mathrm{Dom}(M),$$

we conclude that

$$\Psi_M\left(\sum_{i=1}^{n} p_i \, \Phi_M\big(\mu_{\mathcal{U}}(x_i)\big)\right) \in \mathrm{Dom}(M).$$

By definition, this is exactly

$$\mathbb{E}_{M,p}[\mathcal{U}].$$

Hence

$$\mathbb{E}_{M,p}[\mathcal{U}] \in \mathrm{Dom}(M),$$

so the uncertain expected value is well-defined. □

Representative expected values in several uncertainty frameworks are presented in Table 10.1.

Table 10.1: Representative expected values in several uncertainty frameworks.

Framework	Representative expected value
Fuzzy	Fuzzy expected value
Intuitionistic	Intuitionistic expected value
Neutrosophic	Neutrosophic expected value [460, 461]
Plithogenic	Plithogenic expected value

10.2 Uncertain standard deviations

Standard deviations measure how far data values typically spread around their mean, summarizing variability by the square root of variance in a sample or population [462–464]. Fuzzy standard deviations extend classical standard deviations to fuzzy data, representing dispersion as fuzzy numbers obtained through the extension principle from fuzzy samples in statistics.

Definition 10.2.1 (Fuzzy sample variance and fuzzy sample standard deviation)**.** Let

$$x^* = (x_1^*, x_2^*, \dots, x_n^*)$$

be a fuzzy sample of size $n \geq 2$, where each x_i^* is a fuzzy number on $\mathbb{R}$. Assume that the fuzzy sample is represented as a fuzzy vector on $\mathbb{R}^n$, and let

$$\mu_{x^*} : \mathbb{R}^n \to [0, 1]$$

be its membership function.

For a crisp sample

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

define the classical sample mean by

$$\bar{x} := \frac{1}{n} \sum_{i=1}^{n} x_i,$$

the classical sample variance by

$$s^2(x) := \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2,$$

and the classical sample standard deviation by

$$s(x) := \sqrt{s^2(x)}.$$

The *fuzzy sample variance* of x^*, denoted by $S^2_{x^*}$, is defined as the fuzzy image of the mapping

$$s^2 : \mathbb{R}^n \to [0, \infty)$$

under the extension principle; that is, its membership function is

$$\mu_{S^2_{x^*}}(u) := \sup\Big\{ \mu_{x^*}(x_1, \dots, x_n) \ : \ u = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \Big\}, \qquad u \in [0, \infty).$$

Similarly, the *fuzzy sample standard deviation* of x^*, denoted by S_{x^*}, is defined as the fuzzy image of the mapping

$$s : \mathbb{R}^n \to [0, \infty)$$

under the extension principle; that is, its membership function is

$$\mu_{S_{x^*}}(v) := \sup\Big\{\mu_{x^*}(x_1, \dots, x_n) \ : \ v = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}\Big\}, \qquad v \in [0, \infty).$$

Remark 10.2.2. The above definition is the natural fuzzy extension of the ordinary sample variance and sample standard deviation. It is obtained by propagating the fuzziness of the data through the classical statistical mappings

$$x \mapsto s^2(x) \qquad \text{and} \qquad x \mapsto s(x).$$

Definition 10.2.3 (α-cut form of fuzzy standard deviation)**.** Let

$$C(x^*)_\alpha \subseteq \mathbb{R}^n \qquad (\alpha \in (0, 1])$$

denote the α-cut of the fuzzy sample x^*. Then the α-cuts of the fuzzy sample variance and fuzzy sample standard deviation are given by

$$C(S^2_{x^*})_\alpha = \left[\min_{x \in C(x^*)_\alpha} s^2(x), \ \max_{x \in C(x^*)_\alpha} s^2(x)\right],$$

and

$$C(S_{x^*})_\alpha = \left[\min_{x \in C(x^*)_\alpha} s(x), \ \max_{x \in C(x^*)_\alpha} s(x)\right].$$

Definition 10.2.4 (Neutrosophic sample variance and neutrosophic sample standard deviation)**.** Let

$$A_1, A_2, \dots, A_n \qquad (n \geq 2)$$

be a sample of single-valued neutrosophic numbers, where

$$A_i = (T_i, I_i, F_i) \in [0, 1]^3 \qquad (i = 1, \dots, n).$$

Define the neutrosophic sample mean by

$$\bar{A} := (\bar{T}, \bar{I}, \bar{F}),$$

where

$$\bar{T} := \frac{1}{n}\sum_{i=1}^{n} T_i, \qquad \bar{I} := \frac{1}{n}\sum_{i=1}^{n} I_i, \qquad \bar{F} := \frac{1}{n}\sum_{i=1}^{n} F_i.$$

The *neutrosophic sample variance* of the sample

$$(A_1, \dots, A_n)$$

is defined by

$$S^2_A := \big(S^2_T, S^2_I, S^2_F\big),$$

where

$$S_T^2 := \frac{1}{n-1}\sum_{i=1}^{n}(T_i - \bar{T})^2,$$

$$S_I^2 := \frac{1}{n-1}\sum_{i=1}^{n}(I_i - \bar{I})^2,$$

and

$$S_F^2 := \frac{1}{n-1}\sum_{i=1}^{n}(F_i - \bar{F})^2.$$

The *neutrosophic sample standard deviation* of the sample

$$(A_1, \ldots, A_n)$$

is defined by

$$S_A := \big(S_T, S_I, S_F\big),$$

where

$$S_T := \sqrt{S_T^2}, \qquad S_I := \sqrt{S_I^2}, \qquad S_F := \sqrt{S_F^2}.$$

Remark 10.2.5. The above definition is the componentwise extension of the classical sample variance and sample standard deviation to single-valued neutrosophic data. It measures the dispersion of the truth, indeterminacy, and falsity components separately.

Remark 10.2.6. Since

$$0 \le T_i, I_i, F_i \le 1 \qquad (i = 1, \ldots, n),$$

one has

$$0 \le S_T, S_I, S_F \le 1.$$

Hence

$$S_A = (S_T, S_I, S_F) \in [0,1]^3,$$

so the neutrosophic sample standard deviation is again a single-valued neutrosophic number.

In the uncertain-set framework, however, one cannot in general subtract or square elements of

$$\mathrm{Dom}(M) \subseteq [0,1]^k$$

directly. Therefore, a mathematically consistent definition of uncertain standard deviation must be based on a coordinate representation of uncertain degrees together with a reconstruction map back to the admissible degree-domain.

Definition 10.2.7 (Standard-deviation-admissible representation of an uncertain model)**.** Let M be an uncertain model with admissible degree-domain

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

A *standard-deviation-admissible representation* of M consists of:

(i) a nonempty set

$$C_M \subseteq [0,1]^q$$

for some integer $q \ge 1$;

(ii) a representation map

$$\Phi_M : \mathrm{Dom}(M) \to C_M;$$

(iii) a nonempty set

$$\Delta_M \subseteq [0, \infty)^q,$$

(iv) a reconstruction map

$$\Psi_M^{\mathrm{sd}} : \Delta_M \to \mathrm{Dom}(M),$$

(v) the closure property that, for every integer $n \geq 2$, the coordinatewise sample standard deviation map

$$\mathbf{SD}_n : C_M^n \to [0, \infty)^q$$

given by

$$\mathbf{SD}_n\big(u^{(1)}, \dots, u^{(n)}\big) := (\sigma_1, \dots, \sigma_q),$$

where, for

$$u^{(i)} = (u_1^{(i)}, \dots, u_q^{(i)}) \in C_M \qquad (i = 1, \dots, n),$$

one sets

$$\bar{u}_j := \frac{1}{n} \sum_{i=1}^{n} u_j^{(i)}, \qquad \sigma_j := \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \big(u_j^{(i)} - \bar{u}_j\big)^2} \qquad (j = 1, \dots, q),$$

satisfies

$$\mathbf{SD}_n(C_M^n) \subseteq \Delta_M.$$

Remark 10.2.8. The set C_M is the coordinate domain in which standard deviations are computed componentwise. The set Δ_M is the admissible dispersion domain, and Ψ_M^{sd} converts coordinatewise dispersions back into admissible uncertain values.

Definition 10.2.9 (Uncertain sample standard deviation of uncertain values)**.** Let M be an uncertain model, let

$$n \geq 2,$$

and assume that a standard-deviation-admissible representation

$$(C_M, \Phi_M, \Delta_M, \Psi_M^{\mathrm{sd}})$$

of M has been fixed.

For

$$a_1, \dots, a_n \in \mathrm{Dom}(M),$$

the *uncertain sample standard deviation* of

$$(a_1, \dots, a_n)$$

is defined by

$$\mathrm{USD}_{M,n}(a_1, \dots, a_n) := \Psi_M^{\mathrm{sd}}\big(\mathbf{SD}_n\big(\Phi_M(a_1), \dots, \Phi_M(a_n)\big)\big)\,.$$

Theorem 10.2.10 (Well-definedness of the uncertain sample standard deviation)**.** *Let M be an uncertain model and let*

$$(C_M, \Phi_M, \Delta_M, \Psi_M^{\mathrm{sd}})$$

be a standard-deviation-admissible representation of M. Then, for every integer $n \geq 2$, the mapping

$$\mathrm{USD}_{M,n} : \mathrm{Dom}(M)^n \to \mathrm{Dom}(M)$$

is well-defined.

Proof. Let
$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$
Since
$$\Phi_M : \mathrm{Dom}(M) \to C_M,$$
one has
$$\Phi_M(a_i) \in C_M \qquad (i = 1, \ldots, n).$$
Hence
$$\big(\Phi_M(a_1), \ldots, \Phi_M(a_n)\big) \in C_M^n.$$
By the closure property in the definition of a standard-deviation-admissible representation,
$$\mathbf{SD}_n\big(\Phi_M(a_1), \ldots, \Phi_M(a_n)\big) \in \Delta_M.$$
Since
$$\Psi_M^{\mathrm{sd}} : \Delta_M \to \mathrm{Dom}(M),$$
it follows that
$$\Psi_M^{\mathrm{sd}}\big(\mathbf{SD}_n\big(\Phi_M(a_1), \ldots, \Phi_M(a_n)\big)\big) \in \mathrm{Dom}(M).$$
Therefore
$$\mathrm{USD}_{M,n}(a_1, \ldots, a_n) \in \mathrm{Dom}(M)$$
for every
$$(a_1, \ldots, a_n) \in \mathrm{Dom}(M)^n.$$
Hence the mapping $\mathrm{USD}_{M,n}$ is well-defined. □

Definition 10.2.11 (Uncertain standard deviation of U-sets)**.** Let X be a nonempty set, let M be an uncertain model, and let
$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$
be uncertain sets of type M on X, where
$$\mu_i : X \to \mathrm{Dom}(M).$$
Assume that $n \geq 2$ and that a standard-deviation-admissible representation of M has been fixed.

The *uncertain standard deviation* of
$$\mathcal{U}_1, \ldots, \mathcal{U}_n$$
is the uncertain set
$$\mathrm{USD}_{M,n}(\mathcal{U}_1, \ldots, \mathcal{U}_n) := \big(X, \mu_{\mathrm{USD}_{M,n}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}\big),$$
where
$$\mu_{\mathrm{USD}_{M,n}(\mathcal{U}_1,\ldots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M)$$
is defined pointwise by
$$\mu_{\mathrm{USD}_{M,n}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) := \mathrm{USD}_{M,n}\big(\mu_1(x), \ldots, \mu_n(x)\big) \qquad (x \in X).$$

Theorem 10.2.12 (Well-definedness of the uncertain standard deviation of U-sets)**.** *Let X be a nonempty set, let M be an uncertain model, and let*

$$\mathcal{U}_i = (X, \mu_i) \qquad (i = 1, \ldots, n)$$

be uncertain sets of type M on X, with $n \geq 2$. Then

$$\mathrm{USD}_{M,n}(\mathcal{U}_1, \ldots, \mathcal{U}_n)$$

is a well-defined uncertain set of type M on X.

Proof. Since each

$$\mathcal{U}_i = (X, \mu_i)$$

is an uncertain set of type M, one has

$$\mu_i(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X \text{ and all } i = 1, \ldots, n.$$

Fix any $x \in X$. Then

$$\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M)^n.$$

By the preceding theorem,

$$\mathrm{USD}_{M,n}\big(\mu_1(x), \ldots, \mu_n(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_{\mathrm{USD}_{M,n}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}(x) \in \mathrm{Dom}(M) \qquad \text{for all } x \in X.$$

Therefore the pointwise assignment

$$x \longmapsto \mathrm{USD}_{M,n}\big(\mu_1(x), \ldots, \mu_n(x)\big)$$

defines a mapping

$$\mu_{\mathrm{USD}_{M,n}(\mathcal{U}_1,\ldots,\mathcal{U}_n)} : X \to \mathrm{Dom}(M).$$

Consequently,

$$\mathrm{USD}_{M,n}(\mathcal{U}_1, \ldots, \mathcal{U}_n) = \big(X, \mu_{\mathrm{USD}_{M,n}(\mathcal{U}_1,\ldots,\mathcal{U}_n)}\big)$$

is an uncertain set of type M on X. Thus the uncertain standard deviation of U-sets is well-defined. □

Proposition 10.2.13 (Idempotency with respect to dispersion)**.** *Let $a \in \mathrm{Dom}(M)$ and let $n \geq 2$. Then*

$$\mathrm{USD}_{M,n}(a, \ldots, a) = \Psi_M^{\mathrm{sd}}(0, \ldots, 0).$$

Consequently, for any uncertain set

$$\mathcal{U} = (X, \mu_{\mathcal{U}}),$$

one has

$$\mathrm{USD}_{M,n}(\mathcal{U}, \ldots, \mathcal{U}) = \big(X, \mu_0\big),$$

where

$$\mu_0(x) := \Psi_M^{\mathrm{sd}}(0, \ldots, 0) \qquad (x \in X).$$

Proof. Let

$$\Phi_M(a) = (c_1, \dots, c_q) \in C_M.$$

Then for each coordinate $j = 1, \dots, q$, the sample

$$c_j, c_j, \dots, c_j$$

has mean c_j and sample standard deviation 0. Hence

$$\mathbf{SD}_n\big(\Phi_M(a), \dots, \Phi_M(a)\big) = (0, \dots, 0).$$

Therefore

$$\mathrm{USD}_{M,n}(a, \dots, a) = \Psi_M^{\mathrm{sd}}(0, \dots, 0).$$

The U-set statement follows pointwise. □

Representative uncertain standard deviations in several uncertainty frameworks are presented in Table 10.2.

Table 10.2: Representative uncertain standard deviations in several uncertainty frameworks.

Framework	**Representative uncertain standard deviation**
Fuzzy	Fuzzy standard deviation [465–467]
Intuitionistic	Intuitionistic standard deviation
Neutrosophic	Neutrosophic standard deviation [468–471]
Plithogenic	Plithogenic standard deviation

Chapter 11

Conclusion

In this book, we presented a broad and systematic survey of various operations arising in Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic frameworks. In particular, we reviewed representative classes of set-theoretic, logical, aggregation, and integral-type operations, clarified their roles in modeling uncertainty, interaction, and graded information, and highlighted their relationships within broader uncertainty-oriented frameworks. We hope that future research will further investigate their mathematical properties, such as monotonicity, associativity, continuity, and well-definedness, develop new extensions and hybrid models, and explore applications in diverse fields including decision-making, optimization, artificial intelligence, and data analysis. We also expect that studies involving computational experiments, comparative analyses among different operator families, and practical case-based evaluations will continue to advance this research area.

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Data Availability

This work is purely theoretical and mathematical in nature; therefore, no empirical data or computational datasets were used. Future studies may build upon these results through data-driven, computational, or experimental approaches.

Ethical Statement

This study did not involve human participants, animals, or personal data. Accordingly, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the content or publication of this book.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited support tasks, such as English grammar and language refinement. These tools were not used in any manner that would compromise academic integrity or violate ethical standards.

Appendix (List of Tables)

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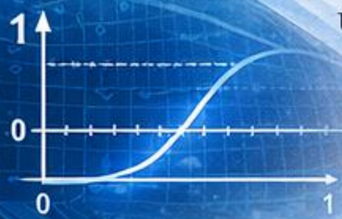
This book presents a comprehensive and systematic survey of fuzzy and uncertain logical operators developed to model reasoning under imprecision, incompleteness, and partial reliability. Building upon foundational frameworks such as fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, vague sets, and hesitant fuzzy sets, the work focuses on the extension of classical logical connectives into uncertainty-aware environments. It provides a structured overview of a wide range of operators, including basic set-theoretic operations, conjunctive-disjunctive families (t-norms, t-conorms, uninorms), relational and inferential operators, aggregation mechanisms, and integral-based models such as Choquet and Sugeno integrals.



The book organizes these operators within a unified uncertain-set framework, clarifying their mathematical properties, interelationships, and functional roles in reasoning processes. By developing a coherent taxonomy and highlighting connections across different uncertainty paradigms, it establishes a solld theoretical foundation for future research. The survey also emphasizes applications in approxmate reasoning, decision-making systems, control theory, and artificial intelligence, making it a valuable reference for both theoretical investigation and practical implementation of uncertainty-based logic.

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