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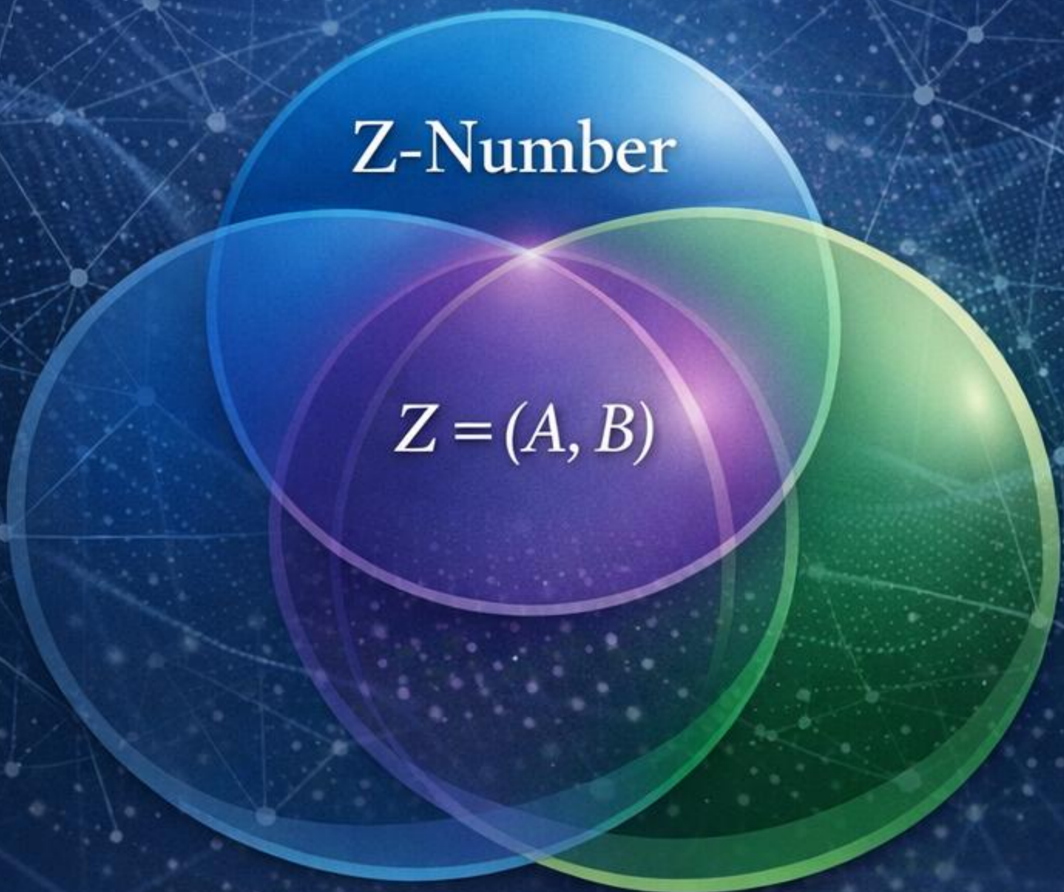
Survey *of* Z-Number, D-Number, *and* ZD-Number

A Comprehensive Study of Advanced Uncertainty Models



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United States of America – Ecuador
2026

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Editor:

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Publishing House
https://fs.unm.edu/NSIA/

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ISBN 978-1-59973-882-6

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Chapter 1

Introduction

1.1 Uncertain Sets

Real-world phenomena are frequently characterized by vagueness, partial truth, indeterminacy, and incomplete information. To describe such uncertainty in a mathematically rigorous manner, many generalized set-theoretic frameworks have been developed over the past decades. Representative examples include Fuzzy Sets [1, 2], Intuitionistic Fuzzy Sets [3, 4], Hesitant Fuzzy Sets [5, 6], Picture Fuzzy Sets [7, 8], Quadripartitioned Neutrosophic Sets [9, 10], Pentapartitioned Neutrosophic Sets [11, 12], Plithogenic Sets [13, 14], HyperFuzzy Sets [15, 16], and HyperNeutrosophic Sets [17, 18].

These frameworks have been applied in a wide range of areas, including decision science, chemistry, control systems, and machine learning [19]. Since different applications require different forms and levels of uncertainty representation, the choice of an appropriate generalized set model depends on the structure of the problem and on the number of uncertainty parameters needed to characterize the underlying phenomenon.

1.2 Z-Number and D-Number

Among the many formalisms proposed for uncertainty modeling, Z-Numbers and D-Numbers occupy an important position because they enrich uncertainty representation in different but complementary ways.

A Z-Number represents an uncertain value by means of two components: the first component specifies the restriction on the value itself, while the second component describes the reliability of that restriction [20–23]. In this way, a Z-Number models not only an assessment, but also the confidence attached to that assessment.

A D-Number, by contrast, assigns belief masses to subsets of a frame that is not necessarily composed of mutually exclusive elements [24–26]. Consequently, D-Numbers allow incomplete information and relax the exclusiveness assumption imposed in classical Dempster–Shafer evidence theory (cf. [27, 28]). This makes them particularly suitable for situations in which the available information is partial, overlapping, or structurally nonexclusive.

Both Z-Numbers and D-Numbers have attracted increasing attention in recent years, and their applications have been investigated in various areas, including Graph Theory [29, 30] and Decision-Making [31–33].

1.3 Our Contributions

This book provides a broad survey of concepts related to Z-Numbers and D-Numbers. In particular, it reviews a variety of associated extensions and related models, including, for example, Neutrosophic Z-Numbers and Neutrosophic D-Numbers. Although the treatment of some of these concepts is necessarily brief, the aim is to present a coherent overview of the existing landscape and to clarify the relationships among the principal frameworks.

In addition to surveying known models, this book also introduces new concepts. In particular, we define the concept of the ZD-Number and discuss it as a natural fusion of the Z-Number viewpoint and the D-Number viewpoint. Through this combination of survey and new development, the book aims to provide both a useful reference on existing theories and a foundation for further research on extended number-based models of uncertainty.

Chapter 2

Preliminaries

This chapter introduces the notation and fundamental concepts used in the sequel.

2.1 Fuzzy Set

Fuzzy set theory generalizes the ordinary notion of a subset by allowing each element to belong to a set with a degree in the unit interval $[0, 1]$ [1,34,35]. We first recall the standard definition.

Definition 2.1.1 (Fuzzy set)**.** [1] Let X be a nonempty set. A *fuzzy set* A on X is determined by a function

$$\mu_A : X \to [0, 1],$$

called the *membership function* of A. Equivalently, one may represent A as

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ expresses the degree to which x belongs to A.

2.2 Intuitionistic Fuzzy Set

Intuitionistic fuzzy sets refine fuzzy sets by assigning to each element both a membership degree and a non-membership degree, thereby leaving room for an explicit hesitation part [4,36]. The usual definition is given below.

Definition 2.2.1 (Intuitionistic fuzzy set)**.** [37] Let E be a nonempty set. An *intuitionistic fuzzy set* (IFS) A on E is of the form

$$A = \big\{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in E\big\},$$

where

$$\mu_A, \nu_A : E \to [0, 1]$$

denote the membership and non-membership functions, respectively, and satisfy

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \qquad \text{for all } x \in E.$$

The quantity

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x)$$

is called the *hesitation degree* of x.

The classical fuzzy-set case is recovered when

$$\nu_A(x) = 1 - \mu_A(x) \qquad \text{for all } x \in E,$$

or equivalently when $\pi_A(x) = 0$ for every $x \in E$.

2.3 Neutrosophic Set

Neutrosophic sets describe uncertainty by assigning to each element three quantities: truth, indeterminacy, and falsity, usually taken in the interval $[0, 1]$ [38–41]. Because the indeterminacy component is handled explicitly, this framework extends both fuzzy sets and intuitionistic fuzzy sets in a flexible way [42].

Definition 2.3.1 (Neutrosophic set)**.** [43, 44] Let X be a nonempty set. A *neutrosophic set* (NS) A on X is specified by three mappings

$$T_A : X \to [0, 1], \qquad I_A : X \to [0, 1], \qquad F_A : X \to [0, 1],$$

where, for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively, of the statement "$x \in A$". These values satisfy

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad \text{for all } x \in X.$$

2.4 Plithogenic Set

Plithogenic set theory extends uncertainty modeling by incorporating attribute-based appurtenance together with contradiction degrees between attribute values [45–48]. A standard formulation is as follows.

Definition 2.4.1 (Plithogenic Set)**.** [45, 46] Let P be a nonempty universe of discourse, and let v be a fixed attribute whose possible values form a nonempty set Pv. Let $s, t \in \mathbb{N}$.

A *plithogenic set* on (P, v, Pv) is a quintuple

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF),$$

where

- $$pdf : P \times Pv \to [0,1]^s$$

  is the *degree of appurtenance function* (DAF); for $x \in P$ and $a \in Pv$, the value $pdf(x,a)$ gives the possibly vector-valued degree to which x belongs relative to the attribute value a;

- $$pCF : Pv \times Pv \to [0,1]^t$$

  is the *degree of contradiction function* (DCF), satisfying

  $$pCF(a,a) = 0, \qquad pCF(a,b) = pCF(b,a) \quad \text{for all } a, b \in Pv.$$

In plithogenic theory, one usually selects a *dominant attribute value* $a^* \in Pv$. Set-theoretic operations such as union and intersection are then constructed by combining appurtenance degrees with contradiction degrees relative to a^*, thereby capturing interaction and opposition among different attribute values.

2.5 Rough Set

Rough set theory treats imprecision by replacing a target set with two approximations: a lower approximation, representing certainty, and an upper approximation, representing possibility. These are derived from an indiscernibility relation [49–52]. The classical Pawlak construction is recalled below.

Definition 2.5.1 (Rough set approximations)**.** [53] Let X be a nonempty universe, and let $R \subseteq X \times X$ be an equivalence relation. For each $x \in X$, define the equivalence class of x by

$$[x]_R := \{\, y \in X \mid (x,y) \in R \,\}.$$

For any subset $U \subseteq X$, define:

1. *Lower approximation*:
   $$\underline{U} := \{\, x \in X \mid [x]_R \subseteq U \,\}.$$
   Thus, $\underline{U}$ consists of those elements whose entire equivalence classes lie inside U.
2. *Upper approximation*:
   $$\overline{U} := \{\, x \in X \mid [x]_R \cap U \neq \varnothing \,\}.$$
   Hence, $\overline{U}$ consists of those elements whose equivalence classes intersect U.

The pair $(\underline{U}, \overline{U})$ is called the *rough approximation* of U, and one always has

$$\underline{U} \subseteq U \subseteq \overline{U}.$$

2.6 Soft Set

Soft sets model uncertainty by means of parameters: each parameter determines a subset of the universe, and the entire family of such subsets forms the soft description. This framework was introduced by Molodtsov and has since been used widely in uncertainty analysis and decision-making [54, 55].

Definition 2.6.1 (Soft set)**.** [55] Let U be a universe, let E be a set of parameters, and let $A \subseteq E$. Denote by $\mathcal{P}(U)$ the power set of U. A pair (F, A) is called a *soft set* over U if

$$F : A \to \mathcal{P}(U).$$

For each parameter $\epsilon \in A$, the subset $F(\epsilon) \subseteq U$ is called the *ϵ-approximation* of (F, A). Thus, a soft set is simply a parameterized family of subsets of the universe U.

2.7 Uncertain set

An *uncertain set* associates with each element a degree taken from a chosen uncertainty model, thereby providing a unifying umbrella for fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and related frameworks [56, 57].

Definition 2.7.1 (Uncertain model)**.** [56] Let U denote the class of all *uncertain models*. Each $M \in U$ is determined by:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0, 1]^k$ of *admissible degree tuples* for some fixed integer $k \geq 1$; and
- model-specific algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \leq 1$ in the intuitionistic fuzzy setting, or $0 \leq T + I + F \leq 3$ in the neutrosophic setting).

Typical instances include:

- **Fuzzy model:** $\mathrm{Dom}(M) = [0, 1]$;
- **Intuitionistic fuzzy model:** $\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 : \ \mu + \nu \leq 1\}$;
- **Neutrosophic model:** $\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 : \ 0 \leq T + I + F \leq 3\}$;
- **Plithogenic model,** and many further extensions.

Definition 2.7.2 (Uncertain set (U-set))**.** [56] Let X be a nonempty universe, and fix an uncertain model M with degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$. An *uncertain set of type M* (briefly, a *U-set*) on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is the *uncertainty-degree function* (membership map) of $\mathcal{U}$. For $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ encodes the degree(s) to which x belongs to $\mathcal{U}$, as prescribed by the model M.

As noted in the remark, various generalizations are possible. For reference, Table 2.1 presents a catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ (cf. [57]).

Table 2.1: A catalogue of uncertainty-set families (U-Sets) by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [57].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [1,2]; N-Fuzzy Set [58–60] Shadowed Set [61–63]
2		Intuitionistic Fuzzy Set [4,36]; Vague Set [64,65]; Bipolar Fuzzy Set (two-component description) [66,67]; Pythagorean Fuzzy Set [24,68]; Fermatean fuzzy Set [69,70]; Variable Fuzzy Set [71–73]; Paraconsistent Fuzzy Set [74, 75]; Bifuzzy Set [76,77]
3		Single-Valued Neutrosophic Set [40,44]; Picture Fuzzy Set [8,78]; Ternary Fuzzy Set [79]; Hesitant Fuzzy Set [6,80]; Spherical Fuzzy Set [81,82]; Tripolar Fuzzy Set (three-component formalisms) [83–85]; Neutrosophic Vague Set [86,87]
4		Quadripartitioned Neutrosophic Set [10,88]; Double-Valued Neutrosophic Set [89,90]; Dual Hesitant Fuzzy Set [91,92]; Ambiguous Set [93–95]; Turiyam Neutrosophic Set [96–99]
5		Pentapartitioned Neutrosophic Set [100–102]; Triple-Valued Neutrosophic Set [103–106]
6		Hexapartitioned Neutrosophic Set [107]; Bipolar Neutrosophic Set [108,109]; Bipolar Picture Fuzzy Sets [110,111]; Quadruple-Valued Neutrosophic Set [105,112]
7		Heptapartitioned Neutrosophic Set [113–115]; Quintuple-Valued Neutrosophic Set [105,116,117]
8		Octapartitioned Neutrosophic Set [107]; Bipolar Quadripartitioned Neutrosophic Set [118,119]; Bipolar Double-valued Neutrosophic Set
9		Nonapartitioned Neutrosophic Set [107]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [120]; MultiFuzzy Set [121]; n-Refined Fuzzy Set [122,123]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [123]; Multi-Intuitionistic Fuzzy Set [121]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [123,124]; Multi-Neutrosophic Set [121,125,126]

Reading guide. In the U-Set scheme [56], each model M is specified by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ and a membership map $\mu_M : X \to \mathrm{Dom}(M)$. The table groups representative families by the ambient dimension k (i.e., how many numerical components are stored per element).
(a) A widely cited viewpoint is that neutrosophic sets provide a unifying umbrella covering several earlier multi-component fuzzy models (and their generalizations); see [42].
(b) Ambiguous sets are commonly presented as subclasses of certain four-component neutrosophic families; see [10,88,95].
(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [127].

2.8 Hypersoft Set

A Hypersoft Set assigns subsets of a universe to tuples of attribute values, enabling multi-attribute parameterization and more detailed soft-set based uncertainty representation [128, 129]. Related hierarchical concepts are also known, such as the SuperHyperSoft Set [129,130] and the TreeSoft Set [131,132].

Definition 2.8.1 (Hypersoft Set)**.** [133] Let U be a nonempty universal set, and let

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$$

be nonempty attribute-value sets. Define

$$\mathcal{C} := \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

A *hypersoft set* over U is a pair

$$(G, \mathcal{C}),$$

where

$$G : \mathcal{C} \to \mathcal{P}(U).$$

Equivalently, it can be written as

$$(G, \mathcal{C}) = \{\, (\gamma, G(\gamma)) \mid \gamma \in \mathcal{C} \,\}.$$

For each tuple

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}, \qquad \gamma_i \in \mathcal{A}_i \quad (i = 1, 2, \ldots, m),$$

the set

$$G(\gamma) \subseteq U$$

represents the collection of elements of U corresponding to the combined attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Chapter 3

Extended Z-Number Models

In this chapter, we examine extended Z-number models. A summary of Z-number variants is presented in Table 3.1.

Table 3.1: Summary of Z-number variants in Chapter 3.

Section	Concept	Brief description
3.1	Z-Number	A pair (A, B) in which A describes a value or restriction and B expresses the reliability of A.
3.2	ZE-number	An extension of the Z-number that incorporates an additional evidence-related or evaluative component beyond the standard value–reliability pair.
3.3	Linguistic Z-number	A Z-number whose value and/or reliability components are represented by linguistic terms rather than only numerical quantities.
3.4	Weighted Z-Number	A Z-number equipped with an additional weight parameter to reflect relative importance, influence, or priority in aggregation and decision analysis.
3.5	Mixed-discrete Z-numbers	Z-number models in which at least one component is represented on a discrete or mixed discrete–continuous domain.
3.6	Hyper Z-Number	A generalized Z-number whose components are set-valued or higher-order, allowing richer uncertainty representation than ordinary fuzzy-valued components.
3.7	SuperHyper Z-Number	A hierarchical extension of the Hyper Z-Number based on iterated set structures for representing multilevel uncertainty and reliability.
3.8	Z-OffNumber	A Z-number constructed using off-set or shifted membership structures, allowing values outside the standard unit-interval interpretation.

Section	Concept	Brief description
3.9	Multipolar Z-Number	A Z-number with multiple poles or aspects, enabling the simultaneous treatment of several evaluative directions or perspectives.
3.10	Augmented Z-number	A Z-number enriched with additional parameters or components in order to capture supplementary semantic or structural information.
3.11	Z-MultiNumbers	A family of multi-component Z-type objects that generalize the standard two-part Z-number to multiple coordinated components.
3.12	Multidimensional Z-number	A Z-number whose value and/or reliability part is vector-valued or multi-attribute, designed for multidimensional assessment settings.
3.13	Probabilistic Z-Number	A Z-number model in which the uncertainty or reliability information is represented probabilistically rather than only fuzzily.
3.14	p-box Z-Number	A Z-number based on probability boxes, using lower and upper cumulative bounds to model distributional uncertainty and reliability imprecision.

3.1 Z-Number

A Z-Number represents an uncertain value and the reliability of that value, combining restriction and confidence in one framework formally [134].

Definition 3.1.1 (Z-number)**.** Let X be a real-valued uncertain variable on a universe U. A *Z-number* is an ordered pair

$$Z = (A, B),$$

where A is a fuzzy subset of U representing a restriction on the values of X, and B is a fuzzy subset of $[0, 1]$ representing the reliability of the statement X is A.

3.2 ZE-number

A ZE-number extends a Z-number by adding an external credibility layer, jointly modeling a fuzzy restriction, subjective reliability, and objective reliability of information under uncertainty [135–138].

Definition 3.2.1 (ZE-number)**.** [135–138] Let $U \subseteq \mathbb{R}$ be a universe of discourse, and let $\mathcal{F}(U)$ denote the family of fuzzy subsets of U. Let X be a real-valued uncertain variable taking values in U.

A *ZE-number* associated with X is a nested ordered pair

$$\mathrm{ZE} = (Z, E) = ((A, B), E),$$

where

$$A \in \mathcal{F}(U), \qquad B \in \mathcal{F}([0,1]), \qquad E \in \mathcal{F}([0,1]).$$

The components are interpreted as follows:

1. A is a fuzzy restriction on the possible values of X;
2. B is the subjective reliability, certainty, or confidence of the statement

$$X \text{ is } A;$$

3. E is the credibility, i.e., the objective reliability of the original Z-number $Z = (A, B)$, and hence acts as an external constraint on the information represented by (A, B).

Thus, a ZE-number enriches a Z-number by introducing a second reliability layer: B expresses the internal or subjective reliability of the assessment, whereas E expresses the external or objective credibility of that assessment.

Remark 3.2.2. If A, B, and E are all fuzzy numbers, then $\mathrm{ZE} = ((A, B), E)$ is called a *fuzzy ZE-number.*

For reference, a comparison of Z-number and ZE-number is presented in Table 3.2.

Table 3.2: A concise comparison of Z-number and ZE-number.

Concept	**Basic form**	**Main idea**	**Key feature**
Z-number	$Z = (A, B)$	A is a fuzzy restriction on the value of an uncertain variable, and B is a fuzzy restriction on the reliability of the statement "X is A."	Represents an uncertain value together with one reliability layer.
ZE-number	$\mathrm{ZE} = ((A, B), E)$	A is a fuzzy value restriction, B is the subjective reliability of the assessment, and E is the external credibility of the original Z-number.	Extends a Z-number by adding a second, external reliability layer.

3.3 Linguistic Z-number

A linguistic Z-number represents a linguistic assessment together with a linguistic reliability term, enabling qualitative uncertain information and its confidence to be modeled jointly [139–141].

Definition 3.3.1 (Linguistic Z-number)**.** [139–141] Let

$$S_1 = \{s_i \mid i = -\tau, -\tau + 1, \ldots, 0, \ldots, \tau - 1, \tau\}$$

and

$$S_2 = \{s'_j \mid j = -\varsigma, -\varsigma + 1, \ldots, 0, \ldots, \varsigma - 1, \varsigma\}$$

be two finite linguistic term sets, where $\tau, \varsigma \in \mathbb{N}$, each set is totally ordered, its subscripts are symmetric with respect to 0, and each set has odd cardinality.

A *linguistic Z-number* (briefly, LZN) is an ordered pair

$$z = (A, B) = (s_i, s'_j),$$

where

$$A \in S_1, \qquad B \in S_2.$$

Here:

1. A is the linguistic assessment (or linguistic restriction) of the object, quantity, or uncertain variable under consideration;
2. B is the linguistic reliability, certainty, or credibility of the statement expressed by A.

Thus, an LZN is a Z-number whose first and second components are both represented by linguistic terms.

Remark 3.3.2. If X is an uncertain variable on a universe of discourse U, then the statement represented by

$$z = (A, B)$$

is interpreted linguistically as

$$\text{“}X \text{ is } A\text{” with reliability } B.$$

Hence, A describes the linguistic value of X, while B describes the linguistic confidence assigned to that description.

3.4 Weighted Z-Number

A weighted Z-number extends a Z-number by attaching importance weights during comparison or aggregation, jointly reflecting uncertain values, reliability, and decision relevance [142–144].

Definition 3.4.1 (Weighted Z-Number)**.** [142–144] Let

$$Z = (A, B)$$

be a Z-number, where

$$A$$

is a fuzzy number on $\mathbb{R}$, representing a fuzzy restriction on the values of a real-valued uncertain variable X, and

$$B$$

is a fuzzy number on $[0, 1]$, representing the reliability of A.

Assume that the membership function $\mu_B : [0, 1] \to [0, 1]$ satisfies

$$\int_0^1 \mu_B(r)\, dr > 0.$$

Define the reliability weight induced by B by

$$\alpha(Z) := \frac{\displaystyle\int_0^1 r\, \mu_B(r)\, dr}{\displaystyle\int_0^1 \mu_B(r)\, dr}.$$

Then $\alpha(Z) \in [0, 1]$.

The *Weighted Z-Number* associated with Z is the fuzzy set

$$Z^w = A^{\alpha(Z)}$$

whose membership function is given by

$$\mu_{Z^w}(x) = \mu_{A^{\alpha(Z)}}(x) := \alpha(Z)\, \mu_A(x), \qquad x \in \mathbb{R}.$$

Equivalently,

$$Z^w = \left\{ \left(x, \alpha(Z)\mu_A(x)\right) \mid x \in \mathbb{R} \right\}.$$

Here, $\alpha(Z)$ is called the *reliability weight* of the Z-number Z. Thus, Z^w is obtained by incorporating the reliability information carried by B into the restriction component A.

3.5 Mixed-discrete Z-numbers

A mixed-discrete Z-number combines a fuzzy value restriction with a discrete fuzzy reliability scale, enabling uncertain quantitative assessments to be represented more efficiently and computationally [145, 146].

Definition 3.5.1 (Mixed-discrete Z-number)**.** Let $U \subseteq \mathbb{R}$ be a universe of discourse, let $\mathcal{FN}(U)$ be the family of fuzzy numbers on U, and let $\mathcal{DFN}(L_m)$ be the family of discrete fuzzy numbers on the finite chain $L_m = \{0, 1, \ldots, m\}$.

A *mixed-discrete Z-number* is an ordered pair

$$Z = (A, B)$$

such that

$$A \in \mathcal{FN}(U), \qquad B \in \mathcal{DFN}(L_m),$$

where A is a fuzzy restriction on an uncertain variable X, and B is a discrete fuzzy representation of the reliability of the statement X is A.

3.6 Hyper Z-Number

A Hyper Z-Number assigns each value a nonempty family of Z-number pairs, capturing multiple admissible fuzzy restrictions and reliability assessments for complex uncertain descriptions jointly [147].

Definition 3.6.1 (Hyper Z-Number)**.** [147] Let $\mathcal{F}$ denote the set of all fuzzy numbers on $\mathbb{R}$, and let

$$\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$$

be the family of all nonempty subsets of a set X.

A *Hyper Z-Number* is a mapping

$$\widetilde{Z} : \mathbb{R} \to \mathcal{P}^*(\mathcal{F} \times \mathcal{F})$$

such that, for each $x \in \mathbb{R}$,

$$\widetilde{Z}(x) = \{(A_i, R_i) \mid i \in I_x\},$$

where I_x is a nonempty index set, and for every $i \in I_x$:

$$A_i \in \mathcal{F}, \qquad R_i \in \mathcal{F}.$$

Here, A_i is a fuzzy-number restriction on the possible value of x, and R_i is a fuzzy-number reliability associated with the restriction A_i.

Thus, each pair (A_i, R_i) is an ordinary Z-number, and $\widetilde{Z}(x)$ represents a nonempty family of admissible Z-number descriptions of x.

3.7 SuperHyper Z-Number

An n-SuperHyper Z-Number assigns each value an n-level nested nonempty family of Z-number pairs, modeling hierarchically organized fuzzy restrictions and reliabilities across multiple structural layers [147].

Definition 3.7.1 (n-th Nonempty Powerset)**.** [148] Let X be a nonempty set. Define recursively

$$\mathcal{P}^{*(0)}(X) := X, \qquad \mathcal{P}^{*(n)}(X) := \mathcal{P}^*\big(\mathcal{P}^{*(n-1)}(X)\big) \quad (n \geq 1).$$

Definition 3.7.2 (n-SuperHyper Z-Number)**.** [147] Let $\mathcal{F}$ be the set of all fuzzy numbers on $\mathbb{R}$, and let $n \geq 2$ be an integer. An *n-SuperHyper Z-Number* is a mapping

$$\widetilde{Z}^{(n)} : \mathbb{R} \to \mathcal{P}^{*(n)}(\mathcal{F} \times \mathcal{F})$$

such that, for each $x \in \mathbb{R}$, the value

$$\widetilde{Z}^{(n)}(x) \in \mathcal{P}^{*(n)}(\mathcal{F} \times \mathcal{F})$$

is an n-level nested nonempty family built from ordered pairs (A, R) with $A, R \in \mathcal{F}$.

Equivalently, an n-SuperHyper Z-Number is a hierarchical collection of Z-number pairs, where each basic pair (A, R) consists of

$$A \in \mathcal{F} \quad \text{and} \quad R \in \mathcal{F},$$

with A representing a fuzzy restriction on the value of x and R representing the fuzzy reliability of that restriction.

Example 3.7.3 (Emergency delivery-time assessment in disaster logistics)**.** Consider a disaster-relief center that must estimate the delivery time of medical supplies to an affected area. Let the real-valued uncertain variable

$$X$$

denote the delivery time, measured in hours. Because the situation is highly dynamic, the estimate is collected in a hierarchical manner from several sources, such as local drivers, traffic-monitoring systems, and field coordinators.

Assume that we use a 2-SuperHyper Z-Number

$$\widetilde{Z}^{(2)} : \mathbb{R} \to \mathcal{P}^{*(2)}(\mathcal{F} \times \mathcal{F})$$

to represent the delivery-time assessment. For the candidate value

$$x = 5,$$

suppose that

$$\widetilde{Z}^{(2)}(5) = \Big\{\mathcal{C}_1, \mathcal{C}_2\Big\},$$

where

$$\mathcal{C}_1 = \{(A_1, R_1), (A_2, R_2)\}, \qquad \mathcal{C}_2 = \{(A_3, R_3), (A_4, R_4)\}.$$

Here:

- $\mathcal{C}_1$ represents the collection of assessments obtained from traffic-related information sources;
- $\mathcal{C}_2$ represents the collection of assessments obtained from field-operation and logistics experts.

Let the basic Z-number pairs be given by the following triangular fuzzy numbers:

$$\begin{aligned} A_1 &= (4.5, 5.0, 5.5), & R_1 &= (0.70, 0.80, 0.90), \\ A_2 &= (4.8, 5.2, 5.8), & R_2 &= (0.60, 0.75, 0.85), \\ A_3 &= (4.0, 5.0, 6.0), & R_3 &= (0.65, 0.78, 0.88), \\ A_4 &= (5.0, 5.5, 6.5), & R_4 &= (0.55, 0.70, 0.82). \end{aligned}$$

Thus,

$$\begin{aligned} \widetilde{Z}^{(2)}(5) = \Big\{ &\big\{\big((4.5, 5.0, 5.5), (0.70, 0.80, 0.90)\big), \big((4.8, 5.2, 5.8), (0.60, 0.75, 0.85)\big)\big\}, \\ &\big\{\big((4.0, 5.0, 6.0), (0.65, 0.78, 0.88)\big), \big((5.0, 5.5, 6.5), (0.55, 0.70, 0.82)\big)\big\}\Big\}. \end{aligned}$$

This means that the delivery time near 5 hours is not described by a single Z-number, but by a hierarchical family of Z-number pairs. At the lower level, each pair (A_i, R_i) represents one concrete assessment together with its reliability. At the higher level, these pairs are grouped according to broader information classes, such as traffic evidence and field-logistics evidence.

Therefore, the 2-SuperHyper Z-Number provides a natural model for real-world situations in which uncertain values and their reliabilities are collected, organized, and interpreted across multiple hierarchical layers.

(m, n)-SuperHyper Z-Number assigns hierarchical collections of Z-number pairs to hierarchical collections of real values, modeling multi-level uncertain values and reliabilities.

Definition 3.7.4 ((m, n)-SuperHyper Z-Number)**.** Let

$$\mathcal{F}_{\mathbb{R}}$$

denote the set of all fuzzy numbers on $\mathbb{R}$, and let

$$\mathcal{F}_{[0,1]}$$

denote the set of all fuzzy numbers on $[0, 1]$. Let

$$m \in \mathbb{N}_0 \qquad \text{and} \qquad n \geq 2.$$

An (m, n)-*SuperHyper Z-Number* is a mapping

$$\widetilde{Z}^{(m,n)} : \mathcal{P}^{*(m)}(\mathbb{R}) \longrightarrow \mathcal{P}^{*(n)}\big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big)$$

such that, for each

$$\xi \in \mathcal{P}^{*(m)}(\mathbb{R}),$$

the value

$$\widetilde{Z}^{(m,n)}(\xi) \in \mathcal{P}^{*(n)}\big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big)$$

is an n-level nested nonempty family built from ordered pairs

$$(A, R) \in \mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]},$$

where

- A is a fuzzy-number restriction on possible values of the quantity under consideration;
- R is a fuzzy-number reliability restriction on the credibility of the statement represented by A.

Equivalently, an (m, n)-SuperHyper Z-Number assigns to each m-level nested nonempty family of real numbers a corresponding n-level nested nonempty family of ordinary Z-number pairs.

Remark 3.7.5. The parameter m controls the hierarchical complexity of the *input side*, while n controls the hierarchical complexity of the *Z-number side*. Thus:

- if $m = 0$, then
$$\mathcal{P}^{*(0)}(\mathbb{R}) = \mathbb{R},$$
and one recovers the usual n-SuperHyper Z-Number,
$$\widetilde{Z}^{(0,n)} : \mathbb{R} \to \mathcal{P}^{*(n)}\big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big);$$
- if $m = 1$, then the input is a nonempty family of real values;
- larger values of m allow iterated hierarchical value-collections on the domain side.

A concise comparison of Z-Number, Hyper Z-Number, SuperHyper Z-Number, and (m, n)-SuperHyper Z-Number is presented in Table 3.3.

Table 3.3: A concise comparison of Z-Number, Hyper Z-Number, SuperHyper Z-Number, and (m, n)-SuperHyper Z-Number.

Concept	**Basic form**	**Hierarchy**	**Main feature**
Z-Number	$Z = (A, R)$, $A \in \mathcal{F}_{\mathbb{R}}$, $R \in \mathcal{F}_{[0,1]}$	None	Represents a fuzzy restriction together with a fuzzy reliability restriction.
Hyper Z-Number	$\widetilde{Z}_H : \mathbb{R} \to \mathcal{P}^{*}\big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big)$	One-level value-side family	Assigns to each real value a nonempty family of ordinary Z-number pairs.
SuperHyper Z-Number	$\widetilde{Z}^{(n)} : \mathbb{R} \to \mathcal{P}^{*(n)}\big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big)$	Depth n on the Z-number side	Assigns an n-level nested nonempty family of ordinary Z-number pairs to each real value.
(m, n)-SuperHyper Z-Number	$\widetilde{Z}^{(m,n)} : \mathcal{P}^{*(m)}(\mathbb{R}) \to \mathcal{P}^{*(n)}\big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big)$	Depth m on the domain side, depth n on the Z-number side	Separates the hierarchical complexity of the input side from that of the Z-number side, yielding the most flexible generalization.

3.8 Z-OffNumber

A Z-OffNumber uses offset fuzzy numbers for both value restriction and reliability, allowing normalized fuzzy interpretation while extending membership grades beyond the standard unit interval [149].

Definition 3.8.1 (Offset fuzzy number)**.** Let $I \subseteq \mathbb{R}$ be an interval, and let $\Psi < 0 < 1 < \Omega$. Define the affine normalization map
$$N_{\Psi,\Omega} : [\Psi, \Omega] \to [0, 1], \qquad N_{\Psi,\Omega}(u) := \frac{u - \Psi}{\Omega - \Psi}.$$

A mapping
$$\mu : I \to [\Psi, \Omega]$$
is called a (Ψ, Ω)-*offset fuzzy number* on I if the normalized membership function
$$\widehat{\mu} := N_{\Psi,\Omega} \circ \mu : I \to [0, 1]$$

is an ordinary fuzzy number on I; that is, $\widehat{\mu}$ is normal, upper semicontinuous, fuzzy convex, and has compact support.

We denote the family of all such offset fuzzy numbers on I by

$$\mathcal{F}_{\text{off}}^{\Psi,\Omega}(I).$$

Definition 3.8.2 (Z-OffNumber)**.** [149] Let

$$\Psi_A < 0 < 1 < \Omega_A, \qquad \Psi_R < 0 < 1 < \Omega_R.$$

A *Z-OffNumber* is an ordered pair

$$Z^{\text{off}} = (A^{\text{off}}, R^{\text{off}})$$

such that

$$A^{\text{off}} \in \mathcal{F}_{\text{off}}^{\Psi_A,\Omega_A}(\mathbb{R}), \qquad R^{\text{off}} \in \mathcal{F}_{\text{off}}^{\Psi_R,\Omega_R}([0,1]).$$

Equivalently, A^{off} and R^{off} are represented by membership functions

$$\mu_{A^{\text{off}}} : \mathbb{R} \to [\Psi_A, \Omega_A], \qquad \mu_{R^{\text{off}}} : [0,1] \to [\Psi_R, \Omega_R],$$

for which the normalized functions

$$\widehat{\mu}_{A^{\text{off}}}(x) = \frac{\mu_{A^{\text{off}}}(x) - \Psi_A}{\Omega_A - \Psi_A}, \qquad \widehat{\mu}_{R^{\text{off}}}(r) = \frac{\mu_{R^{\text{off}}}(r) - \Psi_R}{\Omega_R - \Psi_R}$$

are ordinary fuzzy numbers.

Here:

1. A^{off} is an offset fuzzy restriction on the possible values of a real-valued uncertain variable X;

2. R^{off} is an offset fuzzy reliability restriction on the credibility level $r \in [0,1]$ associated with the restriction A^{off}.

Thus, the statement

$$Z^{\text{off}} = (A^{\text{off}}, R^{\text{off}})$$

means that the variable X is constrained by the offset fuzzy number A^{off}, with offset fuzzy reliability described by R^{off}.

3.9 Multipolar Z-Number

Multipolar Z-Number extends a Z-Number by assigning multiple polarity-specific restrictions and corresponding reliabilities, enabling simultaneous modeling of several evaluative viewpoints within one uncertainty framework coherently. A related concept known in the literature is the Bipolar Z-Number.

Definition 3.9.1 (m-polar fuzzy set)**.** [150] Let U be a nonempty set and let $m \in \mathbb{N}$. An *m-polar fuzzy set* on U is a mapping

$$\mu_A^{(m)} : U \to [0,1]^m.$$

Equivalently, it may be written as

$$A^{(m)} = \left\{ \left(x, \mu_A^{(1)}(x), \mu_A^{(2)}(x), \ldots, \mu_A^{(m)}(x)\right) \,\middle|\, x \in U \right\},$$

where

$$\mu_A^{(k)} : U \to [0,1] \qquad (k = 1, 2, \ldots, m)$$

denotes the k-th polar membership function.

Definition 3.9.2 (Multipolar Z-Number)**.** Let $U \subseteq \mathbb{R}$ be a universe of discourse, let X be a real-valued uncertain variable taking values in U, and let $m \in \mathbb{N}$. A *Multipolar Z-Number of order m* is an ordered pair

$$Z^{(m)} = (A^{(m)}, B),$$

where

1. $A^{(m)}$ is an m-polar fuzzy set on U, called the *multipolar restriction component*;
2. B is a fuzzy subset of $[0,1]$, called the *reliability component*.

More explicitly,

$$A^{(m)} = \left\{ \left(x, \mu_A^{(1)}(x), \mu_A^{(2)}(x), \ldots, \mu_A^{(m)}(x)\right) \,\middle|\, x \in U \right\},$$

with

$$\mu_A^{(k)} : U \to [0,1] \qquad (k = 1, 2, \ldots, m),$$

and

$$B = \{(r, \mu_B(r)) \mid r \in [0,1]\}, \qquad \mu_B : [0,1] \to [0,1].$$

The intended meaning is that $A^{(m)}$ describes the uncertain value of X through m polar viewpoints, while B expresses the reliability of the statement

$$X \text{ is } A^{(m)}.$$

Remark 3.9.3. If $m = 1$, then $A^{(1)}$ is an ordinary fuzzy set, and

$$Z^{(1)} = (A^{(1)}, B)$$

reduces to the classical Z-number.

3.10 Augmented Z-number

An Augmented Z-number extends a Z-number by adding time, context, and affective information, jointly representing uncertain values, reliability, situational meaning, and emotional perspective within statements [151].

Definition 3.10.1 (Augmented Z-number (or Z^*-number))**.** Let U be a universe of discourse for a subject variable X, and let $\mathcal{F}(U)$ denote the family of fuzzy subsets of U. Let

$$\Theta = \{\text{retrospective}, \text{present}, \text{prospective}\}$$

be the set of temporal modes, and let $\mathcal{C}$ be a family of admissible contexts.

Let $\mathcal{E}$ be a finite set of affect labels, let $S \subseteq [0, 1]$ be an affect-strength scale, and let

$$\mathcal{V} = \{+, -\}$$

be the set of affect valences.

An *Augmented Z-number* (briefly, a Z^**-number*) associated with a natural-language statement Y on the subject X is an ordered 5-tuple

$$Z^* = (T, C, A, B, AG),$$

where

$$T \in 2^{\Theta} \setminus \{\varnothing\}, \qquad C \in \mathcal{C}, \qquad A \in \mathcal{F}(U), \qquad B \in \mathcal{F}([0, 1]),$$

and

$$AG \subseteq \mathcal{E} \times S \times \mathcal{V}$$

is a finite set.

The components are interpreted as follows:

1. T is the time component, representing the relevant temporal perspective(s), such as retrospective, present, and/or prospective;
2. C is the context in which the statement Y is used or interpreted;
3. A is the instantiated value of X, equivalently a fuzzy restriction on the possible values of X, determined relative to T and C;
4. B is a fuzzy measure of the reliability, certainty, or confidence of the assessment A, relative to X, T, and C;
5. AG is the affect group, namely a finite set of ordered triples

   $$(e, s, v) \in \mathcal{E} \times S \times \mathcal{V},$$

   where e is an affect, s is its strength, and v is its valence.

Thus, an Augmented Z-number extends a classical Z-number by enriching the restriction–reliability pair (A, B) with temporal, contextual, and affective information.

3.11 Z-MultiNumbers

A Z-MultiNumbers model represents one fuzzy value restriction together with multiple weighted reliability components, integrating several confidence aspects into a single uncertain assessment framework.

Definition 3.11.1 (Z-MultiNumbers)**.** Let X be a nonempty universe of discourse, and let

$$\mathcal{F}(X) = \{A \mid A : X \to [0,1]\}$$

denote the family of all fuzzy sets on X.

For a fixed integer $m \geq 1$, let

$$R_i = [0,1] \qquad (i = 1, \dots, m)$$

be reliability domains, and let

$$c_i : \mathcal{F}(R_i) \to [0,1]$$

be prescribed reliability scoring functionals.

A *Z-MultiNumbers* object is a tuple

$$\mathcal{Z} = \big(A; B_1, \dots, B_m; w_1, \dots, w_m\big),$$

where

1. $A \in \mathcal{F}(X)$ is the value-restriction component;
2. $B_i \in \mathcal{F}(R_i)$ for each $i = 1, \dots, m$ is the i-th reliability component;
3. $w_i \geq 0$ for all i, and
$$\sum_{i=1}^{m} w_i = 1.$$

The *aggregated reliability degree* of $\mathcal{Z}$ is defined by

$$\rho(\mathcal{Z}) = \sum_{i=1}^{m} w_i \, c_i(B_i).$$

Let $T : [0,1]^2 \to [0,1]$ be a chosen triangular norm. The *effective value membership function* induced by $\mathcal{Z}$ is

$$\mu_{\mathcal{Z}}(x) = T\big(\mu_A(x),\, \rho(\mathcal{Z})\big), \qquad x \in X.$$

If $m = 1$ and $w_1 = 1$, then

$$\mathcal{Z} = (A; B_1; 1)$$

reduces to the ordinary Z-number (A, B_1).

3.12 Multidimensional Z-number

A Multidimensional Z-number extends a classical Z-number by combining multiple fuzzy restriction components with one fuzzy reliability component, modeling uncertainty across several correlated attributes simultaneously [152–154].

Definition 3.12.1 (Multidimensional Z-number)**.** [152–154] Let $n \in \mathbb{N}$, and for each $i \in \{1, \dots, n\}$ let U_i be a nonempty universe corresponding to the i-th attribute (or dimension). Let

$$\mathcal{F}(U_i)$$

denote the family of fuzzy subsets of U_i, and let

$$\mathcal{F}([0, 1])$$

denote the family of fuzzy subsets of the unit interval.

A *multidimensional Z-number* (briefly, *MZ-number*) is an ordered pair

$$\mathcal{Z}^{(n)} = (G, B), \qquad G = (A_1, A_2, \dots, A_n),$$

where

$$A_i \in \mathcal{F}(U_i) \quad (i = 1, 2, \dots, n), \qquad B \in \mathcal{F}([0, 1]).$$

The tuple

$$G = (A_1, A_2, \dots, A_n)$$

is called the *n-dimensional restriction vector*, where each A_i represents a fuzzy restriction on the i-th component of an uncertain object, variable, or evaluation. The fuzzy set B represents the reliability, certainty, or credibility assigned to the whole multidimensional restriction vector G.

Hence, a multidimensional Z-number may be written as

$$\mathcal{Z}^{(n)} = ((A_1, A_2, \dots, A_n), B).$$

Remark 3.12.2. If $n = 1$, then

$$\mathcal{Z}^{(1)} = ((A_1), B)$$

is naturally identified with the ordinary Z-number

$$(A_1, B).$$

Thus, the multidimensional Z-number is a direct extension of the classical Z-number.

3.13 Probabilistic Z-Number

Probabilistic Z-Number models an uncertain value by a probability distribution and attaches another probability distribution describing the reliability of that value assessment under uncertainty explicitly.

Definition 3.13.1 (Probabilistic Z-number (or ZP-number))**.** Let $(U,\mathcal{U})$ be a measurable space, and let $\mathcal{B}([0,1])$ be the Borel σ-algebra on $[0,1]$.

A *Probabilistic Z-number* is an ordered pair

$$\mathcal{Z}^P = (P_X, P_R),$$

where

1. P_X is a probability measure on $(U,\mathcal{U})$, called the *value law*;
2. P_R is a probability measure on $([0,1],\mathcal{B}([0,1]))$, called the *reliability law*.

The *mean reliability* induced by $\mathcal{Z}^P$ is defined by

$$\rho(\mathcal{Z}^P) := \int_{[0,1]} r\, dP_R(r).$$

Since $0 \le r \le 1$ on $[0,1]$, one has

$$0 \le \rho(\mathcal{Z}^P) \le 1.$$

The *reliability-discounted value measure* associated with $\mathcal{Z}^P$ is the set function

$$Q_{\mathcal{Z}^P} : \mathcal{U} \to [0,1]$$

defined by

$$Q_{\mathcal{Z}^P}(E) = \rho(\mathcal{Z}^P)\, P_X(E), \qquad E \in \mathcal{U}.$$

Proposition 3.13.2. *For every Probabilistic Z-number $\mathcal{Z}^P = (P_X, P_R)$, the mapping*

$$Q_{\mathcal{Z}^P} : \mathcal{U} \to [0,1]$$

is a subprobability measure on $(U,\mathcal{U})$.

Proof. First,

$$Q_{\mathcal{Z}^P}(\varnothing) = \rho(\mathcal{Z}^P)P_X(\varnothing) = 0.$$

Next, if $\{E_n\}_{n\ge1}$ is a pairwise disjoint sequence in $\mathcal{U}$, then

$$Q_{\mathcal{Z}^P}\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \rho(\mathcal{Z}^P)P_X\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \rho(\mathcal{Z}^P)\sum_{n=1}^{\infty} P_X(E_n) = \sum_{n=1}^{\infty} Q_{\mathcal{Z}^P}(E_n).$$

Thus $Q_{\mathcal{Z}^P}$ is countably additive. Finally,

$$Q_{\mathcal{Z}^P}(U) = \rho(\mathcal{Z}^P)P_X(U) = \rho(\mathcal{Z}^P) \le 1.$$

Hence $Q_{\mathcal{Z}^P}$ is a subprobability measure. □

3.14 p-box Z-Number

p-box Z-Number represents an uncertain value and its reliability by lower and upper probability distributions, capturing imprecise probabilistic restrictions on both informational layers simultaneously.

Definition 3.14.1 (p-box)**.** A *probability box* (briefly, *p-box*) on $\mathbb{R}$ is a pair

$$\mathcal{P} = (\underline{F}, \overline{F}),$$

where $\underline{F}$ and $\overline{F}$ are cumulative distribution functions on $\mathbb{R}$ satisfying

$$\underline{F}(x) \leq \overline{F}(x) \qquad \text{for all } x \in \mathbb{R}.$$

The *credal family* generated by $\mathcal{P}$ is

$$\mathfrak{K}(\mathcal{P}) := \left\{ F \;\middle|\; F \text{ is a cumulative distribution function and } \underline{F}(x) \leq F(x) \leq \overline{F}(x)\ \forall x \in \mathbb{R} \right\}.$$

Definition 3.14.2 (p-box Z-number (or ZPB-number))**.** Let $\mathcal{P}_X = (\underline{F}_X, \overline{F}_X)$ be a p-box on a value domain $U \subseteq \mathbb{R}$, and let

$$\mathcal{P}_R = (\underline{F}_R, \overline{F}_R)$$

be a p-box on the reliability domain $[0,1]$.

A *p-box Z-number* is an ordered pair

$$\mathcal{Z}^{PB} = (\mathcal{P}_X, \mathcal{P}_R).$$

The *lower* and *upper effective reliabilities* are defined by

$$\underline{\rho}(\mathcal{Z}^{PB}) := \inf_{F \in \mathfrak{K}(\mathcal{P}_R)} \int_{[0,1]} r \, dF(r),$$

and

$$\overline{\rho}(\mathcal{Z}^{PB}) := \sup_{F \in \mathfrak{K}(\mathcal{P}_R)} \int_{[0,1]} r \, dF(r).$$

Proposition 3.14.3. *Assume that*

$$\mathfrak{K}(\mathcal{P}_R) \neq \varnothing.$$

Then

$$0 \leq \underline{\rho}(\mathcal{Z}^{PB}) \leq \overline{\rho}(\mathcal{Z}^{PB}) \leq 1.$$

Hence the reliability of a p-box Z-number is always a well-defined interval in $[0,1]$.

Proof. For every

$$F \in \mathfrak{K}(\mathcal{P}_R),$$

the measure induced by F is supported on $[0,1]$. Therefore

$$0 \leq \int_{[0,1]} r \, dF(r) \leq 1.$$

Taking the infimum and supremum over the nonempty set

$$\mathfrak{K}(\mathcal{P}_R)$$

yields

$$0 \leq \underline{\rho}(\mathcal{Z}^{PB}) \leq \overline{\rho}(\mathcal{Z}^{PB}) \leq 1.$$

□

Chapter 4

Uncertain Z-Number

In this chapter, we investigate the concept of the Uncertain Z-Number.

4.1 Intuitionistic Fuzzy Z-Number

An Intuitionistic Fuzzy Z-Number combines intuitionistic fuzzy membership, nonmembership, and hesitation information with a reliability assessment, representing uncertain evaluations together with confidence in them simultaneously [155–157]. Related concepts are also known, including the Hesitant Fuzzy Z-Number [158], the Picture Fuzzy Z-Number [159,160], and the Spherical Fuzzy Z-Number [161,162].

Definition 4.1.1 (Intuitionistic Fuzzy Z-Number)**.** [155–157] Let $U \subseteq \mathbb{R}$ be a universe of discourse, and let $\mathrm{IF}(U)$ denote the family of intuitionistic fuzzy subsets of U. Let X be an uncertain variable taking values in U.

An *Intuitionistic Fuzzy Z-Number* (briefly, *IFZN* or *IZN*) associated with X is an ordered pair

$$Z^I = (A^I, R^I),$$

where

$$A^I \in \mathrm{IF}(U), \qquad R^I \in \mathrm{IF}([0,1]).$$

More explicitly,

$$A^I = \{(x, \mu_{A^I}(x), \nu_{A^I}(x)) : x \in U\}, \qquad R^I = \{(r, \mu_{R^I}(r), \nu_{R^I}(r)) : r \in [0,1]\},$$

with

$$\mu_{A^I} : U \to [0,1], \qquad \nu_{A^I} : U \to [0,1],$$
$$\mu_{R^I} : [0,1] \to [0,1], \qquad \nu_{R^I} : [0,1] \to [0,1],$$

satisfying

$$0 \le \mu_{A^I}(x) + \nu_{A^I}(x) \le 1 \quad \text{for all } x \in U,$$

and

$$0 \le \mu_{R^I}(r) + \nu_{R^I}(r) \le 1 \quad \text{for all } r \in [0,1].$$

Here:

1. A^I is an intuitionistic fuzzy restriction on the possible values of X;
2. R^I is an intuitionistic fuzzy reliability measure of the statement

$$X \text{ is } A^I.$$

Hence, an intuitionistic fuzzy Z-number refines a classical Z-number by equipping both the restriction component and the reliability component with membership and non-membership information, thereby also allowing an implicit hesitation degree.

4.2 Neutrosophic Z-Number

A Neutrosophic Z-number represents a neutrosophic assessment of a value together with a neutrosophic assessment of the reliability of that value, thereby capturing truth, indeterminacy, and falsity on both sides [163–165].

Definition 4.2.1 (Single-valued neutrosophic number)**.** [166–169] A *single-valued neutrosophic number* is an ordered triple

$$A = (T_A, I_A, F_A) \in [0, 1]^3$$

satisfying

$$0 \le T_A + I_A + F_A \le 3.$$

Here T_A, I_A, and F_A denote the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree, respectively.

Definition 4.2.2 (Neutrosophic Z-number)**.** A *Neutrosophic Z-number* is an ordered pair

$$Z = (A, B),$$

where

$$A = (T_A, I_A, F_A) \in [0, 1]^3, \qquad B = (T_B, I_B, F_B) \in [0, 1]^3$$

are single-valued neutrosophic numbers satisfying

$$0 \le T_A + I_A + F_A \le 3, \qquad 0 \le T_B + I_B + F_B \le 3.$$

The component A is called the *value component*, and B is called the *reliability component*.

4.3 Plithogenic Z-Number

A plithogenic Z-number enriches both the restriction and the reliability components of a Z-number by attaching attribute values and contradiction degrees.

Definition 4.3.1 (Plithogenic fuzzy restriction)**.** Let $U \subseteq \mathbb{R}$ be a universe of discourse. Let V be a finite nonempty set of attribute values, let

$$v^* \in V$$

be a fixed dominant attribute value, and let

$$c : V \times V \to [0, 1]$$

be a contradiction-degree function.

A *plithogenic fuzzy restriction* on U is a quadruple

$$A^P = (U, V, v^*, \mu_{A^P})$$

where

$$\mu_{A^P} : U \times V \to [0, 1]$$

assigns to each pair (x, v) a plithogenic appurtenance degree.

The *effective membership function* induced by A^P is defined by

$$\mu^{\text{eff}}_{A^P}(x) := \max_{v \in V} \Big(\mu_{A^P}(x, v)\big(1 - c(v, v^*)\big) \Big), \qquad x \in U.$$

Definition 4.3.2 (Plithogenic Z-Number)**.** Let $U \subseteq \mathbb{R}$. A *plithogenic Z-number* is an ordered pair

$$Z^P = (A^P, B^P),$$

where

1. A^P is a plithogenic fuzzy restriction on U;
2. B^P is a plithogenic fuzzy restriction on $[0, 1]$.

Thus, if

$$A^P = (U, V_A, a^*, \mu_{A^P}) \quad \text{and} \quad B^P = ([0, 1], V_B, b^*, \mu_{B^P}),$$

then the effective membership functions are

$$\mu^{\text{eff}}_{A^P}(x) = \max_{v \in V_A} \Big(\mu_{A^P}(x, v)\big(1 - c_A(v, a^*)\big) \Big), \qquad x \in U,$$

and

$$\mu^{\text{eff}}_{B^P}(r) = \max_{w \in V_B} \Big(\mu_{B^P}(r, w)\big(1 - c_B(w, b^*)\big) \Big), \qquad r \in [0, 1].$$

Theorem 4.3.3 (Well-definedness of Plithogenic Z-Numbers)**.** *Let*

$$Z^P = (A^P, B^P)$$

be a plithogenic Z-number. Then the pair

$$Z^P_{\text{eff}} := (A^P_{\text{eff}}, B^P_{\text{eff}})$$

induced by the effective membership functions

$$\mu^{\text{eff}}_{A^P} : U \to [0, 1], \qquad \mu^{\text{eff}}_{B^P} : [0, 1] \to [0, 1]$$

is well-defined. In particular, Z^P_{eff} is an ordinary Z-number.

Proof. Fix $x \in U$. For each $v \in V_A$,

$$0 \le \mu_{A^P}(x,v) \le 1 \quad \text{and} \quad 0 \le 1 - c_A(v,a^*) \le 1.$$

Hence

$$0 \le \mu_{A^P}(x,v)\big(1 - c_A(v,a^*)\big) \le 1.$$

Since V_A is finite, the maximum exists, and therefore

$$\mu^{\text{eff}}_{A^P}(x) \in [0,1].$$

Thus $\mu^{\text{eff}}_{A^P}$ is a fuzzy membership function on U.

Similarly, for each $r \in [0,1]$ and each $w \in V_B$,

$$0 \le \mu_{B^P}(r,w)\big(1 - c_B(w,b^*)\big) \le 1.$$

Since V_B is finite, the maximum exists, so

$$\mu^{\text{eff}}_{B^P}(r) \in [0,1].$$

Thus $\mu^{\text{eff}}_{B^P}$ is a fuzzy membership function on $[0,1]$.

Therefore,

$$A^P_{\text{eff}} = \{(x, \mu^{\text{eff}}_{A^P}(x)) \mid x \in U\}$$

is a fuzzy set on U, and

$$B^P_{\text{eff}} = \{(r, \mu^{\text{eff}}_{B^P}(r)) \mid r \in [0,1]\}$$

is a fuzzy set on $[0,1]$. Hence

$$Z^P_{\text{eff}} = (A^P_{\text{eff}}, B^P_{\text{eff}})$$

is an ordinary Z-number. □

Remark 4.3.4. If all contradiction degrees are zero, then the plithogenic Z-number reduces to a usual Z-number through its effective form.

4.4 Uncertain Z-Number

An uncertain Z-number generalizes the classical Z-number by replacing both the restriction component and the reliability component with uncertain sets of a fixed type M.

Definition 4.4.1 (Uncertain Z-Number of type M)**.** Let M be an uncertain model with degree-domain

$$\text{Dom}(M) \subseteq [0,1]^k$$

for some integer $k \ge 1$, and let $U \subseteq \mathbb{R}$ be a nonempty universe of discourse. Let X be a real-valued uncertain variable taking values in U.

An *uncertain Z-number of type* M (briefly, an M*-Z-number* or *Uncertain Z-Number*) associated with X is an ordered pair

$$Z^M = (A^M, R^M),$$

where

$$A^M = (U, \alpha_M)$$

is an uncertain set of type M on U, and

$$R^M = ([0, 1], \beta_M)$$

is an uncertain set of type M on $[0, 1]$.

Equivalently,

$$\alpha_M : U \to \mathrm{Dom}(M), \qquad \beta_M : [0, 1] \to \mathrm{Dom}(M),$$

where:

1. A^M is the *uncertain restriction* on the possible values of X;
2. R^M is the *uncertain reliability* of the statement

$$X \text{ is } A^M.$$

Thus, an uncertain Z-number may be written in the form

$$Z^M = \big((U, \alpha_M), ([0, 1], \beta_M)\big).$$

Remark 4.4.2. If M is the fuzzy model, that is,

$$\mathrm{Dom}(M) = [0, 1],$$

then an uncertain Z-number of type M reduces to the ordinary Z-number. If M is the intuitionistic fuzzy model, neutrosophic model, or another uncertain model, then one obtains the corresponding intuitionistic fuzzy, neutrosophic, or other model-based Z-number.

4.5 Rough Z-Number

A rough Z-number models both the restriction and the reliability components of a Z-number by lower and upper fuzzy approximations [170–172].

Definition 4.5.1 (Rough Z-Number)**.** Let $U \subseteq \mathbb{R}$. A *rough Z-number* is an ordered pair

$$Z^R = \big((\underline{A}, \overline{A}), (\underline{B}, \overline{B})\big),$$

where

1. $\underline{A}$ and $\overline{A}$ are fuzzy sets on U;
2. $\underline{B}$ and $\overline{B}$ are fuzzy sets on $[0, 1]$;
3.

$$\mu_{\underline{A}}(x) \leq \mu_{\overline{A}}(x) \qquad \text{for all } x \in U;$$

4.
$$\mu_{\underline{B}}(r) \le \mu_{\overline{B}}(r) \qquad \text{for all } r \in [0,1].$$

Here, $(\underline{A}, \underline{B})$ is called the *lower Z-component*, and $(\overline{A}, \overline{B})$ is called the *upper Z-component.*

Theorem 4.5.2 (Well-definedness of Rough Z-Numbers)**.** *Let*

$$Z^R = \big((\underline{A}, \overline{A}), (\underline{B}, \overline{B})\big)$$

be a rough Z-number. Then the interval-valued mappings

$$I_A(x) := [\mu_{\underline{A}}(x), \mu_{\overline{A}}(x)] \qquad (x \in U)$$

and

$$I_B(r) := [\mu_{\underline{B}}(r), \mu_{\overline{B}}(r)] \qquad (r \in [0,1])$$

are well-defined. Moreover, both

$$(\underline{A}, \underline{B}) \quad \textit{and} \quad (\overline{A}, \overline{B})$$

are ordinary Z-numbers.

Proof. Since $\underline{A}$ and $\overline{A}$ are fuzzy sets on U, one has

$$0 \le \mu_{\underline{A}}(x) \le 1, \qquad 0 \le \mu_{\overline{A}}(x) \le 1$$

for all $x \in U$. By assumption,

$$\mu_{\underline{A}}(x) \le \mu_{\overline{A}}(x),$$

so

$$I_A(x) = [\mu_{\underline{A}}(x), \mu_{\overline{A}}(x)]$$

is a valid interval in $[0,1]$.

Similarly, since $\underline{B}$ and $\overline{B}$ are fuzzy sets on $[0,1]$,

$$0 \le \mu_{\underline{B}}(r) \le 1, \qquad 0 \le \mu_{\overline{B}}(r) \le 1$$

for all $r \in [0,1]$, and by assumption

$$\mu_{\underline{B}}(r) \le \mu_{\overline{B}}(r).$$

Hence

$$I_B(r) = [\mu_{\underline{B}}(r), \mu_{\overline{B}}(r)]$$

is also a valid interval in $[0,1]$.

Therefore both interval-valued mappings are well-defined. Since each of $\underline{A}, \overline{A}$ is a fuzzy set on U, and each of $\underline{B}, \overline{B}$ is a fuzzy set on $[0,1]$, it follows directly from the definition of a Z-number that

$$(\underline{A}, \underline{B}) \quad \text{and} \quad (\overline{A}, \overline{B})$$

are ordinary Z-numbers. □

Remark 4.5.3. If

$$\underline{A} = \overline{A} \quad \text{and} \quad \underline{B} = \overline{B},$$

then the rough Z-number reduces to an ordinary Z-number.

4.6 Soft Z-number

A soft Z-number is a parameterized family of Z-numbers, allowing different parameters to describe uncertain values and reliabilities from multiple viewpoints.

Definition 4.6.1 (Soft Z-number)**.** Let $U \subseteq \mathbb{R}$ be a universe of discourse, and let E be a finite nonempty set of parameters. Define

$$\mathcal{Z}(U) := \{(A, B) \mid A \text{ is a fuzzy set on } U,\ B \text{ is a fuzzy set on } [0,1]\}.$$

A *soft Z-number* over U with parameter set E is a pair

$$(F, E),$$

where

$$F : E \to \mathcal{Z}(U).$$

Equivalently, for each parameter $\epsilon \in E$,

$$F(\epsilon) = (A_\epsilon, B_\epsilon)$$

is an ordinary Z-number on U.

Theorem 4.6.2 (Well-definedness of Soft Z-Numbers)**.** *Let* (F, E) *be a soft Z-number over* U*, and let*

$$w : E \to [0,1]$$

be a weight function satisfying

$$\sum_{\epsilon \in E} w(\epsilon) = 1.$$

For each $\epsilon \in E$*, write*

$$F(\epsilon) = (A_\epsilon, B_\epsilon).$$

Define

$$\mu_{A_w}(x) := \sum_{\epsilon \in E} w(\epsilon)\mu_{A_\epsilon}(x), \qquad x \in U,$$

and

$$\mu_{B_w}(r) := \sum_{\epsilon \in E} w(\epsilon)\mu_{B_\epsilon}(r), \qquad r \in [0,1].$$

Then

$$Z_w := (A_w, B_w)$$

is a well-defined ordinary Z-number.

Proof. For each $\epsilon \in E$ and each $x \in U$,

$$0 \le \mu_{A_\epsilon}(x) \le 1.$$

Since $w(\epsilon) \ge 0$ and $\sum_{\epsilon \in E} w(\epsilon) = 1$, one obtains

$$0 \le \mu_{A_w}(x) = \sum_{\epsilon \in E} w(\epsilon)\mu_{A_\epsilon}(x) \le 1.$$

Hence $\mu_{A_w} : U \to [0,1]$ is a fuzzy membership function on U.

Similarly, for each $\epsilon \in E$ and each $r \in [0,1]$,

$$0 \le \mu_{B_\epsilon}(r) \le 1.$$

Therefore,

$$0 \le \mu_{B_w}(r) = \sum_{\epsilon \in E} w(\epsilon)\mu_{B_\epsilon}(r) \le 1,$$

so $\mu_{B_w} : [0,1] \to [0,1]$ is a fuzzy membership function on $[0,1]$.

Thus,

$$A_w = \{(x, \mu_{A_w}(x)) \mid x \in U\}$$

is a fuzzy set on U, and

$$B_w = \{(r, \mu_{B_w}(r)) \mid r \in [0,1]\}$$

is a fuzzy set on $[0,1]$. Hence

$$Z_w = (A_w, B_w)$$

is an ordinary Z-number. □

Remark 4.6.3. If $E = \{\epsilon_0\}$ is a singleton, then a soft Z-number reduces to the ordinary Z-number

$$F(\epsilon_0).$$

4.7 Hypersoft Z-Number

A Hypersoft Z-Number assigns a Z-number to each tuple of attribute values, thereby combining multi-attribute parameterization with restriction–reliability based uncertainty modeling.

Definition 4.7.1 (Hypersoft Z-Number)**.** Let $U \subseteq \mathbb{R}$ be a nonempty universe of discourse, and let

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$$

be nonempty attribute-value sets. Define the parameter space

$$\mathcal{C} := \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

Let

$$\mathcal{Z}(U) := \{(A, B) \mid A \text{ is a fuzzy set on } U,\ B \text{ is a fuzzy set on } [0,1]\}$$

denote the family of all Z-numbers on U.

A *Hypersoft Z-Number* over U is a pair

$$(H_Z, \mathcal{C}),$$

where

$$H_Z : \mathcal{C} \to \mathcal{Z}(U).$$

Equivalently, for each tuple

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m) \in \mathcal{C}, \qquad \gamma_i \in \mathcal{A}_i \quad (i = 1, 2, \dots, m),$$

one has

$$H_Z(\gamma) = (A_\gamma, B_\gamma),$$

where

$$A_\gamma$$

is a fuzzy restriction on U and

$$B_\gamma$$

is a fuzzy reliability measure on $[0, 1]$ associated with the statement

$$X \text{ is } A_\gamma.$$

Thus, a Hypersoft Z-Number may be written as

$$(H_Z, \mathcal{C}) = \{ (\gamma, (A_\gamma, B_\gamma)) \mid \gamma \in \mathcal{C} \}.$$

Remark 4.7.2. If $m = 1$, then $\mathcal{C} = \mathcal{A}_1$, and the above definition reduces to a soft Z-number indexed by a single parameter set.

Theorem 4.7.3 (Well-definedness of Hypersoft Z-Numbers)**.** *Let $U \subseteq \mathbb{R}$ be a nonempty universe of discourse, let*

$$\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m$$

be nonempty attribute-value sets, and define

$$\mathcal{C} := \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

Assume that for each $\gamma \in \mathcal{C}$ there are fuzzy sets

$$A_\gamma \text{ on } U \qquad \text{and} \qquad B_\gamma \text{ on } [0, 1].$$

Define

$$H_Z : \mathcal{C} \to \mathcal{Z}(U)$$

by

$$H_Z(\gamma) := (A_\gamma, B_\gamma),$$

where

$$\mathcal{Z}(U) := \{(A, B) \mid A \text{ is a fuzzy set on } U,\ B \text{ is a fuzzy set on } [0, 1]\}.$$

Then H_Z is well-defined. In particular, $(H_Z, \mathcal{C})$ is a Hypersoft Z-Number.

Proof. Fix any $\gamma \in \mathcal{C}$. By assumption, A_γ is a fuzzy set on U and B_γ is a fuzzy set on $[0, 1]$. Hence, by the definition of an ordinary Z-number, the ordered pair

$$(A_\gamma, B_\gamma)$$

is a Z-number on U. Therefore,

$$H_Z(\gamma) = (A_\gamma, B_\gamma) \in \mathcal{Z}(U).$$

Since $\gamma \in \mathcal{C}$ was arbitrary, the map

$$H_Z : \mathcal{C} \to \mathcal{Z}(U)$$

is well-defined. Consequently, the pair $(H_Z, \mathcal{C})$ is a Hypersoft Z-Number. □

Chapter 5

D-number Variants

In this chapter, we examine D-numbers and their extensions. A summary of D-number variants is presented in Table 5.1.

Table 5.1: Summary of D-number variants in Chapter 5.

Section	Concept	Brief description
5.1	D-Number	A mass assignment on subsets of a frame of discernment, allowing incompleteness and not requiring the elements of the frame to be mutually exclusive.
5.2	Linguistic D-number	A D-number whose masses, evaluations, or supporting information are expressed by linguistic terms rather than only precise numerical values.
5.3	Hyper D-Number	A generalized D-number that assigns a nonempty set of admissible mass values to each proposition instead of a single scalar mass.
5.4	SuperHyper D-Number	A hierarchical extension of the D-number that assigns hierarchical mass-values to hierarchical propositions through iterated set structures.
5.5	D-OffNumber	A D-number constructed with off-set or shifted value structures, allowing generalized mass representation beyond the standard unit-interval interpretation.
5.6	Multipolar D-Number	A D-number with multiple poles or aspects, enabling simultaneous evaluation from several distinct perspectives or criteria.

5.1 D-Number

A D-number is an evidence representation assigning masses to subsets of a nonexclusive frame, allowing incompleteness and uncertainty beyond Dempster-Shafer belief functions in complex decision settings [24, 173].

Definition 5.1.1 (D-Number)**.** Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be a finite nonempty set. In the theory of D-numbers, the elements of Θ are not required to be mutually exclusive.

A *D-number* on Θ is a set function

$$D : 2^{\Theta} \to [0, 1]$$

such that

$$D(\varnothing) = 0, \qquad \sum_{B \subseteq \Theta} D(B) \leq 1.$$

A subset $B \subseteq \Theta$ such that $D(B) > 0$ is called a *focal element* of D.

If

$$\sum_{B \subseteq \Theta} D(B) = 1,$$

then D is said to be *information-complete*. Otherwise, if

$$\sum_{B \subseteq \Theta} D(B) < 1,$$

then D is said to be *information-incomplete*.

5.2 Linguistic D-number

A Linguistic D-Number assigns belief masses to linguistic terms, representing incomplete qualitative assessments and uncertain information under a D-number framework without exclusiveness assumptions on sets [25].

Definition 5.2.1 (Linguistic D-number)**.** Let

$$S = \{s_0, s_1, \ldots, s_g\}$$

be a finite linguistic term set, usually endowed with a natural order

$$s_0 \prec s_1 \prec \cdots \prec s_g.$$

A *linguistic D-number* on S is a mapping

$$D_L : S \to [0, 1]$$

such that

$$\sum_{s \in S} D_L(s) \leq 1.$$

Equivalently, a linguistic D-number can be represented as a finite set

$$D_L = \{(s_i, \alpha_i) \mid s_i \in S,\ \alpha_i > 0,\ i = 1, 2, \ldots, m\},$$

where

$$\alpha_i = D_L(s_i), \qquad \sum_{i=1}^{m} \alpha_i \leq 1.$$

Here, each s_i is a linguistic term and α_i denotes the belief degree (or confidence degree) assigned to s_i.

If

$$\sum_{s \in S} D_L(s) = 1,$$

then D_L is said to be *information-complete*; otherwise, if

$$\sum_{s \in S} D_L(s) < 1,$$

then D_L is said to be *information-incomplete*.

5.3 Hyper D-Number

A Hyper D-number assigns to each proposition a nonempty set of admissible belief masses in the unit interval, thereby representing set-valued uncertainty and possibly incomplete information beyond classical D-number models.

Definition 5.3.1 (Hyper D-number)**.** Let $\Omega = \{F_1, \ldots, F_N\}$ be a finite nonempty frame of discernment, and let

$$\mathcal{P}^*([0,1]) = \{\, B \subseteq [0,1] \mid B \neq \emptyset \,\}$$

denote the family of all nonempty subsets of the unit interval.

A *Hyper D-number* on Ω is a mapping

$$D_H : 2^{\Omega} \longrightarrow \mathcal{P}([0,1])$$

satisfying the following conditions:

(i) **Null-set condition:**

$$D_H(\emptyset) = \emptyset.$$

(ii) **Nonemptiness on nonempty propositions:** for every $A \subseteq \Omega$ with $A \neq \emptyset$,

$$D_H(A) \in \mathcal{P}^*([0,1]).$$

(iii) **Total-mass condition:**

$$\sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} \sup D_H(A) \leq 1.$$

Here, for each nonempty $A \subseteq \Omega$, the quantity $\sup D_H(A)$ denotes the supremum of the nonempty bounded set $D_H(A) \subseteq [0,1]$.

Theorem 5.3.2 (Well-definedness of Hyper D-numbers)**.** *Let $\Omega = \{F_1, \ldots, F_N\}$ be a finite nonempty frame of discernment, and let D_H be a Hyper D-number on Ω. Then the following hold:*

(i) For every nonempty $A \subseteq \Omega$, the value $\sup D_H(A)$ is well-defined and satisfies

$$\sup D_H(A) \in [0,1].$$

(ii) The sum

$$\sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} \sup D_H(A)$$

is a finite real number.

(iii) Consequently, the total-mass condition in the definition of a Hyper D-number is mathematically meaningful, and the notion of Hyper D-number is well-defined.

Proof. Let $A \subseteq \Omega$ be nonempty. By condition (ii) in the definition, $D_H(A)$ is a nonempty subset of $[0,1]$. Hence $D_H(A)$ is bounded above by 1. Since every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum, the completeness property of $\mathbb{R}$ implies that $\sup D_H(A)$ exists. Moreover, because $D_H(A) \subseteq [0,1]$, we have

$$0 \leq \sup D_H(A) \leq 1,$$

and therefore

$$\sup D_H(A) \in [0,1].$$

This proves (i).

Next, since Ω is finite with $|\Omega| = N$, its power set 2^Ω contains exactly 2^N subsets, and thus exactly $2^N - 1$ nonempty subsets. Therefore the index set

$$\{A \subseteq \Omega \mid A \neq \emptyset\}$$

is finite. By part (i), each term $\sup D_H(A)$ is a real number in $[0,1]$. Hence

$$\sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} \sup D_H(A)$$

is a finite sum of real numbers, and so it is itself a well-defined real number. This proves (ii).

Finally, because each term in the total-mass condition is well-defined and the index set is finite, the inequality

$$\sum_{\substack{A \subseteq \Omega \\ A \neq \emptyset}} \sup D_H(A) \leq 1$$

is mathematically meaningful. Therefore the definition of a Hyper D-number is internally consistent and well-defined. This proves (iii). $\square$

Remark 5.3.3. The above correction is essential. If one allows $D_H(A) = \emptyset$ for some nonempty proposition A, then $\sup D_H(A)$ is not defined in the usual sense, and the total-mass condition becomes ill-posed. Condition (ii) removes precisely this difficulty.

5.4 SuperHyper D-Number

A SuperHyper D-number assigns hierarchical mass-values to hierarchical propositions, thereby modeling multilevel, possibly incomplete, and nonexclusive evidence in a unified framework.

Definition 5.4.1 (Nonempty iterated powersets)**.** For a nonempty set X, define

$$\mathcal{P}^*(X) := \{A \subseteq X \mid A \neq \emptyset\}.$$

Recursively, for each $k \in \mathbb{N}_0$,

$$\mathcal{P}^{*(0)}(X) := X, \qquad \mathcal{P}^{*(k)}(X) := \mathcal{P}^*\big(\mathcal{P}^{*(k-1)}(X)\big) \quad (k \geq 1).$$

Definition 5.4.2 (Null tower)**.** Let X be a set and let $e \in X$. Define recursively

$$e^{\langle 0\rangle} := e, \qquad e^{\langle k\rangle} := \{e^{\langle k-1\rangle}\} \quad (k \geq 1).$$

In particular, for the D-number setting,

$$\emptyset^{\langle 0\rangle} := \emptyset \in 2^{\Omega}, \qquad 0^{\langle 0\rangle} := 0 \in [0,1],$$

and

$$\emptyset^{\langle m\rangle} \in \mathcal{P}^{*(m)}(2^{\Omega}), \qquad 0^{\langle n\rangle} \in \mathcal{P}^{*(n)}([0,1]).$$

Definition 5.4.3 (Deep supremum)**.** For each $n \in \mathbb{N}_0$, define recursively

$$\mathrm{Sup}^{(n)} : \mathcal{P}^{*(n)}([0,1]) \to [0,1]$$

by

$$\mathrm{Sup}^{(0)}(r) := r \qquad (r \in [0,1]),$$

and for $n \geq 1$,

$$\mathrm{Sup}^{(n)}(A) := \sup\big\{\mathrm{Sup}^{(n-1)}(B) \mid B \in A\big\}, \qquad A \in \mathcal{P}^{*(n)}([0,1]).$$

Definition 5.4.4 ((m,n)-SuperHyper D-number)**.** Let

$$\Omega = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

be a finite nonempty frame of discernment, where the elements of Ω are not required to be mutually exclusive. Let

$$m, n \in \mathbb{N}_0.$$

An (m,n)-*SuperHyper D-number* on Ω is a mapping

$$D_{SH}^{(m,n)} : \mathcal{P}^{*(m)}(2^{\Omega}) \longrightarrow \mathcal{P}^{*(n)}([0,1])$$

such that

(i)

$$D_{SH}^{(m,n)}(\emptyset^{\langle m\rangle}) = 0^{\langle n\rangle};$$

(ii)

$$\sum_{\substack{B \in \mathcal{P}^{*(m)}(2^{\Omega}) \\ B \neq \emptyset^{\langle m\rangle}}} \mathrm{Sup}^{(n)}\big(D_{SH}^{(m,n)}(B)\big) \leq 1.$$

A hierarchical proposition

$$B \in \mathcal{P}^{*(m)}(2^{\Omega})$$

is called a *focal element* if

$$\operatorname{Sup}^{(n)}\big(D_{SH}^{(m,n)}(B)\big) > 0.$$

If

$$\sum_{\substack{B\in\mathcal{P}^{*(m)}(2^{\Omega})\\ B\neq\emptyset^{\langle m\rangle}}} \operatorname{Sup}^{(n)}\big(D_{SH}^{(m,n)}(B)\big) = 1,$$

then $D_{SH}^{(m,n)}$ is said to be *complete*; otherwise it is said to be *incomplete*.

Remark 5.4.5. The parameter m controls the hierarchical depth of the proposition side, while n controls the hierarchical depth of the mass-value side.

Remark 5.4.6. If $m = n = k$, then Definition 5.4.4 yields the *k-SuperHyper D-number*

$$D_{SH}^{(k,k)} : \mathcal{P}^{*(k)}(2^{\Omega}) \to \mathcal{P}^{*(k)}([0,1]).$$

In particular, when $(m,n) = (0,0)$, one recovers the ordinary D-number

$$D : 2^{\Omega} \to [0,1].$$

The comparison of D-number, Hyper D-number, n-SuperHyper D-number, and (m,n)-SuperHyper D-number is summarized in Table 5.2.

Table 5.2: A concise comparison of D-number, Hyper D-number, n-SuperHyper D-number, and (m,n)-SuperHyper D-number.

Concept	**Basic form**	**Hierarchy**	**Main feature**
D-number	$D : 2^{\Omega} \to [0,1]$	None	Assigns scalar masses to subsets of a possibly nonexclusive frame, allowing incompleteness.
Hyper D-number	$D_H : 2^{\Omega} \to \mathcal{P}^*([0,1])$	Value side only	Assigns nonempty set-valued masses instead of single scalar masses.
n-SuperHyper D-number	$D_{SH}^{(n,n)} : \mathcal{P}^{*(n)}(2^{\Omega}) \to \mathcal{P}^{*(n)}([0,1])$	Same depth n on both sides	Models hierarchical propositions and hierarchical mass-values with matched depth.
(m,n)-SuperHyper D-number	$D_{SH}^{(m,n)} : \mathcal{P}^{*(m)}(2^{\Omega}) \to \mathcal{P}^{*(n)}([0,1])$	Depth m on the proposition side, depth n on the mass side	Provides a two-parameter generalization that separates proposition hierarchy from mass hierarchy.

5.5 D-OffNumber

A D-OffNumber assigns offset-valued masses to propositions, permitting values below zero or above one while preserving bounded total mass under a generalized D-number framework setting [173].

Definition 5.5.1 (D-OffNumber)**.** [173] Let $\Psi, \Omega \in \mathbb{R}$ satisfy

$$\Psi < 0 < 1 < \Omega,$$

and let X be a finite nonempty set, called the frame of discernment. A *D-OffNumber* on X is a mapping

$$D_{\text{off}} : 2^X \to [\Psi, \Omega]$$

satisfying the following conditions:

1. $$D_{\text{off}}(\varnothing) = 0.$$

2. $$\sum_{B \subseteq X} D_{\text{off}}(B) \leq 1.$$

3. There exists at least one subset $B \subseteq X$ such that

$$D_{\text{off}}(B) > 1 \quad \text{or} \quad D_{\text{off}}(B) < 0.$$

The family of focal sets of D_{off} is defined by

$$\mathcal{F}(D_{\text{off}}) = \{\, B \subseteq X \mid D_{\text{off}}(B) \neq 0 \,\}.$$

Accordingly, D_{off} may be written in the equivalent form

$$D_{\text{off}} = \{(B, v_B) \mid B \in \mathcal{F}(D_{\text{off}}),\ v_B = D_{\text{off}}(B)\}\,,$$

where

$$\sum_{B \in \mathcal{F}(D_{\text{off}})} v_B \leq 1,$$

and

$$D_{\text{off}}(B) = 0 \qquad \text{for all } B \notin \mathcal{F}(D_{\text{off}}).$$

5.6 Multipolar D-Number

Multipolar D-Number assigns to each subset multiple D-number masses, one for each polarity, enabling parallel representation of incomplete, nonexclusive evidence across several evaluative dimensions jointly.

Definition 5.6.1 (Multipolar D-Number)**.** Let

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

be a finite nonempty set. As in the theory of D-numbers, the elements of Θ are not required to be mutually exclusive. Let $m \in \mathbb{N}$.

A *Multipolar D-Number of order* m on Θ is a mapping

$$D^{(m)} : 2^\Theta \to [0,1]^m, \qquad A \mapsto D^{(m)}(A) = \big(D_1(A), D_2(A), \ldots, D_m(A)\big),$$

such that, for every $k \in \{1, 2, \ldots, m\}$, the k-th component

$$D_k : 2^\Theta \to [0, 1]$$

satisfies

$$D_k(\emptyset) = 0, \qquad \sum_{A \subseteq \Theta} D_k(A) \leq 1.$$

Equivalently, a Multipolar D-Number may be viewed as an ordered m-tuple

$$D^{(m)} = (D_1, D_2, \ldots, D_m),$$

where each D_k is an ordinary D-number on Θ.

A subset $A \subseteq \Theta$ is called a *focal element* of $D^{(m)}$ if

$$D_k(A) > 0 \quad \text{for at least one } k \in \{1, 2, \ldots, m\}.$$

If

$$\sum_{A \subseteq \Theta} D_k(A) = 1 \qquad \text{for all } k = 1, 2, \ldots, m,$$

then $D^{(m)}$ is said to be *componentwise complete*. Otherwise, it is said to be *componentwise incomplete*.

Chapter 6

Uncertain D-Number

In this chapter, we present the concept of the Uncertain D-Number.

6.1 Intuitionistic Fuzzy D-Number

An intuitionistic fuzzy D-number assigns to each proposition an intuitionistic fuzzy value, combining support, rejection, and hesitation under the D-number mass-allocation principle.

Definition 6.1.1 (Intuitionistic Fuzzy D-Number)**.** Let Θ be a finite nonempty set, and let

$$\mathrm{IFV} := \{(\mu, \nu) \in [0,1]^2 \mid \mu + \nu \leq 1\}$$

denote the set of all intuitionistic fuzzy values.

An *intuitionistic fuzzy D-number* on Θ is a mapping

$$D^I : 2^\Theta \to \mathrm{IFV}, \qquad A \mapsto D^I(A) = \big(\mu_{D^I}(A), \nu_{D^I}(A)\big),$$

such that

1. $$D^I(\emptyset) = (0, 1);$$

2. for every $A \subseteq \Theta$,
$$0 \leq \mu_{D^I}(A) + \nu_{D^I}(A) \leq 1;$$

3. $$\sum_{\emptyset \neq A \subseteq \Theta} \mu_{D^I}(A) \leq 1.$$

For each $A \subseteq \Theta$, the quantity

$$\pi_{D^I}(A) := 1 - \mu_{D^I}(A) - \nu_{D^I}(A)$$

is called the *hesitation degree* of A. If

$$\sum_{\emptyset \neq A \subseteq \Theta} \mu_{D^I}(A) = 1,$$

then D^I is said to be *complete*; otherwise it is *incomplete*.

Theorem 6.1.2. *Let D^I be an intuitionistic fuzzy D-number on Θ. Define*

$$m_{D^I} : 2^\Theta \to [0, 1]$$

by

$$m_{D^I}(\emptyset) = 0, \qquad m_{D^I}(A) = \mu_{D^I}(A) \quad (\emptyset \neq A \subseteq \Theta).$$

Then m_{D^I} is a D-number on Θ.

Proof. By definition,

$$m_{D^I}(\emptyset) = 0.$$

Also, for every nonempty $A \subseteq \Theta$,

$$0 \leq \mu_{D^I}(A) \leq 1,$$

hence

$$m_{D^I}(A) \in [0, 1].$$

Finally,

$$\sum_{A \subseteq \Theta} m_{D^I}(A) = \sum_{\emptyset \neq A \subseteq \Theta} \mu_{D^I}(A) \leq 1.$$

Therefore, m_{D^I} satisfies all axioms of a D-number. □

Remark 6.1.3. Thus, an intuitionistic fuzzy D-number may be viewed as a D-number together with additional non-membership and hesitation information attached to each subset.

6.2 Neutrosophic D-Number

A neutrosophic D-number assigns truth, indeterminacy, and falsity degrees to each proposition, while preserving the D-number idea through bounded truth-mass allocation (cf. [25, 173]).

Definition 6.2.1 (Neutrosophic D-Number)**.** Let Θ be a finite nonempty set, and let

$$\mathrm{NSV} := \{(t, i, f) \in [0, 1]^3 \mid 0 \leq t + i + f \leq 3\}$$

denote the set of all neutrosophic values.

A *neutrosophic D-number* on Θ is a mapping

$$D^N : 2^\Theta \to \mathrm{NSV}, \qquad A \mapsto D^N(A) = \big(T_{D^N}(A), I_{D^N}(A), F_{D^N}(A)\big),$$

such that

1. $$D^N(\emptyset) = (0, 0, 1);$$

2. for every $A \subseteq \Theta$,
$$0 \le T_{D^N}(A),\ I_{D^N}(A),\ F_{D^N}(A) \le 1;$$

3. $$\sum_{\emptyset \ne A \subseteq \Theta} T_{D^N}(A) \le 1.$$

If
$$\sum_{\emptyset \ne A \subseteq \Theta} T_{D^N}(A) = 1,$$
then D^N is called *complete*; otherwise it is *incomplete*.

Theorem 6.2.2. *Let D^N be a neutrosophic D-number on Θ. Define*
$$m_{D^N} : 2^{\Theta} \to [0, 1]$$
by
$$m_{D^N}(\emptyset) = 0, \qquad m_{D^N}(A) = T_{D^N}(A) \quad (\emptyset \ne A \subseteq \Theta).$$
Then m_{D^N} is a D-number on Θ.

Proof. Clearly,
$$m_{D^N}(\emptyset) = 0.$$
For each nonempty $A \subseteq \Theta$, one has
$$0 \le T_{D^N}(A) \le 1,$$
so that
$$m_{D^N}(A) \in [0, 1].$$
Moreover,
$$\sum_{A \subseteq \Theta} m_{D^N}(A) = \sum_{\emptyset \ne A \subseteq \Theta} T_{D^N}(A) \le 1.$$
Hence m_{D^N} satisfies the defining conditions of a D-number. □

Remark 6.2.3. A neutrosophic D-number enriches each mass assignment by recording not only a truth degree, but also indeterminacy and falsity degrees.

6.3 Plithogenic D-Number

A plithogenic D-number assigns each proposition a support degree together with an attribute value, whose contradiction to a dominant value modulates the effective mass.

Definition 6.3.1 (Plithogenic D-Number)**.** Let Θ be a finite nonempty set. Let V be a finite nonempty set of attribute values, and let $v^* \in V$ be a fixed *dominant attribute value*. Assume that

$$c : V \times V \to [0,1]$$

is a contradiction-degree function satisfying

$$c(v,v) = 0 \qquad \text{for all } v \in V.$$

A *plithogenic D-number* on Θ (with respect to (V, v^*, c)) is a mapping

$$D^P : 2^\Theta \to [0,1] \times V, \qquad A \mapsto D^P(A) = \big(p_{D^P}(A),\, v_{D^P}(A)\big),$$

such that

1. $$D^P(\emptyset) = (0, v^*);$$
2. for every $A \subseteq \Theta$,
$$p_{D^P}(A) \in [0,1], \qquad v_{D^P}(A) \in V;$$
3. $$\sum_{\emptyset \neq A \subseteq \Theta} p_{D^P}(A) \leq 1.$$

The *effective plithogenic mass* induced by D^P is defined by

$$\widetilde{m}_{D^P}(\emptyset) = 0,$$

and, for every nonempty $A \subseteq \Theta$,

$$\widetilde{m}_{D^P}(A) = p_{D^P}(A)\big(1 - c(v_{D^P}(A), v^*)\big).$$

Theorem 6.3.2. *Let D^P be a plithogenic D-number on Θ. Then the induced map*

$$\widetilde{m}_{D^P} : 2^\Theta \to [0,1]$$

is a D-number on Θ.

Proof. By definition,

$$\widetilde{m}_{D^P}(\emptyset) = 0.$$

For every nonempty $A \subseteq \Theta$, since

$$0 \leq p_{D^P}(A) \leq 1 \quad \text{and} \quad 0 \leq 1 - c(v_{D^P}(A), v^*) \leq 1,$$

it follows that

$$0 \leq \widetilde{m}_{D^P}(A) \leq p_{D^P}(A) \leq 1.$$

Therefore,

$$\sum_{A \subseteq \Theta} \widetilde{m}_{D^P}(A) = \sum_{\emptyset \neq A \subseteq \Theta} p_{D^P}(A)\big(1 - c(v_{D^P}(A), v^*)\big) \leq \sum_{\emptyset \neq A \subseteq \Theta} p_{D^P}(A) \leq 1.$$

Hence $\widetilde{m}_{D^P}$ satisfies all axioms of a D-number. □

Remark 6.3.3. If

$$c(v_{D^P}(A), v^*) = 0 \quad \text{for all nonempty } A \subseteq \Theta,$$

then the plithogenic D-number reduces to an ordinary D-number through the identification

$$\widetilde{m}_{D^P}(A) = p_{D^P}(A).$$

6.4 Uncertain D-Number

An uncertain D-number extends the classical D-number by allowing each mass value to belong to a prescribed uncertainty model rather than being a single scalar in $[0, 1]$.

Definition 6.4.1 (Admissible mass semantics for an uncertain model)**.** Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Assume that a distinguished element

$$\mathbf{0}_M \in \mathrm{Dom}(M)$$

is fixed and interpreted as the *null degree* of the model.

A mapping

$$\sigma_M : \mathrm{Dom}(M) \to [0, 1]$$

is called an *admissible mass semantics map* for M if

$$\sigma_M(\mathbf{0}_M) = 0.$$

The value $\sigma_M(d)$ is interpreted as the effective scalar mass associated with the degree tuple $d \in \mathrm{Dom}(M)$.

Definition 6.4.2 (Uncertain D-Number of type M)**.** Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be a finite nonempty set. As in the theory of D-numbers, the elements of Θ are not required to be mutually exclusive.

Let M be an uncertain model with degree-domain

$$\mathrm{Dom}(M) \subseteq [0, 1]^k,$$

let

$$\mathbf{0}_M \in \mathrm{Dom}(M)$$

be a fixed null degree, and let

$$\sigma_M : \mathrm{Dom}(M) \to [0, 1]$$

be an admissible mass semantics map.

An *uncertain D-number of type M* (briefly, an *M-D-number* or *Uncertain D-Number*) on Θ is a mapping

$$D^M : 2^\Theta \to \mathrm{Dom}(M)$$

such that

$$D^M(\emptyset) = \mathbf{0}_M,$$

and

$$\sum_{\emptyset \neq A \subseteq \Theta} \sigma_M\big(D^M(A)\big) \leq 1.$$

A subset $A \subseteq \Theta$ is called a *focal element* of D^M if

$$\sigma_M\big(D^M(A)\big) > 0.$$

If

$$\sum_{\emptyset \neq A \subseteq \Theta} \sigma_M\big(D^M(A)\big) = 1,$$

then D^M is said to be *information-complete*; otherwise, if

$$\sum_{\emptyset \neq A \subseteq \Theta} \sigma_M\big(D^M(A)\big) < 1,$$

then D^M is said to be *information-incomplete*.

Remark 6.4.3. The admissible mass semantics map σ_M converts a model-specific degree tuple into an effective scalar mass. Typical examples include:

1. for the fuzzy model,
$$\sigma_M(d) = d;$$
2. for the intuitionistic fuzzy model with
$$d = (\mu, \nu),$$
one may take
$$\sigma_M(d) = \mu;$$
3. for the neutrosophic model with
$$d = (T, I, F),$$
one may take
$$\sigma_M(d) = T.$$

Accordingly, the uncertain D-number contains the ordinary D-number, intuitionistic fuzzy D-number, neutrosophic D-number, and many other variants as special cases.

6.5 Rough D-Number

A rough D-number represents each proposition by a lower and an upper D-mass, capturing certainty and possibility in the spirit of rough approximation.

Definition 6.5.1 (Rough D-Number)**.** Let Θ be a finite nonempty set, and let

$$\mathcal{D}(\Theta) := \left\{ D : 2^{\Theta} \to [0,1] \;\middle|\; D(\emptyset) = 0,\ \sum_{A \subseteq \Theta} D(A) \leq 1 \right\}$$

denote the family of all D-numbers on Θ.

A *rough D-number* on Θ is an ordered pair

$$D^R = (\underline{D}, \overline{D}) \in \mathcal{D}(\Theta) \times \mathcal{D}(\Theta)$$

such that

$$\underline{D}(A) \leq \overline{D}(A) \qquad \text{for every } A \subseteq \Theta.$$

Equivalently, a rough D-number may be represented by the interval-valued mapping

$$I_{D^R} : 2^{\Theta} \to \{[a,b] \subseteq [0,1] \mid a \leq b\}, \qquad I_{D^R}(A) := [\underline{D}(A), \overline{D}(A)].$$

Here, $\underline{D}(A)$ and $\overline{D}(A)$ are interpreted, respectively, as the *lower mass* and *upper mass* assigned to the proposition A.

Theorem 6.5.2. *Let $D^R = (\underline{D}, \overline{D})$ be a rough D-number on Θ. Then the interval-valued map*

$$I_{D^R}(A) = [\underline{D}(A), \overline{D}(A)] \qquad (A \subseteq \Theta)$$

is well-defined. Moreover, both $\underline{D}$ and $\overline{D}$ are D-numbers on Θ.

Proof. Since D^R is a rough D-number, one has

$$\underline{D}(A) \leq \overline{D}(A) \qquad \text{for every } A \subseteq \Theta.$$

Also, because $\underline{D}, \overline{D} \in \mathcal{D}(\Theta)$, it follows that

$$0 \leq \underline{D}(A) \leq 1, \qquad 0 \leq \overline{D}(A) \leq 1$$

for all $A \subseteq \Theta$. Hence

$$[\underline{D}(A), \overline{D}(A)] \subseteq [0,1]$$

is a valid closed interval, so $I_{D^R}(A)$ is well-defined.

Further, by the definition of $\mathcal{D}(\Theta)$,

$$\underline{D}(\emptyset) = 0, \qquad \sum_{A \subseteq \Theta} \underline{D}(A) \leq 1,$$

and similarly,

$$\overline{D}(\emptyset) = 0, \qquad \sum_{A \subseteq \Theta} \overline{D}(A) \leq 1.$$

Therefore both $\underline{D}$ and $\overline{D}$ satisfy the axioms of D-numbers. □

Remark 6.5.3. If

$$\underline{D}(A) = \overline{D}(A) \qquad \text{for all } A \subseteq \Theta,$$

then the rough D-number reduces to an ordinary D-number.

6.6 Soft D-number

A soft D-number is a parameterized family of D-numbers, allowing different parameters to describe the same frame of discernment from multiple viewpoints.

Definition 6.6.1 (Soft D-number)**.** Let Θ be a finite nonempty set, let E be a nonempty set of parameters, and let

$$\mathcal{D}(\Theta) := \left\{ D : 2^{\Theta} \to [0,1] \;\middle|\; D(\emptyset) = 0,\ \sum_{A \subseteq \Theta} D(A) \leq 1 \right\}$$

be the family of all D-numbers on Θ.

A *soft D-number* over Θ with parameter set E is a pair

$$(F, E),$$

where

$$F : E \to \mathcal{D}(\Theta).$$

Thus, for each parameter $\epsilon \in E$, the image

$$F(\epsilon) : 2^{\Theta} \to [0,1]$$

is a D-number on Θ. Equivalently, one may write

$$(F, E) = \{ (\epsilon, F(\epsilon)) \mid \epsilon \in E \}.$$

Theorem 6.6.2. *Let (F, E) be a soft D-number over Θ, and let*

$$w : E \to [0,1]$$

be a weight function satisfying

$$\sum_{\epsilon \in E} w(\epsilon) = 1.$$

Define

$$D_{F,w} : 2^{\Theta} \to [0,1]$$

by

$$D_{F,w}(A) := \sum_{\epsilon \in E} w(\epsilon)\, F(\epsilon)(A) \qquad (A \subseteq \Theta).$$

Then $D_{F,w}$ is a D-number on Θ.

Proof. For the empty set, since each $F(\epsilon)$ is a D-number,

$$F(\epsilon)(\emptyset) = 0 \qquad \text{for all } \epsilon \in E.$$

Therefore,

$$D_{F,w}(\emptyset) = \sum_{\epsilon \in E} w(\epsilon)\, F(\epsilon)(\emptyset) = 0.$$

Next, for every $A \subseteq \Theta$, one has

$$0 \le F(\epsilon)(A) \le 1 \qquad \text{for all } \epsilon \in E,$$

and $w(\epsilon) \ge 0$, hence

$$D_{F,w}(A) \ge 0.$$

Finally,

$$\sum_{A \subseteq \Theta} D_{F,w}(A) = \sum_{A \subseteq \Theta} \sum_{\epsilon \in E} w(\epsilon)\, F(\epsilon)(A).$$

Since all sums are finite, we may interchange the order of summation:

$$\sum_{A \subseteq \Theta} D_{F,w}(A) = \sum_{\epsilon \in E} w(\epsilon) \sum_{A \subseteq \Theta} F(\epsilon)(A).$$

Because each $F(\epsilon)$ is a D-number,

$$\sum_{A \subseteq \Theta} F(\epsilon)(A) \le 1.$$

Thus,

$$\sum_{A \subseteq \Theta} D_{F,w}(A) \le \sum_{\epsilon \in E} w(\epsilon) \cdot 1 = \sum_{\epsilon \in E} w(\epsilon) = 1.$$

Hence $D_{F,w}$ satisfies all axioms of a D-number. □

6.7 Hypersoft D-Number

A Hypersoft D-Number assigns a D-number to each tuple of attribute values, combining hypersoft parameterization with nonexclusive and possibly incomplete evidence representation.

Definition 6.7.1 (Hypersoft D-Number)**.** Let

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

be a finite nonempty frame of discernment, whose elements are not required to be mutually exclusive. Let

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$$

be nonempty attribute-value sets, and define

$$\mathcal{C} := \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

Let

$$\mathcal{D}(\Theta) := \left\{ D : 2^{\Theta} \to [0,1] \;\middle|\; D(\emptyset) = 0,\ \sum_{B \subseteq \Theta} D(B) \le 1 \right\}$$

denote the family of all D-numbers on Θ.

A *Hypersoft D-Number* over Θ is a pair

$$(H_D, \mathcal{C}),$$

where

$$H_D : \mathcal{C} \to \mathcal{D}(\Theta).$$

Equivalently, for each tuple

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}, \qquad \gamma_i \in \mathcal{A}_i \quad (i = 1, 2, \ldots, m),$$

the image

$$H_D(\gamma) = D_\gamma$$

is a D-number on Θ, that is,

$$D_\gamma : 2^\Theta \to [0,1], \qquad D_\gamma(\emptyset) = 0, \qquad \sum_{B \subseteq \Theta} D_\gamma(B) \le 1.$$

Hence, a Hypersoft D-Number may be expressed as

$$(H_D, \mathcal{C}) = \{\, (\gamma, D_\gamma) \mid \gamma \in \mathcal{C} \,\}.$$

Remark 6.7.2. If $m = 1$, then the Hypersoft D-Number reduces to a soft D-number indexed by a single parameter set.

Theorem 6.7.3 (Well-definedness of Hypersoft D-Numbers)**.** *Let*

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

be a finite nonempty frame of discernment, and let

$$\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$$

be nonempty attribute-value sets. Define

$$\mathcal{C} := \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

Assume that for each $\gamma \in \mathcal{C}$ there exists a mapping

$$D_\gamma : 2^\Theta \to [0,1]$$

such that

$$D_\gamma(\emptyset) = 0, \qquad \sum_{B \subseteq \Theta} D_\gamma(B) \le 1.$$

Define

$$H_D : \mathcal{C} \to \mathcal{D}(\Theta)$$

by

$$H_D(\gamma) := D_\gamma,$$

where

$$\mathcal{D}(\Theta) := \left\{ D : 2^\Theta \to [0,1] \;\middle|\; D(\emptyset) = 0, \ \sum_{B \subseteq \Theta} D(B) \le 1 \right\}.$$

Then H_D is well-defined. In particular, $(H_D, \mathcal{C})$ is a Hypersoft D-Number.

Proof. Fix any $\gamma \in \mathcal{C}$. By assumption, the mapping

$$D_\gamma : 2^\Theta \to [0,1]$$

satisfies

$$D_\gamma(\emptyset) = 0 \qquad \text{and} \qquad \sum_{B \subseteq \Theta} D_\gamma(B) \leq 1.$$

Thus D_γ satisfies the defining axioms of a D-number on Θ, and hence

$$D_\gamma \in \mathcal{D}(\Theta).$$

Since $\gamma \in \mathcal{C}$ was arbitrary, it follows that

$$H_D(\gamma) = D_\gamma \in \mathcal{D}(\Theta) \qquad \text{for all } \gamma \in \mathcal{C}.$$

Therefore the map

$$H_D : \mathcal{C} \to \mathcal{D}(\Theta)$$

is well-defined. Consequently, the pair $(H_D, \mathcal{C})$ is a Hypersoft D-Number. □

Chapter 7

New Concepts: ZD-Number, ZED-Number, and NZ-Number

7.1 ZD-Number

A ZD-number combines a D-number-valued assessment with a D-number-valued reliability description, jointly representing nonexclusive, incomplete uncertain values and their reliability in one two-layer framework.

Definition 7.1.1 (ZD-number)**.** Let Θ be a finite set of admissible values (or value labels) for an uncertain variable X. Let Λ be a finite set of reliability labels. Let

$$g : 2^{\Lambda} \setminus \{\emptyset\} \longrightarrow [0, 1]$$

be a prescribed *reliability semantics map*, assigning to each nonempty subset of reliability labels a numerical reliability degree.

A *ZD-number* on (Θ, Λ, g) is an ordered pair

$$\mathcal{Z} = (D_X, D_R),$$

where:

1. $D_X : 2^{\Theta} \to [0, 1]$ is a D-number on Θ, called the *value component*;
2. $D_R : 2^{\Lambda} \to [0, 1]$ is a D-number on Λ, called the *reliability component*.

The intended interpretation is as follows:

- D_X describes the restriction on the possible values of X, allowing incompleteness and non-exclusiveness;

- D_R describes the reliability of the assessment D_X, also allowing incompleteness and non-exclusiveness.

The *induced reliability degree* of $\mathcal{Z}$ is defined by

$$\rho(\mathcal{Z}) := \sum_{\emptyset \neq B \subseteq \Lambda} D_R(B)\, g(B).$$

Since $0 \leq g(B) \leq 1$ and $\sum_{B \subseteq \Lambda} D_R(B) \leq 1$, one has

$$0 \leq \rho(\mathcal{Z}) \leq 1.$$

The *reliability-discounted value component* associated with $\mathcal{Z}$ is the mapping

$$D_X^{\mathcal{Z}} : 2^{\Theta} \longrightarrow [0,1]$$

defined by

$$D_X^{\mathcal{Z}}(\emptyset) = 0,$$

and, for every nonempty $A \subseteq \Theta$,

$$D_X^{\mathcal{Z}}(A) = \rho(\mathcal{Z})\, D_X(A).$$

Its incompleteness degree is

$$\iota\big(D_X^{\mathcal{Z}}\big) = 1 - \sum_{\emptyset \neq A \subseteq \Theta} D_X^{\mathcal{Z}}(A) = 1 - \rho(\mathcal{Z}) \sum_{\emptyset \neq A \subseteq \Theta} D_X(A).$$

Hence, a ZD-number is a two-layer uncertainty object in which a D-number-valued assessment is equipped with a D-number-valued reliability description.

Example 7.1.2 (Medical triage assessment)**.** Consider an emergency-room physician who assesses the condition of a newly arrived patient. Let

$$\Theta = \{\text{mild}, \text{moderate}, \text{severe}\}$$

be the set of possible clinical severity labels, and let

$$\Lambda = \{\text{low}, \text{medium}, \text{high}\}$$

be the set of reliability labels.

Assume that the reliability semantics map is induced by the score function

$$s(\text{low}) = 0.3, \qquad s(\text{medium}) = 0.6, \qquad s(\text{high}) = 0.9,$$

through

$$g(B) = \frac{1}{|B|} \sum_{\lambda \in B} s(\lambda) \qquad (\emptyset \neq B \subseteq \Lambda).$$

Suppose that the physician's assessment of the patient's condition is modeled by the D-number

$$D_X(\{\text{mild}\}) = 0.20, \qquad D_X(\{\text{mild}, \text{moderate}\}) = 0.45, \qquad D_X(\{\text{moderate}\}) = 0.25,$$

and

$$D_X(A) = 0 \qquad \text{for all other } A \subseteq \Theta.$$

Here, the subset

$$\{\text{mild}, \text{moderate}\}$$

expresses that the patient's condition lies in an overlapping borderline region between mild and moderate severity. Note that

$$\sum_{A \subseteq \Theta} D_X(A) = 0.90,$$

so the value component is information-incomplete.

Assume further that the reliability of this clinical judgment is given by

$$D_R(\{\text{medium}\}) = 0.50, \qquad D_R(\{\text{medium}, \text{high}\}) = 0.30, \qquad D_R(\{\text{high}\}) = 0.10,$$

and

$$D_R(B) = 0 \qquad \text{for all other } B \subseteq \Lambda.$$

Then

$$\sum_{B \subseteq \Lambda} D_R(B) = 0.90,$$

so the reliability component is also information-incomplete.

The induced reliability degree is

$$\rho(\mathcal{Z}) = 0.50\, g(\{\text{medium}\}) + 0.30\, g(\{\text{medium}, \text{high}\}) + 0.10\, g(\{\text{high}\}).$$

Since

$$g(\{\text{medium}\}) = 0.6, \qquad g(\{\text{medium}, \text{high}\}) = \frac{0.6 + 0.9}{2} = 0.75, \qquad g(\{\text{high}\}) = 0.9,$$

we obtain

$$\rho(\mathcal{Z}) = 0.50(0.6) + 0.30(0.75) + 0.10(0.9) = 0.615.$$

Hence the reliability-discounted value component is

$$D_X^{\mathcal{Z}}(\{\text{mild}\}) = 0.615 \times 0.20 = 0.123,$$
$$D_X^{\mathcal{Z}}(\{\text{mild}, \text{moderate}\}) = 0.615 \times 0.45 = 0.27675,$$
$$D_X^{\mathcal{Z}}(\{\text{moderate}\}) = 0.615 \times 0.25 = 0.15375,$$

and

$$D_X^{\mathcal{Z}}(A) = 0 \qquad \text{for all other } A \subseteq \Theta.$$

Therefore, the ZD-number

$$\mathcal{Z} = (D_X, D_R)$$

describes both the physician's overlapping severity assessment and the uncertain reliability of that assessment in a single two-layer framework.

Example 7.1.3 (Supplier delivery-risk evaluation)**.** Consider a manufacturing company evaluating the delivery risk of a supplier for an urgent order. Let

$$\Theta = \{\text{low risk}, \text{medium risk}, \text{high risk}\}$$

be the set of admissible risk labels, and let

$$\Lambda = \{\text{poor}, \text{fair}, \text{strong}\}$$

be the set of reliability labels for the assessment.

Assume that the reliability semantics map is determined by

$$s(\text{poor}) = 0.2, \qquad s(\text{fair}) = 0.5, \qquad s(\text{strong}) = 0.85,$$

and

$$g(B) = \frac{1}{|B|} \sum_{\lambda \in B} s(\lambda) \qquad (\emptyset \neq B \subseteq \Lambda).$$

Suppose that, based on warehouse reports, transportation status, and past performance, the delivery-risk assessment is represented by the D-number

$$D_X(\{\text{medium risk}\}) = 0.30, \qquad D_X(\{\text{medium risk}, \text{high risk}\}) = 0.40, \qquad D_X(\{\text{high risk}\}) = 0.20,$$

and

$$D_X(A) = 0 \qquad \text{for all other } A \subseteq \Theta.$$

The subset

$$\{\text{medium risk}, \text{high risk}\}$$

indicates that the current evidence does not sharply distinguish between a medium-risk and a high-risk delivery scenario. Moreover,

$$\sum_{A \subseteq \Theta} D_X(A) = 0.90,$$

so the value component is incomplete.

Assume that the reliability of this risk evaluation is modeled by

$$D_R(\{\text{fair}\}) = 0.40, \qquad D_R(\{\text{fair}, \text{strong}\}) = 0.35, \qquad D_R(\{\text{strong}\}) = 0.15,$$

and

$$D_R(B) = 0 \qquad \text{for all other } B \subseteq \Lambda.$$

Then

$$\sum_{B \subseteq \Lambda} D_R(B) = 0.90.$$

The induced reliability degree is

$$\rho(\mathcal{Z}) = 0.40\, g(\{\text{fair}\}) + 0.35\, g(\{\text{fair}, \text{strong}\}) + 0.15\, g(\{\text{strong}\}).$$

Since

$$g(\{\text{fair}\}) = 0.5, \qquad g(\{\text{fair}, \text{strong}\}) = \frac{0.5 + 0.85}{2} = 0.675, \qquad g(\{\text{strong}\}) = 0.85,$$

we obtain

$$\rho(\mathcal{Z}) = 0.40(0.5) + 0.35(0.675) + 0.15(0.85) = 0.56375.$$

Therefore,

$$D_X^{\mathcal{Z}}(\{\text{medium risk}\}) = 0.56375 \times 0.30 = 0.169125,$$
$$D_X^{\mathcal{Z}}(\{\text{medium risk}, \text{high risk}\}) = 0.56375 \times 0.40 = 0.2255,$$
$$D_X^{\mathcal{Z}}(\{\text{high risk}\}) = 0.56375 \times 0.20 = 0.11275,$$

and

$$D_X^{\mathcal{Z}}(A) = 0 \qquad \text{for all other } A \subseteq \Theta.$$

Thus, the ZD-number framework simultaneously captures the supplier's delivery risk and the uncertain reliability of the available evidence, which is particularly useful in procurement and supply-chain decision-making.

Proposition 7.1.4. *For every ZD-number $\mathcal{Z} = (D_X, D_R)$, the mapping $D_X^{\mathcal{Z}}$ is a D-number on Θ.*

Proof. By definition,

$$D_X^{\mathcal{Z}}(\emptyset) = 0.$$

Also, for every nonempty $A \subseteq \Theta$,

$$D_X^{\mathcal{Z}}(A) = \rho(\mathcal{Z}) D_X(A) \geq 0.$$

Therefore,

$$\sum_{A \subseteq \Theta} D_X^{\mathcal{Z}}(A) = \sum_{\emptyset \neq A \subseteq \Theta} \rho(\mathcal{Z}) D_X(A) = \rho(\mathcal{Z}) \sum_{\emptyset \neq A \subseteq \Theta} D_X(A).$$

Since $0 \leq \rho(\mathcal{Z}) \leq 1$ and

$$\sum_{A \subseteq \Theta} D_X(A) \leq 1,$$

it follows that

$$\sum_{A \subseteq \Theta} D_X^{\mathcal{Z}}(A) \leq 1.$$

Thus $D_X^{\mathcal{Z}}$ satisfies all axioms of a D-number. □

Remark 7.1.5. A common special choice of the reliability semantics map is obtained from a score function $s : \Lambda \to [0, 1]$ by setting

$$g(B) = \frac{1}{|B|} \sum_{\lambda \in B} s(\lambda) \qquad (\emptyset \neq B \subseteq \Lambda).$$

In that case,

$$\rho(\mathcal{Z}) = \sum_{\emptyset \neq B \subseteq \Lambda} D_R(B) \left(\frac{1}{|B|} \sum_{\lambda \in B} s(\lambda) \right).$$

Remark 7.1.6. The above definition contains several important special cases:

1. If D_R is concentrated on a single reliability label $\lambda \in \Lambda$, that is,
$$D_R(\{\lambda\}) = 1,$$
then
$$\rho(\mathcal{Z}) = g(\{\lambda\}),$$
and the ZD-number reduces to a reliability-discounted D-number.

2. If both D_X and D_R are complete and concentrated on singleton subsets, then $\mathcal{Z}$ becomes a complete two-component assessment, which is a discrete analogue of a classical Z-number.

3. If the reliability component is ignored, then $\mathcal{Z}$ reduces to the ordinary D-number D_X.

A concise comparison of Z-Number, D-Number, and ZD-Number is presented in Table 7.1.

Table 7.1: A concise comparison of Z-Number, D-Number, and ZD-Number.

Concept	**Basic form**	**Main idea**	**Key feature**
Z-Number	$Z = (A, B)$	A is a fuzzy restriction on the value of an uncertain variable, and B is a fuzzy restriction on the reliability of the statement "X is A."	Jointly models an assessment and the confidence attached to that assessment.
D-Number	$D : 2^{\Theta} \to [0, 1]$	A mass assignment on subsets of a frame Θ, where the elements of Θ are not required to be mutually exclusive.	Allows incomplete information and relaxes the exclusiveness assumption of classical Dempster–Shafer theory.
ZD-Number	$\mathcal{Z} = (D_X, D_R)$	D_X is a D-number describing the value side, and D_R is a D-number describing the reliability side.	Combines the two-layer philosophy of Z-Numbers with the nonexclusive and incomplete evidence structure of D-Numbers.

7.2 ZED-number

ZED-number combines a D-number-valued assessment with subjective reliability and external credibility D-number layers, representing uncertain values, confidence, and credibility within one framework.

Definition 7.2.1 (ZED-number)**.** Let Θ be a finite set of admissible values (or value labels) for an uncertain variable X. Let Λ be a finite set of subjective reliability labels, and let
$$\Xi$$
be a finite set of external credibility labels.

Let
$$g : 2^{\Lambda} \setminus \{\emptyset\} \longrightarrow [0, 1]$$

be a prescribed *subjective reliability semantics map*, and let

$$h : 2^{\Xi} \setminus \{\emptyset\} \longrightarrow [0, 1]$$

be a prescribed *external credibility semantics map.*

A *ZED-number* on $(\Theta, \Lambda, \Xi, g, h)$ is a nested ordered pair

$$\mathcal{Z}_{\mathrm{ED}} = \big((D_X, D_R), D_E\big),$$

where:

1. $D_X : 2^{\Theta} \to [0, 1]$ is a D-number on Θ, called the *value component*;
2. $D_R : 2^{\Lambda} \to [0, 1]$ is a D-number on Λ, called the *subjective reliability component*;
3. $D_E : 2^{\Xi} \to [0, 1]$ is a D-number on Ξ, called the *external credibility component.*

The intended interpretation is as follows:

- D_X describes the restriction on the possible values of X, allowing incompleteness and non-exclusiveness;
- D_R describes the subjective reliability of the assessment D_X;
- D_E describes the external or objective credibility of the pair (D_X, D_R).

The *induced external credibility degree* of $\mathcal{Z}_{\mathrm{ED}}$ is defined by

$$\kappa(\mathcal{Z}_{\mathrm{ED}}) := \sum_{\emptyset \neq C \subseteq \Xi} D_E(C)\, h(C).$$

The *externally discounted subjective reliability component* associated with $\mathcal{Z}_{\mathrm{ED}}$ is the mapping

$$D_R^{\mathcal{Z}_{\mathrm{ED}}} : 2^{\Lambda} \longrightarrow [0, 1]$$

defined by

$$D_R^{\mathcal{Z}_{\mathrm{ED}}}(\emptyset) = 0,$$

and, for every nonempty $B \subseteq \Lambda$,

$$D_R^{\mathcal{Z}_{\mathrm{ED}}}(B) = \kappa(\mathcal{Z}_{\mathrm{ED}})\, D_R(B).$$

The *induced total reliability degree* of $\mathcal{Z}_{\mathrm{ED}}$ is defined by

$$\rho(\mathcal{Z}_{\mathrm{ED}}) := \sum_{\emptyset \neq B \subseteq \Lambda} D_R^{\mathcal{Z}_{\mathrm{ED}}}(B)\, g(B).$$

Equivalently,

$$\rho(\mathcal{Z}_{\mathrm{ED}}) = \kappa(\mathcal{Z}_{\mathrm{ED}}) \sum_{\emptyset \neq B \subseteq \Lambda} D_R(B)\, g(B).$$

The *reliability- and credibility-discounted value component* associated with $\mathcal{Z}_{\mathrm{ED}}$ is the mapping

$$D_X^{\mathcal{Z}_{\mathrm{ED}}} : 2^{\Theta} \longrightarrow [0,1]$$

defined by

$$D_X^{\mathcal{Z}_{\mathrm{ED}}}(\emptyset) = 0,$$

and, for every nonempty $A \subseteq \Theta$,

$$D_X^{\mathcal{Z}_{\mathrm{ED}}}(A) = \rho(\mathcal{Z}_{\mathrm{ED}})\, D_X(A).$$

Its incompleteness degree is

$$\iota\Big(D_X^{\mathcal{Z}_{\mathrm{ED}}}\Big) = 1 - \sum_{\emptyset \neq A \subseteq \Theta} D_X^{\mathcal{Z}_{\mathrm{ED}}}(A) = 1 - \rho(\mathcal{Z}_{\mathrm{ED}}) \sum_{\emptyset \neq A \subseteq \Theta} D_X(A).$$

Hence, a ZED-number is a three-layer uncertainty object in which a D-number-valued assessment is equipped with both a D-number-valued subjective reliability description and a D-number-valued external credibility description.

Theorem 7.2.2 (Well-definedness of ZED-numbers)**.** *Let*

$$\mathcal{Z}_{\mathrm{ED}} = \big((D_X, D_R), D_E\big)$$

be a ZED-number on $(\Theta, \Lambda, \Xi, g, h)$*. Then the following statements hold:*

1. *the induced external credibility degree*

   $$\kappa(\mathcal{Z}_{\mathrm{ED}})$$

   is well-defined and satisfies

   $$0 \leq \kappa(\mathcal{Z}_{\mathrm{ED}}) \leq 1;$$

2. *the externally discounted subjective reliability component*

   $$D_R^{\mathcal{Z}_{\mathrm{ED}}}$$

   is a D-number on Λ*;*

3. *the induced total reliability degree*

   $$\rho(\mathcal{Z}_{\mathrm{ED}})$$

   is well-defined and satisfies

   $$0 \leq \rho(\mathcal{Z}_{\mathrm{ED}}) \leq 1;$$

4. *the reliability- and credibility-discounted value component*

$$D_X^{\mathcal{Z}_{\mathrm{ED}}}$$

is a D-number on Θ*.*

Proof. Since

$$0 \le h(C) \le 1 \qquad \text{for every nonempty } C \subseteq \Xi,$$

and D_E is a D-number on Ξ, we have

$$\kappa(\mathcal{Z}_{\mathrm{ED}}) = \sum_{\emptyset \ne C \subseteq \Xi} D_E(C)\, h(C) \ge 0.$$

Moreover,

$$\kappa(\mathcal{Z}_{\mathrm{ED}}) \le \sum_{\emptyset \ne C \subseteq \Xi} D_E(C) \le 1.$$

Therefore,

$$0 \le \kappa(\mathcal{Z}_{\mathrm{ED}}) \le 1,$$

so the first statement holds.

Next, by definition,

$$D_R^{\mathcal{Z}_{\mathrm{ED}}}(\emptyset) = 0.$$

For each nonempty $B \subseteq \Lambda$,

$$D_R^{\mathcal{Z}_{\mathrm{ED}}}(B) = \kappa(\mathcal{Z}_{\mathrm{ED}})\, D_R(B) \ge 0.$$

Also,

$$\sum_{B \subseteq \Lambda} D_R^{\mathcal{Z}_{\mathrm{ED}}}(B) = \sum_{\emptyset \ne B \subseteq \Lambda} \kappa(\mathcal{Z}_{\mathrm{ED}})\, D_R(B) = \kappa(\mathcal{Z}_{\mathrm{ED}}) \sum_{\emptyset \ne B \subseteq \Lambda} D_R(B).$$

Since

$$0 \le \kappa(\mathcal{Z}_{\mathrm{ED}}) \le 1 \quad \text{and} \quad \sum_{B \subseteq \Lambda} D_R(B) \le 1,$$

it follows that

$$\sum_{B \subseteq \Lambda} D_R^{\mathcal{Z}_{\mathrm{ED}}}(B) \le 1.$$

Thus

$$D_R^{\mathcal{Z}_{\mathrm{ED}}}$$

satisfies all axioms of a D-number on Λ, proving the second statement.

Now, since

$$0 \le g(B) \le 1 \qquad \text{for every nonempty } B \subseteq \Lambda,$$

and

$$D_R^{\mathcal{Z}_{\mathrm{ED}}}$$

is a D-number on Λ, we obtain

$$\rho(\mathcal{Z}_{\mathrm{ED}}) = \sum_{\emptyset \ne B \subseteq \Lambda} D_R^{\mathcal{Z}_{\mathrm{ED}}}(B)\, g(B) \ge 0,$$

and

$$\rho(\mathcal{Z}_{\mathrm{ED}}) \leq \sum_{\emptyset \neq B \subseteq \Lambda} D_R^{\mathcal{Z}_{\mathrm{ED}}}(B) \leq 1.$$

Hence

$$0 \leq \rho(\mathcal{Z}_{\mathrm{ED}}) \leq 1,$$

so the third statement holds.

Finally, by definition,

$$D_X^{\mathcal{Z}_{\mathrm{ED}}}(\emptyset) = 0.$$

For every nonempty $A \subseteq \Theta$,

$$D_X^{\mathcal{Z}_{\mathrm{ED}}}(A) = \rho(\mathcal{Z}_{\mathrm{ED}})\, D_X(A) \geq 0.$$

Also,

$$\sum_{A \subseteq \Theta} D_X^{\mathcal{Z}_{\mathrm{ED}}}(A) = \sum_{\emptyset \neq A \subseteq \Theta} \rho(\mathcal{Z}_{\mathrm{ED}})\, D_X(A) = \rho(\mathcal{Z}_{\mathrm{ED}}) \sum_{\emptyset \neq A \subseteq \Theta} D_X(A).$$

Since

$$0 \leq \rho(\mathcal{Z}_{\mathrm{ED}}) \leq 1 \quad \text{and} \quad \sum_{A \subseteq \Theta} D_X(A) \leq 1,$$

we conclude that

$$\sum_{A \subseteq \Theta} D_X^{\mathcal{Z}_{\mathrm{ED}}}(A) \leq 1.$$

Therefore

$$D_X^{\mathcal{Z}_{\mathrm{ED}}}$$

is a D-number on Θ. This proves the fourth statement. □

A concise comparison of ZD-number and ZED-number is presented in Table 7.2.

Table 7.2: A concise comparison of ZD-number and ZED-number.

Concept	**Basic structure**	**Uncertainty model**	**Main feature**
ZD-number	$Z = (D_X, D_R)$	D-number value + D-number reliability	Represents both the value component and its reliability component by D-numbers, thereby allowing incompleteness and nonexclusive evidence on both sides.
ZED-number	$Z = ((D_X, D_R), D_E)$	D-number value + D-number reliability + D-number external credibility	Extends the ZD-number by adding a third D-number component that models additional external evidential credibility or auxiliary assessment.

7.3 NZ-number

NZ-number is a negative-scale analogue of a Z-number, combining a fuzzy value restriction with a fuzzy unreliability component defined on the interval $[-1, 0]$.

Definition 7.3.1 (NZ-number)**.** Let $U \subseteq \mathbb{R}$ be a universe of discourse, and let

$$\mathcal{F}(U)$$

denote the family of all fuzzy subsets of U. Let X be a real-valued uncertain variable taking values in U.

An *NZ-number* associated with X is an ordered pair

$$\mathrm{NZ} = (A, N),$$

where

$$A \in \mathcal{F}(U), \qquad N \in \mathcal{F}([-1, 0]).$$

The components are interpreted as follows:

1. A is a fuzzy restriction on the possible values of X;
2. N is a fuzzy *negative-reliability* (or unreliability) component of the statement

   $$X \text{ is } A,$$

   taking its grades on the interval $[-1, 0]$ rather than on $[0, 1]$.

Thus, an NZ-number may be viewed as a negative-side analogue of a classical Z-number, in which the second component is modeled on $[-1, 0]$.

Define the affine normalization map

$$T_{-1,0} : [-1, 0] \to [0, 1], \qquad T_{-1,0}(t) := t + 1.$$

The *normalized reliability component* associated with N is the fuzzy set

$$\widehat{N} \in \mathcal{F}([0, 1])$$

whose membership function is given by

$$\mu_{\widehat{N}}(r) := \mu_N(r - 1), \qquad r \in [0, 1].$$

The *normalized Z-number* associated with $\mathrm{NZ} = (A, N)$ is then

$$Z_{\mathrm{NZ}} := (A, \widehat{N}).$$

If, in addition,

$$\int_{-1}^{0} \mu_N(t)\, dt > 0,$$

define the *negative reliability score* of NZ by

$$\eta(\mathrm{NZ}) := \frac{\int_{-1}^{0} t\,\mu_N(t)\,dt}{\int_{-1}^{0} \mu_N(t)\,dt},$$

and define the corresponding *effective reliability degree* by

$$\rho(\mathrm{NZ}) := 1 + \eta(\mathrm{NZ}).$$

The *discounted value component* induced by NZ is the fuzzy set

$$A^{\mathrm{NZ}}$$

on U with membership function

$$\mu_{A^{\mathrm{NZ}}}(x) := \rho(\mathrm{NZ})\,\mu_A(x), \qquad x \in U.$$

Theorem 7.3.2 (Well-definedness of NZ-numbers)**.** *Let*

$$\mathrm{NZ} = (A, N)$$

be an NZ-number. Then the following statements hold:

1. *the normalized reliability component*
$$\widehat{N}$$
is a well-defined fuzzy set on $[0, 1]$*;*
2. *the normalized pair*
$$Z_{\mathrm{NZ}} = (A, \widehat{N})$$
is a well-defined ordinary Z-number;
3. *if*
$$\int_{-1}^{0} \mu_N(t)\,dt > 0,$$
then the negative reliability score
$$\eta(\mathrm{NZ})$$
is well-defined and satisfies
$$-1 \le \eta(\mathrm{NZ}) \le 0;$$
4. *consequently, the effective reliability degree*
$$\rho(\mathrm{NZ}) = 1 + \eta(\mathrm{NZ})$$
is well-defined and satisfies
$$0 \le \rho(\mathrm{NZ}) \le 1;$$
5. *the discounted value component*
$$A^{\mathrm{NZ}}$$
is a well-defined fuzzy set on U*.*

Proof. First, for every

$$r \in [0,1],$$

one has

$$r - 1 \in [-1,0].$$

Since N is a fuzzy set on $[-1,0]$, its membership function satisfies

$$\mu_N(r-1) \in [0,1].$$

Hence

$$\mu_{\widehat{N}}(r) := \mu_N(r-1)$$

defines a mapping

$$\mu_{\widehat{N}} : [0,1] \to [0,1].$$

Therefore,

$$\widehat{N} = \{(r, \mu_{\widehat{N}}(r)) \mid r \in [0,1]\}$$

is a well-defined fuzzy set on $[0,1]$. This proves (1).

Since A is a fuzzy set on U and $\widehat{N}$ is a fuzzy set on $[0,1]$, the pair

$$Z_{\mathrm{NZ}} = (A, \widehat{N})$$

satisfies the defining requirements of an ordinary Z-number. Hence (2) holds.

Now assume that

$$\int_{-1}^{0} \mu_N(t)\, dt > 0.$$

Because

$$-1 \le t \le 0 \qquad \text{for all } t \in [-1,0],$$

and

$$\mu_N(t) \ge 0,$$

we obtain

$$-\int_{-1}^{0} \mu_N(t)\, dt \le \int_{-1}^{0} t\, \mu_N(t)\, dt \le 0.$$

Dividing by the positive quantity

$$\int_{-1}^{0} \mu_N(t)\, dt$$

yields

$$-1 \le \frac{\int_{-1}^{0} t\, \mu_N(t)\, dt}{\int_{-1}^{0} \mu_N(t)\, dt} \le 0.$$

That is,

$$-1 \le \eta(\mathrm{NZ}) \le 0.$$

So (3) holds.

Adding 1 to the above inequality gives

$$0 \le 1 + \eta(\mathrm{NZ}) \le 1,$$

that is,

$$0 \leq \rho(\mathrm{NZ}) \leq 1.$$

Hence (4) follows.

Finally, for every $x \in U$, since A is a fuzzy set on U,

$$0 \leq \mu_A(x) \leq 1.$$

Together with

$$0 \leq \rho(\mathrm{NZ}) \leq 1,$$

this implies

$$0 \leq \rho(\mathrm{NZ})\, \mu_A(x) \leq 1.$$

Therefore,

$$\mu_{A^{\mathrm{NZ}}}(x) = \rho(\mathrm{NZ})\, \mu_A(x)$$

defines a mapping

$$\mu_{A^{\mathrm{NZ}}} : U \to [0,1],$$

so

$$A^{\mathrm{NZ}} = \{(x, \mu_{A^{\mathrm{NZ}}}(x)) \mid x \in U\}$$

is a well-defined fuzzy set on U. This proves (5). □

Remark 7.3.3. The normalization map

$$T_{-1,0}(t) = t + 1$$

shows that an NZ-number is canonically associated with an ordinary Z-number. Thus, NZ-numbers may be regarded as a shifted negative-scale counterpart of Z-numbers.

7.4 Meta Z-Number (Z-Number of Z-Numbers)

A Meta Z-Number is a second-order Z-model in which one Z-number evaluates a value and another Z-number evaluates the reliability of that assessment itself explicitly. Let

$$\mathcal{F}_{\mathbb{R}}$$

denote the set of all fuzzy numbers on $\mathbb{R}$, and let

$$\mathcal{F}_{[0,1]}$$

denote the set of all fuzzy numbers on $[0,1]$.

Recall that an ordinary Z-number is an ordered pair

$$Z = (A, R) \in \mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]},$$

where A is a fuzzy restriction on the value of a real-valued variable and R is a fuzzy reliability restriction on that statement.

Definition 7.4.1 (Meta Z-number)**.** A *Meta Z-number* (or *Z-number of Z-numbers*) is an ordered pair

$$\mathfrak{Z} = (Z_V, Z_R),$$

where

$$Z_V = (A_V, R_V) \in \mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}$$

is an ordinary Z-number, and

$$Z_R = (A_R, R_R) \in \mathcal{F}_{[0,1]} \times \mathcal{F}_{[0,1]}$$

is also an ordinary Z-number on the reliability scale $[0, 1]$.

Equivalently, a Meta Z-number may be written as

$$\mathfrak{Z} = \big((A_V, R_V), (A_R, R_R)\big),$$

where

$$A_V \in \mathcal{F}_{\mathbb{R}}, \qquad R_V, A_R, R_R \in \mathcal{F}_{[0,1]}.$$

Here:

- $Z_V = (A_V, R_V)$ represents the first-level value assessment;
- $Z_R = (A_R, R_R)$ represents a second-level Z-type assessment of the reliability of Z_V.

Theorem 7.4.2 (Well-definedness of Meta Z-numbers)**.** *The class*

$$\mathcal{MZ} := \big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big) \times \big(\mathcal{F}_{[0,1]} \times \mathcal{F}_{[0,1]}\big)$$

is well-defined. Moreover, every element

$$\mathfrak{Z} \in \mathcal{MZ}$$

can be written uniquely in the form

$$\mathfrak{Z} = \big((A_V, R_V), (A_R, R_R)\big),$$

with

$$A_V \in \mathcal{F}_{\mathbb{R}}, \qquad R_V, A_R, R_R \in \mathcal{F}_{[0,1]}.$$

Consequently, the notion of a Meta Z-number is mathematically well-defined.

Proof. By definition, $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{[0,1]}$ are sets of fuzzy numbers. Hence their Cartesian products

$$\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]} \qquad \text{and} \qquad \mathcal{F}_{[0,1]} \times \mathcal{F}_{[0,1]}$$

are well-defined sets. Therefore the Cartesian product of these two sets,

$$\mathcal{MZ} = \big(\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]}\big) \times \big(\mathcal{F}_{[0,1]} \times \mathcal{F}_{[0,1]}\big),$$

is also a well-defined set.

Let

$$\mathfrak{Z} \in \mathcal{MZ}.$$

Then, by the definition of Cartesian product, there exist

$$Z_V \in \mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{[0,1]} \qquad \text{and} \qquad Z_R \in \mathcal{F}_{[0,1]} \times \mathcal{F}_{[0,1]}$$

such that

$$\mathfrak{Z} = (Z_V, Z_R).$$

Again by the definition of Cartesian product, there exist

$$A_V \in \mathcal{F}_{\mathbb{R}}, \quad R_V \in \mathcal{F}_{[0,1]}, \quad A_R \in \mathcal{F}_{[0,1]}, \quad R_R \in \mathcal{F}_{[0,1]}$$

such that

$$Z_V = (A_V, R_V), \qquad Z_R = (A_R, R_R).$$

Hence

$$\mathfrak{Z} = \big((A_V, R_V), (A_R, R_R)\big).$$

The uniqueness of this representation follows from the uniqueness of ordered-pair decomposition. Therefore every Meta Z-number has a unique and unambiguous mathematical form, and the definition is well-defined. □

Remark 7.4.3. A Meta Z-number is a second-order Z-model: the first Z-number assesses the value itself, while the second Z-number assesses the reliability layer of the first one.

7.5 Meta D-Number (D-Number of D-Numbers)

A Meta D-Number is a higher-order D-number defined on a finite family of D-numbers, assigning belief masses to sets of evidential assessments themselves. Recall that if $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$ is a finite nonempty frame of discernment, a *D-number* on Θ is a mapping

$$D : 2^{\Theta} \to [0,1]$$

such that

$$D(\emptyset) = 0, \qquad \sum_{A \subseteq \Theta} D(A) \le 1.$$

Let

$$\mathcal{D}(\Theta) := \left\{ D : 2^{\Theta} \to [0,1] \;\middle|\; D(\emptyset) = 0, \; \sum_{A \subseteq \Theta} D(A) \le 1 \right\}$$

denote the set of all D-numbers on Θ.

Definition 7.5.1 (Meta D-number)**.** Let

$$\Theta$$

be a finite nonempty frame of discernment, and let

$$\Xi = \{D_1, D_2, \ldots, D_m\} \subseteq \mathcal{D}(\Theta)$$

be a finite nonempty family of ordinary D-numbers on Θ.

A *Meta D-number* (or *D-number of D-numbers*) on Ξ is a mapping

$$\mathfrak{D} : 2^{\Xi} \to [0, 1]$$

such that

$$\mathfrak{D}(\emptyset) = 0, \qquad \sum_{B \subseteq \Xi} \mathfrak{D}(B) \leq 1.$$

A subset

$$B \subseteq \Xi$$

such that

$$\mathfrak{D}(B) > 0$$

is called a *meta-focal element.*

Theorem 7.5.2 (Well-definedness of Meta D-numbers)**.** *Let*

$$\Theta$$

be a finite nonempty frame of discernment, and let

$$\Xi = \{D_1, D_2, \ldots, D_m\} \subseteq \mathcal{D}(\Theta)$$

be a finite nonempty family of D-numbers on Θ*. Then the class*

$$\mathcal{MD}(\Xi) := \left\{ \mathfrak{D} : 2^{\Xi} \to [0, 1] \;\middle|\; \mathfrak{D}(\emptyset) = 0, \; \sum_{B \subseteq \Xi} \mathfrak{D}(B) \leq 1 \right\}$$

is mathematically well-defined. Consequently, the notion of a Meta D-number is well-defined.

Proof. Since Ξ is a finite nonempty set, its power set

$$2^{\Xi}$$

is also finite and well-defined. Therefore any mapping

$$\mathfrak{D} : 2^{\Xi} \to [0, 1]$$

is an ordinary set-theoretic function with finite domain.

For each subset

$$B \subseteq \Xi,$$

the value

$$\mathfrak{D}(B) \in [0, 1]$$

is a well-defined real number. Because the index set 2^{Ξ} is finite, the sum

$$\sum_{B \subseteq \Xi} \mathfrak{D}(B)$$

is a finite sum of real numbers and is therefore well-defined. Hence the constraints

$$\mathfrak{D}(\emptyset) = 0 \qquad \text{and} \qquad \sum_{B \subseteq \Xi} \mathfrak{D}(B) \leq 1$$

are mathematically meaningful.

Thus the collection

$$\mathcal{MD}(\Xi)$$

is a well-defined class of functions, and every element of it is a well-defined Meta D-number. □

Remark 7.5.3. A Meta D-number is a higher-level evidential allocation defined on a finite family of ordinary D-numbers. In this sense, its frame of discernment consists of D-numbers themselves.

Chapter 8

Conclusion

This book has presented a broad survey of concepts related to Z-Numbers and D-Numbers, with particular attention to their fundamental definitions, representative extensions, and related developments. Through this survey, we have aimed to clarify the roles of these frameworks in uncertainty modeling and to highlight their theoretical richness and potential for further generalization.

It is expected that future research will continue to expand the range of applications of these concepts in various fields. In particular, further developments in decision-making, information fusion, graph theory, topological structures, and other mathematical and applied domains may reveal new directions and deeper connections for Z-Number- and D-Number-based models.

Disclaimer

Funding

This study received no external funding and was conducted without financial support from any organization or grant.

Acknowledgments

The authors sincerely thank all those who offered valuable insights, support, and encouragement throughout the course of this research. They are also grateful to the readers for their interest and to the authors of the cited works, whose scholarly contributions have significantly informed this study. Finally, sincere appreciation is extended to the publishers and reviewers whose efforts supported the dissemination of this work.

Data Availability

This work is purely theoretical and mathematical in nature; therefore, no empirical data or computational datasets were used. Future studies may build upon these results through data-driven, computational, or experimental approaches.

Ethical Statement

This study did not involve human participants, animals, or personal data. Accordingly, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the content or publication of this book.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited support tasks, such as English grammar and language refinement. These tools were not used in any manner that would compromise academic integrity or violate ethical standards.

Appendix (List of Tables)

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Abstract

This book presents a systematic survey of the concepts of Z-Numbers, D-Numbers, and ZD-Numbers, which represent advanced mathematical frameworks for modeling uncertainty, imprecision, and incomplete information. Building upon classical uncertainty theories such as fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets, the work examines how Z-Numbers and D-Numbers extend traditional approaches by incorporating reliability assessments and flexible belief structures.

The book reviews a wide range of extensions and variants, including linguistic, probabilistic, multidimensional, hyper, and plithogenic Z-Number models, as well as numerous D-Number generalizations designed to address non-exclusive hypotheses and incomplete knowledge.

In addition to surveying existing developments, the book introduces the concept of the ZD-Number, which integrates the perspectives of Z-Numbers and D-Numbers into a unified representation capable of capturing both value–reliability relations and flexible evidence structures.

Through definitions, comparisons, and illustrative examples, the book clarifies the relationships among these models and highlights their relevance in areas such as decision-making, information fusion, artificial intelligence, and complex systems analysis. The resulting survey provides both a reference for researchers and a foundation for further theoretical and applied developments in uncertainty modeling.

ISBN 978-1-59973-882-6

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